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Local existence and uniqueness of increasing positive solutions for non-singular and singular beam equation with a parameter

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Abstract

This paper is concerned with a class of beam equations with a parameter. By using the fixed point theorems of mixed monotone operator and the properties of cone, we study the non-singular and singular case, respectively, and obtain the sufficient conditions which guarantee the local existence and uniqueness of increasing positive solutions. Also, we present an iterative algorithm that converges to the solution. Moreover, we get some pleasant properties of the solutions for the boundary value problem dependent parameter. At last, two examples are given to illustrate the main results.

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Keywords: Existence and uniqueness; Positive solution; Non-singular and singular beam equation; Fixed point theorem of mixed monotone operator

1 Introduction

In this paper, we at first study the following non-singular boundary value problem with a parameter:

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t), (Hu)(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u'(1) = 0, \qquad u'''(1) = \lambda g(u(1)), \end{cases} \tag{1.1}$$

where $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,+\infty) \to (-\infty, 0]$ are continuous, λ is a positive parameter, H is a certain operator(not necessarily linear). Here, if we set $H : C([0,1];\mathbb{R}) \to C([0,1];\mathbb{R})$ is a linear operator defined by $Hu = u$ for every $u \in C([0,1];\mathbb{R})$, then the problem (1.1) reduces to the following beam equation:

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t), u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u'(1) = 0, \qquad u'''(1) = \lambda g(u(1)). \end{cases} \tag{1.2}$$

In this model, we suppose $f:(0,1)\times[0,+\infty)\times(0,+\infty)\to[0,+\infty)$ and $g:[0,+\infty)\to(-\infty,0]$ are continuous, $f(t,x,y)$ may be singular at $t=0$ or 1, and also may be singular at $y=0$. $\lambda>0$ is a parameter.

The above fourth-order Eqs. (1.1) and (1.2) model an elastic beam of length 1 subject to a nonlinear foundation given by the function f, where the boundary condition $u(0)=u'(0)=0$ means that the left end of the beam is fixed, the boundary condition $u'(1)=0$, $u'''(1)=\lambda g(u(1))$ means that the right end of the beam is sliding clamped and attached to a bearing, given by the function g.

As is well known, the fourth-order boundary value problems for elastic beam equations are widely applied to material mechanics and engineering, because it can characterise the deformation of the equilibrium state. These equations with nonzero or nonlinear boundary conditions can model beams resting on elastic bearings located in their extremities; see for instance [1–6] and the references therein. Over the past several decades, some researchers have extensively investigated the existence and multiplicity of positive solutions for the elastic beam equations. In most studies, the results are obtained by using the Leray–Schauder continuation method, the topological degree theory, the shooting method, fixed point theorems on cones, the critical point theory, monotone iteration method, the lower and upper solution method and so on; for examples refer to the literature [7–16]. However, there are a few papers concerned with the existence and uniqueness of positive solutions for the fourth-order boundary value problems with parameter including non-singular and singular case; see [17–23].

In [17], Yao investigated the following fourth-order problem with a parameter:

$$\begin{cases} u^{(4)}(t)=\lambda f(t,u(t)), & 0<t<1, \\ u(0)=u(1)=u'(0)=u'(1)=0. \end{cases} \tag{1.3}$$

By application of the Krasnosel'skii fixed point theorem of cone expansion–compression type, Yao obtain several existence and multiplicity results.

In [18], Wang *et al.* deal with the following problem:

$$\begin{cases} u^{(4)}(t)=\lambda f(t,u(t)), & 0<t<1, \\ u(0)=u'(0)=u''(1)=u'''(1)+g(u(1))=0, \end{cases} \tag{1.4}$$

where $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ and $g:[0,+\infty)\to[0,+\infty)$, $\lambda\geq 0$ is a parameter. From the fixed point theorem of cone expansion, they prove the existence, multiplicity and nonexistence of solutions. Furthermore, by using cone theory, the authors establish some uniqueness criteria of positive solutions and show such solution x_λ depends continuously on the parameter λ.

The following beam equation with a parameter is investigated by Zhai *et al.* in [19]:

$$\begin{cases} u^{(4)}(t)=\lambda f(t,u(t)), & 0<t<1, \\ u(0)=u'(0)=u''(1)=u'''(1)+\lambda g(u(1))=0, \end{cases} \tag{1.5}$$

where $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ and $g:[0,+\infty)\to[0,+\infty)$, $\lambda>0$ is a parameter. Depending on a fixed point theorem and some properties of eigenvalue problems for a class of general mixed monotone operators, the authors present two results on the existence and uniqueness of convex monotone positive solutions and also present some pleasant properties of solutions dependent on the parameter.

In [20], Yuan *et al.* study the following boundary value problem to nonlinear singular fourth-order differential equation:

$$\begin{cases} u^{(4)}(t) - \lambda q(t) f(u(t), u''(t)) = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \\ \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 u''(0) - \beta_2 u'''(0) = 0, \\ \gamma_2 u''(1) + \delta_2 u'''(1) = 0, \end{cases} \tag{1.6}$$

where $\lambda > 0$ and α_i, β_i, γ_i, $\delta_i \geq 0$ ($i = 1, 2$) are constants such that $\beta_i\gamma_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$ ($i = 1, 2$), $q \in C^1((0, 1), (0, +\infty))$ and q may be singular at $t = 0$ and/or 1, $f(u, v)$ may be singular at $u = 0$. By using the mixed monotone method, the authors establish the existence and uniqueness of positive solutions for some fourth-order nonlinear singular continuous and discrete boundary value problem.

Motivated by the above work, in this paper, by using the fixed point theorems of mixed monotone operator, we intend to study the local existence and uniqueness of increasing positive solutions for the non-singular beam Eq. (1.1) and singular beam Eq. (1.2). Furthermore, we construct two sequences approximating the unique positive solution. The main contributions of this paper are: (a) for the non-singular problem (1.1), the nonlinear term f is changed with the choice of the operator H, here, the H is not necessarily linear, which makes the nonlinear term more general. (b) For the singular problem (1.2), there is no result on the uniqueness of positive solution in the existing literature, hence our results are new. (c) After obtaining the unique existence results, we construct two sequences with a parameter λ for approximating the unique solution u_λ^*, and also present some pleasant properties of positive solution with respect to λ.

