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Sharp thresholds of blow-up and global existence for the Schrödinger equation with combined power-type and Choquard-type nonlinearities

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Abstract

In this paper, we consider the sharp thresholds of blow-up and global existence for the nonlinear Schrödinger–Choquard equation

$$i\psi_t + \Delta\psi = \lambda_1|\psi|^{p_1}\psi + \lambda_2(I_\alpha * |\psi|^{p_2})|\psi|^{p_2-2}\psi.$$

We derive some finite time blow-up results. Due to the failure of this equation to be scale invariant, we obtain some sharp thresholds of blow-up and global existence by constructing some new estimates. In particular, we prove the global existence for this equation with critical mass in the L^2-critical case. Our obtained results extend and improve some recent results.

MSC: 35Q55; 35A15

Keywords: Nonlinear Schrödinger–Choquard equation; Sharp thresholds; Blow-up

1 Introduction

In this paper, we study the sharp threshold of blow-up and global existence for the nonlinear Schrödinger–Choquard equation

$$\begin{cases} i\psi_t + \Delta\psi = \lambda_1|\psi|^{p_1}\psi + \lambda_2(I_\alpha * |\psi|^{p_2})|\psi|^{p_2-2}\psi, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.1}$$

where $\psi(t,x) : [0,T^*) \times \mathbb{R}^N \to \mathbb{C}$ and $0 < T^* \le \infty$, $N \ge 3$, $\psi_0 \in H^1$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $0 < p_1 < \frac{4}{N-2}$, $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{2+\alpha}{N-2}$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}},$$

where Γ is the Gamma function and $\max\{0, N-4\} < \alpha < N$.

When $\lambda_2 = 0$, Eq. (1.1) is the classical Schrödinger equation which appears in various areas of physics, such as nonlinear plasmas and nonlinear optics; see [2, 18]. This class of equations received a great deal of attention from mathematicians see [2, 18]. Particularly, from scaling invariance of (1.1) with $\lambda_2 = 0$, Weinstein [19] and Zhang [21] obtained the sharp threshold of blow-up and global existence for the L^2-critical nonlinearity and L^2-supercritical nonlinearity, respectively.

When $\lambda_1 = 0$, $0 < \alpha < N$ and $1 + \frac{\alpha}{N} < p_2 < \frac{N+\alpha}{N-2}$, under the assumption that the local well-posedness holds for (1.1), Chen and Guo [3] derived the existence of blow-up solutions and the instability of standing waves. When $0 < \alpha < N$ and $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{2+\alpha}{N}$, Squassina et al. in [1] studied the soliton dynamics of (1.1) under the assumption that the solution ψ of (1.1) is in $C([0,\infty), H^2) \cap C^1((0,\infty), L^2)$. The dynamical properties of blow-up solutions have been investigated in [11]. In [8], Feng and Yuan systematically studied the Cauchy problem (1.1) for general $\max\{0, N-4\} < \alpha < N$ and $2 \le p_2 < \frac{N+\alpha}{N-2}$. More precisely, they studied the local well-posedness, global existence, the existence of blow-up solutions and the dynamics of blow-up solutions. The sharp threshold of global existence and blow-up, the instability of standing wave of (1.1) with $\lambda_1 = 0$ and a harmonic potential have been investigated in [5].

From the local well-posedness of (1.1) with $\lambda_1 = 0$ or $\lambda_2 = 0$, for small initial data ψ_0, the solution $\psi(t)$ to (1.1) exists globally, and the solution $\psi(t)$ may blow up for some large initial data. Hence, whether there are some sharp thresholds of global existence and blow-up for (1.1) is a very interesting problem. In particular, the sharp thresholds of global existence and blow-up for nonlinear Schrödinger equations are pursued strongly in [2, 4, 6, 7, 9, 12–24]. However, in these papers, the scale invariance plays an important role in the study of the sharp threshold of blow-up and global existence. When $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, there is no any scaling invariance for Eq. (1.1). Therefore, the study of the sharp threshold of blow-up and global existence for (1.1) with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ is of particular interest.

To study this problem, we mainly use the idea of Zhang and Zhu [22], where they studied sharp criteria for the Davey–Stewartson system

$$i\psi_t + \Delta\psi = \lambda_1|\psi|^p\psi + \lambda_2 E(|\psi|^2)\psi. \tag{1.2}$$

Due to the failure of (1.1) to be scale invariant, motivated by the idea in [22], we must construct some new estimates to establish some sharp thresholds of blow-up and global existence for (1.1). We will derive sharp thresholds of blow-up and global existence for (1.1) in the following three cases: (i) $\lambda_1 < 0$ and $\lambda_2 < 0$; (ii) $\lambda_1 > 0$ and $\lambda_2 < 0$; (iii) $\lambda_1 < 0$ and $\lambda_2 > 0$. However, the authors in [22] only studied sharp criteria for (1.2) with $\lambda_1 < 0$ and $\lambda_2 < 0$. Therefore, we extend and improve these sharp thresholds for the Davey–Stewartson system to the Schrödinger–Choquard equation. In particular, we can prove the global existence for this equation with critical mass in the L^2-critical case.

This paper is organized as follows: in Sect. 2, we recall some preliminaries. In Sect. 3, we will derive some sufficient conditions on existence of blow-up solutions. In Sect. 4, we will derive some sharp thresholds of blow-up and global existence for (1.1) by constructing some new estimates. Section 5 is a concluding section.

2 Preliminaries

In order to study the sharp threshold of blow-up and global existence for (1.1), we first make the following assumption about the local well-posedness of (1.1).

Assumption 1 Let $\psi_0 \in H^1$, $0 < p_1 < \frac{4}{N-2}$ and $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{2+\alpha}{N-2}$ with $N \geq 3$. Then, there exist $T^* > 0$ and a unique maximal solution $u \in C([0,T^*),H^1)$. In addition, if $T^* < \infty$, then $\|\psi(t)\|_{H^1} \to \infty$ as $t \uparrow T^*$. Moreover, the solution $\psi(t)$ satisfies

$$\big\|\psi(t)\big\|_{L^2} = \|\psi_0\|_{L^2}, \tag{2.1}$$

$$E\big(\psi(t)\big) = E(\psi_0), \tag{2.2}$$

for all $0 \leq t < T^*$, where $E(\psi(t))$ is defined by

$$\begin{aligned} E\big(\psi(t)\big) :=& \frac{1}{2}\int_{\mathbb{R}^N} \big|\nabla\psi(t,x)\big|^2\,dx + \frac{\lambda_1}{p_1+2}\int_{\mathbb{R}^N}\big|\psi(t,x)\big|^{p_1+2}\,dx \\ &+ \frac{\lambda_2}{2p_2}\int_{\mathbb{R}^N}\big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx. \end{aligned} \tag{2.3}$$

Remark When $0 < p_1 < \frac{4}{N-2}$ and $2 \leq p_2 < 1 + \frac{2+\alpha}{N-2}$, this assumption can be easily proved by Strichartz's estimates and a fixed point argument; see [2, 8]. When $1 + \frac{\alpha}{N} < p_2 < 2$, we deduce from the Hardy–Littlewood–Sobolev inequality that $\int_{\mathbb{R}^N}(I_\alpha * |\psi|^{p_2})|\psi|^{p_2}\,dx$ is well-defined for $\psi \in H^1$. Thus, we assume that the local well-posedness of (1.1) holds for $\frac{N+\alpha}{N} < p_2 < 2$. However, we cannot prove this result since the nonlinearity $(I_\alpha * |\psi|^{p_2})|\psi|^{p_2-2}\psi$ is singular when $\frac{N+\alpha}{N} < p_2 < 2$. Consequently, the case of $\frac{N+\alpha}{N} < p_2 < 2$ will be the object of a future investigation.

