Alzahrani and Zeb *Boundary Value Problems* (2020) 2020:3
https://doi.org/10.1186/s13661-019-01315-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Stability analysis and prevention strategies of tobacco smoking model

Ebraheem Alzahrani[1] and Anwar Zeb[2*]

*Correspondence:
anwar@cuiatd.edu.pk
[2]Department of Mathematics, COMSATS University Islamabad, Abbottabad, Pakistan
Full list of author information is available at the end of the article

Abstract

This research work is related to a tobacco smoking model having a significance class of users of tobacco in the form of snuffing. For this purpose, the formulation of the model containing snuffing class is presented; then the equilibrium points as regards being smoking free and smoking positive are discussed. The Hurwitz theorem is used for finding the local stability of the model and Lyaponov function theory is used for the search of global stability. We use different controls for control of smoking and the Pontryagin maximum principle for characterization of the optimal level. For the solution of the proposed model, a nonstandard finite difference (NSFD) scheme and the Runge–Kutta fourth order method are used. Finally, some numerical results are presented for control and without control systems with the help of MATLAB.

MSC: 92D25; 49J15; 93D20

Keywords: Tobacco smoking model; Local and global stability; Control strategies; Pontryagin maximum principle; Nonstandard finite difference (NSFD) scheme

1 Introduction

Mathematical biology is a wide field with many applications. In this field, researchers are focusing on the description of different types of diseases with controls in the form of mathematical models. In 1909, Brownlee [1] took the initiative for the development of mathematical biology. He focused on the theory of chance, further in 1912, he presented basic laws for epidemic spreading [2]. In 1927, the details of the epidemic study were discussed by Kermark and McKendrik [3]. Later, many researchers discussed different models of many other diseases; see [4–17]. On the other hand, one of the social habits that is spreading throughout the world rapidly as an infectious disease is smoking, causing many harmful diseases. Smoking is the process by which people inhale smoke of tobacco consisting of particles and gas or simply, smoking is the experience in which smoke is taken into mouth and then released using pipes or cigars. In the sixteenth century, Columbus introduced smoking to Europe [18], but before and after this date, many other exotic species were introduced, with great adverse impact on ecosystems and effect on human habits [19, 20]. Nicot spread tobacco as a cash crop in England, he was the first who used it like a business, and that is why the word nicotine derived from his name. The cigarette making machine was invented at the end of the 19th century and the capability of that machine was producing 200 cigarettes per minute and now cigarette production has increased up to 9000 cigarettes per minute. Smoking can cause different types of diseases including lung cancer, mouth cancer, throat cancer and many other diseases that are harmful to human health [21–30].

For the first time in 1997, Castillo-Garsow *et al.* [21] formulated a mathematical model for smoking. In this model, they divided the total population in three different classes (potential smokers, chain smokers and permanently quit smokers). In 2008, their model was modified by Sharomi and Gumel [22]. They introduced a new class (temporarily quit smokers). In 2007, Ham [23] identified the different stages and processes of smoking among students through a survey in different vocational technical schools in Korea. Zaman [24] extended the model by introducing a new category (occasional smokers) and presented a dynamical interaction in an integer order. Zeb *et al.* [28] derived the square root dynamics of a giving up smoking model for the purpose that the system goes to finite time extension. Several others presented the smoking models in integer and fractional order [21–30]. The use of tobacco also occurs in the form of snuffing. Till now, no one has discussed mathematically the snuffing class; by adding the snuffing class in this work, we divided the total population in five classes $X(t)$, $H_1(t)$, $H_2(t)$, $Y(t)$, $Z(t)$ representing the susceptible smokers, snuffing class, irregular smokers, regular smokers and quit smokers, respectively, at time t. First, we formulate the model according to given assumptions; then, by using the Hurwitz theorem, we find the local stability, and with the help of the Lyaponov function the global stability is discussed. For prevention strategies, the Pontryagin's maximum principle is used. Finally, some numerical results are presented for control and without control system by using the nonstandard finite difference (NSFD) method and the Runge–Kutta fourth order method.

2 Formation of model

By adding the snuffing class, we divided the total population into five classes $X(t)$, $H_1(t)$, $H_2(t)$, $Y(t)$, $Z(t)$ standing for susceptible smokers, snuffing class, irregular smokers, regular smokers and quit smokers, respectively at time t. The model is given by

$$\begin{aligned} \frac{dX}{dt} &= \lambda - \beta_1 X H_1 - \mu X + \alpha Y, \\ \frac{dH_1}{dt} &= \beta_1 X H_1 - \beta_2 H_1 H_2 - (\rho + \mu) H_1, \\ \frac{dH_2}{dt} &= \beta_2 H_1 H_2 - (d + \omega + \mu) H_2, \\ \frac{dY}{dt} &= \omega H_2 - (\alpha + \gamma + \mu) Y, \\ \frac{dZ}{dt} &= \gamma Y - \mu Z, \end{aligned} \tag{1}$$

where the parameters used in this model are described in Table 1.

