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The nonlocal inverse problem of the identification of the lowest coefficient and the right-hand side in a second-order parabolic equation with integral conditions

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Abstract

In the present paper a nonlocal inverse boundary-value problem for a second-order parabolic equation is considered. This investigation introduces the identification of the lowest unknown coefficient and time-depend right-hand side in a second-order parabolic equation on overdetermination at the internal points. Sufficient conditions for the existence and uniqueness of the classical solution to inverse problem of a second-order parabolic equation are obtained for small time.

MSC: 35A02; 35A09; 35R30; 35K10

Keywords: Inverse problem; Parabolic equation; Overdetermination conditions; Classical solution

1 Introduction

In this article we study the unique solvability of the inverse problem of determining the triple of functions $\{u(x,t), a(t), b(t)\}$ satisfying the equation

$$c(t)u_t(x,t) = u_{xx}(x,t) + a(t)u(x,t) + b(t)g(x,t) + f(x,t), \tag{1.1}$$

with the nonlocal initial condition

$$u(x,0) + \delta u(x,T) + \int_0^T p(t)u(x,t)\,dt = \varphi(x) \quad (0 \le x \le 1), \tag{1.2}$$

periodic boundary condition

$$u(0,t) = \beta u(1,t) \quad (0 \le t \le T), \tag{1.3}$$

nonlocal integral condition

$$\int_0^1 u(x,t)\,dx = 0 \quad (0 \le t \le T), \tag{1.4}$$

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and the overdetermination conditions

$$u(x_i,t) = h_i(t) \quad (i = 1,2; 0 \le t \le T). \tag{1.5}$$

Here $D_T := \{(x,t) : 0 < x < 1, 0 < t \le T\}$ is a rectangular domain, $T > 0$, $\beta, \delta \ge 0$, $x_i \in (0,1)$ $(i = 1,2; x_1 \ne x_2)$ are fixed numbers, $0 < c(t)$, $g(x,t)$, $f(x,t)$, $0 \le p(t)$, $\varphi(x)$, $h_i(t)$ $(i = 1,2)$ are given functions, and $u(x,t)$, $a(t)$, $b(t)$ are the desired functions.

Recently, in theory of partial differential equations, investigations devoted to the inverse coefficient problems and problems with discontinuous coefficients took an important place. Inverse problems arise in situations, when the structure of the mathematical model of the studying process is known and it is necessary to set the problems of determining the parameters of the mathematical model itself. Such problems include the problems of determining the various coefficients of the equations, external influence, boundary conditions, initial conditions, and so on. Many important applied problems lead to inverse problems.

The theory of inverse problems, by virtue of its theoretical and applied importance, is one of the intensively developing sections of the contemporary theory of partial differential equations. It attracts the attention of many researchers, who are interested in both the theory itself and its applications.

Fundamentals of the theory and practice of research of inverse problems were established and developed in the works published by Tikhonov [1], Lavrent'ev [2], Isakov et al. [3].

A more detailed bibliography and a classification of recent works connected with the investigation of inverse problems for partial differential equations can be found in monographs and in articles [4–13].

Problems of the solvability of inverse problems for equations of parabolic type were considered in the papers by Ivanchov [14, 15], Kozhanov [16], Vasin [17], Pyatkov [18], Kabanikhin [19], Ismailov [20–23], and many others. Regarding recent results on the theory of partial differential equation with discontinuous coefficients, we refer the reader to [24–26]. But the statement of problems and the proof technique used in this study are different from representations in these papers.

Nonlocal boundary-value problems are usually called problems in which, by together specifying the values of the solution or its derivatives on a fixed part of the boundary, a relationship is established between these values and the values of the same functions on other internal or boundary manifolds. The theory of nonlocal boundary value problems is important as a section of the general theory of boundary value problems for partial differential equations, it is also important as a branch of the theory of inverse problems.

Definition 1.1 The triplet $\{u(x,t), a(t), b(t)\}$ is said to be a classical solution of problem (1.1)–(1.5), if the functions $u(x,t)$, $a(t)$, and $b(t)$ satisfy the following conditions:

1. The function $u(x,t)$ and its derivatives $u_t(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, are continuous in the domain D_T;
2. The functions $a(t)$ and $b(t)$ are continuous on the interval $[0,T]$;
3. Eq. (1.1) and conditions (1.2)–(1.5) are satisfied in the classical (usual) sense.

Theorem 1.2 ([27]) *Suppose that the following conditions are satisfied*: $\delta \ge 0$, $0 < c(t) \in C[0,T]$, $0 \le p(t) \in C[0,T]$, $f(x,t) \in C(D_T)$, $\varphi(x) \in C[0,1]$, $\int_0^1 f(x,t)\,dx = 0$ $(0 \le t \le T)$,

$g(x,t) \in C(D_T)$, $\int_0^1 g(x,t)\,dx = 0$ $(0 \le t \le T)$, $h_i(t) \in C^1[0,T]$, $(i=1,2)$, $h(t) \equiv h_1(t)g(x_2,t) - h_2(t)g(x_1,t) \ne 0$ $(0 \le t \le T)$, *and the compatibility conditions*

$$\int_0^1 \varphi(x)\,dx = 0,$$

$$h_i(0) + \delta h_i(T) + \int_0^T p(t)h_i(t)\,dt = \varphi(x_i) \quad (i=1,2).$$

Then, the problem of finding a classical solution of (1.1)–(1.5) *is equivalent to the problem of determining functions* $u(x,t) \in C^{2,1}(D_T)$, $a(t) \in C[0,T]$, *and* $b(t) \in C[0,T]$, *satisfying Eq.* (1.1), *conditions* (1.2) *and* (1.3), *and the conditions*

$$u_x(0,t) = u_x(1,t) \quad (0 \le t \le T), \tag{1.6}$$

$$c(t)h_i'(t) = u_{xx}(x_i,t) + a(t)h_i(t) + b(t)g(x_i,t) + f(x_i,t) \tag{1.7}$$

for $i = 1,2$; $0 \le t \le T$.

