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Multiple homoclinic solutions for p-Laplacian Hamiltonian systems with concave–convex nonlinearities

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Abstract

The multiplicity of homoclinic solutions is obtained for a class of the p-Laplacian Hamiltonian systems $\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t,u(t)) = 0$ via variational methods, where $a(t)$ is neither coercive nor bounded necessarily and $W(t,u)$ is under new concave–convex conditions. Recent results in the literature are generalized even for $p = 2$.

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1 Introduction

Let us consider the p-Laplacian Hamiltonian systems

$$\frac{d}{dt}\left(\left|\dot{u}(t)\right|^{p-2}\dot{u}(t)\right) - a(t)\left|u(t)\right|^{p-2}u(t) + \nabla W\left(t,u(t)\right) = 0, \tag{1}$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $p > 1$, $a \in C(\mathbb{R},[a_0,+\infty))$ with $a_0 > 0$ and $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. As usual, we say that u is a nontrivial homoclinic solution (to 0) if $u \not\equiv 0$, $u(t)$ and $\dot{u}(t) \to 0$ as $|t| \to +\infty$.

If $p \equiv 2$ and $a(t) = L(t)$, (1) reduces to the second order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W\left(t,u(t)\right) = 0,$$

where $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$. In the last 30 years, the existence and multiplicity of solutions for Hamiltonian systems or other differential systems have been investigated in many papers via variational methods (see [1–4, 9, 11, 14–18, 23]). It is well-known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. Since the problem is considered on the whole space, one of the difficulties to find the solutions of Hamiltonian systems is the lack of compactness of the Sobolev embedding. To overcome this difficulty, $L(t)$ and $W(t,x)$ were assumed to be periodic in t. Without periodicity, Rabinowitz and Tanaka [9] introduced the following coercive condition:

(L) there exists a continuous function $\alpha : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$\big(L(t)x, x\big) \geq \alpha(t)|x|^2 \quad \text{and} \quad \alpha(t) \to +\infty \quad \text{as } |t| \to +\infty.$$

The operator $\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t))$ in (1) is said to be p-Laplacian. In the last decade there has been an increasing interest in the study of ordinary differential systems driven by the p-Laplacian. The existence and multiplicity of homoclinic orbits for the p-Laplacian Hamiltonian system were studied in recent papers [5–7, 10, 12, 13, 19, 20, 22] and the references therein. Similarly, to overcome the lack of compactness of the Sobolev embedding, the following coercive assumption on a was assumed in [5]:

(A) a is a positive continuous function such that

$$a(t) \to +\infty \quad \text{as } |t| \to +\infty.$$

It is clear that the coercive conditions are much restrictive. In a recent paper, Zhang et al. [22] proved the existence of two nontrivial homoclinic solutions of problem (1) without coercive conditions. They assumed that a is bounded, that is,

(A') there are two constants τ_1 and τ_2 such that

$$0 < \tau_1 \leq a(t) \leq \tau_2 < +\infty \quad \text{for all } t \in \mathbb{R}.$$

Besides, they considered the concave–convex nonlinearity, which is of the form

$$W(t,x) = W_1(t,x) + W_2(t,x),$$

where W_1 is of super-p growth at infinity and W_2 is of sub-p growth at infinity. Explicitly, the authors supposed the following conditions:

(V_1) there exists a constant $\vartheta > p$ such that

$$0 < \vartheta W_1(t,x) \leq \big(\nabla W_1(t,x), x\big), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\};$$

(V_2) there exists a continuous function $w : \mathbb{R} \to \mathbb{R}^+$ such that

$$\lim_{|t|\to+\infty} w(t) = 0$$

and

$$\big|\nabla W_1(t,x)\big| \leq w(t)|x|^{\vartheta-1} \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(V_3) $W_2(t,0) = 0$ for all $t \in \mathbb{R}$, $W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and there exist a constant $1 < \varrho < 2$ and a continuous function $b : \mathbb{R} \to \mathbb{R}^+$ such that

$$W_2(t,x) \geq b(t)|x|^{\varrho}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$;

(V_4) for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$,

$$\left|\nabla W_2(t,x)\right| \le c(t)|u|^{\varrho-1},$$

where $c : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $c \in L^{\xi}(\mathbb{R},\mathbb{R})$ for some constant $1 \le \xi \le 2$;

(V_5)

$$\left(\frac{p\|c\|_{\xi} C^{\varrho}_{\varrho\xi^*}}{\varrho} \frac{\vartheta-\varrho}{\vartheta-p}\right)^{\vartheta-p} < \left(\frac{\vartheta}{p\|\omega\|_{\infty} C^{\vartheta}_{\vartheta}} \frac{p-\varrho}{\vartheta-\varrho}\right)^{\varrho-p},$$

where ξ^* is the conjugate component of ξ.

Obviously, we can deduce from the conditions (V_1) and (V_2) that

(W_0) there exist constants $c_1, c_2 > 0$ and $\mu > p$ such that

$$\left|\nabla W_1(t,x)\right| \le c_1|x|^{\mu-1} + c_2 \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_1) $\nabla W_1(t,x) = o(|x|^{p-1})$ as $|x| \to 0$ uniformly in t;

(W_2) $W_1(t,x)/|x|^p \to +\infty$ as $|x| \to +\infty$ uniformly in t;

(W_3) there exists $d_1 > 0$ such that $W_1(t,x) \ge -d_1|x|^p$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$;

(W_4) there are constants $\nu > p$ and ρ_0, $d_2 > 0$ such that

$$\left(\nabla W_1(t,x),x\right) - \nu W_1(t,x) \ge -d_2|x|^p, \quad \forall t \in \mathbb{R}, \forall |x| \ge \rho_0.$$

Motivated by the above facts, in this note, we try to drop both conditions (A) and (A') and consider the following conditions:

(A_1) $\int_{\mathbb{R}} a(t)^{-\frac{q}{p}}\, dt < +\infty$, where q is the conjugate component of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$;

(A_2) there exists a constant $\lambda > q^{-1}$ such that

$$\operatorname{meas}\left(t \in \mathbb{R} \mid |t|^{-\lambda p} a(t) < M\right) < +\infty, \quad \forall M > 0,$$

where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure and q is the conjugate component of p.

