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Superconvergence of the function value for pentahedral finite elements for an elliptic equation with varying coefficients

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Abstract

In this article, for an elliptic equation with varying coefficients, we first derive an interpolation fundamental estimate for the $\mathcal{P}_2(x,y) \otimes \mathcal{P}_2(z)$ pentahedral finite element over uniform partitions of the domain. Then combined with the estimate for the $W^{2,1}$-seminorm of the discrete Green function, superconvergence of the function value between the finite element approximation and the corresponding interpolant to the true solution is given.

Keywords: Pentahedral finite element; Interpolation fundamental estimate; Superconvergence; Discrete Green's function

1 Introduction and preliminaries

Superconvergence is a phenomenon in numerical methods that refers to faster than normal convergence for the approximate solutions arising from numerical procedures, and it was first addressed in [1]. The term "superconvergence" was first used in [2]. Since then, it has become an actively researched topic in the domain of finite element methods. So far, numerous studies on superconvergence have been published. For one- and two-dimensions, superconvergence has been extensively investigated. For three and more dimensions, studies on superconvergence are progressing at a slow rate. Recently, we focused on superconvergence of the finite element method for three-dimensional problems, and we found that there have been some studies concerning it. Some books and survey papers have also been published. We refer to [3–25] and the references therein. In general, according to the domain partition, there usually exist three types of finite elements for three-dimensional problems, namely tetrahedral elements, pentahedral elements, and block elements. In this paper, we only consider the pentahedral elements. To the best of our knowledge, superconvergence of pentahedral elements (or prismatic elements) has been investigated in [7, 14, 15, 20, 24]. Of these studies, [7] considered superconvergence of pentahedral elements for the elliptic equation with constant coefficients. The study [15] is concerned with superconvergence for the Poisson equation, and demonstrated accuracy of the order $\mathcal{O}(h^4|\ln h|^{\frac{2}{3}})$ in terms of L^∞-norm for the value of the function between the $\mathcal{P}_2(x,y) \otimes \mathcal{P}_2(z)$ pentahedral finite element approximation and the corresponding interpolant. In this paper, we will generalize the results in [7] and [15] to general elliptic equations with varying coefficients.

Additionally, we will use the symbol C to denote a generic constant, which is independent of the discretization parameters h_{xy} and h_z and which may not be the same for each occurrence. We will also use the standard notations for the Sobolev spaces and their norms.

The model problem considered in the article is as follows:

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^{3} a_i\partial_i u + a_0 u = f \quad \text{in } \Omega, u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here, $\Omega = \Omega_{xy} \times \Omega_z \equiv (0,1)^2 \times (0,1) \subset \mathcal{R}^3$ is the unit cube with boundary, $\partial\Omega$, comprising faces parallel to the x-, y-, and z-axes. The diffusion coefficients a_{ij} satisfy the following condition:

There exists a positive constant C such that, for all $X \in \Omega$, we have

$$\sum_{i,j=1}^{3} a_{ij}(X)\eta_i\eta_j \geq C\sum_{i=1}^{3}\eta_i^2 \quad \forall \eta = (\eta_1, \eta_2, \eta_3)^\top \in \mathcal{R}^3.$$

In addition, we also assume $a_{ij}, a_i \in W^{1,\infty}(\Omega)$, $a_0 \in L^\infty(\Omega), f \in L^2(\Omega)$, $a_0 \geq 0$, and write $\partial_1 u = \frac{\partial u}{\partial x}$, $\partial_2 u = \frac{\partial u}{\partial y}$, and $\partial_3 u = \frac{\partial u}{\partial z}$.

Thus, the weak formulation of (1.1) is as follows:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

where

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{3} a_{ij}\partial_i u \partial_j v + \sum_{i=1}^{3} a_i \partial_i u v + a_0 uv \right) dx\,dy\,dz$$

and

$$(f,v) = \int_\Omega fv\,dx\,dy\,dz.$$

To provide the discrete formulation of (1.2), we should first partition the domain Ω. Denote by $\{\mathcal{T}^h\}$ a uniform family of pentahedral partitions, and thus, $\bar{\Omega} = \bigcup_{e\in\mathcal{T}^h} \bar{e}$. Therefore, we can write $\bar{e} = D \times L$ (see Fig. 1), where D and L are closed, and denote an isosceles right triangle with legs h_{xy} parallel to the xy-plane and a one-dimensional interval with length h_z parallel to the z-axis, respectively. We assume that there exist two positive constants C_1 and C_2 such that $C_1 \leq \frac{h_z}{h_{xy}} \leq C_2$.