The content of this paper is organized as follows. In Sect. 2, we present some definition, lemmas and basic results that will be used in the proofs of our theorems. In Sect. 3, by using the fixed point theorems of mixed monotone operators, we prove the existence and uniqueness of monotone positive solutions for the non-singular boundary value problem (1.1). In Sect. 4, we research the existence and uniqueness of monotone positive solutions for the singular boundary value problem (1.2). In Sect. 5, we give two concrete examples to illustrate these results which can be used in practice.

2 Preliminaries

For the convenience of the reader, in this section we present some definitions in ordered Banach spaces, lemmas and basic results that will be used in the proofs of our theorems [24].

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$, where θ denotes the zero element of E.

A cone P is said to be solid if $\mathring{P} = \{x \in P \mid x \text{ is an interior point of } P\}$ is nonempty. A cone P is called normal if there exists a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in E$

with $\theta \leq x \leq y$, where N is called the normality constant of P. Moreover, we say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay (Ax \geq Ay)$.

Furthermore, for all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2.1 ([25]) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i (i = 1, 2) \in P$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. The element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Lemma 2.1 (See Lemma 2.1 and Theorem 2.1 in [25]) *Let P be a normal cone in E. Assume that $A : P \times P \to P$ is a mixed monotone operator and satisfies*:

(A_1) *there exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$;*

(A_2) *for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that $A(tu, t^{-1}v) \geq \varphi(t) A(u, v)$.*

Then

(1) $A : P_h \times P_h \to P_h$;

(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 < v_0$,* $u_0 \leq A(u_0, v_0) \leq A(v_0, u_0) \leq v_0$;

(3) *A has a unique fixed point x^* in P_h;*

(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

If we suppose the operator $A : P_h \times P_h \to P_h$ with P is a solid cone, then $A(h, h) \in P_h$ is automatically satisfied. Besides, when $\varphi(t) = t^\alpha$ with $\alpha \in (0, 1)$ for $t \in (0, 1)$, we see that the following lemma still holds true.

Lemma 2.2 *Let P be a normal, solid cone of E, and let $A : P_h \times P_h \to P_h$ is a mixed monotone operator. Suppose that: there exists $\alpha \in (0, 1)$ such that*

$$A(tu, t^{-1}v) \geq t^\alpha A(u, v), \quad \forall u, v \in P_h, t \in (0, 1). \tag{2.1}$$

Then operator A has a unique fixed point x^* in P_h. Moreover, for any initial x_0, $y_0 \in P_h$, constructing successively the sequences $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

Lemma 2.3 (See Theorem 2.3 in [25]) *Assume that the operator A satisfies the conditions of Lemma 2.2 or Lemma 2.3. Let $x_\lambda (\lambda > 0)$ denote the unique solution of nonlinear eigenvalue equation $A(x, x) = \lambda x$ in P_h. Then we have the following conclusions*:

(1) *If $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0, 1)$, then x_λ is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$.*

(2) *If there exists $\beta \in (0, 1)$ such that $\varphi(t) \geq t^\beta$ for $t \in (0, 1)$, then x_λ is continuous in λ, that is, $\lambda \to \lambda_0 (\lambda_0 > 0)$ implies $\|x_\lambda - x_{\lambda_0}\| \to 0$.*

(3) *If there exists* $\beta \in (0, \frac{1}{2})$ *such that* $\varphi(t) \geq t^{\beta}$ *for* $t \in (0,1)$, *then* $\lim_{\lambda\to\infty} \|x_\lambda\| = 0$, $\lim_{\lambda\to 0^+} \|x_\lambda\| = \infty$.

Lemma 2.4 *Suppose* f, g *are continuous, then* u *is the solution of problem* (1.1) *if and only if* u *is the solution for the following integral equation.*

$$u(t) = \lambda \int_0^1 G(t,s) f\bigl(s, u(s), (Hu)(s)\bigr)\, ds - \lambda g\bigl(u(1)\bigr)\psi(t), \quad \forall t \in [0,1], \tag{2.2}$$

where

$$G(t,s) = \frac{1}{12} \begin{cases} s^2(6t - 3t^2 - 2s), & 0 \leq s \leq t \leq 1; \\ t^2(6s - 3s^2 - 2t), & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.3}$$

and

$$\psi(t) = \frac{t^2}{4} - \frac{t^3}{6}, \quad \forall t \in [0,1]. \tag{2.4}$$

Proof At first, we prove the necessity. Assuming that $u(t)$ is the solution of equation (1.1), for $u^{(4)}(t) = \lambda f(t, u(t), (Hu)(t))$, combined with the boundary conditions $u'''(1) = \lambda g(u(1))$, we integrate it from t to 1:

$$u'''(t) = \lambda g\bigl(u(1)\bigr) - \lambda \int_t^1 f\bigl(s, u(s), (Hu)(s)\bigr)\, ds, \quad \forall t \in [0,1].$$

Next, we continue to integrate $u'''(t)$ from t to 1:

$$u''(t) = u''(1) - \lambda g\bigl(u(1)\bigr)(1-t) + \lambda \int_t^1 (s-t) f\bigl(s, u(s), (Hu)(s)\bigr)\, ds, \quad \forall t \in [0,1].$$

Then, combined with $u'(0) = u'(1) = 0$, we integrate the above formula from 0 to t:

$$\begin{aligned} u'(t) = {}& -\lambda g\bigl(u(1)\bigr)\left(\frac{t}{2} - \frac{t^2}{2}\right) + \lambda \int_0^t \left(\frac{s^2}{2} - \frac{s^2 t}{2}\right) f\bigl(s, u(s), (Hu)(s)\bigr)\, ds \\ & + \lambda \int_t^1 \left(st - \frac{s^2 t}{2} - \frac{t^2}{2}\right) f\bigl(s, u(s), (Hu)(s)\bigr)\, ds, \quad \forall t \in [0,1]. \end{aligned}$$

At last, integrating $u'(t)$ from 0 to t, and using $u(0) = 0$, we have

$$\begin{aligned} u(t) = {}& -\lambda g\bigl(u(1)\bigr)\left(\frac{t^2}{4} - \frac{t^3}{6}\right) + \lambda \int_0^t \left(\frac{s^2 t}{2} - \frac{s^2 t^2}{4} - \frac{s^3}{6}\right) f\bigl(s, u(s), (Hu)(s)\bigr)\, ds \\ & + \lambda \int_t^1 \left(\frac{st^2}{2} - \frac{s^2 t^2}{4} - \frac{t^3}{6}\right) f\bigl(s, u(s), (Hu)(s)\bigr)\, ds \\ = {}& \lambda \int_0^1 G(t,s) f\bigl(s, u(s), (Hu)(s)\bigr)\, ds - \lambda g\bigl(u(1)\bigr)\psi(t), \quad \forall t \in [0,1]. \end{aligned}$$

Here $G(t,s)$ and $\psi(t)$ are defined by (2.3) and (2.4).