Using conditions (A_1) and (A_2) separately, we prove some new compact embedding theorems and discuss the multiplicity of homoclinic solutions for problem (1) with weaker combined nonlinearities. Now we state our main results.

Theorem 1 *Suppose that* $W(t,x) = W_1(t,x) + W_2(t,x)$. *Assume* (A_1), (W_0)–(W_4) *and the following conditions hold*:

(W_5) $W_2(t,0) = 0$ *for all* $t \in \mathbb{R}$ *and there exist a constant* $1 < \theta < p$ *and a continuous function* $b : \mathbb{R} \to \mathbb{R}^+$ *such that*

$$W_2(t,x) \ge b(t)|x|^{\theta}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$;

(W_6) $W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *and there exists a continuous function* $c : \mathbb{R} \to \mathbb{R}^+$ *such that*

$$\left|\nabla W_2(t,x)\right| \le c(t)|x|^{\theta-1},$$

where $c \in L^{\zeta}(\mathbb{R},\mathbb{R})$ *for some constant* $\zeta > 1$ *and* $\|c\|_{\zeta}$ *is small enough*;

(W_7) *$\zeta^*(\theta-1)\geq p$, where ζ^* is the conjugate component of ζ.*
Then problem (1) possesses at least two nontrivial homoclinic solutions.

Remark 1 From Theorem 1, we see that the conditions related to the sup-p term W_1 are weaker than that in [22]. There are functions satisfying the conditions (W_0)–(W_4) but not (V_1) and (V_2). Moreover, we can also give some examples of a not satisfying the conditions (A) and (A'). For example, let

$$W_1(t,x)=\begin{cases}-|x|^4+|x|^3, & |x|\leq\frac{4}{5}\\ (|x|-\frac{4+4^{\frac{1}{3}}}{5})^4+\frac{64-4^{\frac{4}{3}}}{625}, & |x|\geq\frac{4}{5},\end{cases}\qquad W_2(t,x)=\frac{\epsilon}{(1+t^2)^{\frac{3}{4}}}\frac{2|x|^{\frac{3}{2}}}{3},$$

and

$$a(t)=\begin{cases}(n^2+1)^2(|t|-n)+c_0, & n\leq|t|<n+\frac{1}{n^2+1},\\ (n^2+1)+c_0, & n+\frac{1}{n^2+1}\leq|t|<n+\frac{n^2}{n^2+1},\\ (n^2+1)^2(n+1-|t|)+c_0, & n+\frac{n^2}{n^2+1}\leq|t|<n+1,\end{cases}$$

where $n\in\mathbb{N}$, $c_0\in\mathbb{R}$. A straightforward computation shows that W_1, W_2 and a satisfy the assumptions of Theorem 1 with $p=2$, $\mu=5$, $\theta=\frac{3}{2}$, $\zeta=\frac{4}{3}$ and $\epsilon>0$ small enough.

By replacing the condition (A_1), we have the following theorem.

Theorem 2 *Assume that $W(t,x)=W_1(t,x)+W_2(t,x)$. Suppose that (A_2) and (W_0)–(W_7) hold, then problem (1) possesses at least two nontrivial homoclinic solutions.*

Remark 2 There exist functions that satisfy the condition (A_2) but do not satisfy the conditions (A) and (A'), such as $a(t)=t^4\sin^2 t+1$ with $p=2$ and $\lambda=1$. Thus Theorem 2 is different from the previous results.

2 Proof of Theorem 1

First, we introduce the space in which we can construct the variational framework. Let

$$E=\left\{u\in W^{1,p}(\mathbb{R},\mathbb{R}^N):\int_{\mathbb{R}}\left(\left|\dot{u}(t)\right|^p+a(t)\left|u(t)\right|^p\right)dt<+\infty\right\}$$

with the norm

$$\|u\|=\left(\int_{\mathbb{R}}\left(\left|\dot{u}(t)\right|^p+a(t)\left|u(t)\right|^p\right)dt\right)^{\frac{1}{p}}.$$

Then E is a uniform convex Banach space. Denote by $L^\gamma(\mathbb{R},\mathbb{R}^N)$ $(1\leq\gamma<+\infty)$ the Banach spaces of functions with the norms

$$\|u\|_\gamma=\left(\int_{\mathbb{R}}\left|u(t)\right|^\gamma dt\right)^{\frac{1}{\gamma}},$$

and $L^\infty(\mathbb{R},\mathbb{R}^N)$ is the Banach space of essentially bounded functions under the norm

$$\|u\|_\infty=\operatorname{ess\ sup}\left\{\left|u(t)\right|:t\in\mathbb{R}\right\}.$$

Lemma 1 ([22]) *The embedding $E \hookrightarrow L^{\gamma}(\mathbb{R},\mathbb{R}^N)$ $(p \le \gamma \le +\infty)$ is continuous.*

Lemma 2 *Under the condition (A_1), the embedding $E \hookrightarrow L^1(\mathbb{R},\mathbb{R}^N)$ is continuous and compact.*

Proof By (A_1) and Hölder's inequality, for all $u \in E$ one has

$$\begin{aligned}\int_{\mathbb{R}} |u(t)|\,dt &= \int_{\mathbb{R}} a(t)^{-\frac{1}{p}} a(t)^{\frac{1}{p}} |u(t)|\,dt \\ &\le \left(\int_{\mathbb{R}} a(t)^{-\frac{q}{p}}\,dt\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} a(t)|u(t)|^p\,dt\right)^{\frac{1}{p}} \\ &\le \left(\int_{\mathbb{R}} a(t)^{-\frac{q}{p}}\,dt\right)^{\frac{1}{q}} \|u\|,\end{aligned}$$

which implies that the embedding is continuous.