We introduce an $\mathcal{P}_2(x,y) \otimes \mathcal{P}_2(z)$ polynomial space denoted by $\mathcal{P}$, that is,

$$q(x,y,z) = \sum_{(i,j,k)\in\mathcal{I}} a_{ijk}x^i y^j z^k, \quad a_{ijk} \in \mathcal{R}, q \in \mathcal{P} \equiv \mathcal{P}_2(x,y) \otimes \mathcal{P}_2(z),$$

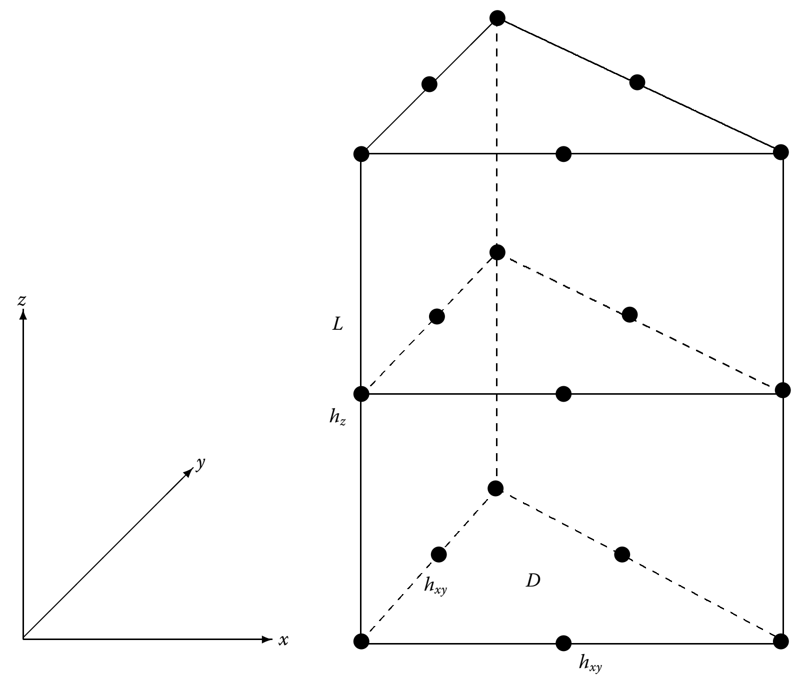

Figure 1 An $\mathcal{P}_2(x,y) \otimes \mathcal{P}_2(z)$ pentahedral element and interpolation nodes

where $\mathcal{P}_2(x,y)$ denotes the quadratic polynomial space with respect to (x,y), and $\mathcal{P}_2(z)$ is the quadratic polynomial space with respect to z. The indexing set $\mathcal{I}$ satisfies

$$\mathcal{I} = \{(i,j,k)|i,j,k \geq 0, i+j \leq 2, k \leq 2\}.$$

An $\mathcal{P}_2(x,y) \otimes \mathcal{P}_2(z)$ interpolation operator is defined by $\Pi^e : H^1(\bar{e}) \cap C(\bar{e}) \to \mathcal{P}(\bar{e})$. Obviously,

$$\Pi^e = \Pi^e_{xy} \otimes \Pi^e_z,$$

where Π^e_{xy} stands for the Lagrange quadratic interpolation operator with respect to $(x,y) \in D$, and Π^e_z stands for the Lagrange quadratic interpolation operator or the quadratic interpolation operator of projection type with respect to $z \in L$.

Furthermore, the $\mathcal{P}_2(x,y) \otimes \mathcal{P}_2(z)$ pentahedral finite element space is defined as follows:

$$S^h_0(\Omega) = \{v \in H^1_0(\Omega) \cap C(\Omega) : v|_e \in \mathcal{P}(e)\ \forall e \in \mathcal{T}^h\}.$$

Thus, the finite element method of (1.2) is

$$\begin{cases} \text{Find } u_h \in S^h_0(\Omega) \text{ satisfying} \\ a(u_h, v) = (f, v) \quad \forall v \in S^h_0(\Omega). \end{cases} \tag{1.3}$$

From (1.2) and (1.3), the following Galerkin orthogonal relation holds:

$$a(u-u_h,v)=0 \quad \forall v\in S_0^h(\Omega). \tag{1.4}$$

In addition, from the definitions of Π^e and $S_0^h(\Omega)$, we can define a global $\mathcal{P}_2(x,y)\otimes\mathcal{P}_2(z)$ interpolation operator $\Pi: H_0^1(\Omega)\cap C(\Omega)\to S_0^h(\Omega)$ such that $(\Pi u)|_e=\Pi^e u$. In next section, we will bound the term $a(u-\Pi u,v)$.

2 An important interpolation fundamental estimate

Lemma 2.1 *Let $\{\mathcal{T}^h\}$ be a uniform family of pentahedral partitions of Ω, $u\in W^{5,\infty}(\Omega)\cap H_0^1(\Omega)$, and $v\in S_0^h(\Omega)$. Subsequently, the interpolation operator Π satisfies the following interpolation fundamental estimate*:

$$\bigl|a(u-\Pi u,v)\bigr|\le C\bigl(h_{xy}^4+h_z^4\bigr)\|u\|_{5,\infty,\Omega}|v|_{2,1,\Omega}^h, \tag{2.1}$$

where $|v|_{2,1,\Omega}^h=\sum_{e\in\mathcal{T}^h}|v|_{2,1,e}$.