Next, we prove the sufficiency. Suppose that $u(t)$ is the solution of the integral equation (2.2), we differentiate the formula (2.2) directly, and obtain $u^{(4)}(t) = \lambda f(t, u(t), (Hu)(t))$. Besides, we also get $u(0) = u'(0) = 0$, $u'(1) = 0$, $u'''(1) = \lambda g(u(1))$. The proof is completed. □

Lemma 2.5 *The functions $G(t,s)$ and $\psi(t)$ satisfy the following properties*:

(1) *$G(t,s)$ is a continuous on the unit square $[0,1] \times [0,1]$, $\psi(t)$ is continuous for any $t \in [0,1]$.*

(2) *$G(t,s) \geq 0$, $\psi(t) \geq 0$ for each $t,s \in [0,1]$.*

(3) *For any $t,s \in [0,1]$, we have*

$$\frac{1}{12}s^2t^2 \leq G(t,s) \leq \frac{1}{2}st^2, \qquad \frac{1}{12}t^2 \leq \psi(t) \leq \frac{1}{4}t^2.$$

(4) *For any $t,s \in [0,1]$, we get*

$$H_t(t,s) \geq 0, \qquad \psi'(t) \geq 0.$$

Proof The property (1) is simple, so we omit its proof. The properties (2) and (3) have been deduced in [22], and by property (3) we can easily see that the property (2) holds true. □

3 Existence of positive solutions for non-singular Eq. (1.1)

In this section, we will work in the Banach apace $E = C[0,1]$ equipped with the norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$ and a partial order given by $x, y \in E$, $x \leq y \Leftrightarrow x(t) \leq y(t)$ for $t \in [0,1]$. The set $P = \{x \in E : x(t) \geq 0, t \in [0,1]\}$, the standard cone. It is clear that P is a normal cone in E and the normality constant is 1.

We first give a main result which is concerned with the non-singular elastic beam Eqs. (1.1).

Theorem 3.1 *Assume that*

(L_1) *$f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,+\infty) \to (-\infty, 0]$ are continuous with $g(1) < 0$;*

(L_2) *$f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, $y \in [0,+\infty)$, and decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$, $x \in [0,+\infty)$, and $g(x)$ is decreasing in $x \in [0,+\infty)$;*

(L_3) *For $\eta \in (0,1)$, there exist $\varphi_i(\eta) \in (\eta, 1)$ $(i = 1,2)$, such that for any $t \in [0,1]$, $x, y \in [0,+\infty)$*

$$f\bigl(t, \eta x, \eta^{-1}y\bigr) \geq \varphi_1(\eta) f(t,x,y), \qquad g(\eta x) \leq \varphi_2(\eta) g(x); \tag{3.1}$$

(L_4) *The operator $H : P \to P$ is an increasing sub-homogeneous operator, i.e.*

(a) *for any $u, v \in P$ with $u \leq v \Rightarrow Hu \leq Hv$;*

(b) *$H(\eta u) \geq \eta Hu$, $\forall u \in P$, $\eta \in (0,1)$.*

Then:

(1) *for any given $\lambda > 0$, problem (1.1) has a unique increasing positive solution u_λ^* in P_h, where $h(t) = t^2$, $t \in [0,1]$;*

(2) *for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$\begin{cases} x_{n+1}(t) = \lambda \int_0^1 G(t,s)f(s,x_n(s),y_n(s))\,ds - \lambda g(x_n(1))\psi(t), & n = 0,1,2,\ldots, \\ y_{n+1}(t) = \lambda \int_0^1 G(t,s)f(s,y_n(s),x_n(s))\,ds - \lambda g(y_n(1))\psi(t), & n = 0,1,2,\ldots, \end{cases}$$

we have $\|x_n - u_\lambda^*\| \to 0$ *and* $\|y_n - u_\lambda^*\| \to 0$ *as* $n \to \infty$;

(3) *if* $\varphi_i(t) > t^{\frac{1}{2}}$ $(i = 1,2)$ *for* $t \in (0,1)$, *then* u_λ^* *is strictly increasing in* λ, *that is*, $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1}^* < u_{\lambda_2}^*$;

(4) *if there exists* $\beta \in (0,1)$ *such that* $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ *for* $t \in (0,1)$, *then* u_λ^* *is continuous in* λ, *that is*, $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$ $(\lambda \to \lambda_0(\lambda_0 > 0))$;

(5) *if there exists* $\beta \in (0,\frac{1}{2})$ *such that* $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ *for* $t \in (0,1)$, *then* $\lim_{\lambda\to\infty} \|u_\lambda\| = \infty$, $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$.

Proof For any $u, v \in P$, we define an operator $A_\lambda : P \times P \to E$ by

$$A_\lambda(u,v)(t) = \lambda \int_0^1 G(t,s)f\bigl(s,u(s),(Hv)(s)\bigr)\,ds - \lambda g\bigl(u(1)\bigr)\psi(t).$$

It is easy to prove that u is the solution of problem (1.1) if and only if $u = A_\lambda(u,u)$. Next, we will divide the proof into several steps to ensure the operator A_λ satisfies all the conditions of Lemma 2.1.

Step 1: We show that $A_\lambda : P \times P \to P$.

From the assumptions (L_1), (L_4) and the nonnegative character, continuity of $G(t,s)$ in Lemma 2.5, it can be easily seen that $A_\lambda : P \times P \to P$ is a well-defined operator.

Step 2: We show that A_λ is a mixed monotone operator.

