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Global existence and blow-up of solution for the semilinear wave equation with interior and boundary source terms

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Abstract

This paper is concerned with semilinear wave equations with nonlinear interior and boundary sources and subject to a nonlinear dynamical boundary condition. By using the potential well method combined with a standard continuous argument, under appropriate assumptions imposed on the source term, we establish global existence of solutions. Moreover, for certain initial data in the unstable set, the finite time blow-up phenomenon is exhibited.

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Keywords: Global existence; Blow-up; Dynamical boundary condition; Wave equation; Potential well theory

1 Introduction

In this paper, we consider the following model of semilinear wave equation with nonlinear interior and boundary sources:

$$u_{tt} - \Delta u_{tt} - \Delta u = g(u), \quad t > 0, \tag{1.1}$$

$$\frac{\partial u_{tt}}{\partial \nu} + \frac{\partial u}{\partial \nu} + u = f(u), \quad x \in \Gamma, t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

in a bounded domain $\Omega \subset R^n$ with smooth boundary $\partial\Omega = \Gamma$, where $f(s)$, $g(s)$ are continuous functions and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative. In this paper, we take $f(s) = b|u|^{k-1}u$, $g(s) = a|u|^{p-1}u$, where a, b are positive constants. For simplicity, we take $a = b = 1$.

Problem (1.1)–(1.3) arises from a model equation of ion-sound waves in 'non-magnetized' plasma taking account of nonlinear sources localized on the boundary [1–6]. This generates a nonlinear dynamical boundary condition which is 'close' to the nonlinear Neumann–Dirichlet condition [1]. Problem (1.1)–(1.3) when $g(u) = 0$ (i.e., without interior source term) has been considered by Korpusov [1]. Korpusov proved that the solution of problem (1.1)–(1.3) exists locally in time for all initial data $u_0(x), u_1(x) \in H^1(\Omega)$ and that the solution blows up in a finite time provided that the initial data (the functions $u_0(x)$ and

$u_1(x)$) are sufficiently 'large', i.e.,

$$\int_\Omega \left[|\nabla u_1|^2 + |u_1|^2\right] dx + \int_\Omega |\nabla u_0|^2 \, dx < 2\int_\Gamma \int_0^{u_0} f(s)\, ds\, dx,$$
$$\int_\Omega [\nabla u_1 \nabla u_0 + u_0 u_1]\, dx > 0,$$

by using a modification of Levine's energy method [7, 8]. Park and Kim [9] discussed the existence and uniform decay rates of the energy of solutions for the following problem:

$$|u_t|^\rho u_{tt} - \beta \Delta u_{tt} - \Delta u - \Delta u_t = 0, \quad \text{in } \Omega \times (0,\infty),$$
$$\beta \frac{\partial u_{tt}}{\partial \nu} + \frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} + u = \int_0^t g(t-s)\big|u_t(s)\big|^\gamma u_t(s)\, ds, \quad \text{on } \Gamma_1 \times (0,\infty),$$
$$u = 0, \quad \text{on } \Gamma_0 \times (0,\infty),$$

with initial condition (1.3), where $\Gamma_1 \cup \Gamma_0 = \partial\Omega = \Gamma$ and $\Gamma_1 \cap \Gamma_0 = \emptyset$. However, as far as we know, until now there have not been many works on this class of problems. In this paper, we will extend the result in [1] for negative initial energy to the semilinear wave equation with positive initial energy and nonlinear interior and boundary sources. It is well known that the presence of the boundary source term in equation (1.2) brings great difficulty due to the fact that the Lopatinskii condition [10] does not hold [11]. A combination of interior and boundary sources with positive initial energy is a much more challenging problem.

To motivate our work, let us recall some results of the following wave equations:

$$u_{tt} - \Delta u + g(u_t) = f(u), \quad \text{in } \Omega \times (0,T), \tag{1.4}$$
$$\frac{\partial u}{\partial \nu} + u + g_1(u_t) = h(u), \quad \text{on } \Gamma \times (0,T). \tag{1.5}$$

This problem has been widely studied. Several results have been established. It is worth noting the pioneering work of Lasiecka and Tataru [12] in which (1.4) with $g = 0$ was conducted under a very weak geometrical condition on $\partial\Omega$. They established the uniform decay rates for the solutions. Vitillaro [13] obtained a full analysis of local and global existence of problem (1.4), (1.5) and (1.3). Recently, Boicu and Lasiecka et al. [11, 14–28] studied problem (1.4), (1.5) and (1.3) with interior and boundary sources and damping terms. They obtained global existence of a unique weak solution and established explicit uniform energy rates. As for blow-up of solutions, they established blow-up results with up-to-critical boundary sources. Especially, at the super-critical level for both interior and boundary sources terms, the blow-up theorem was presented in [15] for initial data of negative energy, and Bociu, Rammaha, and Toundykov [18] proved a blow-up result for weak solutions with nonnegative initial energy.