Let $\{u_n\} \subset E$ be a sequence such that $u_n \rightharpoonup 0$ in E. By Banach–Steinhaus Theorem, there exists $M_0 > 0$ such that

$$\sup_{n\in\mathbb{N}} \|u_n\| \le M_0.$$

Since the embedding is compact on bounded domain, it suffices to show that, for any $\varepsilon > 0$, there exists $r > 0$ such that

$$\int_{|t|>r} |u_n(t)|\,dt < \varepsilon.$$

In fact, we have

$$\begin{aligned}\int_{|t|>r} |u_n(t)|\,dt &\le \int_{|t|>r} a(t)^{-\frac{1}{p}} a(t)^{\frac{1}{p}} |u_n(t)|\,dt \\ &\le \left(\int_{|t|>r} a(t)^{-\frac{q}{p}}\,dt\right)^{\frac{1}{q}} \left(\int_{|t|>r} a(t)|u_n(t)|^p\,dt\right)^{\frac{1}{p}} \\ &\le \left(\int_{|t|>r} a(t)^{-\frac{q}{p}}\,dt\right)^{\frac{1}{q}} \|u_n\| \\ &\le \left(\int_{|t|>r} a(t)^{-\frac{q}{p}}\,dt\right)^{\frac{1}{q}} M_0.\end{aligned}$$

It follows from (A_1) that this can be made arbitrarily small by choosing r large. Hence, we get $u_n \to 0$ in $L^1(\mathbb{R},\mathbb{R}^N)$. □

Remark 3 From Lemma 1 and Lemma 2, for $\gamma = 1$ or $p \le \gamma \le +\infty$, there exists $C_\gamma > 0$ such that

$$\|u\|_\gamma \le C_\gamma \|u\|, \quad \forall u \in E. \tag{2}$$

Lemma 3 *Suppose that the conditions (A_1) and (W_1) hold, then we have $\nabla W_1(t,u_n) \to \nabla W_1(t,u)$ in $L^q(\mathbb{R},\mathbb{R}^N)$ if $u_n \rightharpoonup u$ in E.*

Proof Assume that $u_n \rightharpoonup u$ in E. By the Banach–Steinhaus theorem and (2), there exists $M_1 > 0$ such that

$$\sup_{n\in\mathbb{N}} \|u_n\|_\infty \le M_1 \quad \text{and} \quad \|u\|_\infty \le M_1. \tag{3}$$

We can deduce from (W_0), (W_1) and (3) that there exists $M_2 > 0$ such that

$$\left|\nabla W_1(t,u_n)\right| \le M_2\left|u_n(t)\right|^{p-1} \quad \text{and} \quad \left|\nabla W_1(t,u)\right| \le M_2\left|u(t)\right|^{p-1},$$

which implies that

$$\begin{aligned}\left|\nabla W_1(t,u_n) - \nabla W_1(t,u)\right| &\le M_2\left(\left|u_n(t)\right|^{p-1} + \left|u(t)\right|^{p-1}\right) \\ &\le M_2\left[2^{p-1}\left(\left|u_n(t)-u(t)\right|^{p-1} + \left|u(t)\right|^{p-1}\right) + \left|u(t)\right|^{p-1}\right] \\ &\le M_3\left(\left|u_n(t)-u(t)\right|^{p-1} + \left|u(t)\right|^{p-1}\right), \end{aligned} \tag{4}$$

where M_3 is a positive constant. By (2), (3), (4) and Lemma 2 one gets

$$\begin{aligned} &\int_{\mathbb{R}} \left|\nabla W_1(t,u_n) - \nabla W_1(t,u)\right|^q dt \\ &\quad\le M_3^q \int_{\mathbb{R}} \left(\left|u_n(t)-u(t)\right|^{p-1} + \left|u(t)\right|^{p-1}\right)^q dt \\ &\quad\le 2^{q-1}M_3^q \int_{\mathbb{R}} \left(\left|u_n(t)-u(t)\right|^{p} + \left|u(t)\right|^{p}\right) dt \\ &\quad\le 2^{q-1}M_3^q \|u_n - u\|_\infty^{p-1} \int_{\mathbb{R}} \left|u_n(t)-u(t)\right| dt + 2^{q-1}M_3^q\|u\|_p^p \\ &\quad\le 2^{q-1}M_3^q (2M_1)^{p-1} \int_{\mathbb{R}} \left|u_n(t)-u(t)\right| dt + 2^{q-1}M_3^q C_p^p\|u\|^p \\ &\quad< +\infty. \end{aligned}$$

Using Lebesgue's dominated convergence theorem, we can get the conclusion. □

The corresponding functional of (1) is defined by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \frac{1}{p}\left(\left|\dot{u}(t)\right|^p + a(t)\left|u(t)\right|^p\right) dt - \int_{\mathbb{R}} W\left(t,u(t)\right) dt \\ &= \frac{1}{p}\|u\|^p - \int_{\mathbb{R}} W\left(t,u(t)\right) dt. \end{aligned} \tag{5}$$

For convenience, let

$$\begin{aligned} J(u) &= \int_{\mathbb{R}} \frac{1}{p}\left(\left|\dot{u}(t)\right|^p + a(t)\left|u(t)\right|^p\right) dt, \\ \Phi(u) &= \int_{\mathbb{R}} W_1\left(t,u(t)\right) dt, \\ \Psi(u) &= \int_{\mathbb{R}} W_2\left(t,u(t)\right) dt. \end{aligned}$$

Lemma 4

(i) $J \in C^1(E, \mathbb{R})$ *and*

$$\langle J'(u), v\rangle = \int_{\mathbb{R}} \left[|\dot{u}(t)|^{p-2} (\dot{u}(t), \dot{v}(t)) + a(t)|u(t)|^{p-2}(u(t), v(t)) \right] dt, \quad \forall u, v \in E.$$

(ii) *Under the conditions of Theorem 1*, $I \in C^1(E, \mathbb{R})$. *Moreover, one has*

$$\begin{aligned} \langle I'(u), v\rangle = \int_{\mathbb{R}} &\left[|\dot{u}(t)|^{p-2} (\dot{u}(t), \dot{v}(t)) + a(t)|u(t)|^{p-2}(u(t), v(t)) \right. \\ &\left. - (\nabla W(t, u(t)), v(t)) \right] dt, \quad \forall u, v \in E. \end{aligned} \tag{6}$$

(iii) *The critical points of I in E are homoclinic solutions of* (1) *with* $u(\pm\infty) = \dot{u}(\pm\infty) = 0$.