In fact, for $u_i, v_i \in P$, $i = 1,2$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, $t \in [0,1]$. It follows from (L_4) that $(Hv_1)(t) \leq (Hv_2)(t)$. By (L_2) and property (2) in Lemma 2.5, we get

$$\begin{aligned} A_\lambda(u_1,v_1)(t) &= \lambda \int_0^1 G(t,s)f\bigl(s,u_1(s),(Hv_1)(s)\bigr)\,ds - \lambda g\bigl(u_1(1)\bigr)\psi(t) \\ &\geq \lambda \int_0^1 G(t,s)f\bigl(s,u_2(s),(Hv_2)(s)\bigr)\,ds - \lambda g\bigl(u_2(1)\bigr)\psi(t) = A_\lambda(u_2,v_2)(t). \end{aligned}$$

Hence, $A_\lambda(u_1,v_1) \geq A_\lambda(u_2,v_2)$, which implies A_λ is a mixed monotone operator.

Step 3: We prove $A_\lambda(h,h) \in P_h$. Here we consider the P_h defined by

$$P_h = \{x \in P | x \neq \theta, \text{ there exist } \mu, \xi > 0 \text{ such that } \xi h \leq x \leq \mu h\},$$

in which we take the function $h(t) = t^2$, $t \in [0,1]$. So we have $0 \leq h(t) \leq 1$. It follows from (L_4) that $0 \leq (Hh)(t) \leq (H1)(t)$. According to (L_1), (L_2) and property (3) in Lemma 2.5, we deduce

$$\begin{aligned} A_\lambda(h,h)(t) &= \lambda \int_0^1 G(t,s)f\big(s,h(s),(Hh)(s)\big)\,ds - \lambda g\big(h(1)\big)\psi(t) \\ &\geq \lambda \int_0^1 \frac{1}{12}s^2t^2 f\big(s,0,(H1)(s)\big)\,ds - \lambda g(0)\cdot \frac{1}{12}t^2 \\ &= \left[\frac{\lambda}{12}\int_0^1 s^2 f\big(s,0,(H1)(s)\big)\,ds - \frac{\lambda}{12}g(0)\right]\cdot t^2 \\ &\geq \frac{\lambda}{12}\left[\int_0^1 s^2 f\big(s,0,(H1)(s)\big)\,ds - g(0)\right]\cdot t^2, \quad \forall t\in[0,1], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} A_\lambda(h,h)(t) &\leq \lambda \int_0^1 \frac{1}{2}st^2 f(s,1,0)\,ds - \lambda g(1)\cdot \frac{1}{4}t^2 \\ &= \left[\frac{\lambda}{2}\int_0^1 sf(s,1,0)\,ds - \frac{\lambda}{4}g(1)\right]\cdot t^2 \\ &\leq \frac{\lambda}{2}\left[\int_0^1 sf(s,1,0)\,ds - g(1)\right]\cdot t^2, \quad \forall t\in[0,1]. \end{aligned} \tag{3.3}$$

Let

$$r_1 = \frac{\lambda}{12}\left[\int_0^1 s^2 f\big(s,0,(H1)(s)\big)\,ds - g(0)\right], \qquad r_2 = \frac{\lambda}{2}\left[\int_0^1 sf(s,1,0)\,ds - g(1)\right].$$

Combining (3.2) with (3.3), we have

$$r_1 h \leq A_\lambda(h,h) \leq r_2 h.$$

Since we have the monotonicity property of $f(t,x,y)$ and $g(1) < 0$, we have

$$0 < r_1 = \frac{\lambda}{12}\left[\int_0^1 s^2 f\big(s,0,(H1)(s)\big)\,ds - g(0)\right] \leq \frac{\lambda}{2}\left[\int_0^1 sf(s,1,0)\,ds - g(1)\right] = r_2.$$

As a result, $A_\lambda(h,h)\in P_h$. So the operator A_λ satisfies condition (A_1) in Lemma 2.1.

Step 4: We check that the operator A_λ satisfies condition (A_2).

Let $\varphi(t) = \min\{\varphi_1(t),\varphi_2(t)\}$, $t\in(0,1)$, then $\varphi(t)\in(t,1)$. Using assumptions (L_3), (L_4), for $\eta\in(0,1)$, $u,v\in P$, we obtain

$$\begin{aligned} A_\lambda\big(\eta u,\eta^{-1}v\big)(t) &= \lambda\int_0^1 G(t,s)f\big(s,\eta u(s),H\big(\eta^{-1}v\big)(s)\big)\,ds - \lambda g\big(\eta u(1)\big)\psi(t) \\ &\geq \lambda\int_0^1 G(t,s)f\big(s,\eta u(s),\eta^{-1}(Hv)(s)\big)\,ds - \lambda g\big(\eta u(1)\big)\psi(t) \\ &\geq \varphi_1(\eta)\lambda\int_0^1 G(t,s)f\big(s,u(s),(Hv)(s)\big)\,ds - \varphi_2(\eta)\lambda g\big(u(1)\big)\psi(t) \\ &\geq \varphi(\eta)\left[\lambda\int_0^1 G(t,s)f\big(s,u(s),(Hv)(s)\big)\,ds - \lambda g\big(u(1)\big)\psi(t)\right] \\ &= \varphi(\eta)A_\lambda(u,v)(t), \quad \forall t\in[0,1]. \end{aligned}$$

Consequently, $A_\lambda(\eta u,\eta^{-1}v)\geq\varphi(\eta)A_\lambda(u,v)$, $\forall u,v\in P$, $\eta\in(0,1)$.

Therefore, A_λ satisfies all the conditions of Lemma 2.1. By applying Lemma 2.1, there exists a unique $u_\lambda^* \in P_h$, such that $A_\lambda(u_\lambda^*, u_\lambda^*) = u_\lambda^*$.