Since the first four equations of system (1) are independent of $Z(t)$, without loss of generality, we omit this one and then the system (1) is reduced to the following:

$$\begin{aligned} \frac{dX}{dt} &= \lambda - \beta_1 X H_1 - \mu X + \alpha Y, \\ \frac{dH_1}{dt} &= \beta_1 X H_1 - \beta_2 H_1 H_2 - (\rho + \mu) H_1, \end{aligned} \tag{2}$$

Table 1 Parameters and description

Symbols	Description
λ	Recruitment rate (birth or migration)
β_1	Rate at which susceptible population moves to snuffing class
β_2	Rate at which snuffing class become irregular smokers
ω	Rate at which irregular smokers become regular smokers
γ	Quitting rate
μ	Natural death rate
α	Relapse rate
ρ	Death rate of snuffing class due to tobacco use
d	Death due to tobacco related diseases

$$\frac{dH_2}{dt} = \beta_2 H_1 H_2 - (d + \omega + \mu)H_2,$$
$$\frac{dY}{dt} = \omega H_2 - (\alpha + \gamma + \mu)Y.$$

3 Equilibrium points

3.1 Smoking free equilibrium point

For the smoking free equilibrium point E_0 we use $H_1 = H_2 = Y = 0$ in system (2).

So the smoking free equilibrium point E_0 is

$$E_0 = \left(\frac{\lambda}{\mu}, 0, 0, 0\right).$$

The Jacobian of system (2) is given by

$$J = \begin{pmatrix} -\beta_1 H_1 - \mu & -\beta_1 X & 0 & \alpha \\ \beta_1 H_1 & \beta_1 X - \beta_2 H_2 - (\rho + \mu) & -\beta_2 H_1 & 0 \\ 0 & \beta_2 H_2 & \beta_2 H_1 - (d + \omega + \mu) & 0 \\ 0 & 0 & \omega & -(\alpha + \gamma + \mu) \end{pmatrix},$$

while the Jacobian at free equilibrium point E_0 is

$$J(E_0) = \begin{pmatrix} -\mu & \frac{-\beta_1 \lambda}{\mu} & 0 & \alpha \\ 0 & \frac{\beta_1 \lambda}{\mu} - (\rho + \mu) & 0 & 0 \\ 0 & 0 & -(d + \omega + \mu) & 0 \\ 0 & 0 & \omega & -(\alpha + \gamma + \mu) \end{pmatrix}.$$

For the reproductive number, we consider the following matrices:

$$F = \begin{pmatrix} \frac{\beta_1 \lambda}{\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} (\rho + \mu) & 0 & 0 \\ 0 & (d + \omega + \mu) & 0 \\ 0 & -\omega & (\alpha + \gamma + \mu) \end{pmatrix}.$$

The dominant eigenvalue of FV^{-1} is $\frac{\beta_1\lambda}{\mu(\rho+\mu)}$, so

$$R_0 = \frac{\beta_1\lambda}{\mu(\rho+\mu)} \tag{3}$$

is the required reproductive number [13].

3.2 Smoking present equilibrium point

Theorem 3.1 *For $R_0 > 1$, there exists a positive smoking equilibrium point E^*.*

Proof For smoking present the equilibrium E^* using the left side of system (2) is equal to zero, as follows.

The third equation of system (2) implies that

$$H_1^* = \frac{(d+\omega+\mu)}{\beta_2},$$

from the second equation of system (2), we have

$$X^* = \frac{\beta_2 H_2^* + (\rho+\mu)}{\beta_1},$$

the fourth equation implies that

$$Y^* = \frac{\omega H_2^*}{(\alpha+\gamma+\mu)},$$

similarly, the first equation reveals that

$$X^* = \frac{\lambda - \alpha Y^*}{\beta_1 H_1^* + \mu}.$$

Now, by comparing the values of X^* in terms of H_1^* and H_2^* we find that

$$H_2^* = \frac{(\alpha+\gamma+\mu)(\rho+\mu)[\beta_2\mu(R_0-1) - \beta_1(d+\omega+\mu)]}{(\gamma+\mu)(\beta_1\beta_2\omega) + (\alpha+\gamma+\mu)(\beta_1\beta_2(d+\mu) + \beta_2^2\mu)}.$$

We have $\beta_2\mu(R_0-1) > \beta_1(d+\omega+\mu)$ for $R_0 > 1$. Thus, H_2^* is positive if $R_0 > 1$. So the required positive equilibrium point E^* is

$$\begin{aligned} E^*\left(X^*, H_1^*, H_2^*, Y^*\right) = \Bigg(&\frac{\beta_2 H_2^* + (\rho+\mu)}{\beta_1}, \frac{(d+\omega+\mu)}{\beta_2}, \frac{\omega H_2^*}{(\alpha+\gamma+\mu)}, \\ &\frac{(\alpha+\gamma+\mu)(\rho+\mu)[\beta_2\mu(R_0-1) - \beta_1(d+\omega+\mu)]}{(\gamma+\mu)(\beta_1\beta_2\omega) + (\alpha+\gamma+\mu)(\beta_1\beta_2(d+\mu) + \beta_2^2\mu)} \Bigg). \end{aligned}$$

□

4 Stability of the model

4.1 Local stability

Theorem 4.1 *If $R_0 < 1$, then the system (2) is locally stable and if $R_0 > 1$, then system (2) is unstable.*

