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Existence of ground state solutions for an asymptotically 2-linear fractional Schrödinger–Poisson system

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Abstract

In this paper, we investigate the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\frac{3}{4} < s < 1$, $\frac{1}{2} < t < 1$, and f is a continuous function, which is superlinear at zero, with $f(\tau)\tau \geq 3F(\tau) \geq 0$, $F(\tau) = \int_0^\tau f(s)\,ds$, $\tau \in \mathbb{R}$. We prove that the system admits a ground state solution under the asymptotically 2-linear condition. The result here extends the existing study.

Keywords: Fractional Schrödinger–Poisson system; Ground state solution; Asymptotically 2-linear; Variational methods

1 Introduction

In this paper, we study the existence of ground state solutions for the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\frac{3}{4} < s < 1$, $\frac{1}{2} < t < 1$, $(-\Delta)^s$ and $(-\Delta)^t$ are the fractional Laplace operators, f satisfies the following conditions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim_{\tau \to 0} \frac{f(\tau)}{\tau} = 0$;

(f_2)

$$\lim_{|\tau| \to \infty} \frac{f(\tau)}{|\tau|^2} = \mu \quad \text{with} \quad \sqrt{\frac{54}{\pi} C(3,s)^{-1} S^2 \left(\frac{32\pi}{3}\right)^{\frac{5}{3}}} < \mu < +\infty,$$

where the constant S and the function $C(3,s)$ will be specified in Sect. 2;

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(f_3)

$$f(\tau)\tau \geq 3F(\tau) \geq 0, \quad \forall \tau \in \mathbb{R}, \text{where } F(\tau) = \int_0^\tau f(s)\,ds.$$

In recent years, the nonlinear fractional Schrödinger–Poisson systems have received a lot of attention. In [1], Gao, Tang and Chen studied the existence of ground state solutions of (1.1) in a mild assumption on f with super-quadratic nonlinearity. If u is replaced by $V(x)u$ and $f(u) = \mu|u|^{q-2}u + |u|^{2_s^*-2}u$ ($2_s^* = \frac{6}{3-2s}$) in (1.1), the existence of a nontrivial ground state solution is given by Teng [2]. In [3], based on the symmetric mountain pass theorem, He and Jing investigated a class of fractional Schrödinger–Poisson system with superlinear terms, the existence and multiplicity of nontrivial solutions of such a system are obtained. Wang, Ma and Guan [4] studied the existence of a sign-changing solution of the following nonlinear fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = K(x)f(u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

by means of the constraint variational method and the quantitative deformation lemma.

When $s = t = 1$, system (1.1) reduces to the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + u + \phi u = g(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

Yin, Wu and Tang [5] proved the existence of ground state solutions of (1.2) by using

$$\lim_{|t|\to\infty} \frac{f(t)}{|t|^2} = \nu \quad \text{with} \sqrt{\frac{189}{8\pi S}\left(\frac{32\pi}{3}\right)^{\frac{5}{3}}} < \nu < +\infty,$$

instead of the usual 2-superlinear condition $\lim_{|t|\to\infty} \frac{G(t)}{|t|^3} = +\infty$ ($G(t) = \int_0^t g(s)\,ds$), which relaxed the conditions of nonlinearity in [6–8].

Inspired by [5], the main objective of this paper is to extend the main results of [1], by relaxing the condition of super-quadratic nonlinearity used in [1]. That is, the nonlinearity f is assumed to be asymptotically 2-linear. We deal with the nonlinear fractional Schrödinger–Poisson system (1.1) in view of variational method and some analysis technique. Our result also extends the main results of [5].