By the same argument as that in [2], we can easily derive the following lemma.

Lemma 2.1 *Let $\psi_0 \in \Sigma := \{u \in H^1, xu \in L^2\}$, and the solution $\psi(t)$ to (1.1) exists on the interval $[0,T^*)$. Then, $\psi(t) \in \Sigma$ for all $t \in [0,T^*)$. Moreover, let $F(t) = \int_{\mathbb{R}^N}|x\psi(t,x)|^2\,dx$, then*

$$F'(t) = -4\,\mathrm{Im}\int_{\mathbb{R}^N}\psi(t,x)x\cdot\nabla\bar{\psi}(t,x)\,dx := -4h(t), \tag{2.4}$$

and

$$\begin{aligned} F''(t) &= -4h'(t) \\ &= 8\int_{\mathbb{R}^N}\big|\nabla\psi(t,x)\big|^2\,dx + \frac{4N\lambda_1 p_1}{p_1+2}\int_{\mathbb{R}^N}\big|\psi(t,x)\big|^{p_1+2}\,dx \\ &\quad + \lambda_2\frac{4p_2N - 4N - 4\alpha}{p_2}\int_{\mathbb{R}^N}\big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx. \end{aligned} \tag{2.5}$$

Finally, we recall two important Gagliardo–Nirenberg type inequalities; see [8, 19].

Lemma 2.2 ([19]) *Let Q be the ground state solution of the following elliptic equation*:

$$-\Delta Q + Q - |Q|^{p+2}Q = 0 \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

Then, the optimal constant in the Gagliardo–Nirenberg inequality,

$$\|\psi\|_{L^{p+2}}^{p+2} \leq C_*\|\psi\|_{L^2}^{p+2-\frac{Np}{2}}\|\nabla\psi\|_{L^2}^{\frac{Np}{2}}, \tag{2.7}$$

is

$$C_* = \frac{2(p+2)(2(p+2)-Np)^{\frac{Np-4}{4}}}{(Np)^{\frac{Np}{4}}\|Q\|_{L^2}^p}. \tag{2.8}$$

In particular, in the L^2*-critical case, i.e.,* $p = \frac{4}{N}$, $C_* = \frac{p+2}{2\|Q\|_{L^2}^p}$.

Lemma 2.3 ([8]) *Let R be the ground state solution of the following elliptic equation*:

$$-\Delta R + R - (I_\alpha * |R|^p)|R|^{p-2}R = 0 \quad \text{in } \mathbb{R}^N. \tag{2.9}$$

The best constant in the Gagliardo–Nirenberg type inequality

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi|^p)|\psi|^p \, dx \le C^* \|\nabla\psi\|_{L^2}^{Np-N-\alpha} \|\psi\|_{L^2}^{N+\alpha-Np+2p} \tag{2.10}$$

is

$$C^* = \frac{2p}{2p-Np+N+\alpha}\left(\frac{2p-Np+N+\alpha}{Np-N-\alpha}\right)^{\frac{Np-N-\alpha}{2}} \|R\|_{L^2}^{2-2p}. \tag{2.11}$$

In particular, in the L^2*-critical case, i.e.,* $p = 1 + \frac{2+\alpha}{N}$, $C^* = p\|R\|_{L^2}^{2-2p}$.

This inequality has been extended to the fractional case; see [10].

Finally, we recall the following compactness lemma is vital in the proof of global existence; see [7].

Lemma 2.4 *Let* $N \ge 2$, $0 < p < \frac{4}{N-2}$. *Let* $\{u_n\}$ *be a bounded sequence in* H^1 *such that*

$$\limsup_{n\to\infty} \|u_n\|_{\dot{H}^1} \le M, \qquad \limsup_{n\to\infty} \|u_n\|_{L^{p+2}} \ge m.$$

Then there exist a sequence $(x_n)_{n\ge1}$ *in* $\mathbb{R}^N$ *and* $U \in H^1 \setminus \{0\}$ *such that up to a subsequence,*

$$u_n(\cdot + x_n) \rightharpoonup U \quad \text{weakly in } H^1.$$

3 The existence of blow-up solutions

In this section, we will derive the sufficient conditions about existence of blow-up solutions.

Theorem 3.1 *Let* $\psi_0 \in \Sigma$, $\lambda_1 < 0$, $h_0 := \operatorname{Im}\int_{\mathbb{R}^N} \bar{\psi}_0 x \nabla\psi_0 \, dx > 0$ *and* $\frac{4}{N} < p_1 < \frac{4}{N-2}$ *with* $N \ge 3$. *Then, the solution* $\psi(t)$ *of* (1.1) *blows up in each of the following three cases*:

(1) $\lambda_2 > 0$, $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{Np_1+2\alpha}{2N}$, *and* $E(\psi_0) < 0$;

(2) $\lambda_2 < 0$, $1 + \frac{2+\alpha}{N} < p_2 < 1 + \frac{Np_1+2\alpha}{2N}$, *and* $E(\psi_0) < 0$;

(3) $\lambda_2 < 0$, $1 + \frac{\alpha}{N} < p_2 \le 1 + \frac{2+\alpha}{N}$, *and* $E(\psi_0) + C\|\psi_0\|_{L^2}^{\frac{2Np_1+2p_1\alpha-4Np_2+4N+4\alpha}{Np_1-2Np_2+2N+2\alpha}} < 0$ *for some constant* C.

More precisely, there is $T^* \in (0, C\frac{\|x\psi_0\|_{L^2}^2}{y_0}]$ *such that*

$$\lim_{t\to T^*} \|\nabla\psi(t)\|_{L^2} = \infty.$$

Proof In the following, we will prove $F'(t) < 0$ and $F''(t) < 0$ for all $t \in [0, T^*)$. More precisely, we will prove that

$$h'(t) \ge c\|\nabla\psi(t)\|_{L^2}^2 > 0 \tag{3.1}$$

for some constant $c > 0$, where $h(t)$ is defined by (2.4). Thus, it follows from (2.5) that $F''(t) < 0$ for all $t \in [0, T^*)$. This shows that $F(t)$ is concave and the solution $\psi(t)$ of (1.1) blows up. Indeed, it follows from $y(0) = y_0 > 0$ that $h(t) > h(0) > 0$ for all $t > 0$. On the other hand, we deduce from Hölder's inequality that

$$h(t) \le \|x\psi(t)\|_{L^2}\|\nabla\psi(t)\|_{L^2}$$

for all $t \in [0, T^*)$. This implies

$$\|\nabla\psi(t)\|_{L^2} \ge \frac{h(t)}{\|x\psi_0\|_{L^2}}. \tag{3.2}$$

We deduce from (3.1) and (3.2) that

$$\begin{cases} h'(t) \ge c\frac{h^2(t)}{\|x\psi_0\|_{L^2}^2}, \\ h(0) = h_0 > 0. \end{cases} \tag{3.3}$$

This shows that there is $T^* \in (0, \frac{\|x\psi_0\|_{L^2}^2}{cy_0}]$ such that $\|\nabla\psi(t)\|_{L^2} \to \infty$ as $t \to T^*$.

