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LaSalle stationary oscillation theorem for affine periodic dynamic systems on time scales

Ruichao Guo[1*], Xiaomeng Jiang[2] and Hongren Wang[3]

*Correspondence:
guoruichaomath@outlook.com
[1]School of Applied Mathematics, Jilin University of Finance and Economics, Changchun, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, for affine periodic systems on time scales, we establish LaSalle stationary oscillation theorem to obtain the existence and asymptotic stability of affine periodic solutions on time scales. As applications, we present the existence and asymptotic stability of affine periodic solutions on time scales via Lyapunov's method.

MSC: 34C14; 34N05

Keywords: Time scales; Affine periodic solution; LaSalle stationary oscillation theorem; Lyapunov's method; Nonlinear dynamic equations

1 Introduction and statement of the main result

The research of periodic phenomena has a long history that started with Kepler and Newton when they studied orbits of planets in the solar system. The definition of periodic solution was first introduced by Poincaré in the study of celestial mechanics. As we know, periodicity is a very important property when people study differential equations, but not all the natural phenomena can be described by periodicity only. In fact, some differential equations not only show periodicity in time but also exhibit symmetry in space. Therefore, the concept of affine periodicity was introduced by Li et al. in [9, 20–22, 33, 34, 36]. The affine periodicity is a generalization of pure periodicity. Some results had been proved to be similar to periodic systems. In 1892, Lyapunov introduced the concept of stability of a dynamic system and created Lyapunov's second method in the study of stability. Many authors have developed and applied Lyapunov's method during the past century. One of the important developments in this direction is LaSalle's stationary oscillation theorem [15, 16] which can guarantee the existence of periodic solutions. Li et al. discussed similar LaSalle's stationary oscillation theorem for affine periodic systems and gave more general stationary oscillation conditions (see [35]).

The theory of time scales was initiated by Stefan Hilger in his PhD thesis in 1988 [14] as a means of a unifying structure for the study of continuous and discrete hybrid systems. Time scale is any closed nonempty subset of $\mathbb{R}$ denoted by $\mathbb{T}$. For instance, if $\mathbb{T} = \mathbb{Z}$, dynamic equations are just usual difference equations, while, taking $\mathbb{T} = \mathbb{R}$, they are usual differential equations. Since the theory of time scales can also describe continuous and discrete hybrid processes, it has some important applications, for instance, in the study of option-pricing and stock dynamics in finance, the frequency of markets and duration of market trading in economics, insect population models, neural networks, quantum calculus, among others. See [2, 7, 8, 10, 28] for more details.

Recently, the theory of dynamic equations on time scales has been paid high attention by a lot of mathematicians [1, 3, 4, 11–13, 17, 18, 23, 26, 27, 31, 37]. The existence of solutions of dynamic equations on time scales has been extensively investigated, especially concerning periodicity. On the other hand, almost periodicity on time scales was introduced by Li and Wang in [19, 29, 30]. The theory of almost automorphic functions on time scales was introduced by Lizama and Mesquita (see [24]). After that, Wang and Li considered nonlinear dynamic equations and proved the existence of affine periodic solutions via topological degree theory in [32].

Motivated by these facts, the main goal of this paper is to discuss the existence of affine periodic solutions and asymptotic stability of affine periodic systems on time scales by LaSalle-type stationary oscillation principle. More precisely, we consider the following dynamic system on time scales:

$$x^{\Delta} = f(t,x), \tag{1.1}$$

where $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous function, $\mathbb{T}$ is a T-periodic time scale.

System (1.1) is said to be a (Q,T)-affine-periodic system on time scales if there exists a nonsingular $n \times n$ matrix Q such that

$$f\bigl(\sigma(t) + T, x\bigr) = Qf\bigl(\sigma(t), Q^{-1}x\bigr)$$

for all $(t,x) \in \mathbb{T} \times \mathbb{R}^n$. Obviously, when $Q = I$ (identity), system (1.1) is just usual T-periodic. When $Q = -I$, system (1.1) is anti-periodic. When Q is an orthogonal matrix, system (1.1) is quasi-periodic.

In this paper, we will discuss the existence of (Q,T)-affine-periodic solutions and asymptotic stability of (Q,T)-affine-periodic systems on time scales by constructing a Poincaré map for the solutions of (1.1). The proof is inspired by [35], but some technical details on time scales are more complicated. As applications, the existence and asymptotic stability of (Q,T)-affine-periodic solutions on time scales are obtained via Lyapunov's method.

The present paper is organized as follows. In Sect. 2, we present the preliminary results concerning the theory of time scales. In Sect. 3, we prove the existence of (Q,T)-affine-periodic solutions and asymptotic stability of (Q,T)-affine-periodic systems on time scales. Finally, the last section is devoted to some applications via Lyapunov's method, and an example is given.

2 Preliminaries

In this section, we present some basic definitions, concepts, and results concerning time scales which will be essential to proving our main results. For more details about time scales, see [5, 6]. Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$.

Definition 2.1 For $t \in \mathbb{T}$, we define the forward jump operator and backward jump operator $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\sigma(t) > t$, we say that t is *right-scattered*. If $\sigma(t) = t$, then t is called *right-dense*. Analogously, if $\rho(t) < t$, we say that t is *left-scattered*. If $\rho(t) = t$, then t is called *left-dense*.