Proof Since it is routine to prove that (i) holds, we just need to prove (ii) and (iii). First, we show I in (5) is well defined. By (W_0) and (W_1), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|W_1(t,x)| \le \varepsilon |x|^p + C_\varepsilon |x|^\mu, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{7}$$

Then by (2) and (7) one gets

$$\int_{\mathbb{R}} |W_1(t, u(t))| \, dt \le \varepsilon \int_{\mathbb{R}} |u(t)|^p \, dt + C_\varepsilon \int_{\mathbb{R}} |u(t)|^\mu \, dt \le \varepsilon C_p^p \|u\|^p + C_\varepsilon C_\mu^\mu \|u\|^\mu < +\infty.$$

Besides, by (2), (W_6), (W_7) and Hölder's inequality we have

$$\begin{aligned} \int_{\mathbb{R}} |W_2(t, u(t))| \, dt &\le \frac{1}{\theta} \int_{\mathbb{R}} c(t) |u(t)|^\theta \, dt \\ &\le \frac{1}{\theta} \|c\|_\zeta \|u\|_{\theta\zeta^*}^\theta \\ &\le \frac{C_{\theta\zeta^*}^\theta}{\theta} \|c\|_\zeta \|u\|^\theta < +\infty. \end{aligned} \tag{8}$$

Therefore I is well defined. Next, we show that $I \in C^1(E, \mathbb{R})$. In view of (i), it is sufficient to show that $\Phi \in C^1(E, \mathbb{R})$ and $\Psi \in C^1(E, \mathbb{R})$. Let $\phi(u)$ be as follows:

$$\phi(u)v = \int_{\mathbb{R}} (\nabla W_1(t, u(t)), v(t)) \, dt, \quad \forall v \in E. \tag{9}$$

Obviously, $\phi(u)$ is linear. We show $\phi(u)$ is bounded in the following proof. By (2), (9), (W_0) and Hölder's inequality, one has

$$\begin{aligned} |\phi(u)v| &\le c_1 \int_{\mathbb{R}} |u(t)|^{\mu-1} |v(t)| \, dt + c_2 \int_{\mathbb{R}} |v(t)| \, dt \\ &\le c_1 \left(\int_{\mathbb{R}} |u(t)|^{(\mu-1)\mu^*} \, dt \right)^{\frac{1}{\mu^*}} \left(\int |v(t)|^\mu \, dt \right)^{\frac{1}{\mu}} + c_2 \|v\|_1 \\ &\le c_1 \|u\|_\mu^{\frac{\mu}{\mu^*}} \|v\|_\mu + c_2 C_1 \|v\| \\ &\le \left(c_1 C_\mu^{\frac{\mu}{\mu^*}+1} \|u\|^{\frac{\mu}{\mu^*}} + c_2 C_1 \right) \|v\|, \end{aligned} \tag{10}$$

where μ^* is the conjugate component of μ. It follows from (10) that $\phi(u)$ is bounded. Subsequently, we show that Φ is of C^1 class. For any $u, v \in E$, by the mean value theorem, (W_0) and Hölder's inequality, one gets

$$\begin{aligned}
&\left|\int_{\mathbb{R}} W_1\bigl(t,u(t)+v(t)\bigr)\,dt - \int_{\mathbb{R}} \bigl(W_1\bigl(t,u(t)\bigr)\,dt\right| \\
&\quad = \left|\int_{\mathbb{R}} \bigl(\nabla W_1\bigl(t,u(t)+h(t)v(t),v(t)\bigr)\,dt\right| \\
&\quad \le c_1 \int_{\mathbb{R}} \bigl|u(t)+h(t)v(t)\bigr|^{\mu-1}\bigl|v(t)\bigr|\,dt + c_2\int_{\mathbb{R}}\bigl|v(t)\bigr|\,dt \\
&\quad \le c_1\|u+hv\|_{\mu}^{\frac{\mu}{\mu^*}}\|v\|_{\mu} + c_2C_1\|v\| \\
&\quad \le \bigl(c_1C_{\mu}^{\frac{\mu}{\mu^*}+1}\|u+hv\|^{\frac{\mu}{\mu^*}} + c_2C_1\bigr)\|v\|,
\end{aligned} \tag{11}$$

where $h(t) \in (0,1)$. Combining (10) and (11), we get

$$\int_{\mathbb{R}} W_1\bigl(t,u(t)+v(t)\bigr)\,dt - \int_{\mathbb{R}} W_1\bigl(t,u(t)\bigr)\,dt - \int_{\mathbb{R}}\bigl(\nabla W_1\bigl(t,u(t)\bigr),v(t)\bigr)\,dt \to 0$$

as $v \to 0$ in E, which shows

$$\bigl\langle \Phi'(u),v\bigr\rangle = \int_{\mathbb{R}}\bigl(\nabla W_1\bigl(u(t)\bigr),v(t)\bigr)\,dt$$

for any $u, v \in E$. It remains to prove that Φ' is continuous. Assume that $u \to u_0$ in E and note that

$$\begin{aligned}
&\sup_{\|v\|=1}\bigl|\bigl\langle\Phi'(u),v\bigr\rangle - \bigl\langle\Phi'(u_0),v\bigr\rangle\bigr| \\
&\quad = \sup_{\|v\|=1}\left|\int_{\mathbb{R}}\bigl(\nabla W_1\bigl(t,u(t)\bigr) - \nabla W_1\bigl(t,u_0(t)\bigr),v(t)\bigr)\,dt\right| \\
&\quad \le \sup_{\|v\|=1}\bigl\|\nabla W_1(t,u) - \nabla W_1(t,u_0)\bigr\|_q\left(\int_{\mathbb{R}}\bigl|v(t)\bigr|^p\,dt\right)^{\frac{1}{p}} \\
&\quad \le \sup_{\|v\|=1}\bigl\|\nabla W_1(t,u) - \nabla W_1(t,u_0)\bigr\|_q\left(\int_{\mathbb{R}}\bigl|v(t)\bigr|^p\,dt\right)^{\frac{1}{p}} \\
&\quad \le C_p\sup_{\|v\|=1}\bigl\|\nabla W_1(t,u) - \nabla W_1(t,u_0)\bigr\|_q.
\end{aligned}$$