Case (i): $\lambda_2 > 0$, $Np_1 > 2Np_2 - 2N - 2\alpha$, and $E(\psi_0) < 0$. We deduce from (2.5), (2.2), and our assumptions that

$$\begin{aligned} h'(t) &= -2\|\nabla\psi(t)\|_{L^2}^2 - \frac{N\lambda_1 p_1}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\ &\quad - \lambda_2\frac{p_2N - N - \alpha}{p_2}\int_{\mathbb{R}^N}(I_\alpha * |\psi|^{p_2})(t)|\psi(t)|^{p_2}\,dx \\ &= -2\|\nabla\psi(t)\|_{L^2}^2 \\ &\quad + Np_1\left(\frac{1}{2}\|\nabla\psi(t)\|_{L^2}^2 + \frac{\lambda_2}{2p_2}\int_{\mathbb{R}^N}(I_\alpha * |\psi(t)|^{p_2})|\psi(t)|^{p_2}\,dx - E(\psi_0)\right) \\ &\quad - \lambda_2\frac{p_2N - N - \alpha}{p_2}\int_{\mathbb{R}^N}(I_\alpha * |\psi(t)|^{p_2})|\psi(t)|^{p_2}\,dx \\ &= \frac{Np_1 - 4}{2}\|\nabla\psi(t)\|_{L^2}^2 - Np_1E(\psi_0) \\ &\quad + \frac{\lambda_2}{2p_2}(Np_1 - 2Np_2 + 2N + 2\alpha)\int_{\mathbb{R}^N}(I_\alpha * |\psi(t)|^{p_2})|\psi(t)|^{p_2}\,dx \\ &\ge \frac{Np_1 - 4}{2}\|\nabla\psi(t)\|_{L^2}^2. \end{aligned}$$

This implies that (3.1) holds.

Case (ii): $\lambda_2 < 0$, $Np_1 + 2N + 2\alpha > 2Np_2$, $p_2 > 1 + \frac{\alpha+2}{N}$ and $E(\psi_0) < 0$. We deduce from (2.5), (2.2), and our assumptions that

$$\begin{aligned}
h'(t) &= -2\|\nabla\psi(t)\|_{L^2}^2 - \frac{N\lambda_1 p_1}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\
&\quad - (p_2N - N - \alpha)\left(2E(\psi_0) - \|\nabla\psi(t)\|_{L^2}^2 - \frac{2\lambda_1}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2}\right) \\
&= (p_2N - N - \alpha - 2)\|\nabla\psi(t)\|_{L^2}^2 - 2(p_2N - N - \alpha)E(\psi_0) \\
&\quad - \frac{\lambda_1}{p_1+2}(Np_1 - 2Np_2 + 2N + 2\alpha)\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\
&\geq (p_2N - N - \alpha - 2)\|\nabla\psi(t)\|_{L^2}^2.
\end{aligned}$$

This implies that (3.1) holds.

Case (iii): $\lambda_2 < 0$, $1 + \frac{\alpha}{N} < p_2 \leq 1 + \frac{2+\alpha}{N}$, and $E(\psi_0) + C\|\psi_0\|_{L^2}^{\frac{2Np_1+2p_1\alpha-4Np_2+4N+4\alpha}{Np_1-2Np_2+2N+2\alpha}} < 0$ for some constant C.

We deduce from $p_1 > \frac{4}{N}$ that there is a constant ε such that $p_1 > \frac{2(2+\varepsilon)}{N}$. Let $\theta := \frac{2(2+\varepsilon)}{p_1 N} < 1$. Therefore, it follows from (2.2) and our assumptions that

$$\begin{aligned}
h'(t) &= -2\|\nabla\psi(t)\|_{L^2}^2 - \frac{N\lambda_1 p_1\theta}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} - \frac{N\lambda_1 p_1(1-\theta)}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\
&\quad - \lambda_2\frac{p_2N - N - \alpha}{p_2}\int_{\mathbb{R}^N}\left(I_\alpha * |\psi(t)|^{p_2}\right)|\psi(t)|^{p_2}\,dx \\
&\geq -2\|\nabla\psi(t)\|_{L^2}^2 \\
&\quad + Np_1\theta\left(\frac{1}{2}\|\nabla\psi(t)\|_{L^2}^2 - E(\psi_0) + \frac{\lambda_2}{2p_2}\int_{\mathbb{R}^N}\left(I_\alpha * |\psi|^{p_2}\right)(t)|\psi(t)|^{p_2}\,dx\right) \\
&\quad - \frac{N\lambda_1 p_1(1-\theta)}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} - \lambda_2\theta\frac{p_2N - N - \alpha}{p_2}\int_{\mathbb{R}^N}\left(I_\alpha * |\psi(t)|^{p_2}\right)|\psi(t)|^{p_2}\,dx \\
&\geq \left(-2 + \frac{Np_1\theta}{2}\right)\|\nabla\psi(t)\|_{L^2}^2 - Np_1\theta E(\psi_0) - \frac{N\lambda_1 p_1(1-\theta)}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\
&\quad + \frac{\lambda_2\theta(Np_1 - 2p_2N + 2N + 2\alpha)}{2p_2}\int_{\mathbb{R}^N}\left(I_\alpha * |\psi(t)|^{p_2}\right)|\psi(t)|^{p_2}\,dx.
\end{aligned}$$

Applying Young's inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} I_\alpha * |\psi(t)|^{p_2}|\psi(t)|^{p_2}\,dx &\leq \|\psi(t)\|_{L^{\frac{2Np_2}{N+\alpha}}}^{2p_2} \\
&\leq \|\psi(t)\|_{L^2}^{2p_2 - \frac{2(p_1+2)(Np_2-N-\alpha)}{Np_1}}\|\psi(t)\|_{L^{p_1+2}}^{\frac{2(p_1+2)(Np_2-N-\alpha)}{Np_1}} \\
&\leq C(\delta)\|\psi(t)\|_{L^2}^{\frac{2Np_1+2p_1\alpha-4Np_2+4N+4\alpha}{Np_1-2Np_2+2N+2\alpha}} + \delta\|\psi(t)\|_{L^{p_1+2}}^{p_1+2}.
\end{aligned}$$

Therefore, we can choose $\delta > 0$ enough small such that

$$\delta\frac{|\lambda_2|\theta(Np_1 - 2p_2N + 2N + 2\alpha)}{2p_2} < \frac{N|\lambda_1|p_1(1-\theta)}{p_1+2},$$

which implies

$$y'(t) \geq \varepsilon \|\nabla \psi(t)\|_{L^2}^2 - Np_1\theta E - C(\delta)\frac{|\lambda_2|\theta(Np_1 - 2p_2N + 2N + 2\alpha)}{2p_2}\|\psi_0\|_{L^2}^{\frac{2Np_1+2p_1\alpha-4Np_2+4N+4\alpha}{Np_1-2Np_2+2N+2\alpha}}.$$

Therefore, if $Np_1\theta E + C(\delta)\frac{|\lambda_2|\theta(Np_1-2p_2N+2N+2\alpha)}{2p_2}\|\psi_0\|_{L^2}^{\frac{2Np_1+2p_1\alpha-4Np_2+4N+4\alpha}{Np_1-2Np_2+2N+2\alpha}} < 0$, then

$$y'(t) \geq \varepsilon \|\nabla \psi(t)\|_{L^2}^2.$$

This implies that (3.1) holds. □

4 Sharp conditions of blow-up and global existence

From the local well-posedness of the nonlinear Schrödinger–Choquard equation, for small initial data ψ_0, the solution $\psi(t)$ to (1.1) exists globally, and the solution $\psi(t)$ may blow up for some large initial data. Therefore, whether there are some sharp thresholds of global existence and blow-up for (1.1) is a very interesting problem. For Eq. (1.1), there are two nonlinearities and there is no scaling invariance, which are the main difficulties. We obtain the following sharp conditions of blow-up and global existence for (1.1) by constructing some new estimates.