It is important to observe that similar equations to the one given in (1.1) arise also in the study of viscoelastic plates. Ji and Lasiecka [29] proved that a semilinear Kirchhoff equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u = f(u) \tag{1.6}$$

with nonlinear dissipation acting via moments only is uniform energy decay.

Motivated by these papers, in this paper we aim to investigate the existence and nonexistence of global solutions for problem (1.1)–(1.3) with nonnegative initial energy. More precisely, under appropriate assumptions imposed on the source term, we shall establish global existence of solutions by using the potential well method combined with a standard continuous argument. Moreover, for certain initial data in the unstable set, we will extend the finite time blow-up result in [1] for negative initial energy to the semilinear wave equation with positive initial energy and nonlinear interior and boundary sources. Combining this method with the method of [18, 19, 27], we can also consider the equation

$$u_{tt} - \Delta u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)\,ds + h(u_t) = g(u), \tag{1.7}$$

$$\frac{\partial u_{tt}}{\partial \nu} + \frac{\partial u}{\partial \nu} + u + h_1(u_t) = f(u), \quad x \in \Gamma, \tag{1.8}$$

with initial condition (1.3). The plan of this article is as follows. In Sect. 2, we introduce some notations, assumptions, and preliminaries. In Sect. 3, we show the main results of this article.

2 Preliminaries

In this section, we present some materials needed in the proof of our results. We use the standard Lebesgue space $L^p(\Omega)$ $(1 < p < \infty)$ and the Sobolev space $H^1(\Omega)$ with their usual scalar products and norms. For simplicity, $\|u\|_{L^p(\Omega)} = \|u\|_p$ and $\|u\|_{L^q(\Gamma)} = \|u\|_{q,\Gamma}$ for $1 \le p, q \le \infty$. In particular, we denote $\|u\|_{L^2(\Omega)} = \|u\|$ and $\|u\|_{L^2(\Gamma)} = \|u\|_\Gamma$. It is well known that the norm $(\|\nabla u\|^2 + \|u\|_\Gamma^2)^{\frac{1}{2}}$ is equivalent to the norm $\|u\|_{H^1(\Omega)}$ on the space $H^1(\Omega)$. Thus we put $\|u\|_{H^1(\Omega)} = \|u\|_{1,\Omega} = (\|\nabla u\|^2 + \|u\|_\Gamma^2)^{\frac{1}{2}}$ for $u \in H^1(\Omega)$. The constants C used throughout this paper are positive generic constants, which may be different in various occurrences.

We assume that

$$1 \le p \le \frac{n}{n-2}, \qquad 1 \le k \le \frac{n-1}{n-2} \quad \text{if } n \ge 3; \qquad p \ge 1, \qquad k \ge 1 \quad \text{if } n = 1, 2. \tag{2.1}$$

In this case, we have the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and the Trace–Sobolev embedding $H^1(\Omega) \hookrightarrow L^{k+1}(\Gamma)$. In these cases, the embedding constants are denoted by c_*, B_* respectively, i.e.,

$$\|u\|_{p+1} \le c_*\|u\|_{1,\Omega}, \qquad \|u\|_{k+1,\Gamma} \le B_*\|u\|_{1,\Omega}. \tag{2.2}$$

A function $u(x,t)$ of class $L^\infty(0,T;H^1(\Omega))$ with $u_t, u_{tt} \in L^\infty(0,T;H^1(\Omega))$ is called a weak generalized solution of problem (1.1)–(1.3) [1] if it satisfies the equation

$$(u_{tt},\phi) + (\nabla u_{tt},\nabla\phi) + (\nabla u,\nabla\phi) + \int_\Gamma u\phi\,dx = \int_\Omega |u|^{p-1}u\phi\,dx + \int_\Gamma |u|^{k-1}u\phi\,dx$$

for any $\phi \in H^1(\Omega)$ and for almost all $t \in [0,T]$ and the initial condition

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x).$$

Theorem 2.1 *Let $u_0, u_1 \in H^1$ and p, k satisfy (2.1), then problem (1.1)–(1.3) has a unique weak generalized solution on $[0, T_0)$ for some $T_0 > 0$, and we have either $T_0 = +\infty$ or $T_0 < +\infty$ and*

$$\lim_{t\to T_0} \sup\bigl[\|u\|_{1,\Omega}^2 + \|u_t\|_{1,\Omega}^2\bigr] = +\infty.$$

Remark This theorem can be easily established by combining the argument of [30] and Theorem 2.2 in [1], so we omit it.