Definition 2.2 We define the graininess function $\mu : \mathbb{T} \to [0, \infty)$ by

$$\mu(t) = \sigma(t) - t.$$

For $a, b \in \mathbb{R}$, we use $[a, b]_{\mathbb{T}}$ to denote a closed interval in $\mathbb{T}$, that is, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}; a \leq t \leq b\}$. We denote $\mathbb{T}_+ = \{t \in \mathbb{T}, \mathbb{T} > 0\}$. We define the set $\mathbb{T}^\kappa$ derived from $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.3 Assume that f is an $\mathbb{R}^n$-valued function on time scale $\mathbb{T}$ and $t \in \mathbb{T}^\kappa$. A vector $f^\Delta(t)$ (provided it exists) is said to be the delta (or Hilger) derivative of f at t provided that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f\big(\sigma(t)\big) - f(s) - f^\Delta(t)\big[\sigma(t) - s\big] \right| \leq \varepsilon \left|\sigma(t) - s\right|$$

for all $s \in U$.

Similarly, we can define the nabla-derivative of the function $f : \mathbb{T} \to \mathbb{R}^n$, for details, see [5] and [6].

Definition 2.4 A function $f : \mathbb{T} \to \mathbb{R}^n$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ is denoted by

$$C_{rd} = C_{rd}\big(\mathbb{R}^n\big) = C_{rd}\big(\mathbb{T}, \mathbb{R}^n\big).$$

Next we will introduce the definition of periodic time scale.

Definition 2.5 Let $T > 0$ be a real number. A time scale $\mathbb{T}$ is called T-periodic if $t \in \mathbb{T}$ implies $t + T \in \mathbb{T}$ and $\mu(t) = \mu(t + T)$.

The following lemma shows some useful relationships concerning the delta derivative and continuity. For details, see [5] and [6].

Lemma 2.1 *Assume that $f : \mathbb{T} \to \mathbb{R}^n$ is a function, and let $t \in \mathbb{T}^\kappa$. Then we have the following results.*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If t is right-dense, then f is differentiable at t provided that the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite vector. In this case,

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(iv) *If f is differentiable at t, then*

$$f\bigl(\sigma(t)\bigr)=f(t)+\mu(t)f^{\Delta}(t).$$

Theorem 2.1 (See [5], Theorem 1.70) *Any rd-continuous function $f:\mathbb{T}\to\mathbb{R}^n$ has an anti-derivative, i.e., $F^{\Delta}=f$ on $\mathbb{T}^{\kappa}$.*

Definition 2.6 Let $f\in C_{rd}$ and let F be any function such that $F^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}^{\kappa}$. Then the Cauchy integral by

$$\int_a^b f(t)\Delta t=F(b)-F(a)\quad\text{for all } a,b\in\mathbb{T}.$$

In this sequel, we present the definition of Riemann Δ-integrals. For more details, see [5, 6].

Definition 2.7 A partition of $[a,b]_{\mathbb{T}}$ is a finite sequence of points

$$\{t_0,t_1,\dots,t_m\}\subset[a,b]_{\mathbb{T}},\qquad a=t_0<t_1<\cdots<t_m=b.$$

Given such a partition, we put $\Delta t_i=t_i-t_{i-1}$. A tagged partition consists of a partition and a sequence of tags $\{\xi_1,\dots,\xi_m\}$ such that $\xi_i\in[t_{i-1},t_i)$ for every $i\in\{1,\dots,m\}$. The symbol $D(a,b)$ denotes the set of all tagged partitions of $[a,b]_{\mathbb{T}}$. If $\delta>0$, then the set of all tagged partitions of $[a,b]_{\mathbb{T}}$ such that, for every $i\in\{1,\dots,m\}$, either $\Delta t_i\le\delta$ or $\Delta t_i>\delta$ and $\sigma(t_{i-1})=t_i$ will be denoted by the symbol $D_\delta(a,b)$. For the last case, the only way to choose a tag in $[t_{i-1},t_i)$ is to take $\xi_i=t_{i-1}$.

Definition 2.8 We say that a function f is Riemann Δ-integrable on $[a,b]_{\mathbb{T}}$ if there exists a number I satisfying that, for every $\varepsilon>0$, there exists $\delta>0$ such that

$$\left|\sum_i f(\xi_i)(t_i-t_{i-1})-I\right|<\varepsilon$$

for every $P\in D_\delta(a,b)$ independently of $\xi_i\in[t_{i-1},t_i)_{\mathbb{T}}$ for $1\le i\le m$. It is clear that such a number I is unique and is the Riemann Δ-integral of f from a to b.

Definition 2.9 We say that a function $p:\mathbb{T}\to\mathbb{R}$ is a regressive one provided

$$1+\mu(t)p(t)\neq 0\quad\text{for all } t\in\mathbb{T}^{\kappa}$$

holds. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

In what follows, we present the definition of the generalized exponential function $e_p(t,s)$. For more details, see [5, 6].

Definition 2.10 If $p \in \mathcal{R}$, then we define the exponential function by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big)\Delta\tau\right) \quad \text{for } s,t \in \mathbb{T},$$

where the cylinder transformation $\xi_h(z) : \mathbb{C}_h \to \mathbb{Z}_h$ is given by

$$\xi_h(z) = \begin{cases} \frac{1}{h}\log(1+zh), & h > 0 \\ z, & h = 0 \end{cases}$$

where log is the principal logarithm function.

