Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Linear difference operator with multiple variable parameters and applications to second-order differential equations

Feifan Li[1], Zhonghua Bi[1], Shaowen Yao[1*] and Yun Xin[2*]

*Correspondence:
shaowenyao_78@126.com;
xy_1982@126.com
[1]School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China
[2]College of Computer Science and Technology, Henan Polytechnic University, Jiaozuo, China

Abstract

In this article, we first investigate the linear difference operator $(Ax)(t) := x(t) - \sum_{i=1}^{n} c_i(t)x(t - \delta_i(t))$ in a continuous periodic function space. The existence condition and some properties of the inverse of the operator A are explicitly pointed out. Afterwards, as applications of properties of the operator A, we study the existence of periodic solutions for two kinds of second-order functional differential equations with this operator. One is a kind of second-order functional differential equation, by applications of Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive periodic solutions are established. Another one is a kind of second-order quasi-linear differential equation, we establish the existence of periodic solutions of this equation by an extension of Mawhin's continuous theorem.

MSC: 34B18; 34C25

Keywords: Difference operator; Multiple variable parameters; Periodic solution; Second-order differential equation

1 Introduction

Difference operators play a very important role in solving functional differential equations, which derived from some practical problems, such as biology, economics and population models [11, 20, 25]. In the 1970s, Hale [10] gave a definition for a functional differential equations of an operator. Under the condition that the operator is stable, many researchers obtained the existence of periodic solutions for these functional differential equations by means of some fixed point theorems and topology degree theory. Zhang [26] in 1995 first introduced the properties of the linear autonomous difference operator $(A_1x)(t) := x(t) - cx(t - \delta)$, where c, δ are constants, which became an effective tool for the research on differential equation, since it relieved the above stability restriction. This work has attracted the attention of many scholars in differential equations, for example [2–4, 6–8, 13, 15, 17–19, 21–24, 27]. Lu and Ge [13] in 2004 investigated a linear autonomous difference operator with multiple parameters $(A_2x)(t) := x(t) - \sum_{i=1}^{n} c_i x(t - \delta_i)$ which is an extension of A_1. And they obtained the existence of periodic solutions for the corresponding differential equation. Du et al. [8] in 2009 studied the difference operator $(A_3x)(t) := x(t) - c(t)x(t-\delta)$, where $c(t)$ is a periodic function. By applying Mawhin's continuation theorem and the properties of A_3, they obtained sufficient conditions for the existence of periodic solutions to a kind of Liénard differential equation. Afterwards, Ren et al. [19] in 2011 considered a kind of second-order functional differential equation. By applications of the fixed point index theorem and the properties of the linear difference operator $(A_4x)(t) := x(t) - cx(t-\delta(t))$, where $\delta(t)$ is a periodic function, they obtained sufficient conditions for the existence, multiplicity and nonexistence of positive periodic solutions to the corresponding equation. Subsequently, Cheng and Li in [3] investigated the difference operator $(A_5x)(t) := x(t) - c(t)x(t-\delta(t))$, and applied it to a study of the corresponding functional differential equation.

Naturally, a new question arises: how does the linear difference operator work on multiple variable parameters? Besides practical interests, the topic has obvious intrinsic theoretical significance. To answer this question, in this paper, we discuss properties of the difference operator with multiple variable parameters $(Ax)(t) := x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i(t))$, which is shown in Sect. 2, where $c_i(t)$, $\delta_i(t) \in C(\mathbb{R},\mathbb{R})$, and $c_i(t)$, $\delta_i(t)$ are ω-periodic functions on t, ω is a positive constant. As applications of properties of the difference operator A, we investigate the existence of periodic solutions for two kinds of second-order differential equations as follows.

In Sect. 3, we consider a kind of second-order differential equation with difference operator A:

$$\big((Ax)(t)\big)'' + a(t)x(t) = f\big(t, x\big(t-\tau(t)\big)\big), \tag{1.1}$$

where $\tau(t) \in C(\mathbb{R},\mathbb{R})$, $a(t) \in C(\mathbb{R},(0,+\infty))$, $f(t,x) := f(t,x(t-\tau(t))) \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $\tau(t)$, $a(t)$, $f(t,x)$ are ω-periodic functions on t. By employing properties of A and Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive periodic solutions are established. Meanwhile, we obtain the $f(t,x)$ condition which is weaker than the condition $F(t,x) := f(t,x(t-\tau(t))) - ca(t)x(t-\tau(t))$ in [5, 14]. And we establish the existence of positive periodic solutions of Eq. (1.1) in the cases that $0 < \sum_{i=1}^{n} c_i(t) < 1$ and $-1 < \sum_{i=1}^{n} c_i(t) < 0$, the authors in [19, 22] only discussed the existence of periodic solutions for equations in the case that $-1 < c < 0$.

In Sect. 4, by applications of the extension of Mawhin's continuous theorem due to Ge and Ren [9], we study the following second-order quasi-linear differential equation:

$$\big(\phi_p(Ax)'(t)\big)' = \tilde{f}\big(t, x(t), x'(t)\big), \tag{1.2}$$

where $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$, where $p > 1$ is a constant, $\tilde{f}: [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function, i.e. it is measurable in the first variable and continuous in the second variable, and for every $0 < r < s$ there exists $h_{r,s} \in L^2[0,T]$ such that $|\tilde{f}(t,x(t),x'(t))| \le h_{r,s}$ for all $x \in [r,s]$ and a.e. $t \in [0,T]$. The obvious difficulty of Eq. (1.2) lies in the following two respects. First, although $(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i(t))$ is a natural generalization of the operator $(A_jx)(t)$, $j = 1,2,3,4,5$, this class of differential equation with A typically possesses a more complicated nonlinearity than differential equation with $(A_jx)(t)$. Second, we do not get $(Ax)'(t) = (Ax')(t)$, it means that the *prior bounds* of periodic solutions are not easy to estimate, we get over this problem here.

2 Properties of the difference operator *A*

In this section, we consider properties of the difference operator A. We first give the following notations which will be used in the proofs. Let

$$C_\omega := \{x \in C(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t), t \in \mathbb{R}\}$$

with norm $\|x\| := \max_{t\in[0,\omega]} |x(t)|$. Clearly, $(C_\omega, \|\cdot\|)$ is a Banach space. Define

$$C_\omega^+ := \{x \in C(\mathbb{R},(0,+\infty)) : x(t+\omega) = x(t), t \in \mathbb{R}\},$$

$$c_* := \min_{t\in[0,\omega]} \left|\sum_{i=1}^n c_i(t)\right|, \qquad c^* := \max_{t\in[0,\omega]} \left|\sum_{i=1}^n c_i(t)\right|,$$

$$\|c_i\| := \max_{t\in[0,\omega]} |c_i(t)|, \qquad c_\infty := \sum_{i=1}^n \|c_i\|, \quad i = 1,2,\ldots,n,$$

$$k := \{\hat{k} \mid \|c_{\hat{k}}\| = \max\{\|c_1\|,\|c_2\|,\ldots,\|c_n\|\}\}.$$

Lemma 2.1 ([12]) *If* $c(t) \in C_\omega$, $\delta(t) \in C_\omega^1 := \{x \in C^1(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t), t \in \mathbb{R}\}$ *and* $\delta'(t) < 1$, *then* $c(\mu(t)) \in C_\omega$, *where* $\mu(t)$ *is the inverse function of* $t - \delta(t)$.

Define operators $A, B : C_\omega \to C_\omega$ by

$$(Ax)(t) = x(t) - \sum_{i=1}^n c_i(t)x(t-\delta_i(t)), \qquad (Bx)(t) = \sum_{i=1}^n c_i(t)x(t-\delta_i(t)),$$

then we have the following properties of the difference operator A.

Theorem 2.2

(1) *If* $\sum_{i=1}^n \|c_i\| < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\text{(i)} \quad |(A^{-1}x)(t)| \le \frac{\|x\|}{1-\sum_{i=1}^n \|c_i\|},$$

$$\text{(ii)} \quad \int_0^\omega |(A^{-1}x)(t)|\,dt \le \frac{1}{1-\sum_{i=1}^n \|c_i\|}\int_0^\omega |x(t)|\,dt.$$

(2) *If* $\sum_{i=1}^n \|e_i\| < 1$ *and* $\delta_k'(t) < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\text{(i)} \quad |(A^{-1}x)(t)| \le \frac{\|e_k\|\|x\|}{1-\sum_{i=1}^n \|e_i\|},$$

$$\text{(ii)} \quad \int_0^\omega |(A^{-1}x)(t)|\,dt \le \frac{\|e_k\|}{1-\sum_{i=1}^n \|e_i\|}\int_0^\omega |x(t)|\,dt,$$

where $\sum_{i=1}^n \|e_i\| = \|\frac{1}{c_k}\| + \sum_{i=1, i\neq k}^n \|\frac{c_i}{c_k}\|$, *and for any* $t_0 \in \mathbb{R}$, $c_k(t_0) \neq 0$.

Proof Case 1: $\sum_{i=1}^n \|c_i\| < 1$.

Let $t = D_0$ and $D_j = t - \sum_{i=1}^{j} \delta_{l_i}(D_{i-1})$, $j = 1, 2, \ldots$, then

$$(Bx)(t) = \sum_{l_1=1}^{n} c_{l_1}(D_0)x(D_1),$$
$$\big(B^2x\big)(t) = \sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)x(D_2),$$

therefore, we have

$$\big(B^jx\big)(t) = \sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)\cdots\sum_{l_j=1}^{n} c_{l_j}(D_{j-1})x(D_j)$$

and

$$\sum_{j=0}^{\infty}\big(B^jx\big)(t) = x(t) + \sum_{j=1}^{\infty}\sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)\cdots\sum_{l_j=1}^{n} c_{l_j}(D_{j-1})x(D_j),$$

where $B^0 = I$. Since $A = I - B$ and $\|B\| \le \sum_{i=1}^{n}\|c_i\| < 1$, we see that A has a continuous inverse $A^{-1}: C_\omega \to C_\omega$ with

$$\begin{aligned}\big(A^{-1}x\big)(t) &= \big((I - B)^{-1}x\big)(t) = \sum_{j=0}^{\infty}\big(B^jx\big)(t)\\ &= x(t) + \sum_{j=1}^{\infty}\sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)\cdots\sum_{l_j=1}^{n} c_{l_j}(D_{j-1})x(D_j).\end{aligned} \tag{2.1}$$

Then

$$\begin{aligned}\big|\big(A^{-1}x\big)(t)\big| &= \left|x(t) + \sum_{j=1}^{\infty}\sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)\cdots\sum_{l_j=1}^{n} c_{l_j}(D_{j-1})x(D_j)\right|\\ &\le \|x\| + \sum_{j=1}^{\infty}\left(\sum_{i=1}^{n}\|c_i\|\right)^j\|x\|\\ &\le \frac{\|x\|}{1 - \sum_{i=1}^{n}\|c_i\|}.\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\int_0^\omega\big|\big(A^{-1}x\big)(t)\big|\,dt &= \int_0^\omega\left|\sum_{j=0}^{\infty}\big(B^jx\big)(t)\right|dt\\ &\le \sum_{j=0}^{\infty}\int_0^\omega\big|\big(B^jx\big)(t)\big|\,dt\\ &\le \sum_{j=0}^{\infty}\int_0^\omega\left|\sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)\cdots\sum_{l_j=1}^{n} c_{l_j}(D_{j-1})x(D_j)\right|dt\end{aligned}$$

$$\begin{aligned}&\le \sum_{j=0}^{\infty}\left(\sum_{i=1}^{n}\|c_i\|\right)^j \int_0^{\omega}\left|x(t)\right|dt\\ &\le \frac{1}{1-\sum_{i=1}^{n}\|c_i\|}\int_0^{\omega}\left|x(t)\right|dt.\end{aligned}$$

Case 2: $\sum_{i=1}^{n}\|e_i\|<1$ and $\delta_k'(t)<1$.