Hence u_λ^* ia a unique positive solution of the problem (1.1) for fixed $\lambda > 0$. Furthermore, on the basis of Lemma 2.1, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{cases} x_{n+1}(t) = \lambda \int_0^1 G(t,s)f(s,x_n(s),(Hy_n)(s))\,ds - \lambda g(x_n(1))\psi(t), & n = 0,1,2,\ldots, \\ y_{n+1}(t) = \lambda \int_0^1 G(t,s)f(s,y_n(s),(Hx_n)(s))\,ds - \lambda g(y_n(1))\psi(t), & n = 0,1,2,\ldots, \end{cases}$$

we have $\|x_n - u_\lambda^*\| \to 0$ and $\|y_n - u_\lambda^*\| \to 0$ as $n \to \infty$. In addition, we can show that u_λ^* is an increasing solution. In fact, by

$$u_\lambda^*(t) = \lambda \int_0^1 G(t,s)f\bigl(s,u_\lambda^*(s),\bigl(Hu_\lambda^*\bigr)(s)\bigr)\,ds - \lambda g\bigl(u_\lambda^*(1)\bigr)\psi(t), \quad \forall t \in [0,1],$$

we can compute

$$u_\lambda'^*(t) = \lambda \int_0^1 G_t(t,s)f\bigl(s,u_\lambda^*(s),\bigl(Hu_\lambda^*\bigr)(s)\bigr)\,ds - \lambda g\bigl(u_\lambda^*(1)\bigr)\psi'(t), \quad \forall t \in [0,1].$$

By property (4) in Lemma 2.5 and (L_1), we obtain $u_\lambda'^*(t) \geq 0$, $\forall t \in [0,1]$, which means $u_\lambda^*(t)$ is increasing in $[0,1]$.

Next, if we set $A = \frac{1}{\lambda}A_\lambda$, then A also satisfies all the conditions of Lemma 2.1. So we have $A_\lambda(u_\lambda^*(t), u_\lambda^*(t)) = \lambda A(u_\lambda^*(t), u_\lambda^*(t)) = u_\lambda^*$, that is, $A(u_\lambda^*(t), u_\lambda^*(t)) = \frac{1}{\lambda}u_\lambda^*(t)$. If $\varphi_i(t) > t^{\frac{1}{2}}$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) > t^{\frac{1}{2}}$, Lemma 2.3 (1) implies u_λ^* is strictly decreasing in $\frac{1}{\lambda}$, that is, u_λ^* is strictly increasing in λ, i.e., $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leq u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$; if there exists $\beta \in (0,1)$ such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) \geq t^\beta$ for $t \in (0,1)$, Lemma 2.3 (2) implies u_λ^* is continuous in λ, that is, $\|u_\lambda^* - u_{\lambda 0}^*\| \to 0$ $(\lambda \to \lambda_0(\lambda_0 > 0))$; if there exist $\beta \in (0, \frac{1}{2})$ such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) \geq t^\beta$ for $t \in (0,1)$, Lemma 2.3 (3) implies $\lim_{\lambda\to\infty} \|u_\lambda^*\| = \infty$, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = 0$. The proof is completed. □

Remark 3.1 When $\lambda = 1$, H is a null operator, the similar type of beam equation has been studied by Zhai in [22], in which the existence and uniqueness results are obtained by two fixed point theorems of a sum operator. Note that the functions f, g in [22] only have stationary monotonicity, while in our study they have two different types of monotonicity. Therefore, our study is more general.

4 Existence of positive solutions for singular Eq. (1.2)

In this section, we will present another result which is deal with the singular elastic beam Eqs. (1.2).

Theorem 4.1 *Assume that*

(L_5) $f : (0,1) \times [0,+\infty) \times (0,+\infty) \to [0,+\infty)$ *is continuous*, $f(t,x,y)$ *may be singular at* $t = 0$ *or* 1 *and* $y = 0$. $g : [0,+\infty) \to (-\infty, 0]$ *is continuous*;

(L_6) $f(t,x,y)$ *is increasing in* $x \in [0,+\infty)$ *for fixed* $t \in (0,1)$, $y \in (0,+\infty)$ *and decreasing in* $y \in (0,+\infty)$ *for fixed* $t \in (0,1)$, $x \in [0,+\infty)$; $g(x)$ *is decreasing in* $x \in [0,+\infty)$;

(L_7) *for* $\eta \in (0,1)$, *there exist* $\alpha_i \in (0,1)$ $(i = 1,2)$, *such that*

$$f(t,\eta x,\eta^{-1}y) \geq \eta^{\alpha_1} f(t,x,y), \quad \forall t \in (0,1), x \in [0,+\infty), y \in (0,+\infty). \tag{4.1}$$

$$g(\eta x) \leq \eta^{\alpha_2} g(x), \quad \forall t \in (0,1), x \in [0,+\infty); \tag{4.2}$$

(L_8) *let* $\alpha = \max\{\alpha_1,\alpha_2\}$, $\alpha \in (0,1)$, *then*

$$\int_0^1 s^{1-2\alpha} f(s,1,1)\,ds < +\infty$$

with $f(t,1,1) \not\equiv 0$. *Then the results* (1)–(5) *in Theorem* 3.1 *are still true*.

Proof For any $u, v \in P$, we define an operator $A_\lambda : P \times P \to E$ by

$$A_\lambda(u,v)(t) = \lambda \int_0^1 G(t,s) f\bigl(s,u(s),v(s)\bigr)\,ds - \lambda g\bigl(u(1)\bigr)\psi(t).$$

Evidently, u is the solution of problem (1.2) if and only if $u = A_\lambda(u,u)$. In the sequel we check that A_λ satisfies all the conditions of Lemma 2.2.

At first, we will prove operator $A_\lambda : P_h \times P_h \to P_h$ is a mixed monotone operator. Here we consider the P_h defined by

$$P_h = \left\{ x \in P \,\middle|\, \exists \rho > 1 : \frac{1}{\rho} h \leq x \leq \rho h, \forall t \in [0,1] \right\},$$

with $h(t) = t^2$, $t \in [0,1]$. By (L_7), for all $\eta \in (0,1)$, $t \in (0,1)$, $x \in [0,+\infty)$, $y \in (0,+\infty)$, there exist α_1, $\alpha_2 \in (0,1)$, one has

$$f(t,x,y) = f\bigl(t,\eta\eta^{-1}x,\eta^{-1}\eta y\bigr) \geq \eta^{\alpha_1} f\bigl(t,\eta^{-1}x,\eta y\bigr),$$
$$g(x) = g\bigl(\eta\eta^{-1}x\bigr) \leq \eta^{\alpha_2} g\bigl(\eta^{-1}x\bigr).$$