The following lemma shows Gronwall's inequality on time scales. It can be found in [5].

Lemma 2.2 *Let $y, f \in C_{rd}$ and $p \in \mathbb{R}_+$ and satisfy the inequality*

$$y^\Delta(t) \le p(t)y(t) + f(t) \quad \forall t \in \mathbb{T}.$$

Then

$$y(t) \le y(t_0)e_p(t,t_0) + \int_{t_0}^t e_p\big(\tau, \sigma(\tau)\big)f(\tau)\Delta\tau \quad \forall t \in \mathbb{T}.$$

The following lemma shows the chain rule on time scales (see [25]).

Lemma 2.3 *Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be continuous differentiable and suppose that $g : \mathbb{T} \to \mathbb{R}^1$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}^1$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \int_0^1 \big\{f'\big(g(t) + h\mu(t)g^\Delta(t)\big)\big\}\, dh \cdot g^\Delta(t) \tag{2.1}$$

holds.

3 LaSalle-type stationary oscillation theorem for dynamic equations on time scales

Consider the (Q,T)-affine-periodic dynamic system on time scales

$$x^\Delta = f(t,x), \tag{3.1}$$

where $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous, $\mathbb{T}$ is a T-periodic time scale and ensures the uniqueness of solutions with respect to the initial value (for more details, see Sect. 8.3 in [6]). We always assume Q is an orthogonal matrix in this paper.

Definition 3.1 A function $x : \mathbb{T} \to \mathbb{R}^n$ is called a solution of (3.1) if $x \in \{y : y \in C(\mathbb{T}, \mathbb{R}^n), y^\Delta \in C_{rd}(\mathbb{T}, \mathbb{R}^n)\}$ and $x(t)$ satisfies (3.1) for all $t \in \mathbb{T}$.

Definition 3.2 A function $x : \mathbb{T} \to \mathbb{R}^n$ is said to be an affine periodic solution of (Q,T)-affine-periodic system (3.1) if $x(\sigma(t))$ is a solution of (3.1) and, for any $t \in \mathbb{T}$,

$$x\bigl(\sigma(t) + T\bigr) = Qx\bigl(\sigma(t)\bigr).$$

Remark 3.1 If t is right-dense, then function $x : \mathbb{T} \to \mathbb{R}^n$ is an affine periodic solution of (Q,T)-affine-periodic system (3.1) if $x(t)$ is a solution of (3.1) and, for any $t \in \mathbb{T}$,

$$x(t + T) = Qx(t).$$

Definition 3.3 The solution $x(t, x_0)$ of (3.1) is said to be an asymptotically stable (Q, T)-affine-periodic solution provided that it is stable and if there is $\delta > 0$ such that $|x_0 - y_0| < \delta$ implies that

$$\bigl|x(t, x_0) - x(t, y_0)\bigr| \to 0 \quad \text{as } t \to \infty.$$

Now we will state the following LaSalle-type stationary oscillation theorem on time scales for system (3.1). Without loss of generality, we can always assume $0 \in \mathbb{T}$ in this paper.

Theorem 3.1 *Consider (Q,T)-affine-periodic system (3.1) and assume that the following hypotheses hold for system (3.1).*

(H1) *There exists a solution $z(t)$ of system (3.1) defined on $[0, T]_{\mathbb{T}}$.*

(H2) *There is an rd-continuous function $a : \mathbb{T}_+ \to \mathbb{R}_+ \setminus \{0\}$ with $r = \overline{\lim}_{k\to\infty} a(kT) < 1$. Any two solutions $x(t)$ and $y(t)$ satisfy*

$$\bigl|x(t) - y(t)\bigr| \le a(t)\bigl|x(0) - y(0)\bigr|$$

whenever they exist.

Then system (3.1) has a unique asymptotically stable (Q,T)-affine-periodic solution.

To prove Theorem 3.1, we need the following lemma.

Lemma 3.1 *Assume that (Q,T)-affine-periodic system (3.1) admits the uniqueness with respect to initial value problems. Assume that $x(t, x_0)$ is a solution of (3.1) with the initial value condition $x(0) = x_0$ defined on $\mathbb{T}_+$. Then*

$$Q^{-k}x\bigl(\sigma(t) + kT, x_0\bigr) = x\bigl(\sigma(t), Q^{-k}x(kT, x_0)\bigr).$$

Proof For $t \in \mathbb{T}_+$,

$$\begin{aligned} x\bigl(\sigma(t) + kT, x_0\bigr) &= x_0 + \int_0^{\sigma(t)+kT} f\bigl(s, x(s, x_0)\bigr)\Delta s \\ &= x_0 + \int_0^{kT} f\bigl(s, x(s, x_0)\bigr)\Delta s + \int_{kT}^{\sigma(t)+kT} f\bigl(s, x(s, x_0)\bigr)\Delta s \end{aligned}$$

$$
\begin{aligned}
&= x(kT,x_0) + \int_0^{\sigma(t)} f\bigl(s+kT, x(s+kT,x_0)\bigr)\Delta s \\
&= x(kT,x_0) + Q\int_0^{\sigma(t)} f\bigl(s+(k-1)T, Q^{-1}x(s+kT,x_0)\bigr)\Delta s \\
&= \cdots \\
&= x(kT,x_0) + Q^k\int_0^{\sigma(t)} f\bigl(s, Q^{-k}x(s+kT,x_0)\bigr)\Delta s.
\end{aligned}
$$

The uniqueness of solutions to the initial value problem implies that

$$
Q^{-k}x\bigl(\sigma(t)+kT,x_0\bigr) = x\bigl(\sigma(t), Q^{-k}x(kT,x_0)\bigr). \tag{3.2}
$$

□

Remark 3.2 If t is right-dense, then

$$
Q^{-k}x(t+kT,x_0) = x\bigl(t, Q^{-k}x(kT,x_0)\bigr).
$$

Now we prove Theorem 3.1.