The operator $(Ax)(t)=x(t)-\sum_{i=1}^{n}c_i(t)x(t-\delta_i(t))$ can be converted to

$$\begin{aligned}(Ax)(t)&=x(t)-c_k(t)x\bigl(t-\delta_k(t)\bigr)-\sum_{\substack{i=1\\ i\neq k}}^{n}c_i(t)x\bigl(t-\delta_i(t)\bigr)\\ &=-c_k(t)\left(x\bigl(t-\delta_k(t)\bigr)-\frac{x(t)}{c_k(t)}+\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{c_i(t)}{c_k(t)}x\bigl(t-\delta_i(t)\bigr)\right).\end{aligned}$$

From Lemma 2.1, there exists an inverse function $\mu\in C(R,R)$, such that $\mu(t-\delta_k(t))=t$. Define

$$(Ex)(t)=x(t)-\frac{1}{c_k(\mu(t))}x\bigl(\mu(t)\bigr)+\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{c_i(\mu(t))}{c_k(\mu(t))}x\bigl(\mu(t)-\delta_i\bigl(\mu(t)\bigr)\bigr),$$

$$e_i(t)=\begin{cases}\frac{1}{c_k(\mu(t))}, & \text{for } i=k,\\ -\frac{c_i(\mu(t))}{c_k(\mu(t))}, & \text{for } i\neq k;\end{cases}\qquad \varepsilon_i(t)=\begin{cases}\mu(t), & \text{for } i=k,\\ \mu(t)-\delta_i(\mu(t)), & \text{for } i\neq k.\end{cases}$$

Then $(Ex)(t)=x(t)-\sum_{i=1}^{n}e_i(t)x(\varepsilon_i(t))$. Define $(\hat{B}x)(t)=\sum_{i=1}^{n}e_i(t)x(\varepsilon_i(t))$, let $\hat{D}_0=t$ and $\hat{D}_j=\varepsilon_{l_j}\cdots\varepsilon_{l_2}\varepsilon_{l_1}(t)$, $j=0,1,2,\ldots$, $l_j=1,2,\ldots,n$, we have

$$\begin{aligned}&(\hat{B}x)(t)=\sum_{l_1=1}^{n}e_{l_1}(\hat{D}_0)x(\hat{D}_1),\\ &\bigl(\hat{B}^2x\bigr)(t)=\sum_{l_1=1}^{n}e_{l_1}(\hat{D}_0)\sum_{l_2=1}^{n}e_{l_2}(\hat{D}_1)x(\hat{D}_2),\\ &\ldots,\\ &\bigl(\hat{B}^jx\bigr)(t)=\sum_{l_1=1}^{n}e_{l_1}(\hat{D}_0)\sum_{l_2=1}^{n}e_{l_2}(\hat{D}_1)\cdots\sum_{l_j=1}^{n}e_{l_j}(\hat{D}_{j-1})x(\hat{D}_j).\end{aligned}$$

Since $\|\hat{B}\|\le\sum_{i=1}^{n}\|e_i\|=\|\frac{1}{c_k}\|+\sum_{i\neq k}^{n}\|\frac{c_i}{c_k}\|<1$, we arrive at

$$\bigl(E^{-1}x\bigr)(t)=\sum_{j=0}^{\infty}\bigl(\hat{B}^jx\bigr)(t)=\sum_{j=0}^{\infty}\left(\sum_{l_1=1}^{n}e_{l_1}(\hat{D}_0)\sum_{l_2=1}^{n}e_{l_2}(\hat{D}_1)\cdots\sum_{l_j=1}^{n}e_{l_j}(\hat{D}_{j-1})x(\hat{D}_j)\right).$$

Since $(Ax)(t) = -c_k(t)(Ex)(t-\delta_k(t)) := X(t) \in C_\omega$, we have $(Ex)(t) = -\frac{X(\mu(t))}{c_k(\mu(t))} = -e_k(t) \times X(\mu(t)) := X_0(t) \in C_\omega$. Therefore,

$$\begin{aligned}(A^{-1}X)(t) = x(t) &= (E^{-1}X_0)(t)\\ &= \sum_{j=0}^{\infty}\left(\sum_{l_1=1}^{n} e_{l_1}(\hat{D}_0)\sum_{l_2=1}^{n} e_{l_2}(\hat{D}_1)\cdots\sum_{l_j=1}^{n} e_{l_j}(\hat{D}_{j-1})X_0(\hat{D}_j)\right).\end{aligned}$$

Similar to Case 1, we can get

$$\begin{aligned}\left|(A^{-1}X)(t)\right| &= \left|\sum_{j=0}^{\infty}\left(\sum_{l_1=1}^{n} e_{l_1}(\hat{D}_0)\sum_{l_2=1}^{n} e_{l_2}(\hat{D}_1)\cdots\sum_{l_j=1}^{n} e_{l_j}(\hat{D}_{j-1})X_0(\hat{D}_j)\right)\right|\\ &\le \sum_{j=0}^{\infty}\left(\sum_{i=1}^{n}\|e_i\|\right)^j\|X_0\|\\ &\le \frac{\|X\|\|e_k\|}{1-\sum_{i=1}^{n}\|e_i\|}\end{aligned}$$

and

$$\int_0^\omega \left|(A^{-1}X)(t)\right|dt \le \frac{1}{1-\sum_{i=1}^{n}\|e_i\|}\int_0^\omega \left|X_0(t)\right|dt \le \frac{\|e_k\|}{1-\sum_{i=1}^{n}\|e_i\|}\int_0^\omega \left|X(t)\right|dt. \qquad \square$$

Remark 2.3 Theorem 2.2 extends and improves the corresponding lemmas in [3, 8, 13, 19, 26].

If $\delta_i(t) = \delta_i$, $i = 1, 2, \ldots, n$, here δ_i are constants, then the operator A can be written as

$$(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i),$$

then we have the following corollary.

Corollary 2.4

(1) *If* $\sum_{i=1}^{n}\|c_i\| < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

(i) $$\left|(A^{-1}x)(t)\right| \le \frac{\|x\|}{1-\sum_{i=1}^{n}\|c_i\|},$$

(ii) $$\int_0^\omega \left|(A^{-1}x)(t)\right|dt \le \frac{1}{1-\sum_{i=1}^{n}\|c_i\|}\int_0^\omega \left|x(t)\right|dt.$$

(2) *If* $\sum_{i=1}^{n}\|e_i\| < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

(i) $$\left|(A^{-1}x)(t)\right| \le \frac{\|e_k\|\|x\|}{1-\sum_{i=1}^{n}\|e_i\|},$$

(ii) $$\int_0^\omega \left|(A^{-1}x)(t)\right|dt \le \frac{\|e_k\|}{1-\sum_{i=1}^{n}\|e_i\|}\int_0^\omega \left|x(t)\right|dt,$$

where $\sum_{i=1}^{n}\|e_i\| = \|\frac{1}{c_k}\| + \sum_{i=1, i\neq k}^{n}\|\frac{c_i}{c_k}\|$, *and for any* $t_0 \in \mathbb{R}$, $c_k(t_0) \neq 0$.

If $c_i(t) = c_i$, $i = 1, 2, \dots, n$, i.e., the c_i are constants, then the operator A can be written as

$$(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i x\bigl(t - \delta_i(t)\bigr),$$

therefore, we have the following corollary.

Corollary 2.5

(1) *If* $\sum_{i=1}^{n} |c_i| < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\text{(i)}\quad \bigl|\bigl(A^{-1}x\bigr)(t)\bigr| \le \frac{\|x\|}{1 - \sum_{i=1}^{n} |c_i|},$$

$$\text{(ii)}\quad \int_0^\omega \bigl|\bigl(A^{-1}x\bigr)(t)\bigr|\,dt \le \frac{1}{1 - \sum_{i=1}^{n} |c_i|} \int_0^\omega \bigl|x(t)\bigr|\,dt.$$

(2) *If* $|\frac{1}{c_k}| + \sum_{i=1, i\neq k}^{n} |\frac{c_i}{c_k}| < 1$, *and* $\delta_k'(t) < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\text{(i)}\quad \bigl|\bigl(A^{-1}x\bigr)(t)\bigr| \le \frac{\|x\|}{|c_k| - 1 - \sum_{i=1, i\neq k}^{n} |c_i|},$$

$$\text{(ii)}\quad \int_0^\omega \bigl|\bigl(A^{-1}x\bigr)(t)\bigr|\,dt \le \frac{1}{|c_k| - 1 - \sum_{i=1, i\neq k}^{n} |c_i|} \int_0^\omega \bigl|x(t)\bigr|\,dt,$$

where $|c_k| = \max\{|c_1|, |c_2|, \dots, |c_n|\}$ *and* $c_k \neq 0$.

If $c_i(t) = c_i$, $\delta_i(t) = \delta_i$, $i = 1, 2, \dots, n$, i.e. the c_i, δ_i are constants, then the operator A can be written as

$$(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i x(t - \delta_i),$$

then we obtain the following.

Corollary 2.6

(1) *If* $\sum_{i=1}^{n} |c_i| < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\text{(i)}\quad \bigl|\bigl(A^{-1}x\bigr)(t)\bigr| \le \frac{\|x\|}{1 - \sum_{i=1}^{n} |c_i|},$$

$$\text{(ii)}\quad \int_0^\omega \bigl|\bigl(A^{-1}x\bigr)(t)\bigr|\,dt \le \frac{1}{1 - \sum_{i=1}^{n} |c_i|} \int_0^\omega \bigl|x(t)\bigr|\,dt.$$

(2) *If* $|\frac{1}{c_k}| + \sum_{i=1, i\neq k}^{n} |\frac{c_i}{c_k}| < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\text{(i)}\quad \bigl|\bigl(A^{-1}x\bigr)(t)\bigr| \le \frac{\|x\|}{|c_k| - 1 - \sum_{i=1, i\neq k}^{n} |c_i|},$$

$$\text{(ii)} \quad \int_0^{\omega} \left|(A^{-1}x)(t)\right| dt \le \frac{1}{|c_k| - 1 - \sum_{\substack{i=1\\ i\neq k}}^{n} |c_i|} \int_0^{\omega} \left|x(t)\right| dt,$$

where $|c_k| = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$ *and* $c_k \neq 0$.

Remark 2.7 Corollary 2.6 can be found in [13].

If $n = 1$, then the operator A can be written as

$$(Ax)(t) = x(t) - c_1(t)x\bigl(t - \delta_1(t)\bigr),$$

therefore, we can get the following corollary.

Corollary 2.8

(1) *If* $\|c_1\| < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\text{(i)} \quad \left|(A^{-1}x)(t)\right| \le \frac{\|x\|}{1 - \|c_1\|},$$

$$\text{(ii)} \quad \int_0^{\omega} \left|(A^{-1}x)(t)\right| dt \le \frac{1}{1 - \|c_1\|} \int_0^{\omega} \left|x(t)\right| dt.$$

(2) *If* $c_{1*} > 1$ *and* $\delta_1'(t) < 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\text{(i)} \quad \left|(A^{-1}x)(t)\right| \le \frac{\|x\|}{c_{1*} - 1},$$

$$\text{(ii)} \quad \int_0^{\omega} \left|(A^{-1}x)(t)\right| dt \le \frac{1}{c_{1*} - 1} \int_0^{\omega} \left|x(t)\right| dt.$$

Remark 2.9 Corollary 2.8 can be found in [3].

Remark 2.10 When $n = 1$, (1) if $c_1(t) = c$, where c is constant, we can get the corresponding properties of A_4 in [19]; (2) if $\delta_1(t) = \delta$, where δ is constant, we can get the corresponding properties of A_3 in [8]; (3) if $c_1(t) = c$, $\delta_1(t) = \delta$, we can get the corresponding properties of A_1 in [26].

3 Periodic solutions for Eq. (1.1)

In this section, we discuss the existence of positive periodic solutions for Eq. (1.1) in the cases that $0 < \sum_{i=1}^{n} c_i(t) < 1$ and $-1 < \sum_{i=1}^{n} c_i(t) < 0$. Firstly, we recall Krasnoselskii's fixed point theorem and some lemmas which our proofs are based on.

Theorem 3.1 (Krasnoselskii's fixed point theorem [1]) *Let* C_ω *be a Banach space. Assume that* Ω *is a bounded closed convex subset of* C_ω. *If* $Q, S : \Omega \to C_\omega$ *satisfy*

(i) $Qx_1 + Sx_2 \in \Omega$, $\forall x_1, x_2 \in \Omega$,

(ii) S *is a contractive operator and* Q *is a completely continuous operator.*

Then $Q + S$ *has a fixed point in* Ω.

Lemma 3.2 ([5]) *The equation*

$$\begin{cases} y''(t) + My(t) = h(t), \\ y(0) = y(\omega), \qquad y'(0) = y'(\omega), \end{cases} \tag{3.1}$$

has a unique ω-periodic solution

$$y(t) = \int_0^\omega G(t,s)h(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{\cos\sqrt{M}(t-s-\frac{\omega}{2})}{2\sqrt{M}\sin\frac{\sqrt{M}\omega}{2}}, & 0 \le s \le t \le \omega, \\ \frac{\cos\sqrt{M}(t-s+\frac{\omega}{2})}{2\sqrt{M}\sin\frac{\sqrt{M}\omega}{2}}, & 0 \le t < s \le \omega. \end{cases}$$

Lemma 3.3 ([5]) $\int_0^\omega G(t,s)\,ds = \frac{1}{M}$. *And* $G(t,s)$ *is a differentiable function with* t.