Then, by Lemma 3, we have $\langle\Phi'(u),v\rangle \to \langle\Phi'(u_0),v\rangle$ as $\|u\| \to \|u_0\|$ uniformly with respect to v, which shows that Φ' is continuous. Moreover, by (W_6) and (W_7) one has

$$\begin{aligned}
\left|\int_{\mathbb{R}}\bigl(\nabla W_2\bigl(t,u(t)\bigr),v(t)\bigr)\,dt\right| &\le \int_{\mathbb{R}} c(t)\bigl|u(t)\bigr|^{\theta-1}\bigl|v(t)\bigr|\,dt \\
&\le \|u\|_{\zeta^*(\theta-1)}^{\theta-1}\left(\int_{\mathbb{R}} c^{\zeta}(t)\,dt\right)^{\frac{1}{\zeta}}\|v\|_{\infty}
\end{aligned}$$

for any $u, v \in E$. Similar to the above proof, we can see that

$$\langle \Psi'(u), v \rangle = \int_{\mathbb{R}} \big(\nabla W_2\big(u(t)\big), v(t) \big)\, dt$$

for any $u, v \in E$. Now we prove that Ψ' is continuous. Suppose that $u \to u_0$ in E. By (W_6), for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\left(\int_{|t|>T} c^{\zeta}(t)\, dt \right)^{\frac{1}{\zeta}} < \varepsilon. \tag{12}$$

On account of the continuity of $\nabla W_2(t,x)$ and $u \to u_0$ in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, it follows that

$$\int_{|t|\le T} \big(\nabla W_2\big(t, u(t)\big) - \nabla W_2\big(t, u_0(t)\big), v(t) \big)\, dt < \varepsilon. \tag{13}$$

By (12), (13), (W_6), (W_7) and Hölder's inequality, one gets

$$\begin{aligned}
&\sup_{\|v\|=1} \big| \langle \Psi'(u), v \rangle - \langle \Psi'(u_0), v \rangle \big| \\
&\quad = \sup_{\|v\|=1} \left| \int_{\mathbb{R}} \big(\nabla W_2\big(t, u(t)\big) - \nabla W_2\big(t, u_0(t)\big), v(t) \big)\, dt \right| \\
&\quad \le \sup_{\|v\|=1} \left| \int_{|t|\le T} \big(\nabla W_2\big(t, u(t)\big) - \nabla W_2\big(t, u_0(t)\big), v(t) \big)\, dt \right| \\
&\qquad + \sup_{\|v\|=1} \left| \int_{|t|>T} \big(\nabla W_2\big(t, u(t)\big) - \nabla W_2\big(t, u_0(t)\big), v(t) \big)\, dt \right| \\
&\quad \le \varepsilon + \sup_{\|v\|=1} \left| \int_{|t|>T} c(t) \big(|u(t)|^{\theta-1} + |u_0(t)|^{\theta-1} \big) |v(t)|\, dt \right| \\
&\quad \le \varepsilon + C_{\infty} \left(\int_{|t|>T} c^{\zeta}(t)\, dt \right)^{\frac{1}{\zeta}} \big(\|u\|^{\theta-1}_{(\theta-1)\zeta^*} + \|u_0\|^{\theta-1}_{(\theta-1)\zeta^*} \big) \\
&\quad \le \varepsilon + \varepsilon C_{\infty} \big(\|u\|^{\theta-1}_{(\theta-1)\zeta^*} + \|u_0\|^{\theta-1}_{(\theta-1)\zeta^*} \big),
\end{aligned}$$

which shows that Ψ' is continuous. Thus (ii) holds.

Finally, similar to the proof of Lemma 3.1 in [21], one can check that (iii) holds. □

Subsequently, we display the useful critical points theorem.

Lemma 5 ([8]) *Let E a real Banach space and $I : E \to \mathbb{R}$ be a C^1-smooth functional and satisfy the (C) condition, that is, $\{u_n\}$ has a convergent subsequence in E whenever $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0$ as $n \to +\infty$. If I satisfies the following conditions*:

(i) $I(0) = 0$;

(ii) *there exist constants $\varrho, \alpha > 0$ such that $I|_{\partial B_{\varrho}(0)} \ge \alpha$*;

(iii) *there exists $e \in E \setminus \bar{B}_{\varrho}(0)$ such that $I(e) \le 0$*,

where $B_{\varrho}(0)$ is an open ball in E of radius ϱ centered at 0, then I possesses a critical value $c \ge \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I\big(g(s)\big),$$

where

$$\Gamma = \bigl\{g \in C\bigl([0,1],E\bigr) : g(0) = 0, g(1) = e\bigr\}.$$

Lemma 6 *Assume that the conditions of Theorem 1 hold, then I satisfies the* (C) *condition.*

Proof Suppose that $\{u_n\} \subset E$ is a sequence such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0$ as $n \to +\infty$. Then there exists a constant $M_4 > 0$ such that

$$\bigl|I(u_n)\bigr| \le M_4, \qquad \bigl\|I'(u_n)\bigr\|_{E^*}\bigl(1 + \|u_n\|\bigr) \le M_4. \tag{14}$$

Now we prove that $\{u_n\}$ is bounded in E. Arguing in an indirect way, we assume that $\|u_n\| \to +\infty$ as $n \to +\infty$. Set $z_n = \frac{u_n}{\|u_n\|}$, then $\|z_n\| = 1$, which implies that there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that $z_n \rightharpoonup z_0$ in E. By (2), (5), (8) and (14), we obtain

$$\begin{aligned} \left|\int_{\mathbb{R}} \frac{W_1(t,u_n)}{\|u_n\|^p}\,dt - \frac{1}{p}\right| &= \left|\frac{I(u_n)}{\|u_n\|^p} + \int_{\mathbb{R}} \frac{W_2(t,u_n)}{\|u_n\|^p}\,dt\right| \\ &\le \frac{M_4}{\|u_n\|^p} + \frac{\|c\|_\zeta C_{\theta\zeta^*}^\theta \|u_n\|^\theta}{\theta\|u_n\|^p} \\ &\to 0 \quad \text{as } n \to +\infty. \end{aligned} \tag{15}$$

In the following, we consider two opposite cases.