4.1 L^2-Critical case

Theorem 4.1 *Let $\psi_0 \in H^1$, $\lambda_1 = -1$, $\lambda_2 = 1$, $p_1 = \frac{4}{N}$ and $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{2+\alpha}{N}$. Assume that Q is the ground state solution of* (2.6). *Then, we have the following sharp threshold mass of blow-up and global existence.*

(i) *If $\|\psi_0\|_{L^2} \leq \|Q\|_{L^2}$, then the solution of* (1.1) *exists globally.*

(ii) *If the initial data $\psi_0 = c\rho^{\frac{N}{2}}Q(\rho x)$ satisfies $|x|\psi_0 \in L^2$, where the complex number c satisfying $|c| > 1$, and the real number $\rho > 0$, then the solution ψ of* (1.1) *with initial data ψ_0 blows up in finite time.*

Proof (i) We firstly consider the case $\|\psi_0\|_{L^2} < \|Q\|_{L^2}$. It follows from (2.3) and (2.7) that

$$\begin{aligned} E(\psi_0) &= E(\psi(t)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\psi(t,x)|^2\,dx - \frac{1}{p_1+2}\int_{\mathbb{R}^N}|\psi(t,x)|^{p_1+2}\,dx \\ &\quad + \frac{1}{2p_2}\int_{\mathbb{R}^N}(I_\alpha * |\psi|^{p_2})(t,x)|\psi(t,x)|^{p_2}\,dx \\ &\geq \left(\frac{1}{2} - \frac{\|\psi_0\|_{L^2}^{p_1}}{2\|Q\|_{L^2}^{p_1}}\right)\|\nabla\psi(t)\|_{L^2}^2. \end{aligned}$$

Due to $\|\psi_0\|_{L^2} < \|Q\|_{L^2}$, we find that $\|\nabla\psi(t)\|_{L^2}$ is uniformly bounded for all time t. Therefore, (i) follows from the conservation of mass and Proposition 2.1.

When $\|\psi_0\|_{L^2} = \|Q\|_{L^2}$, if the solution $\psi(t)$ of (1.1) blows up in finite time, then there exists $T^* > 0$ such that $\lim_{t\to T^*}\|\nabla\psi(t)\|_{L^2} = \infty$. Set

$$\rho(t) = \|\nabla Q\|_{L^2}/\|\nabla\psi(t)\|_{L^2} \quad \text{and} \quad v(t,x) = \rho^{\frac{N}{2}}(t)\psi(t,\rho(t)x).$$

Let $\{t_n\}_{n=1}^{\infty}$ be an any time sequence such that $t_n \to T^*$, $\rho_n := \rho(t_n)$ and $v_n(x) := v(t_n, x)$. Then, the sequence $\{v_n\}$ satisfies

$$\|v_n\|_{L^2} = \|\psi(t_n)\|_{L^2} = \|\psi_0\|_{L^2} = \|Q\|_{L^2}, \qquad \|\nabla v_n\|_{L^2} = \rho_n \|\nabla\psi(t_n)\|_{L^2} = \|\nabla Q\|_{L^2}. \quad (4.1)$$

Observe that

$$\begin{aligned} H(v_n) &:= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n(x)|^2\,dx - \frac{1}{p_1+2}\int_{\mathbb{R}^N} |v_n(x)|^{p_1+2}\,dx \\ &= \rho_n^2\left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla\psi(t_n,x)|^2\,dx - \frac{1}{p_1+2}\int_{\mathbb{R}^N} |\psi(t_n,x)|^{p_1+2}\,dx\right) \\ &= \rho_n^2\left(E(\psi_0) - \frac{1}{2p_2}\int_{\mathbb{R}^N} \left(I_\alpha * |\psi|^{p_2}\right)(t,x)|\psi(t,x)|^{p_2}\,dx\right). \end{aligned} \quad (4.2)$$

Thus, we deduce from the Gagliardo–Nirenberg inequality (2.10) and $1+\frac{\alpha}{N} < p_2 < 1+\frac{2+\alpha}{N}$ that

$$\rho_n^2\left(E(\psi_0) - \frac{1}{2p_2}\int_{\mathbb{R}^N} \left(I_\alpha * |\psi|^{p_2}\right)(t,x)|\psi(t,x)|^{p_2}\,dx\right) \to 0, \quad \text{as } n \to \infty.$$

This, together with (4.2) implies that $\int_{\mathbb{R}^N} |v_n(x)|^{p_1+2}\,dx \to (2/N+1)\|\nabla Q\|_{L^2}^2$. Thus, we deduce from (4.1) that there exist a subsequence, still denoted by $\{v_n\}$, and $u \in H^1\backslash\{0\}$ such that

$$u_n := \tau_{x_n} v_n \rightharpoonup u \neq 0 \quad \text{weakly in } H^1,$$

for some $\{x_n\} \subseteq \mathbb{R}^N$. This implies that there exists $C_0 > 0$ such that

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^{p_2}\right)(x)|u_n(x)|^{p_2}\,dx \\ &\quad= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^{p_2}\right)(x)|v_n(x)|^{p_2}\,dx \geq C_0 > 0. \end{aligned} \quad (4.3)$$

On the other hand, we deduce from (2.7) and $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2} = \|Q\|_{L^2}$ that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla\psi(t,x)|^2\,dx - \frac{1}{p_1+2}\int_{\mathbb{R}^N} |\psi(t,x)|^{p_1+2}\,dx \geq 0,$$

for all $t \in [0, T^*)$. This implies that

$$\frac{1}{2p_2}\int_{\mathbb{R}^N} \left(I_\alpha * |\psi|^{p_2}\right)(t,x)|\psi(t,x)|^{p_2}\,dx \leq E(\psi_0),$$

for all $t \in [0, T^*)$. We consequently obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^{p_2}\right)(x)|v_n(x)|^{p_2}\,dx &= \rho_n^{Np_2-N-\alpha}\int_{\mathbb{R}^N} \left(I_\alpha * |\psi|^{p_2}\right)(t_n,x)|\psi(t_n,x)|^{p_2}\,dx \\ &\leq \rho_n^{Np_2-N-\alpha} E(\psi_0) \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

which is a contradiction with (4.3). Thus, the solution $\psi(t)$ of (1.1) exists globally.