Lemma 3.4 ([22]) *If* $M < (\frac{\pi}{\omega})^2$, *then* $0 < l \le G(t,s) \le L$ *for all* $t \in [0,\omega]$ *and* $s \in [0,\omega]$.

Next, we consider the existence of positive periodic solutions for Eq. (1.1) in the case that $c_\infty \in (0, \frac{m}{M+m})$. Let $y(t) = (Ax)(t)$, from Theorem 2.2, we have $x(t) = (A^{-1}y)(t)$. Hence, Eq. (1.1) can be transformed into

$$y''(t) + a(t)y(t) - a(t)H\big(y(t)\big) = f\big(t, x\big(t-\tau(t)\big)\big), \tag{3.2}$$

where $H(y(t)) = -(\sum_{i=1}^n c_i(t)(A^{-1}y)(t-\delta_i(t))) = -(\sum_{i=1}^n c_i(t)x(t-\delta_i(t)))$.

We consider

$$y''(t) + a(t)y(t) - a(t)H\big(y(t)\big) = h(t), \quad h \in C_\omega^+. \tag{3.3}$$

Define the operators $T, N : C_\omega \to C_\omega$ by

$$(Th)(t) = \int_0^\omega G(t,s)h(s)\,ds, \qquad (Ny)(t) = \big(M - a(t)\big)y(t) + a(t)H\big(y(t)\big). \tag{3.4}$$

Clearly, T is completely continuous and N is bounded in C_ω. From Eq. (3.4) and Lemma 3.2, the solution for Eq. (3.3) can be written as

$$y(t) = (Th)(t) + (TNy)(t). \tag{3.5}$$

On the other hand, since $H(y(t)) = -\sum_{i=1}^{n} c_i(t)(A^{-1}y)(t-\delta_i(t))$, from Lemma 2.2, it is clear that

$$\begin{aligned}
|Ny(t)| &\le (M-a(t))|y(t)| + |a(t)|\left|-\sum_{i=1}^{n} c_i(t)(A^{-1}y)(t-\delta_i(t))\right| \\
&\le (M-m)|y(t)| + M\sum_{i=1}^{n}\|c_i\|\cdot\frac{1}{1-\sum_{i=1}^{n}\|c_i\|}\cdot\|y\| \\
&\le \left(M-m+\frac{Mc_\infty}{1-c_\infty}\right)\|y\|.
\end{aligned} \tag{3.6}$$

And it follows that

$$\|N\| \le M-m+\frac{Mc_\infty}{1-c_\infty}. \tag{3.7}$$

In view of $c_\infty \in (0, \frac{m}{M+m})$ and $\|T\| \le \frac{1}{M}$ (see Lemma 3.3), we have from Eq. (3.7)

$$\|TN\| \le \|T\|\|N\| \le \frac{1}{M}\left(M-m+\frac{Mc_\infty}{1-c_\infty}\right) \le \frac{M-m(1-c_\infty)}{M(1-c_\infty)} < 1. \tag{3.8}$$

Therefore,

$$y(t) = (I-TN)^{-1}(Th)(t). \tag{3.9}$$

Define an operator $P: C_\omega \to C_\omega$ by

$$(Ph)(t) = (I-TN)^{-1}(Th)(t). \tag{3.10}$$

Obviously, if $M < (\frac{\pi}{\omega})^2$, for any $h \in C_\omega^+$, $y(t) = (Ph)(t)$ is the unique positive ω-periodic solution of Eq. (1.1). Let

$$k_0 := \frac{\sqrt{(1-c_*^2)^2+4\sigma^2}-(1-c_*^2)}{2\sigma}, \qquad \sigma := \frac{l}{L}.$$

Consider the equation

$$\sigma c_\infty^2 + (1-c_*^2)c_\infty - \sigma = 0, \tag{3.11}$$

it is easy to verify that $\sigma c_\infty^2 + (1-c_*^2)c_\infty - \sigma \le 0$ when $0 < c_\infty \le k_0$, and we have the following lemmas.

Lemma 3.5 *Assume that $M < (\frac{\pi}{\omega})^2$, $c_i(t) \le 0$, $c_\infty \in (0, \frac{m}{M+m})$ and $c_\infty \le k_0$ hold, where $i = 1, 2, \ldots, n$. Then*

$$(Th)(t) \le (Ph)(t) \le \frac{M(1-c_\infty)}{m-(M+m)c_\infty}\|Th\|, \quad \textit{for all } h \in C_\omega^+.$$

Proof From Eq. (3.8), for all $h \in C_\omega^+$, we obtain

$$\begin{aligned}(Ph)(t) &= (I - TN)^{-1}(Th)(t) \le \left\|(I - TN)^{-1}\right\| \|Th\| \\ &\le \frac{\|Th\|}{1 - \|TN\|} \le \frac{M(1 - c_\infty)}{m - (M + m)c_\infty} \|Th\|. \end{aligned} \tag{3.12}$$

Since $\|TN\| < 1$, by Neumann expansions of P, we have

$$\begin{aligned} P &= (I - TN)^{-1} T \\ &= \bigl(I + TN + (TN)^2 + \cdots\bigr)T \\ &= T + TNT + (TN)^2 T + \cdots. \end{aligned} \tag{3.13}$$

From Lemma 3.4, for all $h(t) \in C_\omega^+$, we arrive at

$$\begin{aligned}(Th)(t) &= \int_0^\omega G(t,s)h(s)\,ds \\ &\ge l \int_0^\omega h(s)\,ds = \frac{l}{L} L \int_0^\omega h(s)\,ds \\ &\ge \sigma \max_{t\in[0,\omega]} \int_0^\omega G(t,s)h(s)\,ds = \sigma\|Th\| > 0. \end{aligned}$$

In view of $c_i(t) \le 0$ $(i = 1, 2, \dots, n)$ and $c_\infty \le k_0$, we get by Eq. (2.1)

$$\begin{aligned}\bigl(A^{-1}Th\bigr)(t) &= \sum_{j=0}^{\infty}\left(\sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)\cdots\sum_{l_j=1}^{n} c_{l_j}(D_{j-1})(Th)(D_j)\right) \\ &= (Th)(t) + \sum_{j=\text{even}}^{\infty}\left(\sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)\cdots\sum_{l_j=1}^{n} c_{l_j}(D_{j-1})(Th)(D_j)\right) \\ &\quad + \sum_{j=\text{odd}}^{\infty}\left(\sum_{l_1=1}^{n} c_{l_1}(D_0)\sum_{l_2=1}^{n} c_{l_2}(D_1)\cdots\sum_{l_j=1}^{n} c_{l_j}(D_{j-1})(Th)(D_j)\right) \\ &\ge \sigma\|Th\| + \sigma\|Th\|\sum_{j=\text{even}}^{\infty} c_*^j - \|Th\|\sum_{j=\text{odd}}^{\infty} c^{*j} \\ &\ge \sigma\|Th\| + \sigma\|Th\|\sum_{j=\text{even}}^{\infty} c_*^j - \|Th\|\sum_{j=\text{odd}}^{\infty} c_\infty^j \\ &\ge \frac{\sigma\|Th\|}{1 - c_*^2} - \frac{c_\infty\|Th\|}{1 - c_\infty^2} \ge 0, \end{aligned}$$

therefore, from equality (3.4), we can observe that

$$(NTh)(t) = \bigl(M - a(t)\bigr)(Th)(t) - a(t)\left(\sum_{i=1}^{n} c_i(t)\bigl(A^{-1}Th\bigr)\bigl(t - \delta_i(t)\bigr)\right) \ge 0,$$

clearly, $(TNTh)(t) \geq 0$. Then from the above analysis, we can get

$$\begin{aligned}(Ph)(t) &= (Th)(t) + (TNTh)(t) + \big((TN)^2 Th\big)(t) + \big((TN)^3 Th\big)(t) + \cdots \\ &\geq (Th)(t), \quad \text{for all } h \in C_\omega^+. \end{aligned}$$

□

Lemma 3.6 *Assume that $M < (\frac{\pi}{\omega})^2$, $c_i(t) \geq 0$ and $c_\infty \in (0, \frac{m}{M+m})$ hold, where $i = 1, 2, \ldots, n$. Then*

$$\frac{m - (M+m)c_\infty}{M(1-c_\infty)}(Th)(t) \leq (Ph)(t) \leq \frac{M(1-c_\infty)}{m-(M+m)c_\infty}\|Th\|, \quad \textit{for all } h \in C_\omega^+.$$

Proof Similarly as the proof of Lemma 3.5, it is easy to verify that

$$(Ph)(t) \leq \frac{\|Th\|}{1 - \|TN\|} \leq \frac{M(1-c_\infty)}{m-(M+m)c_\infty}\|Th\|.$$

From Eq. (3.13), we have

$$\begin{aligned} P &= \big(I + TN + (TN)^2 + (TN)^3 + \cdots\big)T \\ &= \big(I + (TN)^2 + (TN)^4 + \cdots\big)T + \big(TN + (TN)^3 + (TN)^5 + \cdots\big)T \\ &= \big(I + (TN)^2 + (TN)^4 + \cdots\big)T + \big(I + (TN)^2 + (TN)^4 + \cdots\big)TNT \\ &= \big(I + (TN)^2 + (TN)^4 + \cdots\big)(I + TN)T. \end{aligned} \tag{3.14}$$

Then we get by Eq. (3.8)

$$\begin{aligned}(Ph)(t) &\geq (I + TN)(Th)(t) \geq \big(I - \|TN\|\big)(Th)(t) \\ &\geq \frac{m-(m+M)c_\infty}{M(1-c_\infty)}(Th)(t) > 0, \quad \text{for all } h \in C_\omega^+. \end{aligned}$$

□

When $n = 1$, then $(Ax)(t) = x(t) - c_1(t)x(t - \delta_1(t))$, if $c_1(t) = c$, here c is a constant, then we have the following corollary.

Corollary 3.7 *Assume that $M < (\frac{\pi}{\omega})^2$ and $|c| \in (0, \frac{m}{M+m})$ hold.*

(i) *If $c < 0$ and $|c| \leq \sigma$, then*

$$(Th)(t) \leq (Ph)(t) \leq \frac{M(1-|c|)}{m-(M+m)|c|}\|Th\|, \quad \textit{for all } h \in C_\omega^+.$$

(ii) *If $c > 0$, then*

$$\frac{m-(M+m)c}{M(1-c)}(Th)(t) \leq (Ph)(t) \leq \frac{M(1-c)}{m-(M+m)c}\|Th\|, \quad \textit{for all } h \in C_\omega^+.$$

Remark 3.8 If $\sum_{i=1}^n \|e_i\| < 1$ and $\delta_k'(t) < 1$, since

$$\|TN\| \leq \|T\|\|N\| \leq 1 - \frac{m}{M} + \frac{c_\infty \|\frac{1}{c_k}\|}{1 - \sum_{i=1}^n \|e_i\|},$$

we cannot get $\|TN\| < 1$, therefore, we cannot get Lemma 3.5 and Lemma 3.6.

Next, we define operators $Q, S : C_\omega \to C_\omega$ by

$$(Qx)(t) = P\big(f\big(t, x\big(t-\tau(t)\big)\big)\big), \qquad (Sx)(t) = \sum_{i=1}^{n} c_i(t)x\big(t-\delta_i(t)\big). \tag{3.15}$$

From the above analysis, the existence of periodic solutions for Eq. (1.1) is equivalent to the existence of solutions for the operator equation

$$Qx + Sx = x \tag{3.16}$$

in C_ω. Moreover, we have the following lemma.

Lemma 3.9 *Q is completely continuous in C_ω.*

Proof Since T is completely continuous and N is bounded in C_ω, from Eq. (3.13), we see that P is completely continuous in C_ω. By Eq. (3.15), it is easy to verify that Q is completely continuous in C_ω. □

Now, we present our results of Eq. (1.1) in the case that $c_\infty \in (0, \frac{m}{M+m})$.

Case 1: $c_i(t) > 0$, $i = 1, 2, \dots, n$.