Proof For local stability at E_0, the Jacobian of system (2) is

$$J(E_0) = \begin{pmatrix} -\mu & \frac{-\beta_1\lambda}{\mu} & 0 & \alpha \\ 0 & \frac{\beta_1\lambda}{\mu} - (\rho+\mu) & 0 & 0 \\ 0 & 0 & -(d+\omega+\mu) & 0 \\ 0 & 0 & \omega & -(\alpha+\gamma+\mu) \end{pmatrix},$$

from which follow the eigenvalues λ_1, λ_2, λ_3 and λ_4,

$$\begin{aligned} \lambda_1 &= -\mu < 0, \\ \lambda_3 &= -(d+\omega+\mu) < 0, \\ \lambda_4 &= -(\alpha+\gamma+\mu) < 0, \\ \lambda_2 &= (\rho+\mu)(R_0 - 1), \end{aligned}$$

implying that $\lambda_2 < 0$ for $R_0 < 1$, $\lambda_2 = 0$ for $R_0 = 1$ and $\lambda_2 > 0$ for $R_0 > 1$. □

Theorem 4.2 *If $R_0 > \frac{\beta_2\lambda}{(d+\omega+\mu)(\rho+\mu)}$, then the system* (2) *is locally stable at E^*, otherwise unstable.*

Proof For local stability at E^* the Jacobian of system (2) is

$$J(E^*) = \begin{pmatrix} -\beta_1 H_1^* - \mu & -\beta_1 X^* & 0 & \alpha \\ \beta_1 H_1^* & \beta_1 X^* - \beta_2 H_2^* - (\rho+\mu) & -\beta_2 H_1^* & 0 \\ 0 & \beta_2 H_2^* & \beta_2 H_1^* - (d+\omega+\mu) & 0 \\ 0 & 0 & \omega & -(\alpha+\gamma+\mu) \end{pmatrix},$$

$$\begin{aligned} J(E^*) &= \begin{pmatrix} -\beta_1 H_1^* - \mu & -\beta_1 X^* & 0 & \alpha \\ \beta_1 H_1^* & 0 & -\beta_2 H_1^* & 0 \\ 0 & \beta_2 H_2^* & 0 & 0 \\ 0 & 0 & \omega & -(\alpha+\gamma+\mu) \end{pmatrix} \\ &= \begin{pmatrix} -\beta_1 H_1^* - \mu & -\beta_1 X^* & 0 & \alpha \\ -\mu & -\beta_1 X^* & -\beta_2 H_1^* & \alpha \\ \beta_1 H_1^* & \beta_2 H_2^* & -\beta_2 H_1^* & 0 \\ 0 & 0 & \omega & -(\alpha+\gamma+\mu) \end{pmatrix} \\ &= \begin{pmatrix} -\beta_1 H_1^* - \mu & -\beta_1 X^* & 0 & \alpha \\ 0 & -\beta_1 X^* + \frac{\mu\beta_2 H_2^*}{\beta_1 H_1^*} & -\beta_2 H_1^* - \frac{\mu\beta_2 H_1^*}{\beta_1 H_1^*} & \alpha \\ \beta_1 H_1^* & \beta_2 H_2^* & -\beta_2 H_1^* & 0 \\ 0 & 0 & \omega & -(\alpha+\gamma+\mu) \end{pmatrix}, \end{aligned}$$

$$J(E^*) = \begin{pmatrix} -\beta_1 H_1^* - \mu & -\beta_1 X^* & \frac{\omega\alpha}{(\alpha+\gamma+\mu)} & 0 \\ 0 & -\beta_1 X^* + \frac{\mu\beta_2 H_2^*}{\beta_1 H_1^*} & -\beta_2 H_1^* - \frac{\mu\beta_2 H_1^*}{\beta_1 H_1^*} + \frac{\omega\alpha}{(\alpha+\gamma+\mu)} & 0 \\ \beta_1 H_1^* & \beta_2 H_2^* & -\beta_2 H_1^* & 0 \\ 0 & 0 & \omega & -(\alpha+\gamma+\mu) \end{pmatrix}.$$

For simplification, this matrix can also be written as

$$J(E^*) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here,

$$A = \begin{pmatrix} -\beta_1 H_1^* - \mu & -\beta_1 X^* \\ 0 & -\beta_1 X^* + \frac{\mu\beta_2 H_2^*}{\beta_1 H_1^*} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{\omega\alpha}{(\alpha+\gamma+\mu)} & 0 \\ -\beta_2 H_1^* - \frac{\mu\beta_2 H_1^*}{\beta_1 H_1^*} + \frac{\omega\alpha}{(\alpha+\gamma+\mu)} & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \beta_1 H_1^* & \beta_2 H_2^* \\ 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} -\beta_2 H_1 & 0 & \\ \omega & (\alpha+\gamma+\mu) & S \end{pmatrix}.$$