Notice that, in the case of $\beta = 1$, the considered problem was investigated in [27]. In this article we assume that $\beta \ne \pm 1$. In particular, for $\beta = 0$, we have an Ionkin type boundary condition [28].

2 The auxiliary spectral problem

Now, in order to investigate problem (1.1)–(1.3) and (1.6), we cite some known facts. Consider the following spectral problem [28, 29]:

$$X''(x) + \lambda X(x) = 0 \quad (0 \le x \le 1), \tag{2.1}$$

$$X(0) = \beta X(1), \qquad X'(0) = X'(1) \quad (\beta \ne \pm 1). \tag{2.2}$$

Obviously, the boundary value problem (2.1)–(2.2) is not self-adjoint and the problem

$$Y''(x) + \lambda Y(x) = 0 \quad (0 \le x \le 1), \tag{2.3}$$

$$Y(0) = Y(1), \qquad Y'(1) = \beta Y'(0) \tag{2.4}$$

will be a conjugated problem.

We denote the system of eigen- and adjoint functions of problem (2.3)–(2.4) in the following [29] as:

$$\begin{aligned} &X_0(x) = ax + b, \qquad \ldots, \qquad X_{2k-1}(x) = (ax+b)\cos\lambda_k x, \\ &X_{2k}(x) = \sin\lambda_k x, \qquad \ldots, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &Y_0(x) = 2, \qquad \ldots, \qquad Y_{2k-1}(x) = 4\cos\lambda_k x, \\ &Y_{2k}(x) = 4(1-b-ax)\sin\lambda_k x, \qquad \ldots, \end{aligned} \tag{2.6}$$

where

$$\lambda_k = 2k\pi \quad (k = 0,1,2,\ldots), \qquad a = (1-\beta)/(1+\beta) \ne 0, \qquad b = \beta/(1+\beta). \tag{2.7}$$

It is straightforward to verify that the biorthogonality conditions

$$(X_i, Y_j) = \int_0^1 X_i(x)Y_j(x)\,dx = \delta_{ij}$$

are fulfilled.

Here, δ_{ij} is Kronecker's symbol.

Theorem 2.1 ([30]) *The system of functions* (2.5) *forms a Riesz basis in the space* $L_2(0,1)$, *and the estimates*

$$r\|\psi(x)\|_{L_2(0,1)} \le \sum_{k=0}^{\infty} \psi_k^2 \le R\|\psi(x)\|_{L_2(0,1)} \tag{2.8}$$

are true for any function $\psi(x) \in L_2(0,1)$, *where*

$$\psi_k = \bigl(\psi(x), Y_k(x)\bigr) = \int_0^1 \psi(x)Y_k(x)\,dx \quad (k = 0,1,\ldots),$$

$$r = \left\{\frac{1}{3}\left(\left(a+\frac{3}{2}b\right)^2 + \frac{3}{4}b^2\right) + \frac{1}{2}\bigl(1 + \|(ax+b)^2\|_{C[0,1]}\bigr)\right\}^{-1},$$

$$R = 8\bigl(1 + \|(1-b-ax)^2\|_{C[0,1]}\bigr).$$

Under the assumptions

$$\psi(x) \in C^{2i-1}[0,1], \qquad \psi^{(2i)}(x) \in L_2(0,1),$$

$$\psi^{(2s)}(0) = \beta\psi^{(2s)}(1), \qquad \psi^{(2s+1)}(0) = \psi^{(2s+1)}(1) \quad (s = \overline{0,i-1}; i \ge 1),$$

we establish the validity of the estimates:

$$\left(\sum_{k=1}^{\infty}\bigl(\lambda_k^{2i}\psi_{2k-1}\bigr)^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\|\psi^{(2i)}(x)\|_{L_2(0,1)}, \tag{2.9}$$

$$\left(\sum_{k=1}^{\infty}\bigl(\lambda_k^{2i}\psi_{2k}\bigr)^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\|\psi^{(2i)}(x)(1-b-ax) - 2ai\psi^{(2i-1)}(x)\|_{L_2(0,1)}. \tag{2.10}$$

Further, under the assumptions

$$\psi(x) \in C^{2i}[0,1], \qquad \psi^{(2i+1)}(x) \in L_2(0,1),$$

$$\psi^{(2s)}(0) = \beta\psi^{(2s)}(1), \qquad \psi^{(2s-1)}(0) = \psi^{(2s-1)}(1) \quad (i \ge 1; s = \overline{0,i}),$$

we obtain

$$\left(\sum_{k=1}^{\infty}\bigl(\lambda_k^{2i+1}\psi_{2k-1}\bigr)^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\|\psi^{(2i+1)}(x)\|_{L_2(0,1)}, \tag{2.11}$$

$$\left(\sum_{k=1}^{\infty}\bigl(\lambda_k^{2i+1}\psi_{2k}\bigr)^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\|\psi^{(2i+1)}(x)(1-b-ax) - a(2i+1)\psi^{(2i)}(x)\|_{L_2(0,1)}. \tag{2.12}$$

Now, denote by $B^{\alpha}_{2,T}$ [30] the space consisting of functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$$

in domain D_T, where the functions $u_k(t)$ $(k=0,1,2,\ldots)$ are continuous on the interval $[0,T]$, and satisfy the condition

$$J(u)\equiv\|u_0(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}<+\infty$$

whenever $\alpha\geq 0$. The norm in the space $B^{\alpha}_{2,T}$ is

$$\|u(x,t)\|_{B^{\alpha}_{2,T}}=J(u).$$

In particular, when $\alpha=3$, the function $u(x,t)$, as an element of the space $B^3_{2,T}$, has the following properties:

$$u(x,t),u_x(x,t),u_{xx}(x,t)\in C(D_T),\qquad u_{xxx}(x,t)\in C\big([0,T];L_2(0,1)\big);$$
$$u(0,t)=\beta u(1,t),\qquad u_x(0,t)=u_x(1,t),\qquad u_{xx}(0,t)=\beta u_{xx}(1,t)\quad(0\leq t\leq T).$$

We denote by E^{α}_T the Banach space $B^{\alpha}_{2,T}\times C[0,T]\times C[0,T]$ of vector functions $z(x,t)=\{u(x,t),a(t),b(t)\}$ with the norm

$$\|z(x,t)\|_{E^{\alpha}_{2,T}}=\|u(x,t)\|_{B^{\alpha}_{2,T}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

It is known that $B^{\alpha}_{2,T}$ and E^{α}_T are Banach spaces.