Proof We divide our proof in three steps.

First, we need to verify that $z(t)$ is extensible on $\mathbb{T}_+$. Once done, according to (H2), it follows that

$$
\bigl|x(t,x_0)-z(t)\bigr| \le a(t)\bigl|x_0-z(0)\bigr| \quad \forall t\in\mathbb{T}_+.
$$

Therefore, all the solutions $x(t,x_0)$ of system (3.1) are defined on $\mathbb{T}_+$. By (H2), $x(t,x_0)$ is uniquely determined by the initial value x_0. By the Peano-type theorem and (H2), system (3.1) has a unique solution $w(t)$ with the initial value condition $w(0) = Q^{-1}z(T)$, and $w(t)$ exists on $[0,T]_{\mathbb{T}}$. Let $u(t) = Qw(t-T)$, by Lemma 2.1:

(a) if t is a right-dense point, then we have

$$
\begin{aligned}
u^{\Delta}(t) &= \lim_{\Delta t\to 0}\frac{u(t+\Delta t)-u(t)}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{Qw(t+\Delta t-T)-Qw(t-T)}{\Delta t} \\
&= Qw^{\Delta}(t-T) \\
&= Qf\bigl(t-T, w(t-T)\bigr) \\
&= Qf\bigl(t-T, Q^{-1}u(t)\bigr) \\
&= f\bigl(t,u(t)\bigr);
\end{aligned}
$$

(b) if t is a right-scattered point, then we have

$$
u^{\Delta}(t) = \frac{u(\sigma(t))-u(t)}{\mu(t)}
$$

$$\begin{aligned}&= \frac{Qw(\sigma(t)-T) - Qw(t-T)}{\mu(t)}\\&= Qw^{\Delta}(t-T)\\&= f\bigl(t,u(t)\bigr).\end{aligned}$$

By (a) and (b), we see that $u(t)$ is the solution of (3.1) with the initial value condition $z(T)$ on $[T,2T]_{\mathbb{T}}$.

Define

$$z(t) = \begin{cases} w(t), & t \in [0,T]_{\mathbb{T}}\\ Qw(t-T), & t \in [T,2T]_{\mathbb{T}}.\end{cases}$$

Thus $z(t)$ exists on $[T,2T]_{\mathbb{T}}$. Repeating this procedure, we know that $z(t)$ exists on $\mathbb{T}_+$.

The next thing is to prove that equation (3.1) has (Q,T)-affine-periodic solutions. We define the Poincaré map $P(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P(x_0) = Q^{-1}x(T,x_0) \quad \forall x_0 \in \mathbb{R}^n,$$

and P is well-defined. Now we prove P is a contraction mapping.

Note that

$$\begin{aligned}x\bigl(\sigma(t)+T,x_0\bigr) &= x_0 + \int_0^{\sigma(t)+T} f\bigl(s,x(s,x_0)\bigr)\Delta s\\&= x(T,x_0) + \int_T^{\sigma(t)+T} f\bigl(s,x(s,x_0)\bigr)\Delta s\\&= x(T,x_0) + \int_0^{\sigma(t)} f\bigl(s+T,x(s+T,x_0)\bigr)\Delta s.\end{aligned}$$

Therefore,

$$Q^{-1}x\bigl(\sigma(t)+T,x_0\bigr) = Q^{-1}x(T,x_0) + \int_0^{\sigma(t)} f\bigl(s,Q^{-1}x(s+T,x_0)\bigr)\Delta s. \tag{3.3}$$

By Lemma 3.1, we obtain

$$x\bigl(\sigma(t),Q^{-1}x(T,x_0)\bigr) = Q^{-1}x(T,x_0) + \int_0^{\sigma(t)} f\bigl(s,x\bigl(s,Q^{-1}x(T,x_0)\bigr)\bigr)\Delta s. \tag{3.4}$$

From (3.3) and (3.4), we can see

$$Q^{-2}x\bigl(\sigma(t)+T,x_0\bigr) = Q^{-1}x\bigl(\sigma(t),Q^{-1}x(T,x_0)\bigr).$$

By induction and the uniqueness of solutions to the initial value problem, we have

$$P^j(x_0) = P \circ P^{j-1}(x_0) = Q^{-j}x(jT,x_0) \quad \forall j \ge 1.$$

By the definition of r, there exist positive integer K and $\tilde{r} \in (r,1)$ such that

$$a(kT) \le \tilde{r} \quad \forall k \ge K.$$

Thus,

$$\begin{aligned}
\left|P^K(x_0) - P^K(y_0)\right| &= \left|Q^{-K}x(KT, x_0) - Q^{-K}x(KT, y_0)\right| \\
&= \left|x(KT, x_0) - x(KT, y_0)\right| \\
&\leq \tilde{r}|x_0 - y_0| \quad \forall x_0, y_0 \in \mathbb{R}^n.
\end{aligned} \tag{3.5}$$