From the above inequalities and the fact that $\alpha = \max\{\alpha_1,\alpha_2\}$ we have

$$f\bigl(t,\eta^{-1}x,\eta y\bigr) \leq \frac{1}{\eta^{\alpha_1}} f(t,x,y) \leq \frac{1}{\eta^{\alpha}} f(t,x,y), \tag{4.3}$$

$$g\bigl(\eta^{-1}x\bigr) \geq \frac{1}{\eta^{\alpha_2}} g(x) \geq \frac{1}{\eta^{\alpha}} g(x). \tag{4.4}$$

Set $x = 1$, $y = 1$ in (4.1)–(4.4), we can easily obtain

$$f\bigl(t,\eta,\eta^{-1}\bigr) \geq \eta^{\alpha} f(t,1,1), \qquad g(\eta) \leq \eta^{\alpha} g(1), \tag{4.5}$$

$$f\bigl(t,\eta^{-1},\eta\bigr) \leq \frac{1}{\eta^{\alpha}} f(t,1,1), \qquad g\bigl(\eta^{-1}\bigr) \geq \frac{1}{\eta^{\alpha}} g(1). \tag{4.6}$$

For any $u,v \in P_h$, we can choose a constant $\rho \geq 1$ to satisfy $\frac{1}{\rho}t^2 \leq u(t), v(t) \leq \rho t^2, \forall t \in [0,1]$. From (L_6) and (4.1)–(4.6), we obtain

$$\begin{aligned} f\bigl(t,u(t),v(t)\bigr) &\leq f\left(t,\rho t^2,\frac{1}{\rho}t^2\right) \leq f\left(t,\rho\frac{1}{t^2},\frac{1}{\rho}t^2\right) \\ &\leq \frac{1}{t^{2\alpha}}f\left(t,\rho,\frac{1}{\rho}\right) \leq \frac{\rho^\alpha}{t^{2\alpha}}f(t,1,1), \quad \forall t\in(0,1), \end{aligned} \tag{4.7}$$

$$g\bigl(u(t)\bigr) \leq g\left(\frac{1}{\rho}t^2\right) \leq t^{2\alpha}g\left(\frac{1}{\rho}\right) \leq \frac{t^{2\alpha}}{\rho^\alpha}g(1), \quad \forall t\in(0,1), \tag{4.8}$$

$$\begin{aligned} f\bigl(t,u(t),v(t)\bigr) &\geq f\left(t,\frac{1}{\rho}t^2,\rho t^2\right) \geq f\left(t,\frac{1}{\rho}t^2,\rho\frac{1}{t^2}\right) \\ &\geq t^{2\alpha}f\left(t,\frac{1}{\rho},\rho\right) \geq \frac{t^{2\alpha}}{\rho^\alpha}f(t,1,1), \quad \forall t\in(0,1), \end{aligned} \tag{4.9}$$

$$g\bigl(u(t)\bigr) \geq g\bigl(\rho t^2\bigr) \geq g\left(\rho\frac{1}{t^2}\right) \geq \frac{1}{t^{2\alpha}}g(\rho) = \frac{\rho^\alpha}{t^{2\alpha}}g(1), \quad \forall t\in(0,1). \tag{4.10}$$

It follows from Lemma 2.5, (4.7), (4.10) and (L_8) that

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda\int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds - \lambda g\bigl(u(1)\bigr)\psi(t) \\ &\leq \lambda\int_0^1 \frac{1}{2}st^2\frac{\rho^\alpha}{s^{2\alpha}}f(s,1,1)\,ds - \lambda\frac{1}{4}t^2\rho^\alpha g(1) \\ &= \rho^\alpha\left[\frac{\lambda}{2}\int_0^1 s^{1-2\alpha}f(s,1,1)\,ds - \frac{\lambda g(1)}{4}\right]t^2 < +\infty, \quad \forall t\in[0,1]. \end{aligned} \tag{4.11}$$

Then Lemma 2.4 implies that $A: P_h \times P_h \to P$ is well defined. By Lemma 2.4, (4.8) and (4.9), we also have

$$\begin{aligned} A_\lambda(u,v)(t) &\geq \lambda\int_0^1 \frac{1}{12}s^2t^2\frac{s^{2\alpha}}{\rho^\alpha}f(s,1,1)\,ds - \lambda\frac{t^2}{12}\frac{1}{\rho^\alpha}g(1) \\ &= \frac{1}{\rho^\alpha}\left[\frac{\lambda}{12}\int_0^1 s^{2+2\alpha}f(s,1,1)\,ds - \frac{\lambda g(1)}{12}\right]t^2, \quad \forall t\in[0,1]. \end{aligned} \tag{4.12}$$

Let ρ be a positive constant defined by

$$\begin{aligned} \rho > \max\Biggl\{&1,\left(\frac{\lambda}{2}\int_0^1 s^{1-2\alpha}f(s,1,1)\,ds - \frac{\lambda g(1)}{4}\right)^{\frac{1}{1-\alpha}}, \\ &\times\left(\frac{\lambda}{12}\int_0^1 s^{2+2\alpha}f(s,1,1)\,ds - \frac{\lambda g(1)}{12}\right)^{\frac{1}{\alpha-1}}\Biggr\}. \end{aligned} \tag{4.13}$$

It follows from (4.11) and (4.12) that

$$\frac{1}{\rho}h(t) = \frac{1}{\rho}t^2 \leq A_\lambda(u,v)(t) \leq \rho t^2 = \rho h(t), \quad \forall t\in[0,1],$$

which means $A_\lambda(u,v) \in P_h$, we prove $A_\lambda: P_h \times P_h \to P_h$.

Next, similar to the proof of Theorem 3.1, from Lemma 2.5 and (L_5), (L_6), we see that $A_\lambda : P_h \times P_h \to P_h$ is a mixed monotone operator.

At last, we show that A_λ satisfies (2.1) in Lemma 2.1. It follows from (L_7) that

$$\begin{aligned} A_\lambda\bigl(\eta u, \eta^{-1}v\bigr)(t) &= \lambda \int_0^1 G(t,s)f\bigl(s, \eta u(s), \eta^{-1}v(s)\bigr)\,ds - \lambda g\bigl(\eta u(1)\bigr)\psi(t) \\ &\geq \eta^{\alpha_1}\lambda \int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds - \eta^{\alpha_2}\lambda g\bigl(u(1)\bigr)\psi(t) \\ &\geq \eta^{\alpha}\biggl[\lambda \int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds - \lambda g\bigl(u(1)\bigr)\psi(t)\biggr] \\ &= \eta^{\alpha}A_\lambda(u,v)(t) \quad \forall u,v \in P_h, \eta \in (0,1), t \in [0,1]. \end{aligned} \tag{4.14}$$

Consequently, $A_\lambda(\eta u, \eta^{-1}v) \geq \eta^{\alpha}A_\lambda(u,v)$, $\forall u,v \in P_h$, $\eta \in (0,1)$.