2 Preliminaries

The fractional Sobolev space $H^s(\mathbb{R}^3)$ can be described by means of the Fourier transform, i.e.

$$H^s(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\xi|^{2s}\left|\widehat{u}(\xi)\right|^2 + \left|\widehat{u}(\xi)\right|^2\right) d\xi < +\infty\right\}$$

endowed with the norm

$$\begin{aligned}\|u\| := \|u\|_{H^s} &= \left(\int_{\mathbb{R}^3} \left(|\xi|^{2s}\left|\widehat{u}(\xi)\right|^2 + \left|\widehat{u}(\xi)\right|^2\right) d\xi\right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\frac{s}{2}} u(x)\right|^2 + \left|u(x)\right|^2\right) dx\right)^{\frac{1}{2}}, \quad \forall u \in H^s\left(\mathbb{R}^3\right).\end{aligned}$$

Since $4s + 2t > 3$, we have $2 \le \frac{12}{3+2t} \le \frac{6}{3-2t}$, thus $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. From [1], we know that there exists a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3) = \{u \in L^{2_t^*}(\mathbb{R}^3) : |\xi|^t \widehat{u}(\xi) \in L^2(\mathbb{R}^3)\}$ which is a weak solution of $(-\Delta)^t \phi_u^t = u^2$, and it has the following representation:

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}}\, dy, \quad x \in \mathbb{R}^3,$$

where $c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3-2t)}{\Gamma(t)}$. Substituting ϕ_u^t in (1.1), we obtain the following fractional Schrödinger equation:

$$(-\Delta)^s u + u + \phi_u^t u = f(u), \quad x \in \mathbb{R}^3. \tag{2.1}$$

For the properties of ϕ_u^t, see [2]. By (2.1), we define the functional $\mathcal{I} : H^s(\mathbb{R}^3) \to \mathbb{R}$ as follows:

$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\frac{s}{2}} u\right|^2 + u^2\right) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2\, dx - \int_{\mathbb{R}^3} F(u)\, dx, \tag{2.2}$$

where $F(u) = \int_0^u f(x)\, dx$. It is easy to see that (f_1) and (f_2) imply that $\mathcal{I}$ is a well-defined C^1-functional, and

$$\left\langle \mathcal{I}'(u), v\right\rangle = \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv\right) dx + \int_{\mathbb{R}^3} \phi_u^t uv\, dx - \int_{\mathbb{R}^3} f(u) v\, dx, \quad \forall v \in H^s\left(\mathbb{R}^3\right).$$

Hence, if u is a critical point of $\mathcal{I}$, then (u, ϕ_u^t) is a solution of (1.1).

Set

$$u_R(x) = \begin{cases} \frac{1}{R}, & |x| \le R, \\ \frac{1}{R}(2 - \frac{|x|}{R}), & R < |x| \le 2R, \\ 0, & |x| > 2R. \end{cases}$$

Hence $u_R \in H^s(\mathbb{R}^3)$. By Proposition 3.4 in [9], we have

$$\|u_R\|_{H^s}^2 = 2C(3,s)^{-1} \int_{\mathbb{R}^3} |\xi|^{2s} \left|\mathcal{F} u_R(\xi)\right|^2 d\xi, \tag{2.3}$$

where $\mathcal{F}$ is the usual Fourier transform in $\mathbb{R}^3$, and

$$C(3,s) = \left(\int_{\mathbb{R}^3} \frac{1 - \cos(\zeta_1)}{|\zeta|^{3+2s}}\, d\zeta\right)^{-1},$$

here $\zeta = (\zeta_1, \zeta_2, \zeta_3)$. From the inequality $|\xi|^{2s} \leq 1 + |\xi|^2$, $s \in (0,1]$, together with (2.3), we get

$$\|u_R\|_{H^\alpha}^2 \leq 2C(3,s)^{-1} \int_{\mathbb{R}^3} \left(1 + |\xi|^2\right) \left|\mathcal{F}u_R(\xi)\right|^2 d\xi = 2C(3,s)^{-1} \|u_R\|_{H^1}^2. \tag{2.4}$$

If $t > \frac{1}{2}$, then we have by Lemma 2.3 in [2]

$$\int_{\mathbb{R}^3} \phi_{u_R(x)}^t u_R(x)^2 \, dx \leq S_t^2 |u_R|_{\frac{12}{3+2t}}^4,$$

where

$$S_t = \inf_{u \in \mathcal{D}^{t,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} u|^2 \, dx}{(\int_{\mathbb{R}^3} |u(x)|^{2_t^*} \, dx)^{\frac{2}{2_t^*}}}.$$