3 Existence and uniqueness of the solution of the inverse problem

Since system (2.5) forms a Riesz basis in $L_2(0,1)$ and systems (2.5)–(2.6) form a system of biorthogonal functions in $L_2(0,1)$, we'll seek the first component $u(x,t)$ of classical solution $\{u(x,t),a(t),b(t)\}$ of problem (1.1)–(1.3), (1.6), and (1.7) in the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x),\tag{3.1}$$

where

$$u_k(t)=\int_0^1 u(x,t)Y_k(x)\,dx\quad(k=0,1,\ldots).\tag{3.2}$$

Moreover, $X_k(x)$ and $Y_k(x)$ are defined by relations (2.5) and (2.6), respectively.

Then by applying the method of separation of variables, from (1.1) and (1.2) we have

$$c(t)u_0'(t)=F_0(t;u,a,b)\quad(0\leq t\leq T),\tag{3.3}$$

$$c(t)u_{2k-1}'(t)+\lambda_k^2u_{2k-1}(t)=F_{2k-1}(t;u,a,b)\quad(k=1,2\ldots;0\leq t\leq T),\tag{3.4}$$

$$c(t)u'_{2k}(t) + \lambda_k^2 u_{2k}(t) = F_{2k}(t;u,a,b) - 2a\lambda_k u_{2k-1}(t) \quad (k = 1,2\ldots; 0 \le t \le T), \tag{3.5}$$

$$u_k(0) + \delta u_k(T) + \int_0^T p(t)u_k(t)\,dt = \varphi_k \quad (k = 0,1,2,\ldots), \tag{3.6}$$

where

$$\lambda_k = 2k\pi \quad (k = 1,2,\ldots),$$

$$F_k(t;u,a,b) = f_k(t) + b(t)g_k(t) + a(t)u_k(t), \quad (k = 0,1,2,\ldots),$$

$$f_k(t) = \int_0^1 f(x,t)Y_k(x)\,dx, \qquad g_k(t) = \int_0^1 g(x,t)Y_k(x)\,dx,$$

$$\varphi_k = \int_0^1 \varphi(x)Y_k(x)\,dx \quad (k = 0,1,\ldots).$$

Solving problem (3.3)–(3.6), we obtain

$$u_0(t) = (1+\delta)^{-1}\left(\varphi_0 - \int_0^T p(t)u_0(t)\,dt - \delta\int_0^T \frac{1}{c(t)}F_{10}(t;u,a,b)\,dt\right) + \int_0^t \frac{1}{c(\tau)}F_0(\tau;u,a,b)\,d\tau, \tag{3.7}$$

$$\begin{aligned} u_{2k-1}(t) = {} & \frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left(\varphi_{2k-1} - \int_0^T p(t)u_{2k-1}(t)\,dt\right) \\ & + \int_0^t \frac{1}{c(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}\,d\tau \\ & - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{1}{c(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}\,d\tau, \quad (k = 1,2,\ldots), \end{aligned} \tag{3.8}$$

$$\begin{aligned} u_{2k}(t) = {} & \frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left(\varphi_{2k} - \int_0^T p(t)u_{2k}(t)\,dt\right) \\ & + \int_0^t \frac{1}{c(\tau)}F_{2k}(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}\,d\tau \\ & - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{1}{c(\tau)}F_{2k}(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}\,d\tau + q_k(t,T), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} q_k(t,T) = {} & -2a\lambda_k\left(\varphi_{2k-1} - \int_0^T p(t)u_{2k-1}(t)\,dt\right) \\ & \times \frac{\delta e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left[\int_0^t \frac{d\tau}{c(\tau)} - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{d\tau}{c(\tau)}\right] \\ & - 2a\lambda_k\left[\int_0^t \frac{1}{c(\tau)}\left(\int_0^\tau \frac{1}{c(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int_\xi^t \frac{\lambda_k^2}{c(s)}ds}\,d\tau\right)d\xi\right. \end{aligned}$$

$$- \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}} \int_0^T \frac{1}{c(\tau)}\left(\int_0^\tau \frac{1}{c(\xi)} F_{2k-1}(\xi;u,a,b) e^{-\int_\xi^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\xi\right)d\tau\Bigg]$$

$$+ 2a\lambda_k \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}} \int_0^T \frac{1}{c(\xi)} F_{2k-1}(\xi;u,a,b) e^{-\int_\xi^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\xi$$

$$\times \left[\int_0^t \frac{1}{c(\tau)}\,d\tau - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}} \int_0^T \frac{1}{c(\tau)}\,d\tau\right] \quad (k=1,2,\ldots). \tag{3.10}$$

After substituting expressions $u_0(t)$, $u_{2k-1}(t)$, and $u_{2k}(t)$ $(k=1,2,\ldots)$, respectively described by (3.7), (3.8), and (3.9), into (3.1), we have

$$\begin{aligned} u(x,t) ={}& (1+\delta)^{-1}\left(\varphi_0 - \int_0^T p(t)u_0(t)\,dt - \delta\int_0^T \frac{1}{c(t)} F_{10}(t;u,a,b)\,dt\right) \\ &+ \int_0^t \frac{1}{c(\tau)} F_0(\tau;u,a,b)\,d\tau \\ &+ \sum_{k=1}^\infty \left\{ \frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}\left(\varphi_{2k-1} - \int_0^T p(t)u_{2k-1}(t)\,dt\right)\right. \\ &+ \int_0^t \frac{1}{c(\tau)} F_{2k-1}(\tau;u,a,b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\tau \\ &\left. - \frac{e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}} \int_0^T \frac{1}{c(\tau)} F_{2k-1}(\tau;u,a,b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\tau\right\} X_{2k-1} \\ &+ \sum_{k=1}^\infty \left\{ \frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}\left(\varphi_{2k} - \int_0^T p(t)u_{2k}(t)\,dt\right)\right. \\ &+ \int_0^t \frac{1}{c(\tau)} F_{2k}(\tau;u,a,b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\tau \\ &\left. - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}} \int_0^T \frac{1}{c(\tau)} F_{2k}(\tau;u,a,b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\tau + q_k(t,T)\right\} X_{2k}(x). \end{aligned} \tag{3.11}$$