Theorem 3.10 *Assume that $M < (\frac{\pi}{\omega})^2$, $c_i(t) > 0$ and $0 < c_* \le \sum_{i=1}^{n} c_i(t) \le c_\infty < \frac{m}{M+m}$ hold. Furthermore, suppose the following condition is satisfied:*

(F_1) *There exist two positive constants r and R such that*

$$\frac{M^2(1-c_*)(1-c_\infty)r}{(m-(M+m)c_\infty)^2} < R$$

and

$$\frac{M^2(1-c_*)(1-c_\infty)r}{m-(M+m)c_\infty} \le f(t,x) \le \big(m-(M+m)c_\infty\big)R,$$

for all $t \in [0, \omega]$ and $x \in [r, R]$.

Then Eq. (1.1) has at least one positive ω-periodic solution $x(t)$ with $r \le x(t) \le R$.

Proof Let

$$\Omega = \{x \in C_\omega : r \le x \le R, \text{for all } t \in \mathbb{R}\}.$$

Obviously, Ω is a bounded closed convex set in C_ω.

For any $x \in \Omega$, $t \in \mathbb{R}$, we get by Eq. (3.15)

$$(Qx)(t+\omega) = P\big(f\big(t+\omega, x\big(t+\omega-\tau(t+\omega)\big)\big)\big) = P\big(f\big(t, x\big(t-\tau(t)\big)\big)\big) = (Qx)(t)$$

and

$$(Sx)(t+\omega) = \sum_{i=1}^{n} c_i(t+\omega)x\big(t+\omega-\delta_i(t+\omega)\big) = \sum_{i=1}^{n} c_i(t)x\big(t-\delta_i(t)\big) = (Sx)(t),$$

which show that $(Qx)(t)$ and $(Sx)(t)$ are ω-periodic. Thus, $Q(\Omega) \subset C_\omega$, $S(\Omega) \subset C_\omega$.

For all $x_1, x_2 \in \Omega$ and $t \in \mathbb{R}$, from Lemma 3.3, Lemma 3.6 and condition (F_1), we have

$$\begin{aligned}
(Qx_1)(t) + (Sx_2)(t) &= P\bigl(f\bigl(t, x_1\bigl(t - \tau(t)\bigr)\bigr)\bigr) + \sum_{i=1}^{n} c_i(t)x_2\bigl(t - \delta_i(t)\bigr) \\
&\leq \frac{M(1 - c_\infty)}{m - (M + m)c_\infty} \bigl\| T\bigl(f\bigl(t, x_1\bigl(t - \tau(t)\bigr)\bigr)\bigr) \bigr\| + \sum_{i=1}^{n} c_i(t)x_2\bigl(t - \delta_i(t)\bigr) \\
&\leq \frac{M(1 - c_\infty)}{m - (M + m)c_\infty} \max_{t \in [0,\omega]} \int_0^\omega G(t,s)f\bigl(s, x_1\bigl(s - \tau(s)\bigr)\bigr)\,ds \\
&\quad + \sum_{i=1}^{n} c_i(t)x_2\bigl(t - \delta_i(t)\bigr) \\
&\leq \frac{M(1 - c_\infty)}{m - (M + m)c_\infty} \bigl(m - (M + m)c_\infty\bigr)R \cdot \frac{1}{M} + R\sum_{i=1}^{n} c_i(t) \\
&\leq \frac{M(1 - c_\infty)}{m - (M + m)c_\infty} \bigl(m - (M + m)c_\infty\bigr)R \cdot \frac{1}{M} + c_\infty R \\
&= R
\end{aligned}$$

and

$$\begin{aligned}
(Qx_1)(t) + (Sx_2)(t) &= P\bigl(f\bigl(t, x_1\bigl(t - \tau(t)\bigr)\bigr)\bigr) + \sum_{i=1}^{n} c_i(t)x_2\bigl(t - \delta_i(t)\bigr) \\
&\geq \frac{m - (M + m)c_\infty}{M(1 - c_\infty)} \int_0^\omega G(t,s)f\bigl(s, x_1\bigl(s - \tau(s)\bigr)\bigr)\,ds \\
&\quad + \sum_{i=1}^{n} c_i(t)x_2\bigl(t - \delta_i(t)\bigr) \\
&\geq \frac{m - (M + m)c_\infty}{M(1 - c_\infty)} \cdot \frac{M^2(1 - c_\infty)}{m - (M + m)c_\infty}(1 - c_*)r \cdot \frac{1}{M} + r\sum_{i=1}^{n} c_i(t) \\
&\geq (1 - c_*)r + c_* r \\
&= r,
\end{aligned}$$

which imply that $r \leq Qx_1 + Sx_2 \leq R$, for all $x_1, x_2 \in \Omega$. Therefore, $Qx_1 + Sx_2 \in \Omega$.

For all $x_1, x_2 \in \Omega$, we obtain

$$\begin{aligned}
\bigl|Sx_1(t) - Sx_2(t)\bigr| &= \left| \sum_{i=1}^{n} c_i(t)x_1\bigl(t - \delta_i(t)\bigr) - \sum_{i=1}^{n} c_i(t)x_2\bigl(t - \delta_i(t)\bigr) \right| \\
&\leq \sum_{i=1}^{n} \bigl| c_i(t)\bigl(x_1\bigl(t - \delta_i(t)\bigr) - x_2\bigl(t - \delta_i(t)\bigr)\bigr) \bigr| \\
&\leq \sum_{i=1}^{n} \|c_i\| \|x_1 - x_2\| = c_\infty \|x_1 - x_2\|,
\end{aligned}$$

then from $c_\infty \in (0, \frac{m}{M+m})$, we conclude that S is contractive.

Since Q is completely continuous, by Theorem 3.1, there is an $x \in \Omega$ such that $Qx + Sx = x$. Therefore, Eq. (1.1) has at least one positive ω-periodic solution $x(t)$ with $r \le x(t) \le R$. □

Case 2: $c_i(t) < 0$, $i = 1, 2, \dots, n$.

We consider the existence of periodic solutions for Eq. (1.1) in the case that $-\frac{m}{M+m} < \sum_{i=1}^{n} c_i(t) < 0$. To conclude the main result, firstly, we consider the equation

$$Mc^2 - (2M + m)c + m = 0. \tag{3.17}$$

It is obvious that Eq. (3.17) has a solution $\zeta = \frac{2M+m-\sqrt{(2M+m)^2-4Mm}}{2M}$ and $0 < \zeta < \frac{m}{m+M}$. If $c_\infty < \zeta$, we have $Mc_\infty^2 - (2M + m)c_\infty + m > 0$.

On the other hand, for any $0 < c_1, c_2 < \frac{m}{m+M}$, we obtain

$$\begin{aligned}
&(M + m)c_1c_2 - mc_1 - Mc_2 + M \\
&\quad > (M + m)c_1c_2 - m\frac{m}{m + M} - M\frac{m}{m + M} + M \\
&\quad = (M + m)c_1c_2 - m + M > 0.
\end{aligned}$$

Then if $r > 0$, we can get $\frac{(M+m)c_\infty c_* - Mc_\infty - mc_* + M}{Mc_\infty^2 - (2M+m)c_\infty + m} r > 0$, since $c_* \le c_\infty < \frac{m}{M+m}$.

Therefore, we have the following theorem.

Theorem 3.11 *Assume that* $M < (\frac{\pi}{\omega})^2$, $c_i(t) < 0$ *and* $c_\infty < \min\{k_0, \zeta\}$ *hold. Furthermore, suppose the following condition is satisfied*:

(F_2) *There exist two positive constants* r, R *such that*

$$\frac{(M + m)c_\infty c_* - Mc_\infty - mc_* + M}{Mc_\infty^2 - (2M + m)c_\infty + m} r < R$$

and

$$M(r + c_\infty R) \le f(t, x) \le \frac{m - (M + m)c_\infty}{1 - c_\infty}(R + c_* r),$$

for all $t \in [0, \omega]$ *and* $x \in [r, R]$.

Then Eq. (1.1) *has at least one positive* ω*-periodic solution* $x(t)$ *with* $r \le x(t) \le R$.

Proof We follow the same notations as in the proof of Theorem 3.10. For all $x_1, x_2 \in \Omega$, from Lemma 3.3, Lemma 3.5 and condition (F_2), we see that

$$\begin{aligned}
(Qx_1)(t) + (Sx_2)(t) &= P\bigl(f\bigl(t, x_1\bigl(t - \tau(t)\bigr)\bigr)\bigr) + \sum_{i=1}^{n} c_i(t)x_2\bigl(t - \delta_i(t)\bigr) \\
&\le \frac{M(1 - c_\infty)}{m - (M + m)c_\infty}\bigl\| T\bigl(f\bigl(t, x_1\bigl(t - \delta(t)\bigr)\bigr)\bigr)\bigr\| + \sum_{i=1}^{n} c_i(t)x_2\bigl(t - \delta_i(t)\bigr) \\
&\le \frac{M(1 - c_\infty)}{m - (M + m)c_\infty} \max_{t\in[0,\omega]} \int_0^\omega G(t, s)f\bigl(s, x_1\bigl(s - \delta(s)\bigr)\bigr)\, ds + r\sum_{i=1}^{n} c_i(t)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{M(1-c_\infty)}{m-(M+m)c_\infty} \cdot \frac{m-(M+m)c_\infty}{1-c_\infty}(R+c_* r) \cdot \frac{1}{M} - c_* r \\
&= R
\end{aligned}$$

and

$$\begin{aligned}
(Qx_1)(t) + (Sx_2)(t) &= P\bigl(f\bigl(t, x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr) + \sum_{i=1}^{n} c_i(t) x_2\bigl(t-\delta_i(t)\bigr) \\
&\geq \int_0^\omega G(t,s) f\bigl(s, x_1\bigl(s-\tau(s)\bigr)\bigr)\,ds + \sum_{i=1}^{n} c_i(t) x_2\bigl(t-\delta_i(t)\bigr) \\
&\geq M(r + c_\infty R) \cdot \frac{1}{M} - c_\infty R \\
&= r.
\end{aligned}$$

From the above two inequalities, it is clear that $Qx_1 + Sx_2 \in \Omega$, for all $x_1, x_2 \in \Omega$.

We use a similar argument as in the proof of Theorem 3.10, we can observe that $Q(\Omega) \subset C_\omega$, $S(\Omega) \subset C_\omega$, S is contractive. Since Q is completely continuous, we get by a direct application of Theorem 3.1 that Eq. (1.1) has at least one positive ω-periodic solution $x(t)$ with $r \leq x(t) \leq R$. □

Remark 3.12 If $n = 1$, then $(Ax)(t) = x(t) - c_1(t)x(t-\delta_1(t))$, we can also get Theorem 3.10 and Theorem 3.11 in a similar way.

If $n = 1$ and $c_1(t) = c$, where c is a constant, from Corollary 3.7, we can get the following corollaries, which improve and extend the corresponding results from [5].

Corollary 3.13 *Assume that* $M < (\frac{\pi}{\omega})^2$ *and* $0 < c < \frac{m}{M+m}$ *hold. Furthermore, suppose the following condition is satisfied*:

(F_1^*) *There exist two positive constants r and R such that*

$$\frac{M^2(1-c)^2 r}{(m-(M+m)c)^2} < R$$

and

$$M^2(1-c)^2 r \leq f(t,x) \leq \bigl(m-(M+m)c\bigr)^2 R,$$

for all $t \in [0,\omega]$ *and* $x \in [r,R]$.

Then Eq. (1.1) *has at least one positive* ω*-periodic solution* $x(t)$ *with* $r \leq x(t) \leq R$.

Remark 3.14 Corollary 3.13 extends and improves Theorem 2.1 in [5].

Corollary 3.15 *Suppose that* $M < (\frac{\pi}{\omega})^2$, $c < 0$ *and* $|c| < \min\{\sigma, \zeta\}$ *hold. Furthermore, assume that the following condition is satisfied*:

(F_2^*) *There exist two non-negative constants* r, R *such that*

$$\frac{(M+m)|c|^2 - (M+m)|c| + M}{M|c|^2 - (2M+m)|c| + m} r < R$$

and

$$M\big(r+|c|R\big)\le f(t,x)\le \frac{m-(M+m)|c|}{1-|c|}\big(R+|c|r\big),$$

for all $t\in[0,\omega]$ *and* $x\in[r,R]$.

Then Eq. (1.1) *has at least one ω-periodic solution* $x(t)$ *with* $r\le x(t)\le R$.

Remark 3.16 Corollary 3.15 extends and improves Theorem 2.3 in [5].

4 Periodic solution for Eq. (1.2)

In this section, we investigative the existence of periodic solutions for Eq. (1.2) by applications of the extension of Mawhin's continuous theorem [9], in order to use this theorem, we recall it first.