It follows that P is an $\tilde{r}$-contraction mapping of order K, then by Banach's contraction mapping fixed point theorem, P has a unique fixed point $x_* \in \mathbb{R}^n$, that is,

$$Q^{-1}x(T, x_*) = x_*.$$

Again by the uniqueness of solutions to the initial value problem, we have

$$x\bigl(\sigma(t) + T, x_*\bigr) = Qx\bigl(\sigma(t), x_*\bigr) \quad \forall t \in \mathbb{T}_+.$$

Therefore, system (3.1) has a unique (Q, T)-affine-periodic solution.

Finally, we show the stability and the asymptotic stability. We need to prove the following estimate inductively:

$$\left|x\bigl(\sigma(t) + mKT, x_*\bigr) - x\bigl(\sigma(t) + mKT, x_0\bigr)\right| \leq a(t)|x_* - x_0|\tilde{r}^m \quad \forall t \in \mathbb{T}_+, m \geq 1. \tag{3.6}$$

The frame of the proof is the same as the one of Theorem 2.1 in [3] and so is proved briefly. Let $t \in \mathbb{T}_+$ be right-scattered. Firstly, for $m = 1$, by Lemma 3.1 and assumption (H2), we have

$$\begin{aligned}
&\left|x\bigl(\sigma(t) + KT, x_*\bigr) - x\bigl(\sigma(t) + KT, x_0\bigr)\right| \\
&\quad = \left|Q^{-1}x\bigl(\sigma(t) + KT, x_*\bigr) - Q^{-1}x\bigl(\sigma(t) + KT, x_0\bigr)\right| \\
&\quad = \left|x\bigl(\sigma(t) + (K-1)T, Q^{-1}x(T, x_*)\bigr) - x\bigl(\sigma(t) + (K-1)T, Q^{-1}x(T, x_0)\bigr)\right| \\
&\quad = \dots \\
&\quad = \left|x\bigl(\sigma(t), Q^{-K}x(KT, x_*)\bigr) - x\bigl(\sigma(t), Q^{-K}x(KT, x_0)\bigr)\right| \\
&\quad \leq a(t)\left|x(KT, x_*) - x(KT, x_0)\right| \\
&\quad \leq a(t)a(KT)|x_* - x_0| \\
&\quad \leq a(t)|x_* - x_0|\tilde{r}.
\end{aligned}$$

Secondly, we claim that, for $k = m - 1$, the assumption holds. Then, for $k = m$, according to Lemma 3.1 and (ii), we have

$$\begin{aligned}
&\left|x\bigl(\sigma(t) + mKT, x_*\bigr) - x\bigl(\sigma(t) + mKT, x_0\bigr)\right| \\
&\quad = \left|Q^{-mK}x\bigl(\sigma(t) + mKT, x_*\bigr) - Q^{-mK}x\bigl(\sigma(t) + mKT, x_0\bigr)\right| \\
&\quad = \left|x\bigl(\sigma(t), Q^{-mK}x(mKT, x_*)\bigr) - x\bigl(\sigma(t), Q^{-mK}(mKT, x_0)\bigr)\right| \\
&\quad \leq a(t)\left|x(mKT, x_*) - x(mKT, x_0)\right|
\end{aligned}$$

$$
\begin{aligned}
&= a(t)\big|Q^{-(m-1)K}x\big(KT, Q^{-(m-1)K}x\big((m-1)KT, x_*\big)\big) \\
&\quad - Q^{-(m-1)K}x\big(KT, Q^{-(m-1)K}x\big((m-1)KT, x_*\big)\big)\big| \\
&= a(t)\big|x\big(KT, Q^{-(m-1)K}x\big((m-1)KT, x_*\big)\big) \\
&\quad - x\big(KT, Q^{-(m-1)K}x\big((m-1)KT, x_*\big)\big)\big| \\
&\leq a(t)a(KT)\big|x\big((m-1)KT, x_*\big) - x\big((m-1)KT, x_0\big)\big| \\
&\leq a(t)a(KT)|x_* - x_0|\tilde{r}^{m-1} \\
&\leq a(t)|x_* - x_0|\tilde{r}^{m}.
\end{aligned}
$$

If t is a right-dense point, then we have

$$
\big|x(t + mKT, x_*) - x(t + mKT, x_0)\big| \leq a(t)|x_* - x_0|\tilde{r}^m \quad \forall t \in \mathbb{T}_+, m \geq 1.
$$

At last, we take $t \in [0, T]_{\mathbb{T}}$. By the arbitrariness of m and $\tilde{r} \in (0, 1)$, we can obtain the stability and the asymptotic stability by inequality (3.6), which completes the proof of Theorem 3.1. □

4 Applications

In this section, we give Lyapunov's method to discuss the existence and asymptotic stability of (Q, T)-affine-periodic solutions of (3.1).