Using (1.7) gives

$$\begin{aligned} a(t) ={}& \big[h(t)\big]^{-1}\Bigg\{ \big(c(t)h_1'(t) - f(x_1,t)\big)g(x_2,t) - \big(c(t)h_2'(t) - f(x_2,t)\big)g(x_1,t) \\ &+ \sum_{k=1}^\infty \lambda_k^2 u_{2k-1}(t)\big(g(x_2,t)X_{2k-1}(x_1) - g(x_1,t)X_{2k-1}(x_2)\big) \\ &+ \sum_{k=1}^\infty \lambda_k^2 u_{2k}(t)\big(g(x_2,t)X_{2k}(x_1) - g(x_1,t)X_{2k}(x_2)\big)\Bigg\}, \end{aligned} \tag{3.12}$$

$$b(t)=\left[h(t)\right]^{-1}\Bigg\{h_1(t)\big(c(t)h_2'(t)-f(x_2,t)\big)-h_2(t)\big(c(t)h_1'(t)-f(x_1,t)\big)$$
$$+\sum_{k=1}^{\infty}\lambda_k^2u_{2k-1}(t)\big(h_1(t)X_{2k-1}(x_2)-h_2(t)X_{2k-1}(x_1)\big)$$
$$+\sum_{k=1}^{\infty}\lambda_k^2u_{2k}(t)\big(h_1(t)X_{2k}(x_2)-h_2(t)X_{2k}(x_1)\big)\Bigg\},\qquad(3.13)$$

where

$$h(t)\equiv h_1(t)g(x_2,t)-h_2(t)g(x_1,t)\neq 0\quad(0\leq t\leq T).$$

We substitute relations (3.8) and (3.9) into (3.12) and (3.13), respectively, to obtain

$$a(t)=\left[h(t)\right]^{-1}\Bigg\{\big(c(t)h_1'(t)-f(x_1,t)\big)g(x_2,t)-\big(c(t)h_2'(t)-f(x_2,t)\big)g(x_1,t)$$
$$+\sum_{k=1}^{\infty}\lambda_k^2\Bigg[\frac{e^{-\int_0^t\frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{c(s)}\,ds}}\left(\varphi_{2k-1}-\int_0^T p(t)u_{2k-1}(t)\,dt\right)$$
$$+\int_0^t\frac{1}{c(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{c(s)}\,ds}\,d\tau$$
$$-\frac{\delta e^{-\int_0^T\frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{c(s)}\,ds}}\int_0^T\frac{1}{c(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{c(s)}\,ds}\,d\tau\Bigg]$$
$$\times\big(g(x_2,t)X_{2k-1}(x_1)-g(x_1,t)X_{2k-1}(x_2)\big)$$
$$+\sum_{k=1}^{\infty}\lambda_k^2\Bigg[\frac{e^{-\int_0^t\frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{c(s)}\,ds}}\left(\varphi_{2k}-\int_0^T p(t)u_{2k}(t)\,dt\right)$$
$$+\int_0^t\frac{1}{c(\tau)}F_{2k}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{c(s)}\,ds}\,d\tau$$
$$-\frac{\delta e^{-\int_0^T\frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{c(s)}\,ds}}\int_0^T\frac{1}{c(\tau)}F_{2k}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{c(s)}\,ds}\,d\tau+q_k(t,T)\Bigg]$$
$$\times\big(g(x_2,t)X_{2k-1}(x_1)-g(x_1,t)X_{2k-1}(x_2)\big)\Bigg\},\qquad(3.14)$$

$$b(t)=\left[h(t)\right]^{-1}\Bigg\{h_1(t)\big(c(t)h_2'(t)-f(x_2,t)\big)-h_2(t)\big(c(t)h_1'(t)-f(x_1,t)\big)$$
$$+\sum_{k=1}^{\infty}\lambda_k^2\Bigg[\frac{e^{-\int_0^t\frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{c(s)}\,ds}}\left(\varphi_{2k-1}-\int_0^T p(t)u_{2k-1}(t)\,dt\right)$$
$$+\int_0^t\frac{1}{c(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{c(s)}\,ds}\,d\tau$$

$$
\begin{aligned}
&- \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}} \int_0^T \frac{1}{c(\tau)} F_{2k-1}(\tau;u,a,b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\tau \Bigg] \\
&\times \big(h_1(t)X_{2k-1}(x_2) - h_2(t)X_{2k-1}(x_1)\big) \\
&+ \sum_{k=1}^{\infty} \lambda_k^2 \Bigg[\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}} \left(\varphi_{2k} - \int_0^T p(t)u_{2k}(t)\,dt\right) \\
&+ \int_0^t \frac{1}{c(\tau)} F_{2k}(\tau;u,a,b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\tau \\
&- \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}\,ds}} \int_0^T \frac{1}{c(\tau)} F_{2k}(\tau;u,a,b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}\,ds}\,d\tau + q_k(t,T) \Bigg] \\
&\times \big(h_1(t)X_{2k}(x_2) - h_2(t)X_{2k}(x_2)\big) \Bigg\}. \qquad (3.15)
\end{aligned}
$$

Thus, the problem of finding the solutions (1.1)–(1.3), (1.6), and (1.7) reduces to finding a solution of system (3.11), (3.14), and (3.15) with unknown functions $u(x,t)$, $a(t)$, and $b(t)$, respectively.