Proof Clearly, the interpolation remainder is

$$\begin{aligned} u-\Pi u &= (u-\Pi_{xy}u)+(u-\Pi_z u)+\bigl(\Pi_{xy}(u-\Pi_z u)-(u-\Pi_z u)\bigr)\\ &= R_{xy}+R_z+R^*, \end{aligned} \tag{2.2}$$

where $(\Pi_{xy}u)|_e=\Pi_{xy}^e u$, $(\Pi_z u)|_e=\Pi_z^e u$, and R^* is a high-order term. Thus, it suffices to analyze R_{xy} and R_z. We first have the bound

$$a(R_{xy},v)=\int_\Omega\left(\sum_{i,j=1}^3 a_{ij}\partial_i R_{xy}\partial_j v+\sum_{i=1}^3 a_i\partial_i R_{xy}v+a_0R_{xy}v\right)dx\,dy\,dz. \tag{2.3}$$

We set

$$I_1=\int_\Omega\left(\sum_{i,j=1}^2 a_{ij}\partial_i R_{xy}\partial_j v+\sum_{i=1}^2 a_i\partial_i R_{xy}v+a_0R_{xy}v\right)dx\,dy\,dz, \tag{2.4}$$

$$I_2=\int_\Omega\left(\sum_{j=1}^2 a_{3j}\partial_3 R_{xy}\partial_j v+a_3\partial_3 R_{xy}v\right)dx\,dy\,dz, \tag{2.5}$$

$$I_3=\int_\Omega\sum_{i=1}^2 a_{i3}\partial_i R_{xy}\partial_3 v\,dx\,dy\,dz, \tag{2.6}$$

$$I_4=\int_\Omega a_{33}\partial_3 R_{xy}\partial_3 v\,dx\,dy\,dz. \tag{2.7}$$

Clearly,

$$a(R_{xy},v)=I_1+I_2+I_3+I_4. \tag{2.8}$$

By the two-dimensional interpolation fundamental estimate of triangular quadratic elements [26], we have

$$|I_1| \le Ch_{xy}^4 \int_{\Omega_z} \|u\|_{4,\infty,\Omega_{xy}} |v|_{2,1,\Omega_{xy}}^h \, dz \le Ch_{xy}^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.9}$$

As for I_2, by Green's formula, we have

$$I_2 = \int_\Omega \left(-\sum_{j=1}^{2} a_{3j}\partial_j\partial_3 R_{xy} v + (a_3 - \partial_1 a_{31} - \partial_2 a_{32})\partial_3 R_{xy} v \right) dx\, dy\, dz.$$

Obviously, $\partial_3 R_{xy} = \partial_3 u - \Pi_{xy}\partial_3 u$. Thus, by the two-dimensional interpolation fundamental estimate of triangular quadratic elements [26], we have

$$|I_2| \le Ch_{xy}^4 \int_{\Omega_z} \|\partial_3 u\|_{4,\infty,\Omega_{xy}} |v|_{2,1,\Omega_{xy}}^h \, dz \le Ch_{xy}^4 \|u\|_{5,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.10}$$

As for I_3, we first bound the integral

$$\int_\Omega a_{13}\partial_1 R_{xy}\partial_3 v \, dx\, dy\, dz.$$

By Green's formula and $v = 0$ on $\partial\Omega$, we get

$$\begin{aligned}
&\int_\Omega a_{13}\partial_1 R_{xy}\partial_3 v \, dx\, dy\, dz \\
&\quad = \int_{\Omega_z} \sum_D \left(\int_D a_{13}\partial_1 R_{xy}\partial_3 v \, dx\, dy \right) dz \\
&\quad = \int_{\Omega_z} \sum_D \left(\int_{\partial D} a_{13} R_{xy}\partial_3 v \, dy \right) dz - \int_{\Omega_z} \sum_D \left(\int_D R_{xy}\partial_1(a_{13}\partial_3 v)\, dx\, dy \right) dz \\
&\quad = \int_{\Omega_z \times \partial\Omega_{xy}} a_{13} R_{xy}\partial_3 v \, dy\, dz - \int_\Omega R_{xy}\partial_1(a_{13}\partial_3 v)\, dx\, dy\, dz \\
&\quad = -\int_\Omega \partial_1 a_{13} R_{xy}\partial_3 v \, dx\, dy\, dz - \int_\Omega a_{13} R_{xy}\partial_1\partial_3 v \, dx\, dy\, dz \\
&\quad = K_1 + K_2.
\end{aligned}$$

Let $S_{0,2}^h(\Omega_{xy})$ be the triangular quadratic finite element space in the domain Ω_{xy}, and $\{\psi_j\}$ be the basis of this space. Obviously, the support S_j of ψ_j is a patch of elements that share an internal edge or internal node. Moreover, because the partition of the domain is uniform, each S_j is point-symmetric. Subsequently, for all cubic polynomials p_3 on S_j, we have

$$\int_{S_j} (p_3 - \Pi_{xy} p_3)\psi_j \, dx\, dy = 0. \tag{2.11}$$

The proof of (2.11) is similar to Lemma 3.2 in [5].