(ii) Since $|x|\psi_0 \in L^2$, $J(t) = \int_{\mathbb{R}^N} |x\psi(t,x)|^2\,dx$ is well-defined, and it follows from Lemma 2.1 that

$$J''(t) = 16E(\psi_0) + \frac{4Np_2 - 4N - 4\alpha - 8}{p_2} \int_{\mathbb{R}^N} \big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx. \tag{4.4}$$

By the definition of initial data $\psi_0(x) = c\rho^{\frac{N}{2}}Q(\rho x)$ and the Pohožaev identity for Eq. (2.6), i.e., $\frac{1}{2}\|\nabla Q\|_{L^2}^2 = \frac{1}{p_1+2}\|Q\|_{L^{p_1+2}}^{p_1+2}$, we deduce that

$$\begin{aligned}
E(\psi_0) &= \frac{|c|^2\rho^2}{2}\int_{\mathbb{R}^N}\big|\nabla Q(x)\big|^2\,dx - \frac{|c|^{p_1+2}\rho^2}{p_1+2}\int_{\mathbb{R}^N}\big|Q(x)\big|^{p_1+2}\,dx \\
&\quad + \frac{|c|^{2p_2}\rho^{Np_2-N-\alpha}}{2p_2}\int_{\mathbb{R}^N}\big(I_\alpha * |Q|^{p_2}\big)(x)\big|Q(x)\big|^{p_2}\,dx \\
&= -\frac{|c|^2\rho^2}{2}\big(|c|^{p_1}-1\big)\|\nabla Q\|_{L^2}^2 \\
&\quad + \frac{|c|^{2p_2}\rho^{Np_2-N-\alpha}}{2p_2}\int_{\mathbb{R}^N}\big(I_\alpha * |Q|^{p_2}\big)(x)\big|Q(x)\big|^{p_2}\,dx.
\end{aligned}\tag{4.5}$$

Thanks to $Np_2 - N - \alpha < 2$, we can take ρ large enough such that

$$E(\psi_0) < 0.$$

It follows from (4.4) that $F''(t) < 16E(\psi_0) < 0$. By the standard concave argument, the solution ψ of (1.1) with the initial data ψ_0 blows up in finite time. □

4.2 L^2-Supercritical case

Theorem 4.2 *Let $\lambda_1 = \lambda_2 = -1$, $p_1 > \frac{4}{N}$, and $\psi \in C([0,T^*),H^1)$ be a solution of (1.1). Then we have the following sharp criteria of blow-up and global existence for (1.1).*

(1) *$\|\psi_0\|_{L^2} < \|R\|_{L^2}$, $p_2 = 1 + \frac{2+\alpha}{N}$, and $E(\psi_0) < \frac{Np_1-4}{2Np_1}\big(1 - \frac{\|\psi_0\|_{L^2}^{2p_2-2}}{\|R\|_{L^2}^{2p_2-2}}\big)y_0^2$. If $\|\nabla\psi_0\|_{L^2} < y_0$, then the solution $\psi(t)$ of (1.1) exists globally; If $\|\nabla u_0\|_{L^2} > y_0$, then the solution $\psi(t)$ of (1.1) blows up, where R is the ground state solution of (2.9) with $p = 1 + \frac{2+\alpha}{N}$, y_0 is defined by (4.8).*

(2) *$1 + \frac{\alpha+2}{N} < p_2 < 1 + \frac{Np_1+2\alpha}{2N}$ and $E(\psi_0) < \frac{Np_2-N-\alpha-2}{2(Np_2-N-\alpha)}y_1^2$. If $\|\nabla\psi_0\|_{L^2} < y_1$, then the solution $\psi(t)$ of (1.1) exists globally; If $\|\nabla\psi_0\|_{L^2} > y_1$, then the solution $\psi(t)$ of (1.1) blows up, where y_1 is the unique positive solution of the equation $f(y) = 0$ and $f(y)$ is defined in (4.13) with $1 + \frac{\alpha+2}{N} < p_2 < 1 + \frac{Np_1+2\alpha}{2N}$.*

(3) *$1 + \frac{Np_1+2\alpha}{2N} < p_2 < 1 + \frac{2+\alpha}{N-2}$ and $E(\psi_0) < \frac{Np_1-4}{2Np_1}y_2^2$. If $\|\nabla\psi_0\|_{L^2} < y_2$, then the solution $\psi(t)$ of (1.1) exists globally; If $\|\nabla\psi_0\|_{L^2} > y_2$, then the solution $\psi(t)$ of (1.1) blows up, where y_2 is the unique positive solution of the equation $f(y) = 0$ and $f(y)$ is defined in (4.13) with $1 + \frac{Np_1+2\alpha}{2N} < p_2 < 1 + \frac{2+\alpha}{N-2}$.*

Proof Case (1): $p_2 = 1 + \frac{2+\alpha}{N}$. First, we deduce from (2.7) and (2.10) that

$$\begin{aligned}
E\big(\psi(t)\big) &\geq \frac{1}{2}\big\|\nabla u(t)\big\|_{L^2}^2 - \frac{C_*}{p_1+2}\big\|\nabla\psi(t)\big\|_{L^2}^{\frac{Np_1}{2}}\big\|\psi(t)\big\|_{L^2}^{p_1+2-\frac{Np_1}{2}} \\
&\quad - \frac{C^*}{2p_2}\big\|\nabla\psi(t)\big\|_{L^2}^2\big\|\psi(t)\big\|_{L^2}^{2p_2-2} \\
&\geq h\big(\big\|\nabla\psi(t)\big\|_{L^2}\big),
\end{aligned}\tag{4.6}$$

where C_* and C^* are defined by (2.8) and (2.11), respectively, $h(y)$ is defined by

$$h(y)=\frac{1}{2}y^2-\frac{C_*}{p_1+2}\|\psi_0\|_{L^2}^{p_1+2-\frac{Np_1}{2}}y^{\frac{Np_1}{2}}-\frac{C^*}{2p_2}\|\psi_0\|_{L^2}^{2p_2-2}y^2,\quad y\in[0,\infty).$$

By a simple computation, we find that $h(y)$ is continuous on $[0,\infty)$ and

$$h'(y)=\left(1-\frac{C^*}{p_2}\|\psi_0\|_{L^2}^{2p_2-2}\right)y-\frac{C_*}{p_1+2}\frac{Np_1}{2}\|\psi_0\|_{L^2}^{p_1+2-\frac{Np_1}{2}}y^{\frac{Np_1}{2}-1}. \tag{4.7}$$

By the assumption $\|\psi_0\|_{L^2}<\|R\|_{L^2}$, $1-\frac{C^*}{p_2}\|\psi_0\|_{L^2}^{2p_2-2}>0$. Thus, the equation $h'(y)=0$ has a unique positive root:

$$y_0=\left(\frac{1-\frac{C^*}{p_2}\|\psi_0\|_{L^2}^{2p_2-2}}{\frac{C_*}{p_1+2}\frac{Np_1}{2}\|\psi_0\|_{L^2}^{p_1+2-\frac{Np_1}{2}}}\right)^{\frac{2}{Np_1-4}}. \tag{4.8}$$

This implies that $h(y)$ is increasing on the interval $[0,y_0)$, decreasing on the interval $[y_0,\infty)$ and

$$h_{\max}=h(y_0)=\frac{Np_1-4}{2Np_1}\left(1-\frac{\|\psi_0\|_{L^2}^{2p_2-2}}{\|R\|_{L^2}^{2p_2-2}}\right)y_0^2. \tag{4.9}$$