Since the eigenvalues of $J(E^*)$ depend on the eigenvalues of A and D, the eigenvalues of A are given as follows:

$$\lambda_1 = -\beta_1 H_1^* - \mu < 0,$$

$$\lambda_2 = -(\rho+\mu) + \frac{\mu\beta_2 H_2^*}{\beta_1 H_1^* \lambda}\bigl(\lambda - H_1^*(\rho+\mu)R_0\bigr),$$

if $R_0 > \frac{\beta_2\lambda}{(d+\omega+\mu)(\rho+\mu)}$, then $\lambda_2 < 0$. Now, the eigenvalues of D are

$$\lambda_3 = -\beta_2 H_1^* < 0,$$

$$\lambda_4 = -(\alpha+\gamma+\mu) < 0,$$

which is the required proof. □

4.2 Global stability

Theorem 4.3 *If $R_0 < 1$, then the system (2) is globally stable.*

Proof For the proof of this theorem, first we construct the Lyapunov function L as

$$L = \ln\frac{X}{X_0} + \ln\frac{H_1}{H_{1_0}} + H_2 + Y. \tag{4}$$

Differentiating Eq. (4) with respect to time

$$L' = \frac{\lambda}{X} - \beta_1 H_1 + \frac{\alpha Y}{X} - \mu + \beta_X - \beta_2 H_2 - (\rho+\mu) - (d+\omega+\mu)H_2,$$

using the values of E_0 in the above equation,

$$L' = \mu - \mu + \frac{\beta_1\lambda}{\mu} - (\rho+\mu),$$

$$L' = R_0(\rho+\mu) - (\rho+\mu),$$

$$L' = (\rho+\mu)(R_0 - 1),$$

therefore, if $R_0 < 1$, then $L' < 0$, which implies that the system (2) is globally stable for $R_0 < 1$. □

5 Numerical method

The NSFD method is used for the numerical solution of the proposed model (1). Basically, NSFD is an iterative method in which we get closer to the solution through iteration [31, 32]. Let the nonstandard ODEs be given by

$$y_k' = f[t, y_1, y_2, \ldots, y_n],$$

where $k = 1, 2, \ldots, n$, then by the NFSD method

$$\begin{aligned}
y_1' &= \frac{y_{1,k+1} - y_{1,k}}{h}, \\
y_2' &= \frac{y_{2,k+1} - y_{2,k}}{h}, \\
&\vdots \\
y_n' &= \frac{y_{n,k+1} - y_{n,k}}{h}.
\end{aligned}$$

Now, using the NSFD method for the numerical solution of system (1) it follows that

$$X_{k+1} = \frac{h(\lambda + \alpha Y_k) + X_k}{1 + h(\beta_1 H_{1,k} + \mu)}, \tag{5}$$

$$H_{1,(k+1)} = \frac{H_{1,k}}{1 + h(-\beta_1 X_k + \beta_2 H_{2,k} + (\rho + \mu))}, \tag{6}$$

$$H_{2,(k+1)} = \frac{H_{2,k}}{1 + h(-\beta_2 H_{1,k} + (d + \omega + \mu))}, \tag{7}$$

$$Y_{k+1} = \frac{h\omega H_{2,k} + Y_k}{1 + h(\alpha + \gamma + \mu)}, \tag{8}$$

and

$$Z_{k+1} = \frac{h\gamma Y_k + Z_k}{1 + h\mu}. \tag{9}$$

6 Summary and simulation

In this section, we give approximate values to the parameter of system (1) in Table 2 and with the help of MATLAB we draw the graph of model (1).

Figure 1 shows the result of system (1) graphically. In these graphs, we used the NSFD and RK4 methods. According to these figures, the population of each class gradually decreases, while the population of quit smokers increases gradually.

7 Control strategies

For reducing the ratio of smokers to non-smokers in the world, we apply the optimal control scheme on the system (2) presented in this section in a similar way to that used by many authors for different diseases and smoking [33–36]. For this purpose, two control variables u_1 and u_2 representing education campaign and anti-nicotine gum/medicine, respectively, are used and by utilizing the Pontryagin maximum principle for the control strategies. Finally, we will show graphically both the systems with control and without

Table 2 Values of parameters for numerical solution

Parameter	Reference	Value
X	[16]	68
S_1	[16]	40
S_2	[16]	30
Y	[16]	20
Z	[16]	15
λ	Assumption	0.1
β_1	Assumption	0.003
β_2	Assumption	0.002
ω	Assumption	0.004
γ	Assumption	0.05
μ	Assumption	0.002
α	Assumption	0.003
ρ	Assumption	0.003
d	Assumption	0.003

control. Using these control variables on system (1), we have

$$\begin{aligned}
\frac{dX}{dt} &= \lambda - \beta_1 X H_1 - \mu X + \alpha Y, \\
\frac{dH_1}{dt} &= \beta_1 X H_1 - \beta_2 H_1 H_2 - (\rho + \mu) H_1 + u_2 Y, \\
\frac{dH_2}{dt} &= \beta_2 H_1 H_2 - (d + \omega + \mu) H_2 - u_1 H_2, \\
\frac{dY}{dt} &= \omega H_2 - (\alpha + \gamma + \mu) Y - u_2 Y, \\
\frac{dZ}{dt} &= \gamma Y - \mu Z + u_1 H_2.
\end{aligned} \tag{10}$$

Now, we construct the objective function for the system (10), which is given by

$$J(u_1, u_2) = \int_0^{t_f} \left[X(t) + H_1(t) + H_2(t) + Y(t) + Z(t) + \frac{c_1 u_1^2(t)}{2} + \frac{c_2 u_2^2(t)}{2} \right] dt,$$

with initial conditions

$$\begin{aligned}
&X(0) = X^0, \\
&H_1(0) = H_1^0, \qquad H_2(0) = H_2^0, \qquad Y(0) = Y^0 \quad \text{and} \quad Z(0) = Z^0.
\end{aligned}$$