Theorem 4.1 *Let* (H1) *hold. Consider the* (Q, T)*-affine-periodic system* (3.1). *Moreover, suppose that there exists a Lyapunov function* $V : \mathbb{T}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ *satisfying the following hypotheses*:

(H3) $V(\sigma(t), x, y)$ *is a* C^1_{rd} *function with*

$$
\begin{aligned}
&V\big(\sigma(t), x, y\big) = V\big(\sigma(t), Q^{\pm 1}x, Q^{\pm 1}y\big), \\
&V\big(\sigma(t), x, y\big) = V\big(\sigma(t) + kT, x, y\big) \quad \forall k \in \mathbb{N}, \\
&a\big(|x - y|\big) \leq V\big(\sigma(t), x, y\big) \leq b\big(|x - y|\big),
\end{aligned}
$$

where $a, b : \mathbb{R}^n \to \mathbb{R}^n$ *are strictly increasing continuous functions with* $a(0) = b(0) = 0$, $a(\infty) = b(\infty) = \infty$ *and constant* C *satisfies that, for some* $\beta \in (0, 1)$,

$$
\inf_{r>0} \frac{a(\beta r)}{Cb(r)} > 0. \tag{4.1}
$$

(H4) *For any two solutions* $x(t)$ *and* $y(t)$ *of* (3.1),

$$
V^{\Delta}\big(\sigma(t), x, y\big)|_{(x,y)=(x(t),y(t))} \leq \alpha V\big(\sigma(t), x, y\big)|_{(x,y)=(x(t),y(t))} \quad \forall t \in \mathbb{T}_+, \tag{4.2}
$$

where $\alpha : \mathbb{T}_+ \to \mathbb{R}^1$ *is locally* Δ*-integrable and satisfies*

$$
\int_0^{kT} \xi_{\mu(\tau)}\big(\alpha(\tau)\big)\Delta\tau \leq -\eta \quad (\eta > 0)
$$

for any integer $k \geq k_0 > 0$.

Then system (3.1) *admits a unique asymptotically stable* (Q, T)*-affine-periodic solution.*

Proof Firstly, we prove that $z(\sigma(t))$ is extensible on $\mathbb{T}_+$. Let $x(t) = z(\sigma(t))$, $y(t) = Q^{-1}z(\sigma(t) + T)$. If $x(t) \equiv y(t)$, $z(\sigma(t))$ is a (Q, T)-affine-periodic solution of (1.1), then $z(\sigma(t))$ exists on $\mathbb{T}$. If $x(t) \neq y(t)$. According to (H3), (H4), and Lemma 2.2, we have

$$\begin{aligned} a\bigl(\bigl|x(t) - y(t)\bigr|\bigr) &\le V\bigl(\sigma(t), x(t), y(t)\bigr) \\ &\le V\bigl(0, x(0), y(0)\bigr)e_\alpha\bigl(\sigma(t), 0\bigr) \\ &\le b\bigl(\bigl|x(0) - y(0)\bigr|\bigr)e_\alpha\bigl(\sigma(t), 0\bigr) \quad \forall t \in [0, T]_{\mathbb{T}}, \end{aligned}$$

which shows that

$$\bigl|x(t) - y(t)\bigr| \le a^{-1}\bigl(b\bigl(\bigl|x(0) - y(0)\bigr|\bigr)e_\alpha\bigl(\sigma(t), 0\bigr)\bigr) \quad \forall t \in [0, T]_{\mathbb{T}}.$$

In the same way as Step 1 in the proof of Theorem 3.1, we can obtain that $z(\sigma(t))$ exists on $\mathbb{T}_+$. Repeating the above arguments, we can prove that all the solutions of (3.1) exist on $\mathbb{T}_+$.

Next we show that the (Q, T)-affine-periodic solution of (3.1) is asymptotically stable. By Lemma 3.1, definition of the exponential function on time scales, and the properties of V, similar to the proof of Theorem 3.1, for any $m \ge 1$, $t \in \mathbb{T}_+$ and $x_0, y_0 \in \mathbb{R}^n$, we have