As $v \in S_0^h(\Omega)$, $\partial_3 v \in S_{0,2}^h(\Omega_{xy})$. Thus, $\partial_3 v = \sum_j \alpha_j(z)\psi_j(x,y) \equiv \sum_j \alpha_j \psi_j$. To bound the term K_1, we also assume $\partial_1 a_{13} \in W^{1,\infty}(\Omega)$. Then

$$\partial_1 a_{13}(Q) = \partial_1 a_{13}(Q_0) + \mathcal{O}(h_{xy}) \equiv a_{13}^0 + \mathcal{O}(h_{xy}) \quad \forall Q \in S_j, \tag{2.12}$$

where Q_0 is the center of S_j. Thus, by (2.11) and (2.12), we have

$$\begin{aligned}
|K_1| &= \left| \int_\Omega \partial_1 a_{13}(u - \Pi_{xy} u)\partial_3 v \, dx\, dy\, dz \right| \\
&\leq \int_{\Omega_z} \sum_j |\alpha_j| \left| \int_{S_j} \bigl(a_{13}^0 + \mathcal{O}(h_{xy})\bigr)(u - \Pi_{xy} u)\psi_j \, dx\, dy \right| dz \\
&\leq \int_{\Omega_z} \sum_j |\alpha_j| \left| \int_{S_j} a_{13}^0 \bigl(u - p_3 - \Pi_{xy}(u - p_3)\bigr)\psi_j \, dx\, dy \right| dz \\
&\quad + \int_{\Omega_z} \sum_j |\alpha_j| \left| \int_{S_j} \mathcal{O}(h_{xy})(u - \Pi_{xy} u)\psi_j \, dx\, dy \right| dz \\
&= K_1' + K_1''.
\end{aligned}$$

Similar to the arguments in [15], we may obtain $\sum_j |\alpha_j| \leq C(z)h_{xy}^{-2}|v|_{2,1,\Omega_{xy}}^h$. Therefore, we obtain

$$K_1' \leq C h_{xy}^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.13}$$

Furthermore, we easily obtain

$$K_1'' \leq C h_{xy}^4 \|u\|_{3,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.14}$$

From (2.13) and (2.14),

$$|K_1| \leq C h_{xy}^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.15}$$

Let $S_{0,2}^h(\Omega_z)$ be the quadratic finite element space in Ω_z, and $\{\phi_i\}$ be basis of this space. Clearly, $S_0^h(\Omega) = S_{0,2}^h(\Omega_z) \otimes S_{0,2}^h(\Omega_{xy})$. Thus, for $v \in S_0^h(\Omega)$, we have $v = \sum_{i,j} v(x_j, y_j, z_i)\phi_i(z) \times \psi_j(x,y) \equiv \sum_{i,j} v_{ij}\phi_i\psi_j$, and $\partial_1\partial_3 v = \sum_{i,j} v_{ij}\partial_3\phi_i\partial_1\psi_j$. Note that the support S_{ij} of $\phi_i\psi_j$ is a patch of elements that share an internal node, an edge, or a face. Moreover, as the partition of the domain is uniform, each S_{ij} is point-symmetric. Thus, similar to (2.11), we have for all cubic polynomials $\tilde{p}_3$ on S_{ij}

$$\int_{S_{ij}} (\tilde{p}_3 - \Pi_{xy}\tilde{p}_3)\partial_3\phi_i\partial_1\psi_j \, dx\, dy\, dz = 0. \tag{2.16}$$

Similar to (2.12), we have

$$a_{13}\bigl(Q^*\bigr) = a_{13}\bigl(Q_0^*\bigr) + \mathcal{O}(h_{xy}) \equiv a_{13}' + \mathcal{O}(h_{xy}) \quad \forall Q^* \in S_{ij}, \tag{2.17}$$

where Q_0^* is the center of S_{ij}. Hence, by (2.16) and (2.17), we get

$$\begin{aligned}
|K_2| &= \left|\int_\Omega a_{13}(u-\Pi_{xy}u)\partial_1\partial_3 v\,dx\,dy\,dz\right| \\
&\le \sum_{i,j}|v_{ij}|\left|\int_{S_{ij}}\left(a'_{13}+\mathcal{O}(h_{xy})\right)(u-\Pi_{xy}u)\partial_3\phi_i\partial_1\psi_j\,dx\,dy\,dz\right| \\
&\le \sum_{i,j}|v_{ij}|\left|\int_{S_{ij}}a'_{13}\left(u-\tilde{p}_3-\Pi_{xy}(u-\tilde{p}_3)\right)\partial_3\phi_i\partial_1\psi_j\,dx\,dy\,dz\right| \\
&\quad+\sum_{i,j}|v_{ij}|\left|\int_{S_{ij}}\mathcal{O}(h_{xy})(u-\Pi_{xy}u)\partial_3\phi_i\partial_1\psi_j\,dx\,dy\,dz\right| \\
&= K'_2+K''_2.
\end{aligned}$$