Lemma 3.1 *If* $\{u(x,t),a(t),b(t)\}$ *is a classical solution of problem* (1.1)–(1.3), (1.6), *and* (1.7), *then the functions*

$$
\begin{aligned}
&u_0(t) = \int_0^1 u(x,t)Y_0(x)\,dx, \\
&u_{2k-1}(t) = \int_0^1 u(x,t)Y_{2k-1}(x)\,dx \quad (k=1,2,\ldots), \\
&u_{2k}(t) = \int_0^1 u(x,t)Y_{2k}(x)\,dx \quad (k=1,2,\ldots)
\end{aligned}
$$

satisfy system (3.7), (3.8), *and* (3.9) *on the interval* $[0,T]$.

Corollary 3.2 *Suppose that system* (3.11), (3.14) *and* (3.15) *has a unique solution. Then the problem* (1.1)–(1.3), (1.6), *and* (1.7), *cannot have more than one solution*; *in other words*, *if problem* (1.1)–(1.3), (1.6), *and* (1.7) *has a solution*, *then it is unique*.

Now, consider the operator

$$
\Phi(u,a,b) = \big\{\Phi_1(u,a,b), \Phi_2(u,a,b), \Phi_3(u,a,b)\big\}
$$

in the space E_T^3, where

$$
\begin{aligned}
&\Phi_1(u,a,b) = \tilde{u}(x,t) = \sum_{k=0}^{\infty} \tilde{u}_k(t)X_k(x), \\
&\Phi_2(u,a,b) = \tilde{a}(t), \qquad \Phi_3(u,a,b) = \tilde{b}(t),
\end{aligned}
$$

and the functions $\tilde{u}_0(t)$, $\tilde{u}_{2k-1}(t)$, $\tilde{u}_{2k}(t)$ $(k = 1,2,\ldots)$, $\tilde{a}(t)$, and $\tilde{b}(t)$ are equal to the right-hand sides of (3.7), (3.8), (3.9), (3.14), and (3.15), respectively.

Assume that the data for problem (1.1)–(1.3) and (1.7) satisfy the following conditions:

(A) $\varphi(x) \in C^2[0,1]$, $\varphi'''(x) \in L_2(0,1)$, $\varphi(0) = \beta\varphi(1)$, $\varphi'(0) = \varphi'(1)$, $\varphi''(0) = \beta\varphi''(1)$ $(\beta \neq \pm 1)$;

(B) $f(x,t) \in C^{2,0}_{x,t}(D_T), f_{xxx}(x,t) \in L_2(D_T), f(0,t) = \beta f(1,t), f_x(0,t) = f_x(1,t),$

$$f_{xx}(0,t) = \beta f_{xx}(1,t) \quad (\beta \neq \pm 1), (0 \le t \le T);$$

(C) $g(x,t) \in C^{2,0}_{x,t}(D_T)$, $g_{xxx}(x,t) \in L_2(D_T)$, $g(0,t) = \beta g(1,t)$, $g_x(0,t) = g_x(1,t)$,

$$g_{xx}(0,t) = \beta g_{xx}(1,t) \quad (\beta \neq \pm 1), (0 \le t \le T);$$

(D) $\delta \ge 0$, $c(t) \in C[0,T]$, $c(t) > 0$ $(0 \le t \le T)$, $0 \le p(t) \in C[0,T]$, $h_i(t) \in C^1[0,T]$ $(i = 1,2)$, $h(t) \equiv h_1(t)g(x_2,t) - h_2(t)g(x_1,t) \neq 0$.

From (3.10), it is easy to see that

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|q_k(t,T)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\le 4\sqrt{3}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}(1+\delta)\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\varphi_{2k-1}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right. \\
&\quad\left. + T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right] \\
&\quad + 6\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|^2_{C[0,T]}\sqrt{\|c(t)\|_{C[0,T]}}\,T(1+\delta)^2\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|f_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right. \\
&\quad + T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\quad\left. + \sqrt{T}\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|g_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Hence, taking into account inequalities (2.9)–(2.12), we have

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|q_k(t,T)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\le 4\sqrt{3}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}(1+\delta)\left[2\sqrt{2}\|\varphi'''(x)\|_{L_2(0,1)}\right. \\
&\quad\left. + T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right] \\
&\quad + 6\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|^2_{C[0,T]}\sqrt{\|c(t)\|_{C[0,T]}}\,T(1+\delta)^2\left[2\sqrt{2T}\|f_{xxx}(x,t)\|_{L_2(D_T)}\right.
\end{aligned}$$

$$+ T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+ 2\sqrt{2T}\|b(t)\|_{C[0,T]}\|g_{xxx}(x,t)\|_{L_2(D_T)}\Bigg]. \tag{3.16}$$

Now from (3.7)–(3.9), respectively, we have

$$\begin{aligned}
&\|\tilde{u}_0(t)\|_{C[0,T]} \\
&\quad \le (1+\delta)^{-1}\left(|\varphi_0| + T\|p(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}\right) \\
&\qquad + \left(1+\delta(1+\delta)^{-1}\right)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left[\sqrt{T}\left(\int_0^T |f_0(t)|^2\,dt\right)^{\frac{1}{2}}\right. \\
&\qquad \left. + T\|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]} + \sqrt{T}\|b(t)\|_{C[0,T]}\left(\int_0^T |g_0(t)|^2\,dt\right)^{\frac{1}{2}}\right],
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\tilde{u}_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\quad \le \sqrt{5}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\varphi_{2k-1}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right] \\
&\qquad + \sqrt{5}(1+\delta))\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left[\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty}\left(\lambda_k^3|f_{2k-1}(\tau)|\right)^2\,d\tau\right)^{\frac{1}{2}}\right. \\
&\qquad + T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\qquad \left. + T\|b(t)\|_{C[0,T]}\left(\int_0^T \sum_{k=1}^{\infty}\left(\lambda_k^3\|g_{2k-1}(\tau)\|_{C[0,T]}\right)^2\,d\tau\right)^{\frac{1}{2}}\right],
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\tilde{u}_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\quad \le \sqrt{6}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\varphi_{2k}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right] \\
&\qquad + \sqrt{6}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left[\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty}\left(\lambda_k^3|f_{2k}(\tau)|\right)^2\,d\tau\right)^{\frac{1}{2}}\right. \\
&\qquad + T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\qquad \left. + \sqrt{T}\|b(t)\|_{C[0,T]}\left(\int_0^T \sum_{k=1}^{\infty}\left(\lambda_k^3|g_{2k}(\tau)|\right)^2\,d\tau\right)^{\frac{1}{2}}\right] \\
&\qquad + \sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|q_k(t,T)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.19}$$