By (2.2) and $E(\psi_0)<h(y_0)$, we have

$$h\left(\|\nabla\psi(t)\|_{L^2}\right)\le E\left(\psi(t)\right)=E(\psi_0)<h(y_0),\quad \text{for all } t\in\left[0,T^*\right). \tag{4.10}$$

Now, we claim that if $\|\nabla\psi_0\|_{L^2}<y_0$, then $\|\nabla\psi(t)\|_{L^2}<y_0$, for all $t\in[0,T^*)$. This implies the solution $\psi(t)$ of (1.1) exists globally. Let us prove this result by contradiction. If not, by the continuity of $\|\nabla\psi(t)\|_{L^2}$, there exists $t_0\in[0,T^*)$ such that $\|\nabla\psi(t_0)\|_{L^2}=y_0$. Thus, $h(\|\nabla\psi(t_0)\|_{L^2})=h(y_0)=h_{\max}$. Moreover, taking $t=t_0$ in (4.10), it follows that

$$h\left(\|\nabla\psi(t_0)\|_{L^2}\right)=h(y_0)=h_{\max}\le E\left(\psi(t)\right)=E(\psi_0)<h_{\max},$$

which is a contradiction. Thus, the solution $\psi(t)$ of (1.1) exists globally.

On the other hand, if $\|\nabla\psi_0\|_{L^2}>y_0$, by the same argument, it follows that $\|\nabla\psi(t)\|_{L^2}>y_0$ for all $t\in[0,T^*)$. Thus, by (2.2), (2.5), (2.7), and the assumption $E(\psi_0)<\frac{Np_1-4}{2Np_1}(1-\frac{\|\psi_0\|_{L^2}^{2p_2-2}}{\|R\|_{L^2}^{2p_2-2}})y_0^2$, we deduce that

$$\begin{aligned}F''(t)&=8\|\nabla\psi(t)\|_{L^2}^2-\frac{4Np_1}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2}-\frac{8}{p_2}\int_{\mathbb{R}^N}\left(I_\alpha*|\psi|^{p_2}\right)(t,x)\left|\psi(t,x)\right|^{p_2}\,dx\\&=4Np_1E(\psi_0)-2(Np_1-4)\|\nabla\psi(t)\|_{L^2}^2\\&\quad+\frac{2(Np_1-4)}{p_2}\int_{\mathbb{R}^N}\left(I_\alpha*|\psi|^{p_2}\right)(t,x)\left|u(t,x)\right|^{p_2}\,dx\\&\le 4Np_1E(\psi_0)-2(Np_1-4)\|\nabla\psi(t)\|_{L^2}^2+2(Np_1-4)\frac{\|\psi_0\|_{L^2}^{2p_2-2}}{\|R\|_{L^2}^{2p_2-2}}\|\nabla\psi(t)\|_{L^2}^2\end{aligned}$$

$$\begin{aligned}&\leq 2(Np_1-4)\left(1-\frac{\|\psi_0\|_{L^2}^{2p_2-2}}{\|R\|_{L^2}^{2p_2-2}}\right)y_0^2\\&\quad-2(Np_1-4)\left(1-\frac{\|\psi_0\|_{L^2}^{2p_2-2}}{\|R\|_{L^2}^{2p_2-2}}\right)\big\|\nabla\psi(t)\big\|_{L^2}^2<0.\end{aligned} \tag{4.11}$$

Therefore, by the classical argument for Schrödinger equations, the solution $\psi(t)$ of (1.1) blows up.

Case (2): $1+\frac{\alpha+2}{N}<p_2<1+\frac{Np_1+2\alpha}{2N}$ and $E(\psi_0)<\frac{Np_2-N-\alpha-2}{2(Np_2-N-\alpha)}y_1^2$. Similarly, we define a function $g(y)$ on $[0,\infty)$ by

$$g(y)=\frac{1}{2}y^2-\frac{C_*}{p_1+2}\|\psi_0\|_{L^2}^{(p_1+2)-\frac{Np_1}{2}}y^{\frac{Np_1}{2}}-\frac{C^*}{2p_2}\|\psi_0\|_{L^2}^{N+\alpha-Np_2+2p_2}y^{Np_2-N-\alpha},\quad y\in[0,\infty).$$

Thus, it follows that $E(\psi(t))\geq g(\|\nabla\psi(t)\|_{L^2})$, $g(y)$ is continuous on $[0,\infty)$ and

$$\begin{aligned}g'(y)&=y-\frac{C_*}{p_1+2}\frac{Np_1}{2}\|\psi_0\|_{L^2}^{(p_1+2)-\frac{Np_1}{2}}y^{\frac{Np_1}{2}-1}\\&\quad-\frac{C^*}{2p_2}(Np_2-N-\alpha)\|\psi_0\|_{L^2}^{N+\alpha-Np_2+2p_2}y^{Np_2-N-\alpha-1}.\end{aligned} \tag{4.12}$$

Next, we define a function $f(y)$ by

$$\begin{aligned}f(y)&=1-\frac{C_*}{p_1+2}\frac{Np_1}{2}\|\psi_0\|_{L^2}^{(p_1+2)-\frac{Np_1}{2}}y^{\frac{Np_1}{2}-2}\\&\quad-\frac{C^*}{2p_2}(Np_2-N-\alpha)\|\psi_0\|_{L^2}^{N+\alpha-Np_2+2p_2}y^{Np_2-N-\alpha-2}.\end{aligned} \tag{4.13}$$

For the equation $f(y)=0$, there is a unique positive solution y_1. Indeed, by the assumption $1+\frac{\alpha+2}{N}<p_2<1+\frac{Np_1+2\alpha}{2N}$, for $y>0$, we have

$$\begin{aligned}f'(y)&=-\frac{C_*}{p_1+2}\frac{Np_1}{2}\left(\frac{Np_1}{2}-2\right)\|\psi_0\|_{L^2}^{(p_1+2)-\frac{Np_1}{2}}y^{\frac{Np_1}{2}-3}\\&\quad-\frac{C^*}{2p_2}(Np_2-N-\alpha)(Np_2-N-\alpha-2)\|\psi_0\|_{L^2}^{N+\alpha-Np_2+2p_2}y^{Np_2-N-\alpha-3}<0,\end{aligned} \tag{4.14}$$

which implies that $f(y)$ is decreasing on $[0,\infty)$. Due to $f(0)=1$, there exists a unique $y_1>0$ such that $f(y_1)=0$. Therefore, we have

$$f(y)>0\quad\text{for all }y\in[0,y_1)\quad\text{and}\quad f(y)<0\quad\text{for all }y\in(y_1,+\infty).$$

This implies that $g(y)$ is increasing on $[0,y_1)$, decreasing on $(y_1,+\infty)$ and

$$\begin{aligned}g_{\max}&=g(y_1)\\&=\left(\frac{1}{2}-\frac{1}{Np_2-N-\alpha}\right)y_1^2\\&\quad+\frac{C_*}{p_1+2}\frac{Np_1-2(Np_2-N-\alpha)}{2(Np_2-N-\alpha)}\|\psi_0\|_{L^2}^{p_1+2-\frac{Np_1}{2}}y^{\frac{Np_1}{2}}.\end{aligned} \tag{4.15}$$