For simplicity, we write

$$M_{ij}=\left|\int_{S_{ij}}a'_{13}\left(u-\tilde{p}_3-\Pi_{xy}(u-\tilde{p}_3)\right)\partial_3\phi_i\partial_1\psi_j\,dx\,dy\,dz\right|. \tag{2.18}$$

Thus,

$$K'_2=\sum_{i,j}|v_{ij}||\partial_3\phi_i\partial_1\psi_j|_{0,1,\Omega}M_{ij}|\partial_3\phi_i\partial_1\psi_j|^{-1}_{0,1,\Omega}. \tag{2.19}$$

Clearly,

$$M_{ij}\le Ch^3_{xy}\|u-\tilde{p}_3\|_{3,\infty,S_{ij}}|\partial_3\phi_i\partial_1\psi_j|_{0,1,\Omega}. \tag{2.20}$$

Taking $\tilde{p}_3$ a three-degree interpolant to u on S_{ij} in (2.20), we have

$$M_{ij}\le Ch^4_{xy}\|u\|_{4,\infty,\Omega}|\partial_3\phi_i\partial_1\psi_j|_{0,1,\Omega}. \tag{2.21}$$

To obtain the desired result, we need to introduce an affine transformation defined by $F:\hat{P}\in\hat{e}\longrightarrow P=B\hat{P}+b\in e$ such that $e=F(\hat{e})$, where $B=(b_{ij})$ is a matrix of order 3×3. For all $\varphi\in L^2(e)$, we write $\hat{\varphi}(\hat{P})=\varphi(F\hat{P})$. The usual transformation rules between the element e and the reference element $\hat{e}$ (see [5, 26], and [27]) tell us that there exists a constant C independent of the mesh parameters such that

$$|\hat{\varphi}|_{0,1,\hat{e}}\le C|\det B|^{-1}|\varphi|_{0,1,e}\quad\text{and}\quad|\varphi|_{0,1,e}\le C|\det B||\hat{\varphi}|_{0,1,\hat{e}}, \tag{2.22}$$

In addition, we set $w=\partial_1\partial_3 v=\sum_{i,j}v_{ij}\partial_3\phi_i\partial_1\psi_j$. It is easy to prove that

$$\sum_{i=1}^{3}\sum_{j=1}^{6}|v_{ij}||\partial_3\phi_i\partial_1\psi_j|_{0,1,e}$$

is a seminorm of w on e. Using the rightmost rule from (2.22), we find that

$$\sum_{i=1}^{3}\sum_{j=1}^{6}|v_{ij}||\partial_3\phi_i\partial_1\psi_j|_{0,1,e}\le C|\det B|\sum_{i=1}^{3}\sum_{j=1}^{6}|v_{ij}||\hat{\partial_3\phi_i}\hat{\partial_1\psi_j}|_{0,1,\hat{e}}. \tag{2.23}$$

By the equivalence of norms in the finite-dimensional space, there also exists a constant C, depending only on the reference element $\hat{e}$, such that

$$\sum_{i=1}^{3}\sum_{j=1}^{6}|v_{ij}||\partial_3\hat{\phi}_i\partial_1\hat{\psi}_j|_{0,1,\hat{e}} \le C|\hat{w}|_{0,1,\hat{e}}. \tag{2.24}$$

Using the left rule from (2.22), we get

$$|\hat{w}|_{0,1,\hat{e}} \le C|\det B|^{-1}|w|_{0,1,e}. \tag{2.25}$$

Combining (2.23)–(2.25) yields

$$\sum_{i=1}^{3}\sum_{j=1}^{6}|v_{ij}||\partial_3\phi_i\partial_1\psi_j|_{0,1,e} \le C|w|_{0,1,e}.$$

Summing over all e in $\mathcal{T}^h$ results in

$$\sum_{i,j}|v_{ij}||\partial_3\phi_i\partial_1\psi_j|_{0,1,\Omega} \le C|w|_{0,1,\Omega}^h \le C|v|_{2,1,\Omega}^h. \tag{2.26}$$

Combining (2.19), (2.21), and (2.26) yields

$$K_2' \le Ch_{xy}^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.27}$$

Similar to the arguments mentioned above, we also get

$$K_2'' \le Ch_{xy}^4\|u\|_{3,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.28}$$

From (2.27) and (2.28),

$$|K_2| \le Ch_{xy}^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.29}$$

Thus, by (2.15) and (2.29), we have

$$\left|\int_\Omega a_{13}\partial_1 R_{xy}\partial_3 v\,dx\,dy\,dz\right| \le |K_1| + |K_2| \le Ch_{xy}^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.30}$$

Similar to the proof of (2.30), we have

$$\left|\int_\Omega a_{23}\partial_2 R_{xy}\partial_3 v\,dx\,dy\,dz\right| \le Ch_{xy}^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.31}$$