Taking into consideration (3.16), from (3.17)–(3.19), by virtue of (2.9)–(2.12), we have the following estimates:

$$\|\tilde{u}_0(t)\|_{C[0,T]} \le A_1(T) + B_1(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_1(T)\|u(x,t)\|_{B^3_{2,T}} + D_1(T)\|b(t)\|_{C[0,T]}, \tag{3.20}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\tilde{u}_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le A_2(T) + B_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_2(T)\|u(x,t)\|_{B^3_{2,T}} + D_2(T)\|b(t)\|_{C[0,T]}, \tag{3.21}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\tilde{u}_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le A_3(T) + B_3(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_3(T)\|u(x,t)\|_{B^3_{2,T}} + D_3(T)\|b(t)\|_{C[0,T]}, \tag{3.22}$$

where

$$A_1(T) = 2(1+\delta)^{-1}\|\varphi(x)\|_{L_2(0,1)} + 2\sqrt{T}\left(1+\delta(1+\delta)^{-1}\right)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|f(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = \left(1+\delta(1+\delta)^{-1}\right)T\left\|\frac{1}{c(t)}\right\|_{C[0,T]},$$

$$C_1(T) = (1+\delta)^{-1}T\|p(t)\|_{C[0,T]},$$

$$D_1(T) = \left(1+\delta(1+\delta)^{-1}\right)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\|b(t)\|_{C[0,T]}\|g(x,t)\|_{L_2(D_T)},$$

$$A_2(T) = 2\sqrt{10}\|\varphi'''(x)\|_{L_2(0,1)} + 2\sqrt{10T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|f_{xxx}(x,t)\|_{L_2(D_T)},$$

$$B_2(T) = \sqrt{6}T(1+\delta))\left\|\frac{1}{c(t)}\right\|_{C[0,T]},$$

$$C_2(T) = \sqrt{6}T\|p(t)\|_{C[0,T]},$$

$$D_2(T) = 4\sqrt{3T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|g_{xxx}(x,t)\|_{L_2(D_T)},$$

$$\begin{aligned}A_3(T) = {}& 4\sqrt{3}\|\varphi'''(x)(1-b-ax) - 3a\varphi''(x)\|_{L_2(0,1)} \\ &+ 4\sqrt{3T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|f_{xxx}(x,t)(1-b-ax) - 3af_{xx}(x,t)\|_{L_2(D_T)} \\ &+ 12\sqrt{2}|a|(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|\varphi'''(x)\|_{L_2(0,1)} \\ &+ 12\sqrt{3}|a|\left\|\frac{1}{c(t)}\right\|^2_{C[0,T]}\sqrt{\|c(t)\|_{C[0,T]}}T\sqrt{T}(1+\delta)^2\|f_{xxx}(x,t)\|_{L_2(D_T)},\end{aligned}$$

$$B_3(T) = \sqrt{6}T(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]} + 6\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|^2_{C[0,T]}\sqrt{\|c(t)\|_{C[0,T]}}T^2(1+\delta)^2,$$

$$C_3(T) = \sqrt{3}T\|p(t)\|_{C[0,T]}\left[\sqrt{2} + 4|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}(1+\delta)\right],$$

$$D_3(T) = 6\sqrt{2T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\Bigg(\|g_{xxx}(x,t)(1-b-ax) - 3ag_{xx}(x,t)\|_{L_2(D_T)}$$

$$+ |a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{\|c(t)\|_{C[0,T]}}T(1+\delta)\|g_{xxx}(x,t)\|_{L_2(D_T)}\Bigg).$$

We conclude from (3.20)–(3.22) that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \le A_4(T) + B_4(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_4(T)\|u(x,t)\|_{B^3_{2,T}} + D_4(T)\|b(t)\|_{C[0,T]}, \tag{3.23}$$

where

$$A_4(T) = A_1(T) + A_2(T) + A_3(T), B_4(T) = B_1(T) + B_2(T) + B_3(T),$$

$$C_4(T) = C_1(T) + C_2(T) + C_3(T), D_4(T) = D_1(T) + D_2(T) + D_3(T).$$

Now from (3.14) and (3.15) we have

$$\|\tilde{a}(t)\|_{C[0,T]}$$

$$\le \|[h(t)]^{-1}\|_{C[0,T]}\Bigg\{\|(c(t)h_1'(t) - f(x_1,t))g(x_2,t) - (c(t)h_2'(t) - f(x_2,t))g(x_1,t))\|_{C[0,T]}$$

$$+ \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\Bigg[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\varphi_{2k-1}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+ T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+ (1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\Bigg[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|f_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+ T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+ \sqrt{T}\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|g_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\Bigg]$$

$$+ \left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\varphi_{2k}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+ (1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\Bigg[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|f_{2k}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+ T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\left\|b(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|g_{2k}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\Bigg]$$

$$+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|q_k(t,T)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\Bigg]$$

$$\times\left\|\left|g(x_2,t)\right|+\left|g(x_1,t)\right|\right\|_{C[0,T]}\|ax+b\|_{C[0,1]}\Bigg\}, \tag{3.24}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}$$

$$\leq\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\Bigg\{\left\|h_1(t)\left(c(t)h_2'(t)-f(x_2,t)\right)-h_2(t)\left(c(t)h_1'(t)-f(x_1,t)\right)\right\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\varphi_{2k-1}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\times T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_{2k-1}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\Bigg[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_{2k-1}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}$$

$$+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_{2k-1}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+T\left\|b(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|g_{2k-1}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\Bigg]$$

$$+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|\varphi_{2k}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_{2k}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\Bigg[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_{2k}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}$$

$$+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_{2k}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+T\left\|b(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|g_{2k}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\Bigg]$$

$$+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|q_k(t,T)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\left\|\left|h_1(t)\right|+\left|h_2(t)\right|\right\|_{C[0,T]}\|ax+b\|_{C[0,1]}\Bigg\}. \tag{3.25}$$