Now, the Hamiltonian function is defined as

$$\begin{aligned}
H = {} & X(t) + H_1(t) + H_2(t) + Y(t) + Z(t) + \frac{c_1 u_1^2(t)}{2} + \frac{c_2 u_2^2}{2} \\
& + \lambda_1 [\lambda - \beta_1 X H_1 - \mu X + \alpha Y] \\
& + \lambda_2 \big[\beta_1 X H_1 - \beta_2 H_1 H_2 - (\rho + \mu) H_1 + u_2 Y\big] \\
& + \lambda_3 \big[\beta_2 H_1 H_2 - (d + \omega + \mu) H_2 - u_1 H_2\big] \\
& + \lambda_4 \big[\omega H_2 - (\alpha + \gamma + \mu) Y - u_2 Y\big] \\
& + \lambda_5 [\gamma - \mu Z + u_1 H_2].
\end{aligned}$$

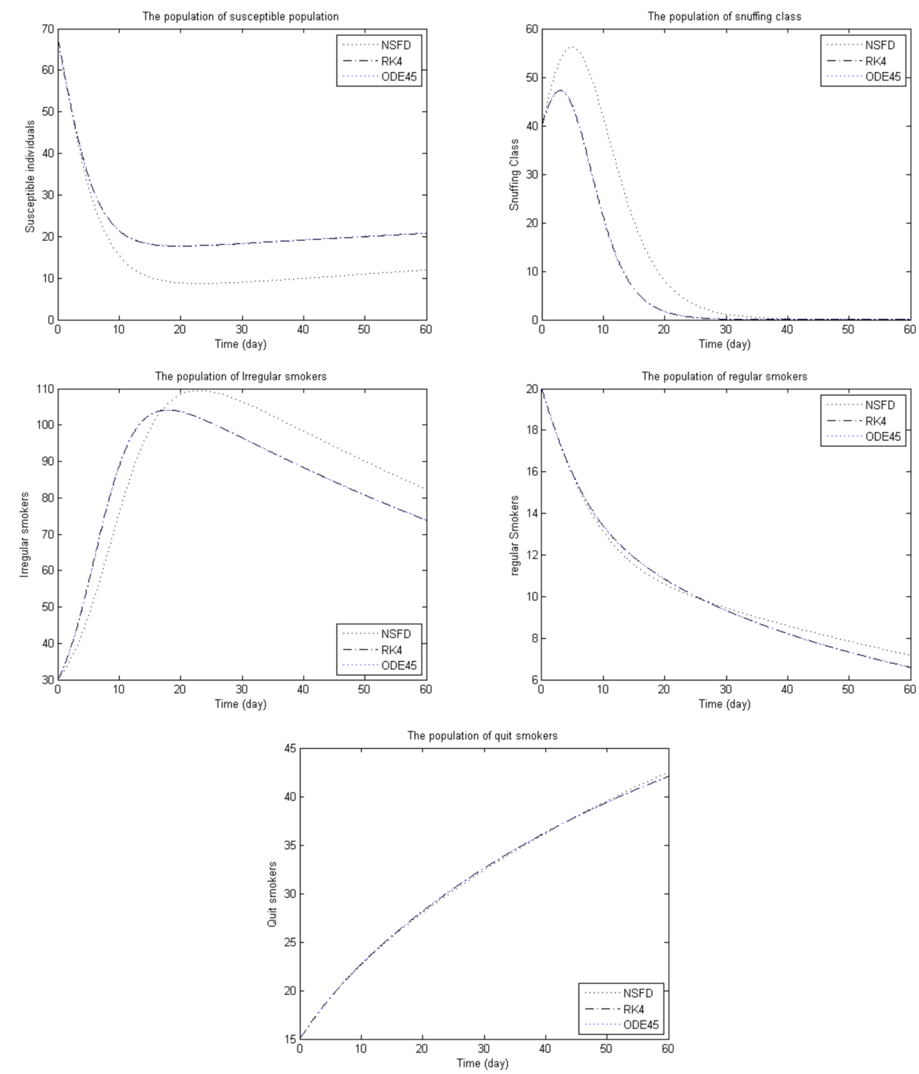

Figure 1 Plots present the susceptible, irregular, regular and quit smokers

Theorem 7.1 *The system* (10) *satisfies the terminal conditions*

$$u_1^* = \min\left(1, \max\left(0, \frac{(\lambda_3 - \lambda_5)H_2}{c_1}\right)\right) \quad \textit{and} \quad u_2^* = \min\left(1, \max\left(0, \frac{(\lambda_4 - \lambda_2)Y}{c_2}\right)\right).$$

Proof According to the Pontryagin maximum principle for the control of smoking, put

$$\frac{dX}{dt} = \frac{\partial H}{\partial \lambda}, \qquad \frac{\partial H}{\partial U} = 0,$$

and

$$\lambda_1' = \frac{-\partial H}{\partial X}, \qquad \lambda_2' = \frac{-\partial H}{\partial H_1}, \qquad \lambda_3' = \frac{-\partial H}{\partial H_2}, \qquad \lambda_4' = \frac{-\partial H}{\partial Y}, \qquad \lambda_5' = \frac{-\partial H}{\partial Z}.$$