Let $\widetilde{X}$ and $\widetilde{Z}$ be Banach spaces with norms $\|\cdot\|_{\widetilde{X}}$ and $\|\cdot\|_{\widetilde{Z}}$, respectively. A continuous operator $\widetilde{M}:\widetilde{X}\cap\operatorname{dom}\widetilde{M}\to\widetilde{Z}$ is said to be *quasi-linear* if

(1) $\operatorname{Im}\widetilde{M}:=\widetilde{M}(\widetilde{X}\cap\operatorname{dom}\widetilde{M})$ is a closed subset of $\widetilde{Z}$;

(2) $\ker\widetilde{M}:=\{x\in\widetilde{X}\cap\operatorname{dom}\widetilde{M}:\widetilde{M}x=0\}$ is a subspace of $\widetilde{X}$ with $\dim\ker\widetilde{M}<+\infty$.

Let $\widetilde{X}_1=\ker\widetilde{M}$ and $\widetilde{X}_2$ be the complement space of $\widetilde{X}_1$ in $\widetilde{X}$, then $\widetilde{X}=\widetilde{X}_1\oplus\widetilde{X}_2$. Meanwhile, $\widetilde{Z}_1$ is a subspace of $\widetilde{Z}$ and $\widetilde{Z}_2$ is the complement space of $\widetilde{Z}_1$ in $\widetilde{Z}$, so $\widetilde{Z}=\widetilde{Z}_1\oplus\widetilde{Z}_2$. Suppose that $\widetilde{P}:\widetilde{X}\to\widetilde{X}_1$ and $\widetilde{Q}:\widetilde{Z}\to\widetilde{Z}_1$ are two projects and $\widetilde{\Omega}\subset\widetilde{X}$ is an open bounded set with the origin $\tilde{\theta}\in\widetilde{\Omega}$.

Let $\widetilde{N}_{\tilde{\lambda}}:\overline{\widetilde{\Omega}}\to\widetilde{Z}$, $\tilde{\lambda}\in[0,1]$ is a continuous operator. Denote $\widetilde{N}_1$ by $\widetilde{N}$, and let $\sum_\lambda=\{x\in\overline{\widetilde{\Omega}}:\widetilde{M}x=\widetilde{N}_\lambda x\}$. $\widetilde{N}_{\tilde{\lambda}}$ is said to be $\widetilde{M}$-compact in $\overline{\widetilde{\Omega}}$ if

(3) there is a vector subspace $\widetilde{Z}_1$ of $\widetilde{Z}$ with $\dim\widetilde{Z}_1=\dim\widetilde{X}_1$ and an operator $\widetilde{R}:\overline{\widetilde{\Omega}}\times\widetilde{X}_2$ being continuous and compact such that, for $\tilde{\lambda}\in[0,1]$,

$$(\widetilde{I}-\widetilde{Q})\widetilde{N}_{\tilde{\lambda}}(\overline{\widetilde{\Omega}})\subset\operatorname{Im}\widetilde{M}\subset(\widetilde{I}-\widetilde{Q})\widetilde{Z},\tag{4.1}$$

$$\widetilde{Q}\widetilde{N}_{\tilde{\lambda}}x=0,\qquad \tilde{\lambda}\in(0,1)\quad\Leftrightarrow\quad \widetilde{Q}\widetilde{N}\tilde{x}=0,\tag{4.2}$$

$$\widetilde{R}(\cdot,0)\text{ is the zero operator}\quad\text{and}\quad \widetilde{R}(\cdot,\tilde{\lambda})|_{\sum_{\tilde{\lambda}}}=(\widetilde{I}-\widetilde{P})|_{\sum_{\tilde{\lambda}}},\tag{4.3}$$

and

$$\widetilde{M}\big[\widetilde{P}+\widetilde{R}(\cdot,\lambda)\big]=(\widetilde{I}-\widetilde{Q})\widetilde{N}_{\tilde{\lambda}}.\tag{4.4}$$

Lemma 4.1 ([9]) *Let* $\widetilde{X}$ *and* $\widetilde{Z}$ *be Banach space with norm* $\|\cdot\|_{\widetilde{X}}$ *and* $\|\cdot\|_{\widetilde{Z}}$, *respectively, and* $\widetilde{\Omega}\subset\widetilde{X}$ *be an open and bounded set with* $\tilde{\theta}\in\widetilde{\Omega}$. *Suppose that* $\widetilde{M}:\widetilde{X}\cap\operatorname{dom}\widetilde{M}\to\widetilde{Z}$ *is a quasi-linear operator and*

$$\widetilde{N}_{\tilde{\lambda}}:\overline{\widetilde{\Omega}}\to\widetilde{Z},\qquad \tilde{\lambda}\in(0,1)$$

is an $\widetilde{M}$*-compact mapping. In addition, if*

(a) $\widetilde{M}x\ne\widetilde{N}_{\tilde{\lambda}}x$, $\tilde{\lambda}\in(0,1)$, $x\in\partial\widetilde{\Omega}$,

(b) $\deg\{\widetilde{J}\widetilde{Q}\widetilde{N},\widetilde{\Omega}\cap\ker\widetilde{M},0\}\ne0$,

where $\widetilde{N}=\widetilde{N}_1$, *then the abstract equation* $\widetilde{M}x=\widetilde{N}x$ *has at least one solution in* $\overline{\widetilde{\Omega}}$.

Let $\widetilde{J}:\widetilde{Z}_1\to\widetilde{X}_1$ be a homeomorphism with $\widetilde{J}(\tilde{\theta})=\tilde{\theta}$.

Theorem 4.2 *Assume* $\sum_{i=1}^{n}\|c_i\|<1$, *or* $\|\frac{1}{c_k}\|+\sum_{\substack{i=1\\ i\neq k}}^{n}\|\frac{c_i}{c_k}\|<1$, $\widetilde{\Omega}$ *be open bounded set in* C_ω^1. *Suppose the following conditions hold*:

(i) *For each* $\tilde{\lambda}\in(0,1)$, *the equation*

$$\bigl(\phi_p(Ax)'(t)\bigr)'=\tilde{\lambda}\tilde{f}\bigl(t,x(t),x'(t)\bigr) \tag{4.5}$$

has no solution on $\partial\widetilde{\Omega}$.

(ii) *The equation*

$$\widetilde{F}(a):=\frac{1}{\omega}\int_0^\omega \tilde{f}\bigl(a,x(a),0\bigr)\,dt=0$$

has no solution on $\partial\widetilde{\Omega}\cap\mathbb{R}$.

(iii) *The Brouwer degree*

$$\deg\{\widetilde{F},\widetilde{\Omega}\cap\mathbb{R},0\}\neq 0.$$

Then Eq. (1.2) *has at least one periodic solution on* $\overline{\widetilde{\Omega}}$.

Proof In order to use Lemma 4.1 to study the existence of periodic solution to Eq. (1.2). We can set $\widetilde{X}:=\{x\in C[0,\omega]:x(0)=x(\omega)\}$ and $\widetilde{Z}:=C[0,\omega]$,

$$\widetilde{M}:\widetilde{X}\cap\operatorname{dom}\widetilde{M}\to\widetilde{Z},\qquad (\widetilde{M}x)(t)=\bigl(\phi_p(Ax)'(t)\bigr)', \tag{4.6}$$

where $\operatorname{dom}\widetilde{M}:=\{u\in\widetilde{X}:\phi_p(Au)'\in C^1(\mathbb{R},\mathbb{R})\}$. Then $\ker\widetilde{M}=\mathbb{R}$. In fact

$$\begin{aligned}\ker\widetilde{M}&=\bigl\{x\in\widetilde{X}:\bigl(\phi_p(Ax)'(t)\bigr)'=0\bigr\}\\&=\bigl\{x\in\widetilde{X}:\phi_p(Ax)'\equiv\tilde{c}\bigr\}\\&=\bigl\{x\in\widetilde{X}:(Ax)'\equiv\phi_q(c):=\tilde{c}_1\bigr\}\\&=\bigl\{x\in\widetilde{X}:(Ax)(t)\equiv\tilde{c}_1t+\tilde{c}_2\bigr\},\end{aligned}$$

where $q>1$ is a constant with $\frac{1}{p}+\frac{1}{q}=1$ and $\tilde{c}$, $\tilde{c}_1$, $\tilde{c}_2$ are constants in $\mathbb{R}$. Since $(Ax)(0)=(Ax)(\omega)$, we get $\ker\widetilde{M}=\{x\in\widetilde{X}:(Ax)(t)\equiv\tilde{c}_2\}$. In addition,

$$\begin{aligned}\operatorname{Im}\widetilde{M}=\biggl\{&\tilde{y}\in\widetilde{Z},\text{for } x(t)\in\widetilde{X}\cap\operatorname{dom}\widetilde{M},\bigl(\phi_p(Ax)'(t)\bigr)'=\tilde{y}(t),\\&\int_0^\omega\tilde{y}(t)\,dt=\int_0^\omega\bigl(\phi_p(Ax)'(t)\bigr)'\,dt=0\biggr\}.\end{aligned}$$

So $\widetilde{M}$ is *quasi-linear*. Let

$$\begin{aligned}&\widetilde{X}_1=\ker\widetilde{M},\qquad \widetilde{X}_2=\bigl\{x\in\widetilde{X}:x(0)=x(\omega)=0\bigr\},\\&\widetilde{Z}_1=\mathbb{R},\qquad \widetilde{Z}_2=\operatorname{Im}\widetilde{M}.\end{aligned}$$

Clearly, $\dim \widetilde{X}_1 = \dim \widetilde{Z}_1 = 1$, and $\widetilde{X} = \widetilde{X}_1 \oplus \widetilde{X}_2$, $\widetilde{P} : \widetilde{X} \to \widetilde{X}_1$, $\widetilde{Q} : \widetilde{Z} \to \widetilde{Z}_1$, be defined by

$$\widetilde{P}x = x(0), \qquad \widetilde{Q}y = \frac{1}{\omega}\int_0^{\omega} y(s)\,ds.$$

For $\forall \overline{\widetilde{\Omega}} \subset \widetilde{X}$, define $\widetilde{N}_{\tilde{\lambda}} : \overline{\widetilde{\Omega}} \to \widetilde{Z}$ by

$$(\widetilde{N}_{\tilde{\lambda}}x)(t) = \tilde{\lambda}\tilde{f}\bigl(t, x(t), x'(t)\bigr).$$

We claim $(\widetilde{I} - \widetilde{Q})\widetilde{N}_{\tilde{\lambda}}(\overline{\widetilde{\Omega}}) \subset \operatorname{Im}\widetilde{M} = (\widetilde{I} - \widetilde{Q})\widetilde{Z}$ holds. In fact, for $x \in \overline{\widetilde{\Omega}}$, we see that

$$\begin{aligned}
&\int_0^{\omega} (\widetilde{I} - \widetilde{Q})\widetilde{N}_{\tilde{\lambda}}x(t)\,dt \\
&\quad = \int_0^{\omega} (\widetilde{I} - \widetilde{Q})\tilde{\lambda}\tilde{f}\bigl(t, x(t), x'(t)\bigr)\,dt \\
&\quad = \int_0^{\omega} \tilde{\lambda}\tilde{f}\bigl(t, x(t), x'(t)\bigr)\,dt - \int_0^{\omega} \frac{\tilde{\lambda}}{\omega}\int_0^{\omega} \tilde{f}\bigl(s, x(s), x'(s)\bigr)\,ds\,dt \\
&\quad = 0.
\end{aligned}$$

Therefore, we have $(\widetilde{I} - \widetilde{Q})\widetilde{N}_{\tilde{\lambda}}(\overline{\widetilde{\Omega}}) \subset \operatorname{Im}\widetilde{M}$. Moreover, for any $x \in \widetilde{Z}$, we get

$$\begin{aligned}
&\int_0^{\omega} (\widetilde{I} - \widetilde{Q})x(t)\,dt \\
&\quad = \int_0^{\omega} \left(x(t) - \frac{1}{\omega}\int_0^{\omega} x(s)\,ds \right) dt \\
&\quad = \int_0^{\omega} x(t)\,dt - \int_0^{\omega} \frac{1}{\omega}\int_0^{\omega} x(s)\,ds\,dt \\
&\quad = 0.
\end{aligned}$$

So, $(\widetilde{I} - \widetilde{Q})\widetilde{Z} \subset \operatorname{Im}\widetilde{M}$. On the other hand, $x \in \operatorname{Im}\widetilde{M}$ and $\int_0^{\omega} x(t)\,dt = 0$, then we have $x(t) = x(t) - \int_0^{\omega} x(t)\,dt$. Hence, we can get $x(t) \in (\widetilde{I} - \widetilde{Q})\widetilde{Z}$. Therefore, $\operatorname{Im}\widetilde{M} = (\widetilde{I} - \widetilde{Q})\widetilde{Z}$.