Therefore, A_λ satisfies all the conditions of Lemma 2.2. By Lemma 2.2, there exists a unique $u_\lambda^* \in P_h$ such that $A_\lambda(u_\lambda^*, u_\lambda^*) = u_\lambda^*$. It is easy to check that u_λ^* ia a unique positive solution of the problem (1.1) for fixed $\lambda > 0$. Similar to the proof in Theorem 3.1, by (L_5) and property (4) in Lemma 2.5, we can illustrate the unique positive solution $u_\lambda^*(t)$ is increasing on $(0,1)$. Also, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{cases} x_{n+1}(t) = \lambda \int_0^1 G(t,s)f(s,x_n(s),y_n(s))\,ds + \lambda g(x_n(1))\psi(t), & n = 0,1,2,\ldots, \\ y_{n+1}(t) = \lambda \int_0^1 G(t,s)f(s,y_n(s),x_n(s))\,ds + \lambda g(y_n(1))\psi(t), & n = 0,1,2,\ldots, \end{cases}$$

we have $\|x_n - u_\lambda^*\| \to 0$ and $\|y_n - u_\lambda^*\| \to 0$ as $n \to \infty$. Furthermore, if we set $A_\lambda = \lambda A$, on the basis of Lemma 2.3, if $\varphi_i(t) > t^{\frac{1}{2}}$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) > t^{\frac{1}{2}}$, u_λ^* is strictly decreasing in $\frac{1}{\lambda}$, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* > u_{\lambda_2}^*$; and if there exists $\beta \in (0,1)$ such that $\varphi_i(t) \geq t^{\beta}$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) \geq t^{\beta}$ for $t \in (0,1)$, so u_λ^* is continuous in λ, that is, $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$ $(\lambda \to \lambda_0(\lambda_0 > 0))$. And if there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi_i(t) \geq t^{\beta}$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) \geq t^{\beta}$ for $t \in (0,1)$, so $\lim_{\lambda\to\infty}\|u_\lambda^*\| = \infty$, $\lim_{\lambda\to 0^+}\|u_\lambda^*\| = 0$. The proof is completed. □

Remark 4.1 According to the proof, when $g \equiv 0$ in the problem (1.1) and (1.2), our Theorems 3.1 and 4.1 still hold true. In this case, set $\lambda = 1$, this model is the classical sliding clamped beam. So our study is relevant in the field of engineering.

5 Example

In this section, we will give two concrete examples to illustrate those results can be used in practice.

Example 5.1 Consider the following fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = u^{\frac{1}{4}}(t) + \frac{1}{1+(u^{\frac{1}{2}}(t))^{\frac{1}{3}}} + \cos^2(t) \quad 0 < t < 1; \\ u(0) = u'(0) = 0, \\ u'(1) = 0, \qquad u'''(1) = -u^{\frac{1}{3}}(1) - 1. \end{cases} \tag{5.1}$$

Obviously, problem (5.1) fits the framework of problem (1.1) with $\lambda = 1$, and $H : E \to E$ an operator defined by

$$(Hu)(t) = u^{\frac{1}{2}}(t), \qquad f(t,x,y) = x^{\frac{1}{4}} + \frac{1}{1+y^{\frac{1}{3}}} + \cos^2 t, \qquad g(x) = -\big(x^{\frac{1}{3}} + 1\big).$$

It is easy to see that $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous, $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, $y \in [0,+\infty)$ and decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$, $x \in [0,+\infty)$. Besides, $g : [0,+\infty) \to (-\infty,0]$ is continuous and $g(x)$ is decreasing in $x \in [0,+\infty)$, $g(1) = -2 < 0$. So the conditions (L_1), (L_2) hold.

Next, we show the operator H satisfies condition (L_4). In fact, (1) $\forall u \in P$, we have $(Hu)(t) = u^{\frac{1}{2}}(t) \geq 0$, which implies $Hu \in P$, so $H : P \to P$. (2) $\forall u, v \in P$ with $u \leq v$, we have $(Hu)(t) = u^{\frac{1}{2}}(t) \leq v^{\frac{1}{2}}(t) = (Hv)(t)$, $\forall t \in [0,1]$, which means $Hu \leq Hv$, and H is an increasing operator. (3) $\forall u \in P$, $\eta \in (0,1)$, $t \in [0,1]$, we deduce that $H(\eta u)(t) = (\eta u)^{\frac{1}{2}}(t) \geq \eta u^{\frac{1}{2}}(t) = \eta(Hu)(t)$, by which we get $H(\eta u) \geq \eta Hu$, so H is a sub-homogeneous operator.

Furthermore, if we set $\varphi_1(\eta) = \eta^{\frac{1}{3}}$, $\varphi_2(\eta) = \eta^{\frac{1}{3}}$, $\eta \in (0,1)$, then $\varphi_1(\eta), \varphi_2(\eta) \in (\eta, 1)$ and

$$\begin{aligned} f\big(t, \eta x, \eta^{-1} y\big) &= (\eta x)^{\frac{1}{4}} + \frac{1}{1 + (\eta^{-1} y)^{\frac{1}{3}}} + \cos^2 t \geq \eta^{\frac{1}{4}} x^{\frac{1}{4}} + \frac{\eta^{\frac{1}{3}}}{1 + y^{\frac{1}{3}}} + \cos^2 t \\ &\geq \eta^{\frac{1}{3}} \left(x^{\frac{1}{2}} + \frac{1}{1+y^{\frac{1}{3}}} + \cos^2 t \right) = \varphi_1(\eta) f(t,x,y), \\ g(\eta x) &= -\big((\eta x)^{\frac{1}{3}} + 1\big) \leq -\eta^{\frac{1}{3}}\big(x^{\frac{1}{3}} + 1\big) = \varphi_2(\eta) g(x), \end{aligned}$$