Differentiating the Hamiltonian function with respect to X, H_1, H_2, Y and Z, respectively, we get the values of λ_1', λ_2', λ_3', λ_4', and λ_5' in the form of

$$\begin{aligned}
\lambda_1' &= -(1 - \lambda_1\beta_1 H_1 - \lambda_1\mu + \lambda_2\beta_1 H_1) \\
&= -1 + \lambda_1\beta_1 H_1 + \lambda_1\mu - \lambda_2\beta_1 H_1, \\
\lambda_2' &= -\bigl(1 - \lambda_1\beta_1 X + \lambda_2\beta_1 X - \lambda_2\beta_2 H_2 - \lambda_2(\rho + \mu) + \lambda_3\beta_2 H_2\bigr) \\
&= -1 + \lambda_1\beta_1 X - \lambda_2\beta_1 X + \lambda_2\beta_2 H_2 + \lambda_2(\rho + \mu) - \lambda_3\beta_2 H_2, \\
\lambda_3' &= -\bigl(1 - \lambda_2\beta_2 H_1 + \lambda_3\beta_2 H_1 - \lambda_3(d + \omega + \mu) - \lambda_3 u_1 + \lambda_5 u_1\bigr) \\
&= -1 + \lambda_2\beta_2 H_1 - \lambda_3\beta_2 H_1 + \lambda_3(d + \omega + \mu) + \lambda_3 u_1 - \lambda_5 u_1, \\
\lambda_4' &= -\bigl(1 + \lambda_1\alpha + \lambda_2 u_2 - \lambda_4(\alpha + \gamma + \mu) - \lambda_4 u_2 + \lambda_5\gamma\bigr) \\
&= -1 - \lambda_1\alpha - \lambda_2 u_2 + \lambda_4(\alpha + \gamma + \mu) + \lambda_4 u_2 - \lambda_5\gamma, \\
\lambda_5' &= -(1 - \lambda_5\mu) \\
&= -1 + \lambda_5\mu.
\end{aligned}$$

For u_1^* and u_2^* differentiate the Hamiltonian function with respect to u_1 and u_2, respectively, and we have

$$\begin{aligned}
&\frac{\partial H}{\partial u_1} = 0, \\
&c_1 u_1^* - \lambda_3 H_2 + \lambda_5 H_2 = 0, \\
&c_1 u_1^* = \lambda_3 H_2 - \lambda_5 H_2, \\
&u_1^* = \frac{(\lambda_3 - \lambda_5) H_2}{c_1}, \\
&\frac{\partial H}{\partial u_2} = 0, \\
&c_2 u_2^* + \lambda_2 Y - \lambda_4 Y = 0, \\
&c_2 u_2^* = \lambda_4 Y - \lambda_2 Y, \\
&u_2^* = \frac{(\lambda_4 - \lambda_2) Y}{c_2}.
\end{aligned}$$

The optimality conditions are given as follows:

$$\begin{aligned}
u_1^* &= \min\left(1, \max\left(0, \frac{(\lambda_3 - \lambda_5) S_2}{c_1}\right)\right), \\
u_2^* &= \min\left(1, \max\left(0, \frac{(\lambda_4 - \lambda_2) Y}{c_2}\right)\right).
\end{aligned}$$

The terminal conditions for the system (10) are given as follows:

$$\begin{aligned}
&X' = \lambda - \beta_1 X H_1 - \mu X + \alpha Y, \\
&H_1' = \beta_1 X H_1 - \beta_2 H_1 H_2 - (\rho + \mu) H_1 + \min\left(1, \max\left(0, \frac{(\lambda_4 - \lambda_2) Y}{c_2}\right)\right) Y,
\end{aligned}$$

$$
\begin{aligned}
H_2' &= \beta_2 H_1 H_2 - (d+\omega+\mu)H_2 - \min\left(1, \max\left(0, \frac{(\lambda_3-\lambda_5)H_2}{c_1}\right)\right)H_2, \\
Y' &= \omega H_2 - (\alpha+\gamma+\mu)Y - \min\left(1, \max\left(0, \frac{(\lambda_4-\lambda_2)Y}{c_2}\right)\right)Y, \\
Z' &= \gamma Y - \mu Z + \min\left(1, \max\left(0, \frac{(\lambda_3-\lambda_5)H_2}{c_1}\right)\right)H_2, \\
\lambda_1' &= -1 + \lambda_1\beta_1 H_1 + \lambda_1\mu - \lambda_2\beta_1 H_1, \\
\lambda_2' &= -1 + \lambda_1\beta_1 X - \lambda_2\beta_1 X + \lambda_2\beta_2 H_2 + \lambda_2(\rho+\mu) - \lambda_3\beta_2 H_2, \\
\lambda_3' &= -1 + \lambda_2\beta_2 H_1 - \lambda_3\beta_2 H_1 + \lambda_3(d+\omega+\mu) - \lambda_3 \min\left(1, \max\left(0, \frac{(\lambda_3\lambda_5)H_2}{c_1}\right)\right), \\
\lambda_4' &= -1 - \lambda_1\alpha - \lambda_2 \min\left(1, \max\left(0, \frac{(\lambda_4-\lambda_2)Y}{c_2}\right)\right) \\
&\quad + \lambda_4(\alpha+\gamma+\mu) + \lambda_4 \min\left(1, \max\left(0, \frac{(\lambda_4-\lambda_2)Y}{c_2}\right)\right) - \lambda_5\gamma, \\
\lambda_5' &= -1 + \lambda_5\mu. \qquad \square
\end{aligned}
$$

8 Numerical solution

This section is concerned with the investigation of a numerical solution of the smoking model with controls u_1 and u_2. The NSFD method is used for this purpose and system (10) is presented graphically by using the values of parameters given in Table 2 with $u_1 = 0.7$ and $u_2 = 0.9$. Figure 2 shows the results for both systems with and without control.