From $\widetilde{Q}\widetilde{N}_{\tilde{\lambda}}x = 0$, we obtain

$$\frac{\tilde{\lambda}}{\omega}\int_0^{\omega} \tilde{f}\bigl(t, x(t), x'(t)\bigr)\,dt = 0.$$

Since $\tilde{\lambda} \in (0,1)$, then we have $\frac{1}{\omega}\int_0^{\omega} \tilde{f}(t, x(t), x'(t))\,dt = 0$. Therefore, $\widetilde{Q}\widetilde{N}x = 0$, then Eq. (4.4) also holds.

Let $\widetilde{J} : \widetilde{Z}_1 \to \widetilde{X}_1$, $\widetilde{J}(x) = x$, then $\widetilde{J}(0) = 0$. Define $\widetilde{R} : \overline{\widetilde{\Omega}} \times [0,1] \to \widetilde{X}_2$,

$$\begin{aligned}
&\widetilde{R}(x, \tilde{\lambda})(t) \\
&\quad = A^{-1}\int_0^t \phi_p^{-1}\left(\tilde{a} + \int_0^s \lambda\tilde{f}\bigl(u, x(u), x'(u)\bigr)\,du - \frac{\tilde{\lambda}s}{\omega}\int_0^{\omega} \tilde{f}\bigl(u, x(u), x'(u)\bigr)\,du \right) ds,
\end{aligned} \tag{4.7}$$

where $\tilde{a} \in \widetilde{R}$ is a constant such that

$$\widetilde{R}(x,\tilde{\lambda})(\omega) = A^{-1}\int_0^{\omega}\phi_p^{-1}\left(\tilde{a} + \int_0^s \tilde{\lambda}\tilde{f}\big(u,x(u),x'(u)\big)\,du - \frac{\tilde{\lambda}s}{\omega}\int_0^{\omega}\tilde{f}\big(u,x(t),x'(u)\big)\,du\right)ds$$
$$= 0. \tag{4.8}$$

From Lemma 3.1 of [16], we know that $\tilde{a}$ is uniquely defined by

$$\tilde{a} = \bar{\tilde{a}}(x,\lambda),$$

where $\bar{\tilde{a}}(x,\lambda)$ is continuous on $\overline{\widetilde{\Omega}} \times [0,1]$ and bounded sets of $\overline{\widetilde{\Omega}} \times [0,1]$ into bounded sets of $\mathbb{R}$.

From Eq. (4.4), one can find that

$$\widetilde{\mathrm{R}} : \overline{\widetilde{\Omega}} \times [0,1] \to \widetilde{X}_2.$$

Now, for any $x \in \sum_{\tilde{\lambda}} = \{x \in \overline{\widetilde{\Omega}} : \widetilde{M}x = \widetilde{N}_{\tilde{\lambda}}x\} = \{x \in \overline{\widetilde{\Omega}} : (\phi_p(Ax)'(t))' = \tilde{\lambda}\tilde{f}(t,x(t),x'(t))\}$, we have $\int_0^{\omega}\tilde{f}(t,x(t),x'(t))\,dt = 0$, together with Eq. (4.2) gives

$$\begin{aligned}\widetilde{R}(x,\tilde{\lambda})(t) &= A^{-1}\int_0^t \phi_p^{-1}\big(\tilde{a} + \int_0^s \tilde{\lambda}\tilde{f}\big(u,x(u),x'(u)\,du\big)\,ds \\ &= A^{-1}\int_0^t \phi_p^{-1}\left(\tilde{a} + \int_0^s \big(\phi_p(Ax)'(u)\big)'\,du\right)ds \\ &= A^{-1}\int_0^t \phi_p^{-1}\big(\tilde{a} + \phi_p(Ax)'(s) - \phi_p(Ax)'(0)\big)\,ds.\end{aligned}$$

Take $\tilde{a} = \phi_p(Ax)'(0)$, then we can get

$$\begin{aligned}\widetilde{R}(x,\tilde{\lambda})(\omega) &= A^{-1}\int_0^{\omega}\big(\phi_p^{-1}\big(\phi_p(Ax)'(s)\big)\big)\,ds \\ &= A^{-1}\int_0^{\omega}(Ax)'(t)\,ds \\ &= A^{-1}\big((Ax)(\omega) - (Ax)(0)\big) \\ &= x(\omega) - x(0) \\ &= 0,\end{aligned}$$

where $\tilde{a}$ is unique, we see that

$$\tilde{a} = \bar{\tilde{a}}(x,\tilde{\lambda}) = \phi_p(Ax)'(0), \quad \forall\tilde{\lambda} \in [0,1].$$

So, we have

$$\begin{aligned}\widetilde{R}(x,\tilde{\lambda})(t)|_{x\in\sum_{\tilde{\lambda}}} &= A^{-1}\int_0^t\left(\phi_p^{-1}\left(\phi_p(Ax)'(0) + \int_0^s \tilde{\lambda}\tilde{f}\big(u,x(u),x'(u)\big)\,du\right)\right)ds \\ &= A^{-1}\int_0^t \big(\phi_p^{-1}\big(\phi_p(Ax)'(s)\big)\big)\,ds\end{aligned}$$

$$\begin{aligned}&= A^{-1}\int_0^t (Ax)'(s)\,ds\\&= x(t)-x(0)\\&= (\widetilde{I}-\widetilde{P})x(t),\end{aligned}$$

which yields the second part of (4.3). Meanwhile, if $\tilde{\lambda}=0$, the

$$\sum_{\tilde{\lambda}} = \{x\in\overline{\widetilde{\Omega}} : \widetilde{M}x=\widetilde{N}_{\tilde{\lambda}}x\} = \bigl\{x\in\overline{\widetilde{\Omega}} : \bigl(\phi_p(Ax)'(t)\bigr)'=\tilde{\lambda}\tilde{f}\bigl(t,x(t),x'(t)\bigr)\bigr\} = \tilde{c}_3,$$

where $\tilde{c}_3\in\mathbb{R}$ is a constant. Thus, by the continuity of $\bar{\tilde{a}}(x,\tilde{\lambda})$ with respect to $(x,\tilde{\lambda})$, $\tilde{a} = \bar{\tilde{a}}(x,0)=\phi_p(A\tilde{c})'(0)=0$, we have

$$\widetilde{R}(x,0)(t)=A^{-1}\int_0^t \phi_p^{-1}(0)\,ds = 0,\quad \forall x\in\overline{\widetilde{\Omega}},$$

which yields the first part of Eq. (4.3). Furthermore, we consider

$$\widetilde{M}(\widetilde{P}+\widetilde{R})=(\widetilde{I}-\widetilde{Q})\widetilde{N}_{\tilde{\lambda}},$$

in fact,

$$\frac{d}{dt}\phi_p\bigl(A(\widetilde{P}+\widetilde{R})\bigr)'=(\widetilde{I}-\widetilde{Q})\widetilde{N}_{\tilde{\lambda}}. \tag{4.9}$$

Integrating both sides of Eq. (4.9) over $[0,s]$, we have

$$\int_0^s \frac{d}{dt}\phi_p\bigl(A(\widetilde{P}+\widetilde{R})\bigr)'\,ds = \int_0^s (\widetilde{I}-\widetilde{Q})\widetilde{N}_{\tilde{\lambda}}\,ds.$$

Therefore,

$$\begin{aligned}\phi_p\bigl(A(\widetilde{P}+\widetilde{R})\bigr)'(s)-\tilde{a} &= \tilde{\lambda}\int_0^s \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du-\int_0^s\frac{\tilde{\lambda}}{\omega}\int_0^\omega \tilde{f}\bigl(u,x(u),x'(u)\bigr)dudt\\&= \tilde{\lambda}\int_0^s \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du-\frac{\tilde{\lambda}s}{\omega}\int_0^\omega \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du,\end{aligned}$$

where $\tilde{a}:=\phi_p(A(\widetilde{P}+\widetilde{R}))'(0)$. Then we can get

$$\bigl(A(\widetilde{P}+\widetilde{R})\bigr)'(s)=\phi_p^{-1}\biggl(\tilde{a}+\tilde{\lambda}\int_0^s \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du-\frac{\tilde{\lambda}s}{\omega}\int_0^\omega \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du\biggr). \tag{4.10}$$

Integrating both sides of Eq. (4.10) over $[0,t]$, we arrive at

$$\begin{aligned}&\int_0^t \bigl(A(\widetilde{P}+\widetilde{R})\bigr)'(s)\,ds\\&\quad=\int_0^t \phi_p^{-1}\biggl(\tilde{a}+\tilde{\lambda}\int_0^s \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du-\frac{\tilde{\lambda}s}{\omega}\int_0^\omega \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du\biggr)\,ds,\end{aligned}$$

then

$$
\begin{aligned}
&(\widetilde{P}+\widetilde{R})(t)-(\widetilde{P}+\widetilde{R})(0)\\
&\quad =A^{-1}\left(\int_0^t \phi_p^{-1}\left(\tilde{a}+\tilde{\lambda}\int_0^s \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du-\frac{\tilde{\lambda}s}{\omega}\int_0^{\omega}\tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du\right)ds\right).
\end{aligned}
$$

Since $\widetilde{R}(x,\tilde{\lambda})(0)=0$, $\widetilde{P}(t)=\widetilde{P}(0)=0$, we can get

$$
\begin{aligned}
&\widetilde{R}(x,\tilde{\lambda})(t)\\
&\quad =A^{-1}\left(\int_0^t \phi_p^{-1}\left(\tilde{a}+\tilde{\lambda}\int_0^s \tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du-\frac{\tilde{\lambda}s}{\omega}\int_0^{\omega}\tilde{f}\bigl(u,x(u),x'(u)\bigr)\,du\right)ds\right).
\end{aligned}
$$

Hence, $\widetilde{N}_{\tilde{\lambda}}$ is M-compact on $\overline{\widetilde{\Omega}}$. Obviously, the equation

$$
\bigl(\phi_p(Ax)'(t)\bigr)'=\tilde{\lambda}\tilde{f}\bigl(t,x(t),x'(t)\bigr)
$$

can be converted to

$$
\widetilde{M}x=\widetilde{N}_{\tilde{\lambda}}x,\quad \tilde{\lambda}\in(0,1),
$$

where $\widetilde{M}$ and $\widetilde{N}_{\tilde{\lambda}}$ are defined by Eqs. (4.2) and (4.6), respectively. As proved above,

$$
\widetilde{N}_{\tilde{\lambda}}:\overline{\widetilde{\Omega}}\to\widetilde{Z},\quad \tilde{\lambda}\in(0,1)
$$

is an $\widetilde{M}$-compact mapping. From assumption (i), one finds

$$
\widetilde{M}x\neq\widetilde{N}_{\tilde{\lambda}}x,\quad \tilde{\lambda}\in(0,1),x\in\partial\widetilde{\Omega},
$$

and assumptions (ii) and (iii) imply that $\deg\{\widetilde{J}\widetilde{Q}\widetilde{N},\widetilde{\Omega}\cap\ker\widetilde{M},\tilde{\theta}\}$ is valid and

$$
\deg\{\widetilde{J}\widetilde{Q}\widetilde{N},\widetilde{\Omega}\cap\ker\widetilde{M},\tilde{\theta}\}\neq 0.
$$

So by applications of Lemma 4.2, we see that Eq. (4.5) has one ω-periodic solution. □

4.1 Application of Theorem 4.2: quasi-linear equation

As an application, we consider the following p-Laplacian neutral equation:

$$
\left(\phi_p\left(x(t)-\sum_{i=1}^{n}c_i(t)x\bigl(t-\delta_i(t)\bigr)\right)'\right)'+g\bigl(t,x(t)\bigr)=p(t),\tag{4.11}
$$

where $g(t,x(t))\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is an ω-periodic function about t, $p\in C(\mathbb{R},\mathbb{R})$ is an ω-periodic function and $\int_0^{\omega}p(t)\,dt=0$. By application of Theorem 4.2, we will investigate the existence of periodic solution for Eq. (4.11) satisfying $\sum_{i=1}^{n}\|c_i\|<1$, or $\|\frac{1}{c_k}\|+\sum_{i=1, i\neq k}^{n}\|\frac{c_i}{c_k}\|<1$.