Combining (2.6), (2.30), and (2.31), we get

$$|I_3| \le Ch_{xy}^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.32}$$

As for I_4, we write $a_{33}(Q) = a_{33}(Q_0) + \mathcal{O}(h_{xy}) \equiv a_{33}^0 + \mathcal{O}(h_{xy})$ $\forall Q \in S_j$, and $\partial_3 v = \sum_j \alpha_j \psi_j$. Thus,

$$\begin{aligned}
|I_4| &= \left| \int_\Omega a_{33} \partial_3 R_{xy} \sum_j \alpha_j \psi_j \, dx\, dy\, dz \right| \\
&\le \int_{\Omega_z} \sum_j |\alpha_j| \left| \int_{S_j} a_{33} \partial_3 R_{xy} \psi_j \, dx\, dy \right| dz \\
&\le \int_{\Omega_z} \sum_j |\alpha_j| \left| \int_{S_j} a_{33}^0 \big(\partial_3 u - p_3 - \Pi_{xy}(\partial_3 u - p_3)\big) \psi_j \, dx\, dy \right| dz \\
&\quad + \int_{\Omega_z} \sum_j |\alpha_j| \left| \int_{S_j} \mathcal{O}(h_{xy}) \big(\partial_3 u - \Pi_{xy}(\partial_3 u)\big) \psi_j \, dx\, dy \right| dz \\
&= K_3 + K_4.
\end{aligned}$$

Similar to the proof of (2.13), we have

$$K_3 \le C h_{xy}^4 \|u\|_{5,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.33}$$

Clearly,

$$K_4 \le C h_{xy}^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.34}$$

Combining (2.33) and (2.34) yields

$$|I_4| \le C h_{xy}^4 \|u\|_{5,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.35}$$

From (2.8)–(2.10), (2.32), and (2.35),

$$\big|a(R_{xy}, v)\big| \le C h_{xy}^4 \|u\|_{5,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.36}$$

Now, we can bound the term

$$a(R_z, v) = \int_\Omega \left(\sum_{i,j=1}^3 a_{ij} \partial_i R_z \partial_j v + \sum_{i=1}^3 a_i \partial_i R_z v + a_0 R_z v \right) dx\, dy\, dz. \tag{2.37}$$

Additionally, we set

$$J_1 = \int_\Omega \left(\sum_{i,j=1}^2 a_{ij} \partial_i R_z \partial_j v + \sum_{i=1}^2 a_i \partial_i R_z v + a_0 R_z v \right) dx\, dy\, dz, \tag{2.38}$$

$$J_2 = \int_\Omega \left(\sum_{j=1}^2 a_{3j} \partial_3 R_z \partial_j v + a_3 \partial_3 R_z v \right) dx\, dy\, dz, \tag{2.39}$$

$$J_3 = \int_\Omega \sum_{i=1}^2 a_{i3} \partial_i R_z \partial_3 v \, dx\, dy\, dz, \tag{2.40}$$

$$J_4 = \int_\Omega a_{33}\partial_3 R_z \partial_3 v\,dx\,dy\,dz. \tag{2.41}$$

Clearly,

$$a(R_z, v) = J_1 + J_2 + J_3 + J_4. \tag{2.42}$$

To simply bound the aforementioned terms, we may use the so-called interpolation operator of projection type (see [15]).

Let $\{l_j(z)\}_{j=0}^{\infty}$ be the normalized orthogonal Legendre polynomial system from the space $\mathcal{L}^2(L)$, and $\partial_z u \in \mathcal{L}^2(L)$. For a fixed point $(x, y) \in D$, we have the following expansion:

$$u(x,y,z) = \sum_{j=0}^{\infty} \beta_j(x,y)\omega_j(z), \quad (x,y,z) \in \bar{e} = D \times L,$$

where

$$\omega_0(z) = 1, \qquad \omega_{j+1}(z) = \int_{z_{i-1}}^{z} l_j(\xi)\,d\xi = \mathcal{O}\bigl(h_z^{\frac{1}{2}}\bigr), \qquad l_j(z) = \mathcal{O}\bigl(h_z^{-\frac{1}{2}}\bigr), \quad j \geq 0. \tag{2.43}$$

The coefficients $\beta_j(x, y)$ satisfy $\beta_0(x, y) = u(x, y, z_{i-1})$, and for $j \geq 1$,

$$\beta_j(x,y) = \int_L \partial_z u l_{j-1}(z)\,dz = \mathcal{O}\bigl(h_z^{j-\frac{1}{2}}\bigr). \tag{2.44}$$

Let Π_z^e be the quadratic interpolation operator of projection type with respect to z defined by

$$\Pi_z^e u = \sum_{j=0}^{2} \beta_j(x,y)\omega_j(z), \quad (x,y,z) \in \bar{e} = D \times L.$$

Thus, the interpolation remainder is

$$R_z = u - \Pi_z^e u = \sum_{j=3}^{\infty} \beta_j(x,y)\omega_j(z), \quad (x,y,z) \in \bar{e}. \tag{2.45}$$