Using (3.16), by virtue of (2.9)–(2.12), from the relations (3.24) and (3.25), we obtain

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\leq A_5(T)+B_5(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3}$$
$$+C_5(T)\left\|u(x,t)\right\|_{B_{2,T}^3}+D_5(T)\left\|b(t)\right\|_{C[0,T]}, \tag{3.26}$$

$$\begin{aligned}\|\tilde{b}(t)\|_{C[0,T]} &\le A_6(T) + B_6(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} \\ &\quad + C_6(T)\|u(x,t)\|_{B^3_{2,T}} + D_6(T)\|b(t)\|_{C[0,T]},\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
A_5(T) &= \|[h(t)]^{-1}\|_{C[0,T]}\Bigg\{\|h_1(t)(c(t)h_2'(t) - f(x_2,t)) - h_2(t)(c(t)h_1'(t) - f(x_1,t))\|_{C[0,T]} \\
&\quad + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\Bigg[2\sqrt{2}\Bigg[\left(1 + 4\sqrt{3}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}(1+\delta)\right)\|\varphi'''(x)\|_{L_2(0,1)} \\
&\quad + \|\varphi'''(x)(1-b-ax) - 3a\varphi''(x)\|_{L_2(0,1)}\Bigg] + 2\sqrt{2T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]} \\
&\quad \times \Bigg[\left(1 + 6\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{\|c(t)\|_{C[0,T]}}T(1+\delta)\right)\|f_{xxx}(x,t)\|_{L_2(D_T)} \\
&\quad + \|f_{xxx}(x,t)(1-b-ax) - 3af_{xx}(x,t)\|_{L_2(D_T)}\Bigg]\Bigg] \\
&\quad \times \big\||g(x_2,t)| + |g(x_1,t)|\big\|_{C[0,T]}\|ax+b\|_{C[0,1]}\Bigg\}, \\
B_5(T) &= 2T\|[h(t)]^{-1}\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]} \\
&\quad \times \left(1 + 3\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{\|c(t)\|_{C[0,T]}}T(1+\delta)\right) \\
&\quad \times \big\||g(x_2,t)| + |g(x_1,t)|\big\|_{C[0,T]}\|ax+b\|_{C[0,1]}, \\
C_5(T) &= 2T\|[h(t)]^{-1}\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|p(t)\|_{C[0,T]}\left(1 + 2\sqrt{3}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\right) \\
&\quad \times \big\||g(x_2,t)| + |g(x_1,t)|\big\|_{C[0,T]}\|ax+b\|_{C[0,1]}, \\
D_5(T) &= 2\sqrt{2T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|[h(t)]^{-1}\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} \\
&\quad \times \Bigg[\left(1 + 6\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{\|c(t)\|_{C[0,T]}}T(1+\delta)\right)\|g_{xxx}(x,t)\|_{L_2(D_T)} \\
&\quad + \|g_{xxx}(x,t)(1-b-ax) - 3ag_{xx}(x,t)\|_{L_2(D_T)}\Bigg] \\
&\quad \times \big\||g(x_2,t)| + |g(x_1,t)|\big\|_{C[0,T]}\|ax+b\|_{C[0,1]}, \\
A_6(T) &= \|[h(t)]^{-1}\|_{C[0,T]} \\
&\quad \times \Bigg\{\|(c(t)h_1'(t) - f(x_1,t))g(x_2,t) - (c(t)h_2'(t) - f(x_2,t))g(x_1,t)\|_{C[0,T]} \\
&\quad + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\Bigg[2\sqrt{2}\Bigg[\left(1 + 4\sqrt{3}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}(1+\delta)\right)\|\varphi'''(x)\|_{L_2(0,1)}
\end{aligned}$$

$$+\left\|\varphi'''(x)(1-b-ax)-3a\varphi''(x)\right\|_{L_2(0,1)}\Bigg]+2\sqrt{2T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}$$
$$\times\Bigg[\left(1+6\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{\left\|c(t)\right\|_{C[0,T]}}T(1+\delta)\right)\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}$$
$$+\left\|f_{xxx}(x,t)(1-b-ax)-3af_{xx}(x,t)\right\|_{L_2(D_T)}\Bigg]\Bigg]$$
$$\times\left\|\left|h_1(t)\right|+\left|h_2(t)\right|\right\|_{C[0,T]}\|ax+b\|_{C[0,1]}\Bigg\},$$

$$B_6(T)=2T\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}$$
$$\times\left(1+3\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{\left\|c(t)\right\|_{C[0,T]}}T(1+\delta)\right)$$
$$\times\left\|\left|h_1(t)\right|+\left|h_2(t)\right|\right\|_{C[0,T]}\|ax+b\|_{C[0,1]},$$

$$C_6(T)=2T\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\|p(t)\right\|_{C[0,T]}\left(1+2\sqrt{3}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\right)$$
$$\times\left\|\left|h_1(t)\right|+\left|h_2(t)\right|\right\|_{C[0,T]}\|ax+b\|_{C[0,1]},$$

$$D_6(T)=2\sqrt{2T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$
$$\times\Bigg[\left(1+6\sqrt{2}|a|\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{\left\|c(t)\right\|_{C[0,T]}}T(1+\delta)\right)\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}$$
$$+\left\|g_{xxx}(x,t)(1-b-ax)-3ag_{xx}(x,t)\right\|_{L_2(D_T)}\Bigg]$$
$$\times\left\|\left|h_1(t)\right|+\left|h_2(t)\right|\right\|_{C[0,T]}\|ax+b\|_{C[0,1]}.$$