$$\begin{aligned} &V\bigl(\sigma(t) + mKT, x(t + mKT, y_0), x(t + mKT, x_0)\bigr) \\ &\quad = V\bigl(\sigma(t) + mKT, Q^{-1}x(t + mKT, y_0), Q^{-1}x(t + mKT, x_0)\bigr) \\ &\quad = V\bigl(\sigma(t) + mKT, Q^{-1}x\bigl(t + (m-1)KT, Q^{-1}x(T, y_0)\bigr), \\ &\qquad Q^{-1}x\bigl(t + (m-1)KT, Q^{-1}x(T, x_0)\bigr)\bigr) \\ &\quad = \cdots \\ &\quad = V\bigl(\sigma(t) + mKT, x\bigl(t, Q^{-mK}x(mKT, y_0)\bigr), x\bigl(t, Q^{-mK}x(mKT, x_0)\bigr)\bigr) \\ &\quad \le e_\alpha\bigl(\sigma(t), 0\bigr)V\bigl(0, x(mKT, y_0), x(mKT, x_0)\bigr) \\ &\quad = e_\alpha\bigl(\sigma(t), 0\bigr)V\bigl(KT, x(mKT, y_0), x(mKT, x_0)\bigr) \\ &\quad = \exp\biggl(\int_0^{\sigma(t)} \xi_{\mu(\tau)}\bigl(\alpha(\tau)\bigr)\Delta\tau\biggr) \cdot V\bigl(KT, Q^{-(m-1)K}x\bigl(KT, \\ &\qquad Q^{-(m-1)K}x\bigl((m-1)KT, y_0\bigr)\bigr), Q^{-(m-1)K}x\bigl(KT, Q^{-(m-1)K}x\bigl((m-1)KT, x_0\bigr)\bigr)\bigr) \\ &\quad \le \exp\biggl(\int_0^{\sigma(t)} \xi_{\mu(\tau)}\bigl(\alpha(\tau)\bigr)\Delta(\tau)\biggr) \cdot \biggl\{\exp\biggl(\int_0^{KT} \xi_{\mu(\tau)}\bigl(\alpha(\tau)\bigr)\Delta\tau\biggr)\biggr\}^m \\ &\qquad \cdot V\bigl(0, x\bigl((m-1)KT, y_0\bigr), x\bigl((m-1)KT, x_0\bigr)\bigr) \\ &\quad \le \cdots \\ &\quad \le V(0, y_0, x_0)\exp\biggl(\int_0^{\sigma(t)} \xi_\mu(\tau)\bigl(\alpha(\tau)\bigr)\Delta\tau\biggr) \cdot \biggl\{\exp\biggl(\int_0^{KT} \xi_{\mu(\tau)}\bigl(\alpha(\tau)\bigr)\Delta\tau\biggr)\biggr\}^m \\ &\quad \le b\bigl(|y_0 - x_0|\bigr)\exp\biggl(\int_0^{\sigma(t)} \xi_{\mu(\tau)}\bigl(\alpha(\tau)\bigr)\Delta\tau\biggr) \cdot \biggl\{\exp\biggl(\int_0^{KT} \xi_{\mu(\tau)}\bigl(\alpha(\tau)\bigr)\Delta\tau\biggr)\biggr\}^m. \end{aligned}$$

By (H4), there is a constant C such that

$$\exp\left(\int_0^{\sigma(t)} \xi_{\mu(\tau)}\big(\alpha(\tau)\big)\Delta\tau\right) \le C,$$

then

$$V(\sigma(t)+mKT, x(t+mKT,y_0), x(t+mKT,x_0)) \le C\big(e^{-\eta}\big)^m b(y_0-x_0).$$

That is, fixing integers $K \ge k_0$ and $m = N \in \mathbb{N}^+$ such that

$$\big(e^{-\eta}\big)^N \le \inf_{r>0} \frac{a(\beta r)}{Cb(r)},$$

then

$$V\big(\sigma(t)+NKT, x(t+NKT,y_0), x(t+NKT,x_0)\big) \le \inf_{r>0} \frac{a(\beta r)}{b(r)} b\big(|y_0-x_0|\big). \tag{4.3}$$

Next, by the definition of P in Theorem 3.1, we show that

$$P^{NK}(x_0) = Q^{-(NK)}x(NKT,x_0)$$

is a contraction map. According to the assumptions and (4.3), we derive that

$$\begin{aligned}\big|P^{NK}(y_0)-P^{NK}(x_0)\big| &= \big|Q^{-(NK)}x(NKT,y_0)-Q^{-(NK)}x(NKT,x_0)\big|\\ &= \big|x(NKT,y_0)-x(NKT,x_0)\big|\\ &\le \beta|y_0-x_0| \quad \forall y_0,x_0\in\mathbb{R}^n,\end{aligned}$$

which implies P^{NK} is a contraction map. Thus, we can get desired stability and asymptotic stability by the above inequality. This completes the proof. □

Remark 4.1 Li et al. in [35] studied the existence of affine periodic solutions for affine periodic (functional) differential systems. We prove analogously a result for dynamic equations on time scales. Comparing with their results, condition (H4) in Theorem 4.1 is a little bit different from the conditions of Theorem 4.2 in [35]. In fact, the condition with respect to function $\alpha(t)$ in Theorem 4.1 is more complex than the one in continuous systems. Choosing $\mathbb{T} = \mathbb{R}$ in Theorem 4.1, we obtain the continuous result in [35].

Example 4.1 Let $\mathbb{T}$ be a 1-periodic time scale. Consider the following system:

$$x^{\Delta} = -a(t)x + (\sin t, \cos t, \sin 2\pi t, \cos 2\pi t)^{\top}, \tag{4.4}$$

where $a : \mathbb{T} \to \mathbb{R}$ is rd-continuous satisfying $a(\sigma(t)+1) = a(\sigma(t))$ and

$$\int_0^{kT} \xi_{\mu(\tau)}\big(-2a(\tau)+\mu(\tau)a^2(\tau)\big)\Delta\tau \le -\eta \quad (\eta > 0). \tag{4.5}$$

Set

$$Q = \begin{pmatrix} \cos 1 & \sin 1 & 0 & 0 \\ -\sin 1 & \cos 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.6}$$

Let

$$f(t,x) = -a(t)x + (\sin t, \cos t, \sin 2\pi t, \cos 2\pi t)^\top.$$

Then

$$f\bigl(\sigma(t)+T, x\bigr) = Qf\bigl(\sigma(t), Q^{-1}x\bigr).$$

Thus, (4.4) is a $(Q,1)$-affine-periodic system on time scales. Choose a Lyapunov function

$$V(x) = \frac{1}{2}|x|^2.$$

It is easy to verify that condition (H3) holds for (4.4).