Remark 2.1 If $t = 1$, then the above inequalities modifies to the following inequalities:

$$\int_{\mathbb{R}^3} \phi_{u_R(x)} u_R(x)^2 \, dx \leq S^2 |u_R|_{\frac{12}{5}}^4, \tag{2.5}$$

where

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^3} |u(x)|^6 \, dx)^{\frac{1}{3}}}.$$

From [5], we have

$$\int_{\mathbb{R}^3} \left|\nabla u_R(x)\right|^2 dx = \frac{28\pi}{3R}, \qquad \int_{\mathbb{R}^3} \left|u_R(x)\right|^3 dx \geq \frac{4\pi}{3}, \tag{2.6}$$

and

$$|u_R|_{\frac{12}{5}}^4 = \left(\frac{32\pi}{3}\right)^{\frac{5}{3}} R. \tag{2.7}$$

Lemma 2.1 *If* (f_1) *and* (f_2) *hold, then*

(i) *there exists a* $v \in H^s(\mathbb{R}^3) \setminus \{0\}$ *such that* $\mathcal{I}(v) \leq 0$;

(ii) $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}(\gamma(t)) > 0$, *where*

$$\Gamma = \left\{\gamma \in C\left([0,1], H^s\left(\mathbb{R}^3\right)\right) : \gamma(0) = 0, \gamma(1) = v\right\}.$$

Proof Set $R = \frac{8\pi\mu_0}{3S^2(\frac{32\pi}{3})^{\frac{5}{3}}}$, where

$$\mu_0 = \sqrt{\frac{45}{8\pi} C(3,s)^{-1} S^2 \left(\frac{32\pi}{3}\right)^{\frac{5}{3}}} < \sqrt{\frac{6}{\pi} C(3,s)^{-1} S^2 \left(\frac{32\pi}{3}\right)^{\frac{5}{3}}}.$$

Denote $u_{R,\theta} = \theta^2 u_R(\theta x)$, from (2.2), (2.4), Fatou's lemma, (f_2) and (2.5)–(2.7), we obtain

$$\begin{aligned}
\lim_{\theta\to+\infty}\frac{\mathcal{I}(u_{R,\theta})}{\theta^3} &= \lim_{\theta\to+\infty}\frac{1}{\theta^3}\left(\frac{1}{2}\int_{\mathbb{R}^3}\left(\left|(-\Delta)^{\frac{s}{2}}u_{R,\theta}\right|^2 + u_{R,\theta}^2\right)dx\right.\\
&\quad\left. + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_{R,\theta}}^t u_{R,\theta}^2\,dx - \int_{\mathbb{R}^3}F(u_{R,\theta})\,dx\right)\\
&\leq \lim_{\theta\to+\infty}\frac{1}{\theta^3}\left(C(3,s)^{-1}\int_{\mathbb{R}^3}\left(|\nabla u_{R,\theta}|^2 + u_{R,\theta}^2\right)dx\right.\\
&\quad\left. + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_{R,\theta}}^t u_{R,\theta}^2\,dx - \int_{\mathbb{R}^3}F(u_{R,\theta})\,dx\right)\\
&= \lim_{\theta\to+\infty}\frac{1}{\theta^3}\left(C(3,s)^{-1}\left[\theta^3\int_{\mathbb{R}^3}\left|\nabla u_R(x)\right|^2dx + \theta\int_{\mathbb{R}^3}u_R(x)^2\,dx\right]\right.\\
&\quad\left. + \frac{\theta^{1+2t}}{4}\int_{\mathbb{R}^3}\phi_{u_R(x)}^t u_R(x)^2\,dx - \int_{\mathbb{R}^3}F\left(\theta^2 u_R(\theta x)\right)dx\right)\\
&= C(3,s)^{-1}\int_{\mathbb{R}^3}\left|\nabla u_R(x)\right|^2dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_R(x)}^t u_R(x)^2\,dx\cdot\lim_{\theta\to+\infty}\frac{1}{\theta^{2(1-t)}}\\
&\quad - \lim_{\theta\to+\infty}\int_{\mathbb{R}^3}\frac{F(\theta^2 u_R)}{|\theta^2 u_R|^3}|u_R|^3\,dx\\
&\leq \begin{cases}\frac{28\pi}{3R}C(3,s)^{-1} + \frac{S^2}{4}|u_R|_{\frac{12}{5}}^4 - \frac{\mu}{3}\int_{\mathbb{R}^3}|u_R|^3\,dx, & t=1,\\ \frac{28\pi}{3R}C(3,s)^{-1} - \frac{\mu}{3}\int_{\mathbb{R}^3}|u_R|^3\,dx, & t<1\end{cases}\\
&< \frac{28\pi}{3R}C(3,s)^{-1} + \frac{S^2}{4}|u_R|_{\frac{12}{5}}^4 - \sqrt{\frac{6}{\pi}C(3,s)^{-1}S^2\left(\frac{32\pi}{3}\right)^{\frac{5}{3}}}\int_{\mathbb{R}^3}|u_R|^3\,dx\\
&< \frac{28\pi}{3R}C(3,s)^{-1} + \frac{S^2}{4}\left(\frac{32\pi}{3}\right)^{\frac{5}{3}}R - \frac{4\pi}{3}\mu_0\\
&= -\frac{2\pi}{3R}C(3,s)^{-1} < 0.
\end{aligned}$$