Theorem 4.3 *Assume the following conditions are satisfied*:

(H_1) *There exist two positive constants* $\widetilde{D}_1$ *and* $\widetilde{D}_2$ *with* $\widetilde{D}_1 < \widetilde{D}_2$, *such that* $g(t,x(t)) > 0$ *for* $x(t) > \widetilde{D}_2$ *and* $g(t,x(t)) < 0$ *for* $x(t) < -\widetilde{D}_1$.

(H_2) *There exist positive constants* m, n *and* $\widetilde{B}$ *such that*

$$\bigl|g\bigl(t,x(t)\bigr)\bigr| \le m|x|^{p-1} + n, \quad \text{for } |x| > \widetilde{B} \text{ and } t \in \mathbb{R}.$$

Then Eq. (4.11) *has at least one solution with period* ω *if*

$$\frac{\tilde{\sigma}\omega(m(1+\sum_{i=1}^n \|c_i\|))^{\frac{1}{p}}}{2} + \frac{\tilde{\sigma}\omega\sum_{i=1}^n \|c_i'\|}{2} + \tilde{\sigma}\sum_{i=1}^n \frac{\|c_i\|\|\delta_i'\|}{1-\delta_i'(t)} < 1,$$

where $\delta_i'(t) < 1$ *for* $i = 1, 2, \ldots, n$ *and*

$$\tilde{\sigma} = \begin{cases} \frac{1}{1-\sum_{i=1}^n \|c_i\|}, & \text{for } \sum_{i=1}^n \|c_i\| < 1, \\ \frac{1}{1-\|\frac{1}{c_k}\|-\sum_{i=1, i\neq k}^n \|\frac{c_i}{c_k}\|}, & \text{for } \|\frac{1}{c_k}\| + \sum_{i=1, i\neq k}^n \|\frac{c_i}{c_k}\| < 1. \end{cases}$$

Proof Consider the homotopic equation

$$\left(\phi_p\left(x(t) - \sum_{i=1}^n c_i(t)x\bigl(t-\delta_i(t)\bigr)\right)'\right)' + \tilde{\lambda} g\bigl(t,x(t)\bigr) = \tilde{\lambda} p(t). \tag{4.12}$$

Firstly, we claim that the set of all ω-periodic solutions of Eq. (4.12) is bounded. Let $x(t) \in C_\omega$ be an arbitrary ω-periodic solution of Eq. (4.12). Integrating both sides of Eq. (4.12) over $[0,\omega]$, we have

$$\int_0^\omega g\bigl(t,x(t)\bigr)\,dt = 0. \tag{4.13}$$

From the mean value theorem, there is a constant $\xi \in (0,\omega)$ such that

$$g\bigl(\xi,x(\xi)\bigr) = 0,$$

then we get by condition (H_1)

$$-\widetilde{D}_1 \le x(\xi) \le \widetilde{D}_2.$$

Therefore,

$$\begin{aligned} \|x\| &= \max_{t\in[0,\omega]} \bigl|x(t)\bigr| = \max_{t\in[\xi,\xi+\omega]} \bigl|x(t)\bigr| \\ &= \frac{1}{2} \max_{t\in[\xi,\xi+\omega]} \bigl(\bigl|x(t)\bigr| + \bigl|x(t-\omega)\bigr|\bigr) \\ &= \frac{1}{2} \max_{t\in[\xi,\xi+\omega]} \left(\left|x(\xi) + \int_\xi^\omega x'(s)\,ds\right| + \left|x(\xi) - \int_{t-\omega}^\xi x'(s)\,ds\right|\right) \end{aligned}$$

$$\le \widetilde{D}_2 + \frac{1}{2}\left(\int_{\xi}^{t} \left|x'(s)\right| ds + \int_{t-\omega}^{\xi} \left|x'(s)\right| ds\right)$$
$$\le \widetilde{D}_2 + \frac{1}{2}\int_{0}^{\omega} \left|x'(s)\right| ds. \tag{4.14}$$

Multiplying both sides of Eq. (4.12) by $(Ax)(t)$ and integrating over the interval $[0,\omega]$, we get

$$\int_0^{\omega} \left(\phi_p(Ax)'(t)\right)'(Ax)(t)\,dt + \tilde{\lambda}\int_0^{\omega} g\big(t,x(t)\big)(Ax)(t)\,dt = \tilde{\lambda}\int_0^{\omega} p(t)(Ax)(t)\,dt. \tag{4.15}$$

Substituting $\int_0^{\omega}(\phi_p(Ax)'(t))'(Ax)(t)\,dt = -\int_0^{\omega}|(Ax)'(t)|^p\,dt$ into Eq. (4.15), it is clear that

$$-\int_0^{\omega} \left|(Ax)'(t)\right|^p dt = -\tilde{\lambda}\int_0^{\omega} g\big(t,x(t)\big)(Ax)(t)\,dt + \tilde{\lambda}\int_0^{\omega} p(t)(Ax)(t)\,dt.$$

So, we have

$$\begin{aligned}\int_0^{\omega} \left|(Ax)'(t)\right|^p dt &\le \int_0^{\omega} \left|g\big(t,x(t)\big)\right| \left|x(t) - \sum_{i=1}^{n} c_i(t)x\big(t-\delta_i(t)\big)\right| dt \\ &\quad + \int_0^{\omega} \left|p(t)\right| \left|x(t) - \sum_{i=1}^{n} c_i(t)x\big(t-\delta_i(t)\big)\right| dt \\ &\le \left(1+\sum_{i=1}^{n}\|c_i\|\right)\|x\|\int_0^{\omega}\left|g\big(t,x(t)\big)\right| dt \\ &\quad + \left(1+\sum_{i=1}^{n}\|c_i\|\right)\|x\|\int_0^{\omega}\left|p(t)\right| dt.\end{aligned} \tag{4.16}$$

Define

$$E_1 := \left\{t\in[0,\omega] \mid \left|x(t)\right| \le \widetilde{B}\right\}, \qquad E_2 := \left\{t\in[0,\omega] \mid \left|x(t)\right| > \widetilde{B}\right\}.$$

From condition (H_2), we obtain

$$\begin{aligned}\int_0^{\omega} \left|(Ax)'(t)\right|^p dt &\le \left(1+\sum_{i=1}^{n}\|c_i\|\right)\|x\|\int_{E_1+E_2}\left|g\big(t,x(t)\big)\right| dt \\ &\quad + \left(1+\sum_{i=1}^{n}\|c_i\|\right)\|x\|\int_0^{\omega}\left|p(t)\right| dt \\ &\le \left(1+\sum_{i=1}^{n}\|c_i\|\right)\|x\|\left(m\|x\|^{p-1}\omega + n\omega + \|g_{\widetilde{B}}\|\omega\right) \\ &\quad + \left(1+\sum_{i=1}^{n}\|c_i\|\right)\|p\|\omega\|x\| \\ &= m\omega\left(1+\sum_{i=1}^{n}\|c_i\|\right)\|x\|^p + \widetilde{N}_1\|x\|,\end{aligned} \tag{4.17}$$

where $\|g_{\widetilde{B}}\| := \max_{|x|\le\widetilde{B}} |g(t,x(t))|$, $\|p\| := \max_{t\in[0,\omega]} |p(t)|$ and $\widetilde{N}_1 := (1+\sum_{i=1}^n \|c_i\|)(\|g_{\widetilde{B}}\|\omega + n\omega + \|p\|\omega)$. Substituting Eq. (4.14) into Eq. (4.17), we get

$$\int_0^\omega \big|(Ax)'(t)\big|^p\,dt \le m\omega\left(1+\sum_{i=1}^n \|c_i\|\right)\left(\widetilde{D}_2 + \frac{1}{2}\int_0^\omega \big|x'(t)\big|\,dt\right)^p + \widetilde{N}_1\left(\widetilde{D}_2 + \frac{1}{2}\int_0^\omega \big|x'(t)\big|\,dt\right). \tag{4.18}$$

Since $(Ax)(t) = x(t) - \sum_{i=1}^n x(t-\delta_i(t))$, we arrive at

$$\begin{aligned}(Ax)'(t) &= \left(x(t) - \sum_{i=1}^n c_i(t)x\big(t-\delta_i(t)\big)\right)' \\ &= x'(t) - \sum_{i=1}^n c_i'(t)x\big(t-\delta_i(t)\big) - \sum_{i=1}^n c_i(t)x'\big(t-\delta_i(t)\big)\big(1-\delta_i'(t)\big) \\ &= x'(t) - \sum_{i=1}^n c_i'(t)x\big(t-\delta_i(t)\big) - \sum_{i=1}^n c_i(t)x'\big(t-\delta_i(t)\big) \\ &\quad + \sum_{i=1}^n c_i(t)x'\big(t-\delta_i(t)\big)\delta_i'(t)\end{aligned}$$

and

$$\big(Ax'\big)(t) = x'(t) - \sum_{i=1}^n c_i(t)x'\big(t-\delta_i(t)\big).$$

Thus,

$$\big(Ax'\big)(t) = (Ax)'(t) + \sum_{i=1}^n c_i'(t)x\big(t-\delta_i(t)\big) - \sum_{i=1}^n c_i(t)x'\big(t-\delta_i(t)\big)\delta_i'(t).$$

By applying Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned}\int_0^\omega \big|x'(t)\big|\,dt &= \int_0^\omega \big|\big(A^{-1}Ax'\big)(t)\big|\,dt \le \tilde{\sigma}\int_0^\omega \big|\big(Ax'\big)(t)\big|\,dt \\ &= \tilde{\sigma}\int_0^\omega \left|(Ax)'(t) + \sum_{i=1}^n c_i'(t)x\big(t-\delta_i(t)\big) - \sum_{i=1}^n c_i(t)x'\big(t-\delta_i(t)\big)\delta_i'(t)\right|\,dt \\ &\le \tilde{\sigma}\int_0^\omega \big|(Ax)'(t)\big|\,dt + \tilde{\sigma}\int_0^\omega \left|\sum_{i=1}^n c_i'(t)x\big(t-\delta_i(t)\big)\right|\,dt \\ &\quad + \tilde{\sigma}\int_0^\omega \left|\sum_{i=1}^n c_i(t)x'\big(t-\delta_i(t)\big)\delta_i'(t)\right|\,dt \\ &\le \tilde{\sigma}\,\omega^{\frac{1}{q}}\left(\int_0^\omega \big|(Ax)'(t)\big|^p\,dt\right)^{\frac{1}{p}} + \tilde{\sigma}\,\omega\sum_{i=1}^n \big\|c_i'\big\|\,\|x\| \\ &\quad + \tilde{\sigma}\sum_{i=1}^n \|c_i\|\,\big\|\delta_i'\big\|\int_0^\omega \big|x'\big(t-\delta_i(t)\big)\big|\,dt\end{aligned}$$

$$\le \tilde{\sigma}\omega^{\frac{1}{q}}\left(\int_0^\omega \left|(Ax)'(t)\right|^p dt\right)^{\frac{1}{p}} + \tilde{\sigma}\omega\sum_{i=1}^n \left\|c_i'\right\|\|x\| + \tilde{\sigma}\sum_{i=1}^n \frac{\|c_i\|\|\delta_i'\|}{1-\delta_i'}\int_0^\omega \left|x'(t)\right| dt, \tag{4.19}$$

where $\|c_i'\| = \max_{t\in[0,\omega]}|c_i'(t)|$, $\|\delta_i'\| = \max_{t\in[0,\omega]}|\delta_i'(t)|$, for $i = 1,2,\ldots,n$. Substituting Eq. (4.14) and Eq. (4.18) into Eq. (4.19), since $(a+b)^k \le a^k + b^k$, $0 < k < 1$, we get

$$\begin{aligned}\int_0^\omega \left|x'(t)\right| dt &\le \tilde{\sigma}\omega^{\frac{1}{q}}\left(m\omega\left(1+\sum_{i=1}^n \|c_i\|\right)\right)^{\frac{1}{p}}\left(\widetilde{D}_2 + \frac{1}{2}\int_0^\omega \left|x'(t)\right| dt\right) \\ &\quad + \tilde{\sigma}\omega^{\frac{1}{q}}(\widetilde{N}_1)^{\frac{1}{p}}\left(\widetilde{D}_2 + \frac{1}{2}\int_0^\omega \left|x'(t)\right| dt\right)^{\frac{1}{p}} \\ &\quad + \tilde{\sigma}\sum_{i=1}^n \left\|c_i'\right\|\omega\left(\widetilde{D}_2 + \frac{1}{2}\int_0^\omega \left|x'(t)\right| dt\right) + \tilde{\sigma}\sum_{i=1}^n \frac{\|c_i\|\|\delta_i'\|}{1-\delta_i'}\int_0^\omega \left|x'(t)\right| dt \\ &\le \left(\frac{\tilde{\sigma}\omega(m(1+\sum_{i=1}^n \|c_i\|))^{\frac{1}{p}}}{2} + \frac{\tilde{\sigma}\sum_{i=1}^n \|c_i'\|\omega}{2} + \tilde{\sigma}\sum_{i=1}^n \frac{\|c_i\|\|\delta_i'\|}{1-\delta_i'}\right) \\ &\quad \times \int_0^\omega \left|x'(t)\right| dt + \tilde{\sigma}\omega^{\frac{1}{q}}\left(m\omega\left(1+\sum_{i=1}^n \|c_i\|\right)\right)^{\frac{1}{p}}\widetilde{D}_2 + \tilde{\sigma}\omega^{\frac{1}{q}}(\widetilde{N}_1\widetilde{D}_2)^{\frac{1}{p}} \\ &\quad + \tilde{\sigma}\omega^{\frac{1}{q}}(\widetilde{N}_1)^{\frac{1}{p}}\left(\frac{1}{2}\int_0^\omega \left|x'(t)\right| dt\right)^{\frac{1}{p}} + \tilde{\sigma}\sum_{i=1}^n \left\|c_i'\right\|\omega\widetilde{D}_2. \end{aligned}\tag{4.20}$$