for $t \in [0,1], x, y \in [0,+\infty)$. As a result, condition (L_3) holds. Hence all the conditions of Theorem 3.1 are satisfied. By the application of Theorem 3.1, we can see that the problem (5.1) has a unique increasing positive solution $u_\lambda^* \in P_h$. And for any initial values $x_0, y_0 \in P_h$, constructing two sequences

$$\begin{cases} x_{n+1}(t) = \lambda \int_0^1 G(t,s)(\sqrt[4]{x_n} + \frac{1}{1+\sqrt[3]{y_n}} + \cos^2 s)\,ds - \lambda(\sqrt[3]{x_n} + 1)\psi(t), & n = 0,1,2,\ldots, \\ y_{n+1}(t) = \lambda \int_0^1 G(t,s)(\sqrt[4]{y_n} + \frac{1}{1+\sqrt[3]{x_n}} + \cos^2 s)\,ds - \lambda(\sqrt[3]{y_n} + 1)\psi(t), & n = 0,1,2,\ldots, \end{cases}$$

we have $\|x_n - u_\lambda^*\| \to 0$ and $\|y_n - u_\lambda^*\| \to 0$ as $n \to \infty$. Moreover, we have $\varphi_1(t) = \varphi_2(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then u_λ^* is increasing in λ. Also, setting $\beta = \frac{1}{3}$, we easily obtain u_λ^* is continuous in λ and $\lim_{\lambda\to\infty} \|u_\lambda^*\| = \infty$, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = 0$.

Remark 5.1 The operator $H : E \to E$ which satisfies the assumption (L_4) of Theorem 3.1 includes the linear or the nonlinear cases. For example, we may set a composition operator defined by

$$(Hu)(t) = u\big(\varphi(t)\big), \quad \forall t \in [0,1], u \in E,$$

and a multiplication operator defined by

$$(Hu)(t) = \varphi(t) u(t), \quad \forall t \in [0,1], u \in E,$$

where $\varphi : [0,1] \to [0,+\infty)$ is a continuous function. We use an integral operator defined by

$$(Hu)(t) = \int_0^t u(s)\,ds, \quad \forall t \in [0,1], u \in E.$$

It is easy to see that the above three operators are linear and satisfy the required assumption. Besides, we may present some nonlinear operators that meet the conditions (L_4). An example is the operator defined by

$$(Hu)(t) = \max\{|u(s)| : 0 \le s \le t\}, \quad \forall t \in [0,1], u \in E.$$

We can define another nonlinear operator by

$$(Hu)(t) = u^{\gamma}(t), \quad \forall t \in [0,1], u \in E, \gamma \in (0,1).$$

Example 5.2 Consider the following fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = \sqrt[3]{\frac{u(t)+1}{t(1-t)u(t)}} & 0 < t < 1; \\ u(0) = u'(0) = 0, \\ u'(1) = 0, \qquad u'''(1) = -u^{\frac{1}{2}}(1), \end{cases} \tag{5.2}$$

where $\lambda = 1$,

$$f(t,x,y) = \sqrt[3]{\frac{x+1}{t(1-t)y}}, \qquad g(x) = -x^{\frac{1}{2}}.$$

It is easy to see that $f : (0,1) \times [0,+\infty) \times (0,+\infty) \to [0,+\infty)$ is continuous with $f(t,1,1) = \sqrt[3]{\frac{2}{t(1-t)}} \not\equiv 0$. $f(t,x,y)$ is singular at $t = 0, 1$ and $y = 0$. $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in (0,1)$, $y \in (0,+\infty)$ and decreasing in $y \in (0,+\infty)$ for fixed $t \in (0,1)$, $x \in [0,+\infty)$. Besides, $g : [0,+\infty) \to (-\infty,0]$ is continuous and $g(x)$ is decreasing in $x \in [0,+\infty)$.

Set $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{2}$, then $\alpha = \frac{2}{3}$, and

$$f(t,\eta x,\eta^{-1}y) = \sqrt[3]{\frac{\eta x+1}{t(1-t)\eta^{-1}y}} \ge \eta^{\frac{2}{3}}\sqrt[3]{\frac{x+1}{t(1-t)y}} = \eta^{\alpha_1} f(t,x,y),$$
$$g(\eta x) = -(\eta x)^{\frac{1}{2}} \le -\eta^{\frac{1}{2}}x^{\frac{1}{2}} = \eta^{\alpha_2} g(x),$$

for $\eta \in (0,1)$, $t \in (0,1)$, $x \in [0,+\infty)$, $y \in (0,+\infty)$. Also we have

$$\int_0^1 s^{1-2\alpha} f(s,1,1)\,ds = \int_0^1 s^{-\frac{1}{3}}\sqrt[3]{\frac{2}{s(1-s)}}\,ds = 4.57 < +\infty,$$
$$\left(\frac{\lambda}{2}\int_0^1 s^{1-2\alpha} f(s,1,1)\,ds - \frac{\lambda g(1)}{12}\right)^{\frac{1}{1-\alpha}} = \left(\frac{1}{2} \times 4.57 + \frac{1}{4}\right)^3 = 16.29,$$
$$\left(\frac{\lambda}{12}\int_0^1 s^{2+2\alpha} f(s,1,1)\,ds - \frac{\lambda g(1)}{4}\right)^{\frac{1}{\alpha-1}} = \left(\frac{1}{12}\int_0^1 s^{\frac{10}{3}}\sqrt[3]{\frac{2}{s(1-s)}}\,ds + \frac{1}{12}\right)^{-3} = 354.33.$$

Hence

$$\rho > \{1, 16.29, 354.33\}.$$

As a result, all the conditions of Theorem 4.1 are satisfied. By the application of Theorem 4.1, we can see that the problem (5.2) has a unique positive solution $u_\lambda^* \in P_h$. Here if we set $\rho = 355$, we get $u_\lambda^*(t) \in (\frac{1}{355}t^2, 355t^2)$, then the property of the unique positive solution is clearer.

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The authors declare that they have no competing interests.

Authors' contributions
HW participated in the design of the study and drafted the manuscript. LZ carried out the theoretical studies and helped to draft the manuscript. All authors read and approved the final manuscript.

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