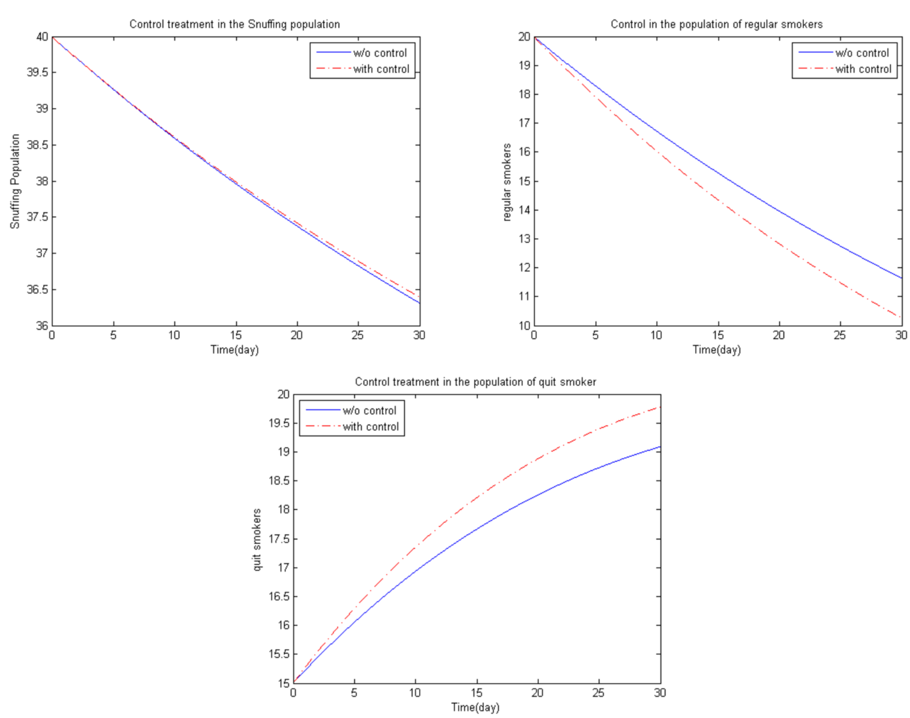

Figure 2 Plots present the susceptible, irregular, regular and quit smokers with and without control

9 Conclusion

In this work, the formulation of a model containing a snuffing class is presented; then the equilibrium points that are smoking free and smoking positive are discussed. The Hurwitz theorem is used for finding the local stability of the model and Lyaponov function theory is used for the search of global stability. For control of smoking we use different controls and for a characterization of the optimal level we use the Pontryagin maximum principle. For the solution of the proposed model, a nonstandard finite difference (NSFD) scheme and the Runge–Kutta fourth order method are used. Finally, some numerical results are presented for systems with and without control and by using the nonstandard finite difference (NSFD) method and Runge–Kutta fourth order method with the help of MATLAB.

Acknowledgements
This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. D-149-130-39. The authors, therefore, gratefully acknowledge DSR for technical and financial support.

Funding
This article is supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah.

Availability of data and materials
The authors confirm that the data supporting the findings of this study are available within the article cited therein.

Competing interests
The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' contributions
The authors equally contributed in preparing this manuscript. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia. [2]Department of Mathematics, COMSATS University Islamabad, Abbottabad, Pakistan.

Publisher's Note

Received: 4 October 2019 Accepted: 17 December 2019 Published online: 03 January 2020