The above-mentioned statements are presented in [15]. Obviously, we only need to consider the main term $r_3 = \beta_3(x, y)\omega_3(z)$ in (2.45). As for J_1, we first bound

$$\int_\Omega a_{11}\partial_1 r_3 \partial_1 v\,dx\,dy\,dz.$$

By integration by parts, the Poincaré inequality, (2.43), and (2.44), we get

$$\begin{aligned}
&\left|\int_\Omega a_{11}\partial_1 r_3 \partial_1 v\,dx\,dy\,dz\right| \\
&\quad \leq \sum_e \left|\int_e a_{11}\partial_1 r_3 \partial_1 v\,dx\,dy\,dz\right|
\end{aligned}$$

$$\begin{aligned}
&\le \sum_e \left| \int_e a_{11}^0 \partial_1 \beta_3(x,y)\omega_3(z)\partial_1 v \, dx\, dy\, dz \right| \\
&\quad + \sum_e \left| \int_e \mathcal{O}(h_z)\partial_1 \beta_3(x,y)\omega_3(z)\partial_1 v \, dx\, dy\, dz \right| \\
&\le \sum_e \left| \int_e a_{11}^0 \partial_1 \beta_3(x,y)\tilde{D}^{-1}\omega_3(z)\partial_3\partial_1 v \, dx\, dy\, dz \right| \\
&\quad + \sum_e \left| \int_e \mathcal{O}(h_z)\partial_1 \beta_3(x,y)\omega_3(z)\partial_1 v \, dx\, dy\, dz \right| \\
&\le Ch_z^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h + Ch_z^4 \|u\|_{4,\infty,\Omega} \sum_e \int_e |\partial_1 v| \, dx\, dy\, dz \\
&\le Ch_z^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h + Ch_z^4 \|u\|_{4,\infty,\Omega} \int_{\Omega_{xy}} |\partial_1 v|_{1,1,\Omega_z} \, dx\, dy \\
&\le Ch_z^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h,
\end{aligned}$$

where $\frac{d(\tilde{D}^{-1}\omega_3(z))}{dz} = \omega_3(z)$, $\tilde{D}^{-1}\omega_3 = \mathcal{O}(h_z^{1.5})$, $a_{11}(N) = a_{11}(N_0) + \mathcal{O}(h_z) \equiv a_{11}^0 + \mathcal{O}(h_z)$ for every $N \in \bar{e}$, and N_0 is the center of $\bar{e}$.

Similarly, for the rightmost term from (2.38), we can easily obtain

$$\left| \int_\Omega a_0 r_3 v \, dx\, dy\, dz \right| \le Ch_z^4 \|u\|_{3,\infty,\Omega} |v|_{2,1,\Omega}^h.$$

As for the other terms from (2.38), using arguments similar to the ones mentioned above, we derive their bounds as follows:

$$Ch_z^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h.$$

Thus, we have

$$|J_1| \le Ch_z^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.46}$$

As for J_2, we first analyze the case of $j = 1$. By Green's formula, we get

$$\int_\Omega a_{31}\partial_3 r_3 \partial_1 v \, dx\, dy\, dz = -\sum_e \int_e \partial_1\big(a_{31}\beta_3(x,y)\big) l_2(z) v \, dx\, dy\, dz. \tag{2.47}$$

For the right term from (2.47), integration by parts yields

$$\begin{aligned}
\int_\Omega a_{31}\partial_3 r_3 \partial_1 v \, dx\, dy\, dz = -\sum_e \int_e \tilde{D}^{-2} l_2 \big(&\partial_3\partial_3 v \partial_1(a_{31}\beta_3) \\
&+ 2\partial_3 v \partial_3\partial_1(a_{31}\beta_3) + v\partial_3\partial_3\partial_1(a_{31}\beta_3)\big) \, dx\, dy\, dz,
\end{aligned}$$

where $\frac{d^2(\tilde{D}^{-2}l_2(z))}{dz^2} = l_2(z)$. From (2.43) and (2.44),

$$\tilde{D}^{-2} l_2 = \mathcal{O}\big(h_z^{1.5}\big), \qquad |\beta_3| \le Ch_z^{2.5} \|u\|_{3,\infty,\Omega}, \qquad |\partial_1 \beta_3| \le Ch_z^{2.5} \|u\|_{4,\infty,\Omega}. \tag{2.48}$$

Hence,

$$\left|\int_\Omega a_{31}\partial_3 r_3 \partial_1 v\,dx\,dy\,dz\right| \le Ch_z^4\|u\|_{4,\infty,\Omega}\sum_e\int_e\bigl(|\partial_3\partial_3 v| + |\partial_3 v| + |v|\bigr)\,dx\,dy\,dz. \tag{2.49}$$

By the Poincaré inequality in (2.49), we get

$$\left|\int_\Omega a_{31}\partial_3 r_3 \partial_1 v\,dx\,dy\,dz\right| \le Ch_z^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.50}$$

Similarly, in the case of $j = 2$, we also have

$$\left|\int_\Omega a_{32}\partial_3 r_3 \partial_2 v\,dx\,dy\,dz\right| \le Ch_z^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.51}$$