On the other hand, we deduce from (2.2) and the assumption $E(u_0) < \frac{Np_2-N-\alpha-2}{2(Np_2-N-\alpha)}y_1^2$ that

$$\begin{aligned} g(\|\nabla\psi(t)\|_{L^2}) &\le E(\psi(t)) = E(\psi_0) \\ &< \left(\frac{1}{2} - \frac{1}{Np_2-N-\alpha}\right)y_1^2 \\ &\quad + \frac{C_*}{p_1+2}\frac{Np_1-2(Np_2-N-\alpha)}{2(Np_2-N-\alpha)}\|\psi_0\|_{L^2}^{p_1+2-\frac{Np_1}{2}}y^{\frac{Np_1}{2}} = g(y_1). \end{aligned} \tag{4.16}$$

By the same argument as Case (1), we find that if $\|\nabla\psi_0\|_{L^2} < y_1$, then, for all $t \in [0, T^*)$, $\|\nabla\psi(t)\|_{L^2} < y_1$, which implies the solution $\psi(t)$ of (1.1) exists globally.

And if $\|\nabla\psi_0\|_{L^2} > y_1$, by the same way, it follows that $\|\nabla\psi(t)\|_{L^2} > y_1$ for all $t \in [0, T^*)$. Thus, it follows from (2.2) and (2.5) that

$$\begin{aligned} F''(t) &= 8\|\nabla\psi(t)\|_{L^2}^2 - \frac{4Np_1}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\ &\quad - \frac{4p_2N-4N-4\alpha}{p_2}\int_{\mathbb{R}^N}(I_\alpha * |\psi|^{p_2})(t,x)|\psi(t,x)|^{p_2}\,dx \\ &= 8(Np_2-N-\alpha)E(\psi_0) - 4(Np_2-N-\alpha-2)\|\nabla\psi(t)\|_{L^2}^2 \\ &\quad - \frac{4(Np_1-2(Np_2-N-\alpha))}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\ &< 4(Np_2-N-\alpha-2)y_1^2 - 4(Np_2-N-\alpha-2)y_1^2 = 0. \end{aligned} \tag{4.17}$$

This implies that the solution of (1.1) blows up.

Case (3): $1 + \frac{Np_1+2\alpha}{2N} < p_2 < 1 + \frac{2+\alpha}{N-2}$ and $E(\psi_0) < \frac{Np_1-4}{2Np_1}y_2^2$. By the same argument as Case (2), we have

$$\begin{aligned} g_{\max} &= g(y_2) \\ &= \left(\frac{1}{2} - \frac{2}{Np_1}\right)y_2^2 \\ &\quad + \frac{C^*}{2p_2}\frac{2(Np_2-N-\alpha)-Np_1}{Np_1}\|\psi_0\|_{L^2}^{N+\alpha-Np_2+2p_2}y_2^{Np_2-N-\alpha}, \end{aligned} \tag{4.18}$$

where y_2 is the unique positive solution of (4.13). Thus, we deduce from (2.2) and the assumption $E(\psi_0) < \frac{Np_1-4}{2Np_1}y_2^2$ that

$$\begin{aligned} g(\|\nabla\psi(t)\|_{L^2}) &\le E(\psi(t)) = E(\psi_0) \\ &< \left(\frac{1}{2} - \frac{2}{Np_1}\right)y_2^2 + \frac{C^*}{2p_2}\frac{2(Np_2-N-\alpha)-Np_1}{Np_1}\|\psi_0\|_{L^2}^{N+\alpha-Np_2+2p_2}y_2^{Np_2-N-\alpha} \\ &= g(y_2). \end{aligned}$$

By the same argument as Case (1), we find that if $\|\nabla\psi_0\|_{L^2} < y_1$, then, for all $t \in [0, T^*)$, $\|\nabla\psi(t)\|_{L^2} < y_1$, which implies the solution $\psi(t)$ of (1.1) exists globally.

And if $\|\nabla\psi_0\|_{L^2} > y_2$, in the same way, it follows that $\|\nabla\psi(t)\|_{L^2} > y_2$ for all $t \in [0, T^*)$. Thus, it follows from (2.2) and (2.5) that

$$\begin{aligned}
F''(t) &= 8\big\|\nabla\psi(t)\big\|_{L^2}^2 - \frac{4Np_1}{p_1+2}\big\|\psi(t)\big\|_{L^{p_1+2}}^{p_1+2} \\
&\quad - \frac{4p_2N - 4N - 4\alpha}{p_2}\int_{\mathbb{R}^N}\big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx \\
&= 4Np_1E(\psi_0) - 2(Np_1-4)\big\|\nabla\psi(t)\big\|_{L^2}^2 \\
&\quad - \frac{4(Np_2 - N - \alpha) - 2Np_1}{p_2}\big\|\psi(t)\big\|_{L^{p_1+2}}^{p_1+2} \\
&< 2(Np_1-4)y_2^2 - 2(Np_1-4)y_2^2 = 0.
\end{aligned} \tag{4.19}$$

This implies that the solution $\psi(t)$ of (1.1) blows up. □

Theorem 4.3 *Let $\lambda_1 = 1$, $\lambda_2 = -1$, $1 + \frac{Np_1+2\alpha}{2N} < p_2 < 1 + \frac{2+\alpha}{N-2}$, and $E(\psi_0) < \frac{Np_2-N-\alpha-2}{2(Np_2-N-\alpha)}x_0^2$, and $\psi \in C([0,T^*), H^1)$ be a solution of (1.1). If $\|\nabla\psi_0\| < x_0$, then the solution $\psi(t)$ of (1.1) exists globally; If $\|\nabla\psi_0\| > x_0$, then the solution $\psi(t)$ of (1.1) blows up, where x_0 is defined by (4.21).*

Proof Applying (2.10), it follows that

$$\begin{aligned}
E\big(\psi(t)\big) &= \frac{1}{2}\big\|\nabla\psi(t)\big\|_{L^2}^2 + \frac{1}{p_1+2}\big\|\psi(t)\big\|_{L^{p_1+2}}^{p_1+2} - \frac{1}{2p_2}\int_{\mathbb{R}^N}\big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx \\
&\geq \frac{1}{2}\big\|\nabla\psi(t)\big\|_{L^2}^2 - \frac{C^*}{2p_2}\big\|\nabla\psi(t)\big\|_{L^2}^{Np_2-N-\alpha}\big\|\psi(t)\big\|_{L^2}^{N+\alpha-Np_2+2p_2} \\
&= f\big(\big\|\nabla\psi(t)\big\|_{L^2}\big),
\end{aligned} \tag{4.20}$$

where the C^* are defined by (2.11) and

$$f(x) := \frac{1}{2}x^2 - \frac{C^*}{2p_2}\|\psi_0\|_{L^2}^{N+\alpha-Np_2+2p_2}x^{Np_2-N-\alpha}.$$

By a simple computation, we find that the unique positive solution x_0 of $f'(x) = 0$ is given by

$$x_0 = \left(\frac{2p_2}{C^*(Np_2-N-\alpha)\|\psi_0\|_{L^2}^{N+\alpha-Np_2+2p_2}}\right)^{\frac{1}{Np_2-N-\alpha-2}}. \tag{4.21}$$

This implies that f is increasing on $(0, x_0)$ and decreasing on (x_0, ∞). By a simple computation, it follows that

$$f(x_0) = \frac{Np_2 - N - \alpha - 2}{2(Np_2 - N - \alpha)}x_0^2.$$

By (2.2) and the assumption $E(\psi_0) < f(x_0)$, it follows that

$$f\big(\big\|\nabla\psi(t)\big\|_{L^2}\big) \leq E(\psi_0) < f(x_0), \quad \forall t \in \big[0, T^*\big).$$

If $\|\nabla\psi_0\|_{L^2} < x_0$, it follows from the continuity argument that $\|\nabla\psi(t)\|_{L^2} < x_0$ for all $t \in [0, T^*)$. Therefore, the solution $\psi(t)$ of (1.1) exists globally.