Case 1: $z_0 \not\equiv 0$. Let $\Omega = \{t \in \mathbb{R} | |z_0(t)| > 0\}$. Then we can see that $\operatorname{meas}(\Omega) > 0$, where meas denotes the Lebesgue measure. Then there exists $\chi > 0$ such that $\operatorname{meas}(\Lambda) > 0$, where $\Lambda = \Omega \cap P_\chi$ and $P_\chi = \{t \in \mathbb{R} | |t| \le \chi\}$. Since $\|u_n\| \to +\infty$ as $n \to +\infty$, we have $|u_n(t)| \to +\infty$ as $n \to +\infty$ for a.e. $t \in \Lambda$. By (W_2), (W_3) and Fatou's lemma, one can get

$$\begin{aligned} &\lim_{n\to+\infty} \int_{\mathbb{R}} \frac{W_1(t,u_n(t))}{\|u_n\|^p}\,dt \\ &\quad= \lim_{n\to+\infty} \int_{\Lambda} \frac{W_1(t,u_n(t))}{\|u_n\|^p}\,dt + \lim_{n\to+\infty} \int_{\mathbb{R}\setminus\Lambda} \frac{W_1(t,u_n(t))}{\|u_n\|^p}\,dt \\ &\quad\ge \lim_{n\to+\infty} \int_{\Lambda} \frac{W_1(t,u_n(t))}{|u_n(t)|^p}\bigl|z_n(t)\bigr|^p\,dt - d_1 \int_{\mathbb{R}\setminus\Lambda} \bigl|z_n(t)\bigr|^p\,dt \\ &\quad\ge \lim_{n\to+\infty} \int_{\Lambda} \frac{W_1(t,u_n(t))}{|u_n(t)|^p}\bigl|z_n(t)\bigr|^p\,dt - d_1 C_p^p \|z_n\|^p \\ &\quad= +\infty, \end{aligned}$$

which contradicts (15). So $\|u_n\|$ is bounded in this case.

Case 2: $z_0 \equiv 0$. Set

$$\widetilde{W_1}(t,x) = \bigl(\nabla W_1(t,x), x\bigr) - \nu W_1(t,x),$$

where ν is defined in (W_4). From (W_1), we can deduce that $\widetilde{W_1}(t,x) = o(|x|^p)$ as $|x| \to 0$, then there exists $\rho_1 \in (0,\rho_0)$ such that

$$\bigl|\widetilde{W_1}(t,x)\bigr| \le |x|^p \tag{16}$$

for all $|x| \leq \rho_1$, where ρ_0 is defined in (W_4). It follows from (6), (8), (14), (16), (W_4) and (W_6) that

$$\begin{aligned}
o(1) &= \frac{\nu M_4 + M_4}{\|u_n\|^p} \\
&\geq \frac{\nu I(u_n) - \langle I'(u_n), u_n\rangle}{\|u_n\|^p} \\
&\geq \left(\frac{\nu}{p} - 1\right) + \frac{1}{\|u_n\|^p} \int_{\mathbb{R}} \widetilde{W}_1\big(t, u_n(t)\big)\,dt - \frac{\nu + \theta}{\theta \|u_n\|^p} \|c\|_\zeta C^\theta_{\theta\zeta^*} \|u_n\|^\theta \\
&\geq \left(\frac{\nu}{p} - 1\right) + \frac{1}{\|u_n\|^p} \int_{|u_n| \leq \rho_1} \widetilde{W}_1\big(t, u_n(t)\big)\,dt + \frac{1}{\|u_n\|^p} \int_{\rho_1 < |u_n| \leq \rho_0} \widetilde{W}_1\big(t, u_n(t)\big)\,dt \\
&\quad + \frac{1}{\|u_n\|^p} \int_{|u_n| > \rho_0} \widetilde{W}_1\big(t, u_n(t)\big)\,dt - o(1) \\
&\geq \left(\frac{\nu}{p} - 1\right) - \frac{1}{\|u_n\|^p} \left(\int_{|u_n| \leq \rho_1} \big|u_n(t)\big|^p\,dt + d_2 \int_{|u_n| > \rho_0} \big|u_n(t)\big|^p\,dt\right) \\
&\quad - \frac{\max_{\rho_1 < |x| \leq \rho_0} |\widetilde{W}_1(t,x)|}{\rho_1^p} \int_{\rho_1 < |u_n| \leq \rho_0} \frac{|u_n(t)|^p}{\|u_n\|^p}\,dt - o(1) \\
&\geq \left(\frac{\nu}{p} - 1\right) - \left(1 + d_2 + \frac{\max_{\rho_1 < |x| \leq \rho_0} |\widetilde{W}_1(t,x)|}{\rho_1^p}\right) \int_{\mathbb{R}} \big|z_n(t)\big|^p\,dt - o(1) \\
&\to \frac{\nu}{p} - 1 \quad \text{as } n \to +\infty,
\end{aligned}$$

which is a contradiction. Therefore, $\|u_n\|$ is bounded.

Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E, which yields

$$\begin{aligned}
\big\langle I'(u_n) - I'(u), u_n - u\big\rangle &= \|u_n - u\|^p \\
&\quad - \int_{\mathbb{R}} \big(\nabla W_1\big(t, u_n(t)\big) - \nabla W_1\big(t, u(t)\big), u_n(t) - u(t)\big)\,dt \\
&\quad - \int_{\mathbb{R}} \big(\nabla W_2\big(t, u_n(t)\big) - \nabla W_2\big(t, u(t)\big), u_n(t) - u(t)\big)\,dt \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{17}$$

It follows from (2), (W_0) and Lemma 2 that

$$\begin{aligned}
&\int_{\mathbb{R}} \big(\nabla W_1\big(t, u_n(t)\big) - \nabla W_1\big(t, u(t)\big), u_n(t) - u(t)\big)\,dt \\
&\quad \leq \int_{\mathbb{R}} \big(c_1\big|u_n(t)\big|^{\mu-1} + c_1\big|u(t)\big|^{\mu-1} + 2c_2\big)\big|u_n(t) - u(t)\big|\,dt \\
&\quad \leq \big(c_1 C_\infty^{\mu-1}\|u_n\|^{\mu-1} + c_1 C_\infty^{\mu-1}\|u\|^{\mu-1} + 2c_2\big)\|u_n - u\|_1 \\
&\quad \to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{18}$$

On account of the continuity of $\nabla W_2(t,x)$ and $u_n \to u$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{|t| \leq T} \big(\nabla W_2\big(t, u_n(t)\big) - \nabla W_2\big(t, u(t)\big), u_n(t) - u(t)\big)\,dt < \varepsilon, \quad \forall n \geq n_0, \tag{19}$$

where T is defined in (12). In addition, by (12), (W_7) and Hölder's inequality, we have

$$\begin{aligned}
&\int_{|t|>T} \bigl(\nabla W_2\bigl(t,u_n(t)\bigr) - \nabla W_2\bigl(t,u(t)\bigr), u_n(t) - u(t)\bigr)\,dt \\
&\quad\le \int_{|t|>T} c(t)\bigl(\bigl|u_n(t)\bigr|^{\theta-1} + \bigl|u(t)\bigr|^{\theta-1}\bigr)\bigl|u_n(t) - u(t)\bigr|\,dt \\
&\quad\le \|u_n - u\|_\infty \biggl(\int_{|t|>T} c^\zeta(t)\,dt\biggr)^{\frac{1}{\zeta}} \bigl(\|u_n\|_{\zeta^*(\theta-1)}^{\theta-1} + \|u\|_{\zeta^*(\theta-1)}^{\theta-1}\bigr) \\
&\quad\le \varepsilon\|u_n - u\|_\infty \bigl(\|u_n\|_{\zeta^*(\theta-1)}^{\theta-1} + \|u\|_{\zeta^*(\theta-1)}^{\theta-1}\bigr). \qquad (20)
\end{aligned}$$

Hence, by (17)–(20) we conclude that $\|u_n - u\| \to 0$ as $n \to +\infty$, which means that the (C) condition is fulfilled. □

Lemma 7 *Suppose that the conditions of Theorem 1 hold, then there exist $\varrho_1, \alpha_1 > 0$ such that $I|_{\partial B_{\varrho_1}} \ge \alpha_1$, where $B_{\varrho_1} = \{u \in E : \|u\| \le \varrho_1\}$.*

Proof In view of (7) and (8), for any $u \in E$ and $0 < \varepsilon < (pC_p^p)^{-1}$, we have

$$\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \int_{\mathbb{R}} W_1(t,u)\,dt - \int_{\mathbb{R}} W_2(t,u)\,dt \\
&\ge \frac{1}{p}\|u\|^p - \varepsilon\int_{\mathbb{R}} |u|^p\,dt - C_\varepsilon \int_{\mathbb{R}} |u|^\mu\,dt - \frac{C_{\theta\zeta^*}^\theta}{\theta}\|c\|_\zeta \|u\|^\theta \\
&\ge \frac{1}{p}\|u\|^p - \varepsilon C_p^p\|u\|^p - C_\varepsilon C_\mu^\mu \|u\|^\mu - \frac{C_{\theta\zeta^*}^\theta}{\theta}\|c\|_\zeta \|u\|^\theta \\
&\ge \biggl(\frac{1}{p} - \varepsilon C_p^p\biggr)\|u\|^p - C_\varepsilon C_\mu^\mu \|u\|^\mu - \frac{C_{\theta\zeta^*}^\theta}{\theta}\|c\|_\zeta \|u\|^\theta,
\end{aligned}$$

which combined with (W_6) implies that there exist positive constants ϱ_1 and α_1 such that $I|_{\partial B_{\varrho_1}} \ge \alpha_1$. □

Lemma 8 *Assume that the conditions of Theorem 1 hold, then there exists $v_1 \in E$ such that $\|v_1\| > \varrho_1$ and $I(v_1) \le 0$, where ϱ_1 is defined in Lemma 7.*

Proof We choose $v_0 \in C_0^\infty([-1,1],\mathbb{R}^N)$ such that $\|v_0\| = 1$. For $\beta > (p\int_{-1}^1 |v_0(t)|^p\,dt)^{-1}$, it follows from (W_2) that there exists $\tau > 0$ such that

$$W(t,x) \ge \beta|x|^p$$

for all $|x| \ge \tau$. By (W_3), we get

$$W(t,x) \ge \beta\bigl(|x|^p - \tau^p\bigr) - d_1\tau^p \qquad (21)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. For $\eta > 0$, by (21) and (W_5) we have

$$\begin{aligned} I(\eta v_0) &= \frac{\eta^p}{p} - \int_{-1}^{1} W_1\big(t, \eta v_0(t)\big)\,dt - \int_{-1}^{1} W_2\big(t, \eta v_0(t)\big)\,dt \\ &\le \frac{\eta^p}{p} - \int_{-1}^{1} W_1\big(t, \eta v_0(t)\big)\,dt \\ &\le \frac{\eta^p}{p} - \int_{-1}^{1} \beta\big|\eta v_0(t)\big|^p\,dt + \beta\int_{-1}^{1} \tau^p\,dt + d_1\int_{-1}^{1} \tau^p\,dt \\ &\le \left(\frac{1}{p} - \beta\int_{-1}^{1}\big|v_0(t)\big|^p\,dt\right)\eta^p + 2\beta\tau^p + 2d_1\tau^p, \end{aligned}$$

which implies that

$$I(\eta v_0) \to -\infty \quad \text{as } \eta \to +\infty.$$

Therefore, there exists $\eta_0 > 0$ such that $I(\eta_0 v_0) < 0$. Let $v_1 = \eta_0 v_0$, we can see $I(v_1) < 0$, which proves this lemma. □