Thus, $\mathcal{I}(u_{R,\theta}) \leq 0$ if θ is sufficiently large.

(ii) By (f_1) and (f_2), for $\varepsilon = \frac{1}{4} > 0$, there exists $C > 0$ such that

$$f(\theta) \leq \frac{1}{4}\theta + C\theta^2. \tag{2.8}$$

From (2.8) and by using the Sobolev inequality, we obtain

$$\mathcal{I}(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{4}|u|_2^2 - C|u|_3^3 \geq \frac{1}{4}\|u\|^2 - CS_{s,3}^{-\frac{3}{2}}\|u\|^3,$$

where

$$S_{s,3} = \inf_{u\in\mathcal{D}^{s,2}}\frac{\int_{\mathbb{R}^3}\left(|(-\Delta)^{\frac{s}{2}}u|^2 + u^2\right)dx}{\left(\int_{\mathbb{R}^3}|u(x)|^3\,dx\right)^{\frac{2}{3}}}.$$

For sufficiently small $\rho > 0$, we have $I(u) > 0$ with $\|u\| = \rho$. □

Similar to the proof of Lemma 2.2 in [5], we have the following lemma.

Lemma 2.2 *Suppose that* (f_1), (f_2) *hold. If* $\{u_n\} \subset H^s(\mathbb{R}^3)$ *is a bounded* $(PS)_{c\neq 0}$ *sequence of* $\mathcal{I}$, *then there exists* $u_0 \neq 0$ *such that* $\mathcal{I}'(u_0) = 0$.

3 Main result

Theorem 3.1 *Assume the conditions* (f_1)–(f_3) *are satisfied, then system* (1.1) *has at least a ground state solution.*

Proof For convenience, we introduce a functional on $H^s(\mathbb{R}^3)$ as follows:

$$\mathcal{J}(u) = \frac{1+2s}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} u^2\,dx + \frac{5-2t}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\,dx + \int_{\mathbb{R}^3}\left[(1+2t)F(u) - 2f(u)u\right]dx. \tag{3.1}$$

Inspired by the idea of Jeanjean [10], we define the map $\Psi : \mathbb{R}\times H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3)$ by $\Omega(\lambda, w)(x) = e^{2\lambda}w(e^{\lambda}x)$. For each λ and $w \in H^s(\mathbb{R}^3)$, we can compute the functional $\mathcal{I}\circ\Omega$ as follows:

$$\mathcal{I}\big(\Omega(\lambda,w)\big) = \frac{e^{(1+2s)}\lambda}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}w\right|^2 + \frac{e^{\lambda}}{2}\int_{\mathbb{R}^3} w^2 + \frac{e^{(5-2t)\lambda}}{4}\int_{\mathbb{R}^3}\phi_w^t w^2 - \frac{1}{e^{3\lambda}}\int_{\mathbb{R}^3}F\big(e^{2\lambda}w\big). \tag{3.2}$$