Since $\frac{\tilde{\sigma}\omega(m(1+\sum_{i=1}^n \|c_i\|))^{\frac{1}{p}}}{2} + \frac{\tilde{\sigma}\omega\sum_{i=1}^n \|c_i'\|}{2} + \tilde{\sigma}\sum_{i=1}^n \frac{\|c_i\|\|\delta_i'\|}{1-\delta_i'} < 1$, it is easily seen that there exists a constant $M_1' > 0$ (independent of $\tilde{\lambda}$) such that

$$\int_0^\omega \left|x'(t)\right| dt \le M_1'. \tag{4.21}$$

From Eq. (4.14), we have

$$\|x\| \le \widetilde{D}_2 + \frac{1}{2}\int_0^\omega \left|x'(s)\right| ds \le \widetilde{D}_2 + \frac{1}{2}M_1' := M_1. \tag{4.22}$$

As $(Ax)(0) = (Ax)(\omega)$, there exists a point $t_0 \in (0,\omega)$ such that $(Ax)'(t_0) = 0$, while $\phi_p(0) = 0$, from Eq. (4.12), we see that

$$\begin{aligned}\left|\phi_p(Ax)'(t)\right| &= \left|\int_{t_0}^t \left(\phi_p(Ax)'(s)\right)' ds\right| \\ &\le \tilde{\lambda}\int_0^\omega \left|g\big(t,x(t)\big)\right| dt + \tilde{\lambda}\int_0^\omega \left|p(t)\right| dt \\ &\le \omega\|g_{M_1}\| + \omega\|p\| := M_2',\end{aligned}$$

where $\|g_{M_1}\| := \max_{|x(t)|\le M_1} |g(t,x(t))|$. Next we claim that there exists a positive constant $M_2^* > M_2' + 1$, such that, for all $t \in \mathbb{R}$, we obtain

$$\left\|(Ax)'\right\| \le M_2^*. \tag{4.23}$$

In fact, if $(Ax)'$ is not bounded, there exists a positive constant M_2'' such that $\|(Ax)'\| > M_2''$ for some $(Ax)' \in \mathbb{R}$, therefore, we have $\|\phi_p(Ax)'\| = \|(Ax)'\|^{p-1} \ge (M_2'')^{p-1}$. Then it is a contradiction, so Eq. (4.23) holds. From Lemma 2.2 and Eq. (4.23), we arrive at

$$\begin{aligned} \left\|x'\right\| &= \left\|A^{-1}Ax'\right\| \\ &= \left\|A^{-1}\left(Ax'\right)(t)\right\| \\ &\le \tilde{\sigma}\left\|(Ax)'(t) + \sum_{i=1}^{n} c_i'(t)x\left(t-\delta_i(t)\right) - \sum_{i=1}^{n}\|c_i\|\left\|\delta_i'\right\|\int_0^{\omega}\left|x'\left(t-\delta_i(t)\right)\right|dt\right\| \\ &\le \tilde{\sigma}\left\|(Ax)'\right\| + \tilde{\sigma}\left(\sum_{i=1}^{n}\left\|c_i'\right\|\|x\|\right) + \tilde{\sigma}\sum_{i=1}^{n}\|c_i\|\left\|\delta_i'\right\|\int_0^{\omega}\left|x'\left(t-\delta_i(t)\right)\right|dt \\ &\le \tilde{\sigma}M_2^* + \tilde{\sigma}\sum_{i=1}^{n}\left\|c_i'\right\|M_1 + \tilde{\sigma}\sum_{i=1}^{n}\frac{\|c_i\|\|\delta_i'\|}{1-\delta_i'}M_1' := M_2. \end{aligned} \tag{4.24}$$

Set $M^* = \sqrt{M_1^2 + M_2^2} + 1$, we have

$$\widetilde{\Omega} = \left\{x \in C_\omega^1(\mathbb{R},\mathbb{R}) \mid \|x\| \le M^* + 1, \left\|x'\right\| \le M^* + 1\right\},$$

and we know that Eq. (4.11) has no solution on $\partial\widetilde{\Omega}$ as $\tilde{\lambda} \in (0,1)$ and when $x(t) \in \partial\widetilde{\Omega} \cap \mathbb{R}$, $x(t) = M^* + 1$ or $x(t) = -M^* - 1$. So, from condition (H_1), we see that

$$\begin{aligned} &\frac{1}{\omega}\int_0^{\omega} g\left(M^* + 1\right)dt > 0, \\ &\frac{1}{\omega}\int_0^{\omega} g\left(-M^* - 1\right)dt < 0, \end{aligned}$$

since $\int_0^{\omega} e(t)\,dt = 0$. So condition (ii) of Theorem 4.2 is also satisfied. Obviously, we can get

$$\begin{aligned} \deg\{\widetilde{F}, \widetilde{\Omega} \cap \mathbb{R}, 0\} &= \deg\left\{\frac{1}{\omega}\int_0^{\omega} g\left(t,x(t)\right)dt, \partial\widetilde{\Omega} \cap \mathbb{R}, 0\right\} \\ &= \deg\{x, \partial\widetilde{\Omega} \cap \mathbb{R}, 0\} \ne 0. \end{aligned}$$

So condition (iii) of Theorem 4.2 is satisfied. In view of Theorem 4.2, there exists at least one ω-periodic solution. □

5 Conclusions

In this paper, we first investigate some properties of the neutral operator with multiple variable parameters $(Ax)(t)$. Afterwards, applying Krasnoselskii's fixed point theorem and properties of the operator A, we prove the existence of a positive periodic solution for a second-order neutral differential equation with multiple variable parameters. On the other hand, we find that the second-order quasi-linear neutral differential equation has a periodic solution by using the extension of Mawhin's continuous theorem.

Acknowledgements
FFL, ZHB, SWY and YX are grateful to anonymous referees for their constructive comments and suggestions which have greatly improved this paper.

Funding
This work was supported by National Natural Science Foundation of China (No. 71601072), Education Department of Henan Province project (Nos. 16B110006, 20B110006), Fundamental Research Funds for the Universities of Henan Province (NSFRF170302), Young backbone teachers of colleges and universities in Henan Province (2017GGJS057).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
FFL, ZHB, SWY and YX contributed to each part of this study equally and declare that they have no competing interests.

Competing interests
FL, ZHB, SWY and YX declare that they have no competing interests.

Consent for publication
FFL, ZHB, SWY and YX read and approved the final version of the manuscript.

Authors' contributions
FFL, ZHB, SWY and YX contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Publisher's Note

Received: 3 August 2019 Accepted: 17 December 2019 Published online: 10 January 2020

References

1. Burton, T.: A fixed-point theorem of Krasnoselskii. Appl. Math. Lett. **11**, 85–88 (1998)
2. Cheng, Z., Bi, Z.: Study on a kind of *p*-Laplacian neutral differential equation with multiple variable coefficients. J. Appl. Anal. Comput. **9**, 501–525 (2019)
3. Cheng, Z., Li, F.: Positive periodic solutions for a kind of second-order neutral differential equations with variable coefficient and delay. Mediterr. J. Math. **15**, 1–19 (2018)
4. Cheng, Z., Li, F.: Weak and strong singularities fro second-order nonlinear differential equations with a linear difference operator. J. Fixed Point Theory Appl. **21**, 1–23 (2019)
5. Cheung, W., Ren, J., Han, W.: Positive periodic solution of second-order neutral functional differential equations. Nonlinear Anal. **71**, 3948–3955 (2009)
6. Danca, M., Feckan, M., Pospisil, M.: Difference equations with impulses. Opusc. Math. **39**, 5–22 (2019)
7. Du, B.: Anti-periodic solutions problems for inertial competitive neutral-type neutral networks via Wirtinger inequality. J. Inequal. Appl. **2019**, 187 (2019)
8. Du, B., Guo, L., Ge, W., Lu, S.: Periodic solutions for generalized Liénard neutral equation with variable parameter. Nonlinear Anal. TMA **70**, 2387–2394 (2009)
9. Ge, W., Ren, J.: An extension of Mawhin's continuation theorem and its application to boundary value problems with a *p*-Laplacian. Nonlinear Anal. TMA **58**, 477–488 (2004)
10. Hale, J.: Theory of Functional Differential Equations. Springer, Berlin (1977)
11. Kuang, Y.: Delay Differential Equations with Applications in Population Dynamics. Academic Press, San Diego (1993)
12. Lu, S., Ge, W.: Periodic solutions for a kind of second order differential equation with multiple deviating arguments. Appl. Math. Comput. **146**, 195–209 (2003)
13. Lu, S., Ge, W.: Existence of periodic solutions for a kind of second order neutral functional differential equation. Appl. Math. Comput. **157**, 433–448 (2004)
14. Luo, Y., Wei, W., Shen, J.: Existence of positive periodic solutions for two kinds of neutral functional differential equations. Appl. Math. Lett. **21**, 581–587 (2008)
15. Lv, L., Cheng, Z.: Positive periodic solution to superlinear neutral differential equation with time-dependent parameter. Appl. Math. Lett. **98**, 271–277 (2019)
16. Manasevich, R., Mawhin, J.: Periodic solutions for nonlinear systems with *p*-Laplacian-like operators. J. Differ. Equ. **145**, 367–393 (1998)
17. Pinelas, S., Dix, J.: Oscillation of solutions to non-linear difference equations with several advanced arguments. Opusc. Math. **37**, 889–898 (2017)
18. Radulescu, V., Repovs, D.: Partial Differential Equations with Variable Exponents, Variational Methods and Qualitative Analysis. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton (2015)

19. Ren, J., Cheng, Z., Siegmund, S.: Neutral operator and neutral differential equation. Abstr. Appl. Anal. **2011**, 969276 (2011)
20. Ren, J., Zhu, D., Wang, H.: Spreading-vanishing dichotomy in information diffusion in online social networks with intervention. Discrete Contin. Dyn. Syst., Ser. B **24**, 1843–1865 (2019)
21. Stevic, S.: Solvability of a product-type system of difference equations with six parameters. Adv. Nonlinear Anal. **8**, 29–51 (2019)
22. Wu, J., Wang, Z.: Two periodic solutions of second-order neutral functional differential equations. J. Math. Anal. Appl. **329**, 677–689 (2007)
23. Xu, Y., Zhu, D., Ren, J.: On a reaction–diffusion–advection system: fixed boundary or free boundary. Electron. J. Qual. Theory Differ. Equ. **2018**, 26 (2018)
24. Yao, S., Cheng, Z.: The homotopy perturbation method for a nonlinear oscillator with a damping. J. Low Freq. Noise Vib. Act. Control **38**, 1110–1112 (2019)
25. Yao, S., Ma, Z., Cheng, Z.: Pattern formation of a diffusive predator–prey model with strong Allee effect and nonconstant death rate. Physica A **527**, 1–11 (2019)
26. Zhang, M.: Periodic solution of linear and quasilinear neutral functional differential equation. J. Math. Anal. Appl. **189**, 378–392 (1995)
27. Zhou, T., Du, B., Du, H.: Positive periodic solution for indefinite singular Liénard equation with *p*-Laplacian. Adv. Differ. Equ. **2019**, 158 (2019)