For the right term from (2.39), integration by parts yields

$$\begin{aligned}\int_\Omega a_3\partial_3 r_3 v\,dx\,dy\,dz &= \sum_e\int_e a_3\beta_3(x,y)l_2(z)v\,dx\,dy\,dz\\ &= \sum_e\int_e \tilde{D}^{-2}l_2(\partial_3\partial_3 v a_3\beta_3 + 2\partial_3 v\partial_3 a_3\beta_3 + v\partial_3\partial_3 a_3\beta_3)\,dx\,dy\,dz.\end{aligned}$$

Using (2.48) and the Poincaré inequality, we obtain

$$\left|\int_\Omega a_3\partial_3 r_3 v\,dx\,dy\,dz\right| \le Ch_z^4\|u\|_{3,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.52}$$

Combining (2.50)–(2.52) yields

$$|J_2| \le Ch_z^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h. \tag{2.53}$$

As for J_3, we first consider the case of $i = 1$. Clearly, integration by parts yields

$$\begin{aligned}&\int_\Omega a_{13}\partial_1 r_3\partial_3 v\,dx\,dy\,dz\\ &\quad= \sum_e\int_e a_{13}\partial_1\beta_3(x,y)\omega_3(z)\partial_3 v\,dx\,dy\,dz\\ &\quad= -\sum_e\int_e \partial_1\beta_3(x,y)\tilde{D}^{-1}\omega_3(z)\partial_3(a_{13}\partial_3 v)\,dx\,dy\,dz.\end{aligned}$$

Furthermore, by (2.48), the Poincaré inequality and $\tilde{D}^{-1}\omega_3 = \mathcal{O}(h_z^{1.5})$, we have

$$\left|\int_\Omega a_{13}\partial_1 r_3\partial_3 v\,dx\,dy\,dz\right| \le Ch_z^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h.$$

Similarly, when $i = 2$, we also get

$$\left|\int_\Omega a_{23}\partial_2 r_3\partial_3 v\,dx\,dy\,dz\right| \le Ch_z^4\|u\|_{4,\infty,\Omega}|v|_{2,1,\Omega}^h.$$

Thus, we have

$$|J_3| \le Ch_z^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.54}$$

Finally, for J_4, integration by parts yields

$$\begin{aligned}
&\int_\Omega a_{33}\partial_3 r_3 \partial_3 v \, dx\, dy\, dz \\
&\quad = \sum_e \int_e a_{33}\beta_3(x,y) l_2(z) \partial_3 v \, dx\, dy\, dz \\
&\quad = \sum_e \int_e \beta_3 \tilde{D}^{-2} l_2 (2\partial_3\partial_3 v \partial_3 a_{33} + \partial_3 v \partial_3\partial_3 a_{33}) \, dx\, dy\, dz.
\end{aligned}$$

Thus,

$$\left| \int_\Omega a_{33}\partial_3 r_3 \partial_3 v \, dx\, dy\, dz \right| \le Ch_z^4 \|u\|_{3,\infty,\Omega} |v|_{2,1,\Omega}^h.$$

Hence,

$$|J_4| \le Ch_z^4 \|u\|_{3,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.55}$$

Combining (2.42), (2.46), and (2.53)–(2.55) results in

$$\left| a(R_z, v) \right| \le Ch_z^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \tag{2.56}$$

From (2.36) and (2.56), the desired result (2.1) is immediately obtained. The proof of Lemma 2.1 is therefore completed. □

3 Pointwise superconvergence estimates

To analyze pointwise superconvergence, for each fixed $Z \in \Omega$, we may introduce the discrete Green function defined by

$$a\left(v, G_Z^h\right) = v(Z) \quad \forall v \in S_0^h(\Omega). \tag{3.1}$$

As for G_Z^h, we have the following result.

Lemma 3.1 *For $G_Z^h \in S_0^h(\Omega)$ the discrete Green function, we have the following estimate*:

$$\left| G_Z^h \right|_{2,1,\Omega}^h \le C |\ln h|^{\frac{2}{3}}. \tag{3.2}$$

The proof of Lemma 3.1 can be found in [16].

From (1.4), (2.1), (3.1), and (3.2), we immediately obtain the following theorem.

Theorem 3.1 *Let $\{\mathcal{T}^h\}$ be a uniform family of pentahedral partitions of Ω, and $u \in W^{5,\infty}(\Omega) \cap H_0^1(\Omega)$. For u_h and Πu, the $\mathcal{P}_2(x,y) \otimes \mathcal{P}_2(z)$ pentahedral finite element approximation and the corresponding interpolant to u, respectively, we have the following* *pointwise superconvergence estimate*:

$$|u_h - \Pi u|_{0,\infty,\Omega} \leq C\left(h_{xy}^4 + h_z^4\right)|\ln h|^{\frac{2}{3}}\|u\|_{5,\infty,\Omega}.$$

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The first author proved Lemma 2.1 and Theorem 3.1, and the second author gave the idea of this article. All authors read and approved the final manuscript.

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