By (3.2), (f_1) and (f_2), we see that $\mathcal{I}\circ\Omega$ is continuously Fréchet-differentiable on $\mathbb{R}\times H^s(\mathbb{R}^3)$. By virtue of Lemma 2.1, there exists $\lambda^*\in\mathbb{R}$ such that $(\mathcal{I}\circ\Omega)(\lambda^*, u_R) < 0$. The mountain pass level of $\mathcal{I}\circ\Omega$ is given as follows:

$$\bar{c} = \inf_{\bar{\gamma}\in\bar{\Gamma}}\sup_{t\in[0,1]}(\mathcal{I}\circ\Omega)\big(\bar{\gamma}(t)\big), \tag{3.3}$$

where the family of paths is denoted by

$$\bar{\Gamma} = \left\{\bar{\gamma}\in C\big([0,1];\mathbb{R}\times H^s(\mathbb{R})\big) : \bar{\gamma}(0) = (0,0),\ (\mathcal{I}\circ\Omega)\big(\bar{\gamma}(1)\big) < 0\right\}.$$

For $\Gamma = \{\Omega\circ\bar{\gamma} : \bar{\gamma}\in\bar{\Gamma}\}$, we have $c \le \bar{c}$. Obviously, $\{0\}\times\Gamma\subset\bar{\Gamma}$ and then $\bar{c}\le c$. Thus, $\bar{c} = c$. It follows for each $(\eta, u)\in\mathbb{R}\times H^s(\mathbb{R}^3)$ that

$$\begin{aligned}\mathcal{I}'\big(\Omega(\lambda_n,w_n)\big)\big[\Omega(\lambda_n,u)\big] &= e^{(1+2s)\lambda_n}\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}w_n(-\Delta)^{\frac{s}{2}}u \\ &\quad + e^{\lambda_n}\int_{\mathbb{R}^3}w_n u + e^{(5-2t)\lambda_n}\int_{\mathbb{R}^3}\phi_{w_n}^t w_n u - \frac{1}{e^{\lambda_n}}\int_{\mathbb{R}^3}f\big(e^{2\lambda_n}w_n\big)u,\\ (\mathcal{I}\circ\Omega)'(\lambda_n,w_n)[\eta,u] &= \mathcal{I}'\big(\Omega(\lambda_n,w_n)\big)\big[\Omega(\lambda_n,u)\big] + \mathcal{J}\big(\Omega(\lambda_n,w_n)\big)\eta.\end{aligned}$$

From Theorem 2.9 of [11], (3.3) and setting $u_n = \Omega(\lambda_n, w_n)$, one has

$$\mathcal{I}(u_n)\to c>0,\qquad \mathcal{I}'(u_n)\to 0,\qquad \mathcal{J}(u_n)\to 0. \tag{3.4}$$

By (3.4) and (f_3), we derive that

$$\begin{aligned} c &\geq \mathcal{I}(u_n) - \frac{1}{5-2t}\mathcal{J}(u_n) + o(1) \\ &= \frac{2-(s+t)}{5-2t}\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx + \frac{2-t}{5-2t}\int_{\mathbb{R}^3} u_n^2\, dx \\ &\quad + \frac{2}{5-2t}\int_{\mathbb{R}^3} \left[f(u_n)u_n - 3F(u_n)\right] dx + o(1) \\ &\geq \frac{2-t}{5-2t}\int_{\mathbb{R}^3} u_n^2\, dx + o(1), \end{aligned} \tag{3.5}$$

which implies that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^3)$. According to (3.4), we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} f(u_n)u_n\, dx + o(\|u_n\|) &= \int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 + u_n^2\right) dx + \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2\, dx \\ &\geq \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx. \end{aligned} \tag{3.6}$$