By Lemma 2.3, for any two solutions $x(\sigma(t))$ and $y(\sigma(t))$ of system (4.4), we have

$$\begin{aligned}
&V^\Delta\bigl(x\bigl(\sigma(t)\bigr) - y\bigl(\sigma(t)\bigr)\bigr) \\
&\quad = \sum_{i=1}^{4} \int_0^1 \biggl\langle \frac{\partial V}{\partial x_i}\bigl(x\bigl(\sigma(t)\bigr) - y\bigl(\sigma(t)\bigr) + h\mu(t)\bigl(x^\Delta\bigl(\sigma(t)\bigr) - y^\Delta\bigl(\sigma(t)\bigr)\bigr)\bigr)\,dh, \\
&\qquad x_i^\Delta\bigl(\sigma(t)\bigr) - y_i^\Delta\bigl(\sigma(t)\bigr)\biggr\rangle \\
&\quad = \sum_{i=1}^{4} \biggl\langle \int_0^1 \bigl(x_i\bigl(\sigma(t)\bigr) - y_i\bigl(\sigma(t)\bigr) + h\mu(t)\bigl(x_i^\Delta\bigl(\sigma(t)\bigr) - y_i^\Delta\bigl(\sigma(t)\bigr)\bigr)\bigr)\,dh, \\
&\qquad x_i^\Delta\bigl(\sigma(t)\bigr) - y_i^\Delta\bigl(\sigma(t)\bigr)\biggr\rangle \\
&\quad = \sum_{i=1}^{4} \biggl\langle x_i\bigl(\sigma(t)\bigr) - y_i\bigl(\sigma(t)\bigr) + \frac{1}{2}\mu(t)\bigl(x_i^\Delta\bigl(\sigma(t)\bigr) - y_i^\Delta\bigl(\sigma(t)\bigr)\bigr), x_i^\Delta\bigl(\sigma(t)\bigr) - y_i^\Delta\bigl(\sigma(t)\bigr)\biggr\rangle \\
&\quad = \sum_{i=1}^{4} \bigl\langle x_i\bigl(\sigma(t)\bigr) - y_i\bigl(\sigma(t)\bigr), x_i^\Delta\bigl(\sigma(t)\bigr) - y_i^\Delta\bigl(\sigma(t)\bigr)\bigr\rangle \\
&\qquad + \frac{1}{2}\mu(t)\sum_{i=1}^{4}\bigl(x_i^\Delta\bigl(\sigma(t)\bigr) - y_i^\Delta\bigl(\sigma(t)\bigr)\bigr)^2 \\
&\quad = \sum_{i=1}^{4} \bigl\langle x_i\bigl(\sigma(t)\bigr) - y_i\bigl(\sigma(t)\bigr), -a(t)\bigl(x_i\bigl(\sigma(t)\bigr) - y_i\bigl(\sigma(t)\bigr)\bigr)\bigr\rangle \\
&\qquad + \frac{1}{2}\mu(t)a^2(t)\sum_{i=1}^{4}\bigl(x_i\bigl(\sigma(t)\bigr) - y_i\bigl(\sigma(t)\bigr)\bigr)^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{4}\left(-a(t) + \frac{1}{2}\mu(t)a^2(t)\right)\bigl(x_i(\sigma(t)) - y_i(\sigma(t))\bigr)^2 \\
&= \bigl(-2a(t) + \mu(t)a^2(t)\bigr)V\bigl(x(\sigma(t)) - y(\sigma(t))\bigr).
\end{aligned}
$$

Let $\alpha(t) = -2a(t) + \mu(t)a^2(t)$. By Gronwall's inequality on time scales, we obtain

$$
\begin{aligned}
&V\bigl(x(\sigma(t)) - y(\sigma(t))\bigr) \\
&\quad\le e_\alpha\bigl(\sigma(t),0\bigr)V\bigl(x(0) - y(0)\bigr) \\
&\quad= \exp\left(\int_0^{\sigma(t)} \xi_{\mu(\tau)}\bigl(\alpha(\tau)\bigr)\Delta\tau\right)V\bigl(x(0) - y(0)\bigr) \\
&\quad= \exp\left(\int_0^{\sigma(t)} \xi_{\mu(\tau)}\bigl(-2a(\tau) + \mu(\tau)a^2(\tau)\bigr)\Delta\tau\right)V\bigl(x(0) - y(0)\bigr).
\end{aligned}
$$

By (4.5), we have

$$
\int_0^{kT} \xi_{\mu(\tau)}\bigl(-2a(\tau) + \mu(\tau)a^2(\tau)\bigr)\Delta\tau \le -\eta \quad (\eta > 0).
$$

This means that condition (H4) is fulfilled. Hence, (4.4) has a unique asymptotically stable (Q,T)-affine-periodic solution by Theorem 4.1.

5 Conclusion

The main goal of this paper is to discuss the existence of affine periodic solutions and asymptotic stability of affine periodic systems on time scales by LaSalle-type stationary oscillation principle. As applications, we present the existence and asymptotic stability of affine periodic solutions on time scales via Lyapunov's method. Comparing with continuous systems, some technical details on time scales are more complicated.

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Author details
[1]School of Applied Mathematics, Jilin University of Finance and Economics, Changchun, P.R. China. [2]College of Mathematics, Jilin University, Changchun, P.R. China. [3]College of Mathematics, Jilin Normal University, Siping, China.

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