If $\|\nabla\psi_0\|_{L^2} > x_0$, we deduce from the continuity argument that $\|\nabla\psi(t)\|_{L^2} > x_0$ for all $t \in [0, T^*)$. We choose $\delta > 0$ small enough so that

$$E(\psi_0) \leq (1-\delta)f(x_0).$$

This implies that

$$\begin{aligned} 8(Np_2 - N - \alpha)E(\psi_0) &\leq 8(Np_2 - N - \alpha)(1-\delta)f(x_0) \\ &= 4(Np_2 - N - \alpha - 2)(1-\delta)x_0^2. \end{aligned} \tag{4.22}$$

Thus, we deduce from (2.2), (2.5) and (4.22) that

$$\begin{aligned} F''(t) &= 8\|\nabla\psi(t)\|_{L^2}^2 + \frac{4Np_1}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\ &\quad - \frac{4p_2N - 4N - 4\alpha}{p_2}\int_{\mathbb{R}^N}\big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx \\ &= 4(2 - Np_2 + N + \alpha)\|\nabla\psi(t)\|_{L^2}^2 + \frac{4Np_1 - 8(Np_2 - N - \alpha)}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} \\ &\quad + 8(Np_2 - N - \alpha)E(\psi_0) \\ &\leq -4(Np_2 - N - \alpha - 2)\delta x_0^2 < 0. \end{aligned} \tag{4.23}$$

Therefore, by the classical argument for Schrödinger equations, the solution $\psi(t)$ of (1.1) blows up. □

Theorem 4.4 *Let $\lambda_1 = -1$, $\lambda_2 = 1$, $1 + \frac{\alpha+2}{N} < p_2 < 1 + \frac{Np_1+2\alpha}{2N}$, and $E(\psi_0) < \frac{Np_2-N-\alpha-2}{2(Np_2-N-\alpha)}x_0^2$, and $\psi \in C([0,T^*), H^1)$ be a solution of (1.1). If $\|\nabla\psi_0\| < x_1$, then the solution $\psi(t)$ of (1.1) exists globally; If $\|\nabla\psi_0\| > x_1$, then the solution $\psi(t)$ of (1.1) blows up, where x_1 is defined by (4.25).*

Proof Applying (2.7), it follows that

$$\begin{aligned} E\big(\psi(t)\big) &= \frac{1}{2}\|\nabla\psi(t)\|_{L^2}^2 - \frac{1}{p_1+2}\|\psi(t)\|_{L^{p_1+2}}^{p_1+2} + \frac{1}{2p_2}\int_{\mathbb{R}^N}\big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx \\ &\geq \frac{1}{2}\|\nabla\psi(t)\|_{L^2}^2 - \frac{C_*}{p_1+2}\|\nabla\psi(t)\|_{L^2}^{\frac{Np_1}{2}}\|\psi(t)\|_{L^2}^{p_1+2-\frac{Np_1}{2}} \\ &= f\big(\|\nabla\psi(t)\|_{L^2}\big), \end{aligned} \tag{4.24}$$

where the C_* are defined by (2.8) and

$$f(x) := \frac{1}{2}x^2 - \frac{C_*}{p_1+2}\|\psi_0\|_{L^2}^{p_1+2-\frac{Np_1}{2}}x^{\frac{Np_1}{2}}.$$

By a simple computation, we find that the unique positive solution x_1 of $f'(x) = 0$ is given by

$$x_1 = \left(\frac{2(p_1+2)}{C_* N p_1 \|\psi_0\|_{L^2}^{p_1+2-\frac{Np_1}{2}}}\right)^{\frac{2}{Np_1-4}}. \tag{4.25}$$

This implies that f is increasing on $(0, x_1)$ and decreasing on (x_1, ∞). By a simple computation, it follows that

$$f(x_1) = \frac{Np_1 - 4}{2Np_1} x_1^2.$$

By (2.2) and the assumption $E(\psi_0) < f(x_1)$, it follows that

$$f\left(\left\|\nabla\psi(t)\right\|_{L^2}\right) \le E(\psi_0) < f(x_1), \quad \forall t \in \left[0, T^*\right).$$

If $\|\nabla\psi_0\|_{L^2} < x_1$, it follows from the continuity argument that $\|\nabla\psi(t)\|_{L^2} < x_1$ for all $t \in [0, T^*)$. Therefore, the solution $\psi(t)$ of (1.1) exists globally.

If $\|\nabla\psi_0\|_{L^2} > x_1$, we deduce from the continuity argument that $\|\nabla\psi(t)\|_{L^2} > x_1$ for all $t \in [0, T^*)$. We can choose $\delta > 0$ small enough so that

$$E(\psi_0) \le (1-\delta) f(x_1).$$

This implies that

$$4Np_1 E(\psi_0) \le 4Np_1(1-\delta) f(x_1) = 2(Np_1 - 4)(1-\delta) x_1^2. \tag{4.26}$$

Thus, we deduce from (2.2), (2.5) and (4.26) that

$$\begin{aligned}
F''(t) &= 8\left\|\nabla\psi(t)\right\|_{L^2}^2 - \frac{4Np_1}{p_1+2}\left\|\psi(t)\right\|_{L^{p_1+2}}^{p_1+2} \\
&\quad + \frac{4p_2N - 4N - 4\alpha}{p_2}\int_{\mathbb{R}^N}\left(I_\alpha * |\psi|^{p_2}\right)(t,x)\left|\psi(t,x)\right|^{p_2}\,dx \\
&= (8 - 2Np_1)\left\|\nabla\psi(t)\right\|_{L^2}^2 \\
&\quad + \frac{4Np_2 - 4N - 4\alpha - 2Np_1}{p_2}\int_{\mathbb{R}^N}\left(I_\alpha * |\psi|^{p_2}\right)(t,x)\left|\psi(t,x)\right|^{p_2}\,dx \\
&\quad + 4Np_1 E(\psi_0) \\
&\le -2(Np_1 - 4)\delta x_1^2 < 0.
\end{aligned} \tag{4.27}$$

Therefore, by the classical argument for Schrödinger equations, the solution $\psi(t)$ of (1.1) blows up. □

5 Conclusions

In this paper, we obtain some sharp thresholds of blow-up and global existence for the nonlinear Schrödinger–Choquard equation. We firstly obtain some sufficient conditions about existence of blow-up solutions. Due to the loss of scaling invariance for this equation, we derive some sharp thresholds of blow-up and global existence by constructing some new estimates. In particular, we prove the global existence for this equation with critical mass in the L^2-critical case.

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