References

1. Brownlee, J.: Certain considerations on the causation and course of epidemics. Proc. R. Soc. Med. **2**, 243–258 (1909)
2. Brownlee, J.: The mathematical theory of random migration and epidemic distribution. Proc. R. Soc. Edinb. **31**, 262–289 (1912)
3. Kermack, W.O., McKendrick, A.G.: Contributions to the mathematical theory of epidemics, part 1. Proc. R. Soc. Edinb., Sect. A, Math., **115**, 700–721 (1927)
4. Chong, J.-R.: Analysis clarifies route of AIDS. Los Angeles Times (2007)
5. Wang, K., Wang, W., Song, S.: Dynamics of an *HBV* model with difusion and deley. J. Theor. Biol. **253**(1), 36–44 (2008)
6. Huo, H.F., Ma, Z.P.: Dynamics of a delayed epidemic model with non-monotonic incidence rate. Commun. Nonlinear Sci. Numer. Simul. **15**(2), 459–468 (2010)
7. McCluskey, C.C.: Complete global stability for an SIR epidemic model with delay distributed or discrete. Nonlinear Anal., Real World Appl. **11**(1), 55–59 (2010)
8. Xu, R., Ma, Z.: Global stability of a SIR epidemic model with nonlinear incidence rate and time delay. Nonlinear Anal., Real World Appl. **10**(5), 3175–3189 (2009)
9. Xu, R., Ma, Z.: Stability of a delayed SIRS epidemic model with a nonlinear incidence rate. Chaos Solitons Fractals **41**(5), 2319–2325 (2009)
10. Song, X., Cheng, S.: A delay-differential equation model of HIV infection of $CD4^+$ T-cells. J. Korean Math. Soc. **42**(5), 1071–1086 (2005)
11. Guglielmi, N., Hairer, E.: Implementing Radau IIA methods for stiff delay differential equations. J. Comput. Math. **67**(1), 1–12 (2001)
12. Takeuchi, Y., Ma, W., Beretta, E.: Global asymptotic properties of a delay SIR epidemic model with finite incubation times. Nonlinear Anal., Theory Methods Appl. **42**(6), 931–947 (2010)
13. Van den Driessche, P., Watmough, J.: Further notes on the basic reproduction number. In: Mathematical Epidemiology. Lecture Notes in Mathematics, vol. 1945, pp. 159–178. Springer, Berlin (2008)
14. d'Onofrio, A., Manfredi, P., Salinelli, E.: Bifurcation thresholds in an SIR model with information-dependent vaccination. Math. Model. Nat. Phenom. **2**(1), 26–43 (2007)
15. Yi, N., Zhao, Z., Zhang, Q.: Bifurcations of an SEIQS epidemic model. Int. J. Inf. Syst. Sci. **5**(3–4), 296–310 (2009)

16. Fathalla, A.R., Anwar, M.N.: Qualitative analysis of delayed SIR epidemic model with saturated incidence rate. Int. J. Differ. Equ. **2012**, Article ID 408637 (2012)
17. Hethcote, H.W.: Qualitative analyses of communicable disease models. Math. Biosci. **7**, 335–356 (1976)
18. Smith, E.A., Malone, R.E.: Everywhere the soldier will be, wartime tobacco promotion in the US military. Am. J. Publ. Health **99**(9), 1595–1602 (2009)
19. Costa, R.M.S., Pavone, P.: Diachronic biodiversity analysis of a metropolitan area in the Mediterranean region. Acta Hortic. **1215**, 49–52 (2018)
20. Costa, R.M.S., Avone, P.: Invasive plants and natural habitats: the role of alien species in the urban vegetation. Acta Hortic. **1215**, 57–60 (2018)
21. Castillo-Garsow, C., Jordan-Salivia, G., Rodriguez Herrera, A.: Mathematical models for dynamics of tobacco use, recovery and relapse. Technical Report Series BU-1505-M, Cornell University (2000)
22. Sharomi, O., Gumel, A.B.: Curtailing smoking dynamics: a mathematical modeling approach. Appl. Math. Comput. **195**(2), 475–499 (2008)
23. Ham, O.K.: Stages and processes of smoking cessation among adolescents. West. J. Nursing Res. **29**(3), 301–315 (2007)
24. Zaman, G.: Qualitative behavior of giving up smoking models. Bull. Malays. Math. Sci. Soc. **34**(2), 403–415 (2011)
25. Swartz, J.B.: Use of a multistage model to predict time trends in smoking induced lung cancer. J. Epidemiol. Community Health **46**(3), 311–315 (1992)
26. Choi, H., Jung, I., Kang, Y.: Giving up smoking dynamic on adolescent nicotine dependence: a mathematical modeling approach. In: KSIAM 2011 Spring Conference, Daejeon, Korea (2011)
27. Huo, H., Zhu, C.: Influence of relapse in a giving up smoking model. Abstr. Appl. Anal. **2013**, Article ID 525461 (2013)
28. Zeb, A., Zaman, G., Momani, S.: Square-root dynamics of a giving up smoking model. Appl. Math. Model. **37**(7), 5326–5334 (2013)
29. Erturk, V.S., Momani, S., Zaman, G.: A numeric–analytic method for approximating a giving up smoking model containing fractional derivatives. Comput. Math. Appl. **64**(10), 3065–3074 (2012)
30. Alkhudhari, Z., Al- Sheikh, S., Al-Tuwarirqi, S.: Global dynamics of mathematical model on smoking. ISRN Appl. Math. **2011**, Article ID 847075 (2011)
31. Mickens, R.E.: Nonstandard Finite Difference Methods for Differential Equations. World Scientific, Singapore (1994)
32. Mickens, R.E.: Calculation of denominator functions for nonstandard finite difference schemes for differential equations satisfying a positivity condition. Numer. Methods Partial Differ. Equ. **23**(3), 672–691 (2007)
33. Pontryagin, L.S., Botyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E.F.: The Mathematical Theory of Optimal Processes. Wiley, Hoboken (1962)
34. Joshi, H.R., Lenhart, S., Li, M.Y., Wang, L.: Optimal control methods applied to diseases models. Contemp. Math. **410**, 187–208 (2006)
35. Zaman, G., Kang, Y.H., Jung, I.H.: Stability analysis and optimal vaccination of an SIR model. Biosystems **93**, 240–249 (2008)
36. Zeb, A., Zaman, G., Jung, I.H., Khan, M.: Optimal compaign strategies in fractional-order smoking dynamics. Z. Naturforsch. A **69**(5–6), 225–231 (2014)