Combining (f_1) and (f_2), we have

$$f(\tau)\tau \leq C|\tau|^3 + \varepsilon\tau^2. \tag{3.7}$$

By means of (3.7), the interpolation and Sobolev inequalities, we get

$$\begin{aligned} \int_{\mathbb{R}^3} f(u_n)u_n\, dx &\leq C\int_{\mathbb{R}^3} |u_n|^3\, dx + \varepsilon\int_{\mathbb{R}^3} u_n^2\, dx \\ &\leq C\left(\int_{\mathbb{R}^3} |u_n|^2\, dx\right)^{\frac{6s-3}{4s}} \cdot \left(\int_{\mathbb{R}^3} |u_n|^{2_s^*}\, dx\right)^{\frac{3-2s}{4s}} + C\varepsilon \\ &\leq C\left(\int_{\mathbb{R}^3} |u_n|^2\, dx\right)^{\frac{6s-3}{4s}} \cdot \left(\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx\right)^{\frac{3}{4s}} + C\varepsilon. \end{aligned} \tag{3.8}$$

Since $s > \frac{3}{4}$, by (3.6) and (3.8), we know that $\{(-\Delta)^{\frac{s}{2}} u_n\}$ is bounded in $L^2(\mathbb{R}^3)$. Hence, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Define

$$m = \inf_M \mathcal{I}(u), \qquad M = \left\{u \in H^s\left(\mathbb{R}^3\right) \setminus \{0\} | \mathcal{I}'(u) = 0\right\}.$$

In view of Lemma 2.2 and $\{u_n\}$ being bounded, we obtain $u_0 \neq 0$ and $\mathcal{I}'(u_0) = 0$. Thus M is not empty and $0 \leq m \leq \mathcal{I}(u_0)$. In the following we will prove m can be achieved in M. Suppose that $\{u_n\}$ is a sequence of nontrivial critical points of $\mathcal{I}$ satisfying $\mathcal{I}(u_n) \to m$. Similar to the proofs of (3.5), (3.6) and (3.8), we find that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. By $\mathcal{I}'(u_n)u_n = 0$, (2.8) and the Sobolev inequality, we have

$$\begin{aligned} \|u_n\|^2 &= \int_{\mathbb{R}^3} f(u_n)u_n\, dx - \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2\, dx \\ &\leq \frac{1}{4}|u_n|_2^2 + C|u_n|_3^3 \leq \frac{1}{4}\|u_n\|^2 + CS_{s,3}^{-\frac{3}{2}}\|u_n\|^3. \end{aligned} \tag{3.9}$$

From (3.9), there exists a positive $\rho > 0$ such that

$$\liminf_{n\to\infty} \|u_n\| \geq \rho > 0. \tag{3.10}$$

Applying the Lions lemma in [11], if $\{u_n\}$ vanishes, one has $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for all $q \in (2,6)$. Thus, if $t > \frac{1}{2}$, then it follows (12) in [2] that

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2\,dx \leq S_t^{\frac{1}{2}} |u_n|^4_{\frac{12}{3+2t}} \to 0.$$

By (3.7) and $u_n \to 0$ in $L^q(\mathbb{R}^3)$, we get $\int_{\mathbb{R}^3} f(u_n)u_n\,dx \to 0$. Combining with (3.9), we can easily deduce that $\|u_n\| \to 0$ in a contradiction with (3.10). Hence there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that $\lim_{n\to\infty} \sup_{y_n\in\mathbb{R}^3} \int_{B_r(y_n)} |u_n|^2 \geq \delta > 0$. Set $\bar{u}_n = u_n(x + y_n)$, then we have $\bar{u}_n \rightharpoonup u \not\equiv 0$ in $H^s(\mathbb{R}^3)$, $\mathcal{I}(\bar{u}_n) \to m$ and $\mathcal{I}'(\bar{u}_n) = 0$. Thus, we get $\mathcal{I}'(u) = 0$ and $\mathcal{I}(u) \geq m$. Since $\mathcal{I}'(u) = 0$, one has

$$\int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\frac{s}{2}} u\right|^2 + u^2\right) dx + \int_{\mathbb{R}^3} \phi_u^t u^2\,dx - \int_{\mathbb{R}^3} f(u)u\,dx = 0, \tag{3.11}$$

and by the Pohožaev identity [2, 12], we have

$$\frac{3-2s}{2} \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} u^2\,dx + \frac{3+2t}{4} \int_{\mathbb{R}^3} \phi_u^t u^2\,dx = 3 \int_{\mathbb{R}^3} F(u)\,dx. \tag{3.12}$$