Proof of Theorem 1 By Lemmas 4–8, we can see that I possesses at least one nontrivial critical point. Then the critical point is the first homoclinic solution to (1). To get the second solution, we just need to prove that $\inf_{u\in B_{\varrho_1}} I(u) < 0$, where B_{ϱ_1} is defined in Lemma 7. We choose $v_2 \in C_0^\infty([-1,1], \mathbb{R}^N) \setminus \{0\}$. Then, by (W_3) and (W_5), for any $l > 0$ we get

$$\begin{aligned} I(lv_2) &= \frac{l^p}{p}\|v_2\|^p - \int_{-1}^{1} W_1\big(t, lv_2(t)\big)\,dt - \int_{-1}^{1} W_2\big(t, lv_2(t)\big)\,dt \\ &\le \frac{l^p}{p}\|v_2\|^p + d_1 l^p\int_{-1}^{1}\big|v_2(t)\big|^p\,dt - l^\theta\int_{-1}^{1} b(t)\big|v_2(t)\big|^\theta\,dt \\ &\le \frac{l^p}{p}\|v_2\|^p + d_1 l^p\int_{-1}^{1}\big|v_2(t)\big|^p\,dt - l^\theta\Big(\min_{t\in[-1,1]} b(t)\Big)\int_{-1}^{1}\big|v_2(t)\big|^\theta\,dt \\ &< 0 \end{aligned}$$

for l small enough, which implies that $\delta_1 = \inf_{u\in B_{\varrho_1}} I(u) < 0$. Then it follows from Ekeland's variational principle that there exists a minimizing sequence $\{v_n\} \subset B_{\varrho_1}$ such that

$$\delta_1 \le I(v_n) < \delta_1 + \frac{1}{n} \quad \text{and} \quad I(u) \ge I(v_n) - \frac{1}{n}\|u - v_n\| \quad \text{for } u \in B_{\varrho_1}.$$

Thus, $\{v_n\}$ is a bounded (PS) sequence, which means that it is also a (C) sequence. Then from Lemma 6, there exists $u_1 \in E$ such that $I'(u_1) = 0$ and $I(u_1) < 0$. In conclusion, problem (1) possesses at least two nontrivial homoclinic solutions. □

3 Proof of Theorem 2

In this section, we still work in the Banach space

$$E = \left\{u \in W^{1,p}\big(\mathbb{R}, \mathbb{R}^N\big) : \int_{\mathbb{R}}\big(\big|\dot{u}(t)\big|^p + a(t)\big|u(t)\big|^p\big)\,dt < +\infty\right\}$$

with the norm

$$\|u\| = \left(\int_{\mathbb{R}} \left(\left|\dot{u}(t)\right|^p + a(t)\left|u(t)\right|^p \right) dt \right)^{\frac{1}{p}}.$$

Lemma 9 *Suppose that the condition (A_2) holds, the embedding $E \hookrightarrow L^1(\mathbb{R},\mathbb{R}^N)$ is continuous and compact.*

Proof Assume that $\{u_n\} \subset E$ such that $u_n \rightharpoonup 0$ in E. We will show that $u_n \to 0$ in $L_1(\mathbb{R},\mathbb{R}^N)$. By the Banach–Steinhaus theorem, there exists $M_5 > 0$ such that

$$\sup_{n\in\mathbb{N}} \|u_n\| \le M_5.$$

For any $\varepsilon > 0$, by condition (A_2) there is $r_0 > 0$ such that

$$\operatorname{meas} B_\varepsilon < \varepsilon,$$

where

$$B_\varepsilon = \left\{ t \in \mathbb{R} \setminus (-r_0, r_0) \mid |t|^{-\lambda p} a(t) < \varepsilon^{-1} \right\}.$$

Let

$$D_\varepsilon = \mathbb{R} \setminus \left((-r_0, r_0) \cup B_\varepsilon \right),$$
$$\mu_\varepsilon = \inf_{t\in D_\varepsilon} |t|^{-\lambda p} a(t),$$

then $\frac{1}{\mu_\varepsilon} \le \varepsilon$. On the one hand, one has

$$\begin{aligned}
\int_{|t|\ge r_0} |u_n|\,dt &= \int_{B_\varepsilon} |u_n|\,dt + \int_{D_\varepsilon} |u_n|\,dt \\
&\le \|u_n\|_\infty \cdot \operatorname{meas} B_\varepsilon + \int_{D_\varepsilon} |t|^{\lambda} |u_n| |t|^{-\lambda}\,dt \\
&\le \varepsilon C_\infty M_5 + \left(\int_{D_\varepsilon} |t|^{\lambda p} |u_n|^p\,dt \right)^{\frac{1}{p}} \left(\int_{|t|\ge r_0} |t|^{-\lambda q}\,dt \right)^{\frac{1}{q}} \\
&\le \varepsilon C_\infty M_5 + \delta_2 \mu_\varepsilon^{-\frac{1}{p}} \left(\int_{D_\varepsilon} a(t) |u_n|^p\,dt \right)^{\frac{1}{p}} \\
&\le \varepsilon C_\infty M_5 + \varepsilon^{\frac{1}{p}} \delta_2 M_5,
\end{aligned} \tag{22}$$

where $\delta_2 = (\int_{|t|\ge r_0} |t|^{-\lambda q}\,dt)^{\frac{1}{q}}$. On the other hand, it follows from the Sobolev compact embedding theorem that $u_n \to 0$ in $L^1((-r_0,r_0),\mathbb{R}^N)$. Therefore, the embedding $E \hookrightarrow L^1(\mathbb{R},\mathbb{R}^N)$ is compact.

Now for $\varepsilon = 1$, by (22) we have

$$\int_{|t|\ge r_0} |u|\,dt \le C_\infty \|u\| + \delta_2 \|u\| = (C_\infty + \delta_2)\|u\|, \quad \forall u \in E,$$

which implies that the embedding is also continuous. □

Proof of Theorem 2 By similar steps to the proof of Theorem 1, we can obtain the conclusion of Theorem 2. □

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