From inequalities (3.23), (3.26), and (3.27), we deduce that

$$\begin{aligned}&\left\|\tilde{u}(x,t)\right\|_{B^3_{2,T}}+\left\|\tilde{a}(t)\right\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}\\&\quad\le A(T)+B(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}}\\&\qquad+C(T)\left\|u(x,t)\right\|_{B^3_{2,T}}+D(T)\left\|b(t)\right\|_{C[0,T]},\end{aligned}\tag{3.28}$$

where

$$A(T)=A_4(T)+A_5(T)+A_6(T),\qquad B(T)=B_4(T)+B_5(T)+B_6(T),$$
$$C(T)=C_4(T)+C_5(T)+C_6(T),\qquad D(T)=D_4(T)+D_5(T)+D_6(T).$$

Theorem 3.3 *If conditions* (A)–(D) *and the condition*

$$\bigl(B(T)\bigl(A(T)+2\bigr)+C(T)+D(T)\bigr)\bigl(A(T)+2\bigr)<1\tag{3.29}$$

hold, then problem (1.1)–(1.3), (1.6), *and* (1.7) *has a unique solution in the ball* $K = K_R(\|z\|_{E_T^3} \le R \le A(T) + 2)$ *of the space* E_T^3.

Remark 3.4 Inequality (3.29) is satisfied for sufficiently small values of T.

Proof In the space E_T^3, we consider the equation

$$z = \Phi z, \tag{3.30}$$

where $z = \{u, a, b\}$, and the components $\Phi_i(u, a, b)$ $(i = 1, 2, 3)$ of operator $\Phi(u, a, b)$ are defined by the right-hand side of Eqs. (3.11), (3.14), and (3.15).

Consider the operator $\Phi(u, a, b)$ in the ball $K = K_R$ of the space E_T^3. Similarly, with the aid of (3.28), we obtain that for any $z_1, z_2, z_3 \in K_R$ the following inequalities hold:

$$\begin{aligned}
&\|\Phi z\|_{E_T^3} \\
&\quad \le A(T) + B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3} + C(T)\|u(x,t)\|_{B_{2,T}^3} + D(T)\|b(t)\|_{C[0,T]} \\
&\quad \le A(T) + B(T)\big(A(T)+2\big)^2 + C(T)\big(A(T)+2\big) + D(T)\big(A(T)+2\big) < A(T) + 2, \qquad (3.31)\\
&\|\Phi z_1 - \Phi z_2\|_{E_T^3} \le B(T)R\big(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^3}\big) \\
&\qquad\qquad + C(T)\|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^3} + D(T)\|b_1(t) - b_2(t)\|_{C[0,T]}. \qquad (3.32)
\end{aligned}$$

Then by (3.29), from (3.31) and (3.32) it is clear that operator Φ on the set $K = K_R$ satisfies the conditions of the contraction mapping principle. Therefore, operator Φ has a unique fixed point $z = \{u, a, b\}$, in the ball $K = K_R$, which is a solution of Eq. (3.30), i.e., in the ball $K = K_R$ it is the unique solution of the system (3.11), (3.14), and (3.15). Then the function $u(x,t)$, as an element of space $B_{2,T}^3$, is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Next, from (3.4) and (3.5), it follows that $u_k'(t)$ $(k = 1, 2, \ldots)$ are continuous on $[0, T]$, and consequently we have

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\big(\lambda_k\|u_{2k-1}'(t)\|_{C[0,T]}\big)^2\right)^{\frac{1}{2}} \\
&\quad \le \left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{2}\left[\left(\sum_{k=1}^{\infty}\big(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\big)^2\right)^{\frac{1}{2}}\right. \\
&\qquad \left. + 2\sqrt{2}\big\|\|f_x(x,t) + a(t)u_x(x,t) + b(t)g_x(x,t)\|_{C[0,T]}\big\|_{L_2(0,1)}\right] \\
&\quad < +\infty, \\
&\left(\sum_{k=1}^{\infty}\big(\lambda_k\|u_{2k}'(t)\|_{C[0,T]}\big)^2\right)^{\frac{1}{2}} \\
&\quad \le \left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{3}\left[\left(\sum_{k=1}^{\infty}\big(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\big)^2\right)^{\frac{1}{2}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+ 2\sqrt{2}\Big\| \big\| \big(f_x(x,t) + a(t)u_x(x,t) + b(t)g_x(x,t)\big)(1 - b - ax) \\
&+ a\big(f(x,t) + a(t)u(x,t) + b(t)g(x,t)\big) \big\|_{C[0,T]} \Big\|_{L_2(0,1)} \\
&+ 2a \left(\sum_{k=1}^{\infty} \big(\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]}\big)^2 \right)^{\frac{1}{2}} \Bigg] \\
&< +\infty.
\end{aligned}
$$

Hence we conclude that the function $u_t(t,x)$ is continuous in domain D_T.

Further, it is possible to verify that Eq. (1.1) and conditions (1.2), (1.3), (1.6), and (1.7) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t), b(t)\}$ is a solution of (1.1)–(1.3), (1.6), and (1.7), and by Corollary 3.2 it is unique in the ball $K = K_R$. The proof is complete. □

From Theorems 1.2 and 3.3, the following assertion follows directly.

Theorem 3.5 *Suppose that all assumptions of Theorem 3.3 and the compatibility conditions*

$$
\int_0^1 \varphi(x)\,dx = 0,
$$

$$
h_i(0) + \delta h_i(T) + \int_0^T p(t)h_i(t)\,dt = \varphi(x_i) \quad (i = 1,2)
$$

hold. If

$$
\int_0^1 f(x,t)\,dx = 0, \qquad \int_0^1 g(x,t)\,dx = 0 \quad (0 \le t \le T),
$$

the, problem (1.1)–(1.5) has a unique classical solution in the ball $K = K_R$.

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