According to (3.11)–(3.12), one has $\mathcal{J}(u) = 0$. As $\mathcal{I}'(\bar{u}_n) = 0$, $\mathcal{I}'(u) = 0$ and by Fatou's lemma we have

$$\begin{aligned}
m &= \lim_{n\to\infty} \left(\frac{2-(s+t)}{5-2t} \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} \bar{u}_n\right|^2 dx + \frac{2-t}{5-2t} \int_{\mathbb{R}^3} \bar{u}_n^2\,dx \right. \\
&\quad \left. + \frac{2}{5-2t} \int_{\mathbb{R}^3} \left[f(\bar{u}_n)\bar{u}_n - 3F(\bar{u}_n)\right] dx \right) \\
&\geq \frac{2-(s+t)}{5-2t} \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{2-t}{5-2t} \int_{\mathbb{R}^3} u^2\,dx \\
&\quad + \frac{2}{5-2t} \int_{\mathbb{R}^3} \left[f(u)u - 3F(u)\right] dx \\
&= \mathcal{I}(u) - \frac{1}{5-2t} \mathcal{J}(u) = \mathcal{I}(u) \geq m.
\end{aligned}$$

Hence $\mathcal{I}(u) = m$. The proof is complete. □

Remark 3.1 If $s = t = 1$, Our main result Theorem 3.1 reduces to Theorem 1.1 in [5]. On the other hand, Theorem 3.1 in this paper relaxes the condition of super-quadratic nonlinearity in [1] to being asymptotically 2-linear.

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References

1. Gao, Z., Tang, X., Chen, S.: Ground state solutions for a class of nonlinear fractional Schrödinger–Poisson systems with super-quadratic nonlinearity. Chaos Solitons Fractals **105**, 189–194 (2017)
2. Teng, K.M.: Existence of ground state solutions for the nonlinear fractional Schrödinger–Poisson system with critical Sobolev exponent. J. Differ. Equ. **261**, 3061–3106 (2016)
3. He, Y., Jing, L.: Existence and multiplicity of non-trivial solutions for the fractional Schrödinger–Poisson system with superlinear terms. Bound. Value Probl. **2019**, Article ID 25 (2019)
4. Wang, D.B., Ma, Y.M., Guan, W.: Least energy sign-changing solutions for the fractional Schrödinger–Poisson systems in $\mathbb{R}^3$. Bound. Value Probl. **2019**, Article ID 4 (2019)
5. Yin, L.F., Wu, X.P., Tang, C.L.: Ground state solutions for an asymptotically 2-linear Schrödinger–Poisson system. Appl. Math. Lett. **87**, 7–12 (2019)
6. Azzollini, A., Pomponio, A.: A note on the ground state solutions for the nonlinear Schrödinger–Maxwell equations. Boll. Unione Mat. Ital. (9) **2**(1), 93–104 (2009)
7. Azzollini, A., d'Avenia, P., Pompomio, A.: On the Schrödinger–Maxwell equations under the effiect of a general nonlinear term. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **27**(2), 779–791 (2010)
8. Sun, J.J., Ma, S.W.: Ground state solutions for some Schrödinger–Poisson systems with periodic potentials. J. Differ. Equ. **260**(3), 2119–2149 (2016)
9. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 521–573 (2012)
10. Jeanjean, L.: Existence of solutions with prescribed norm for semilinear elliptic equations. Nonlinear Anal. **28**(10), 1633–1659 (1997)
11. Willem, M.: Minimax Theorems. Progr. Nonlinear Differential Equations Appl., vol. 24. Birkhäuser, Boston (1996)
12. Teng, K.M.: Ground state solutions for the non-linear fractional Schrödinger–Poisson system. Appl. Anal. **98**, 1959–1996 (2019)

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