We define the functional that plays as the "potential energy"

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|u\|_\Gamma^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{k+1}\|u\|_{k+1,\Gamma}^{k+1} \\ &= \frac{1}{2}\|u\|_{1,\Omega}^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{k+1}\|u\|_{k+1,\Gamma}^{k+1}, \end{aligned} \tag{2.3}$$

and the Nehari functional

$$I(u) = \|u\|_{1,\Omega}^2 - \|u\|_{p+1}^{p+1} - \|u\|_{k+1,\Gamma}^{k+1}. \tag{2.4}$$

We have also the following energy identity:

$$E(t) = \frac{1}{2}\|u_t\|_{1,\Omega}^2 + J(u) = E(0). \tag{2.5}$$

In the sequel, a crucial role is played by the Nehari manifold to I, that is,

$$N = \bigl\{u \in H^1(\Omega) | I(u) = 0, \|u\|_{1,\Omega} \neq 0\bigr\},$$

and we can readily give the mountain-pass level d by $d = \inf_{u\in N} J(u)$.

Next, we show some properties related to functions $J(u)$ and $I(u)$ in the following lemma.

Lemma 2.2 *Let $u \in H^1(\Omega)$ and $\|u\|_{1,\Omega} \neq 0$, then*

(i) $\lim_{\lambda\to 0} J(\lambda u) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$;

(ii) *There exists unique $\lambda_0 = \lambda_0(u)$ such that $\frac{d}{d\lambda} J(\lambda u)|_{\lambda=\lambda_0} = 0$;*

(iii) *$J(\lambda u)$ is increasing on $0 < \lambda \le \lambda_0$, decreasing on $\lambda_0 \le \lambda < +\infty$, and takes the maximum at $\lambda = \lambda_0$;*

(iv) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda_0$; $I(\lambda u) < 0$ for $\lambda > \lambda_0$, and $I(\lambda_0 u) = 0$.*

Proof The first conclusion follows from

$$J(\lambda u) = \frac{\lambda^2}{2}\|u\|_{1,\Omega}^2 - \frac{\lambda^{p+1}}{p+1}\|u\|_{p+1}^{p+1} - \frac{\lambda^{k+1}}{k+1}\|u\|_{k+1,\Gamma}^{k+1}.$$

As in [19], let λ_0 be the first positive zero of the function $F'(x)$ where

$$F(x) = \frac{1}{2}x^2 - \frac{c_*^{p+1}}{p+1}x^{p+1} - \frac{B_*^{k+1}}{k+1}x^{k+1}, \tag{2.6}$$

and we can verify that the function $F(x)$ is increasing in $0 < \lambda \le \lambda_0$, decreasing in $\lambda_0 \le \lambda < +\infty$, and F has a maximum at $\lambda = \lambda_0$. Then we have that (ii) and (iii) hold. The conclusion (iv) follows from (iii) and the fact that $I(\lambda u) = \lambda \frac{I(\lambda u)}{d\lambda}$. □

Lemma 2.3 *Let $I(u)$ be the Nehari functional defined in* (2.4) *and λ_0 be the first positive zero of the function $F'(x)$, then* (i) *if $0 < \|u\|_{1,\Omega} < \lambda_0$, then $I(u) > 0$;* (ii) *if $I(u) < 0$, then $\|u\|_{1,\Omega} > \lambda_0$;* (iii) *if $I(u) = 0$ and $\|u\|_{1,\Omega} \neq 0$, i.e., $u \in N$, then $\|u\|_{1,\Omega} \ge \lambda_0$.*

Proof We note that

$$1 = c_*^{p+1}\lambda_0^{p-1} + B_*^{k+1}\lambda_0^{k-1}, \tag{2.7}$$

so we denote $\phi(x) = c_*^{p+1}x^{p-1} + B_*^{k+1}x^{k-1}$, then $\phi(\lambda_0) = 1$.

(i) Since $\phi(x)$ is a strictly increasing function in $(0, \lambda_0)$, from $0 < \|u\|_{1,\Omega} < \lambda_0$, we get $\phi(\|u\|_{1,\Omega}) < \phi(\lambda_0)$ and hence

$$\begin{aligned} I(u) &= \|u\|_{1,\Omega}^2 - \|u\|_{p+1}^{p+1} - \|u\|_{k+1,\Gamma}^{k+1} \\ &\ge \|u\|_{1,\Omega}^2\bigl(1 - c_*^{p+1}\|u\|_{1,\Omega}^{p-1} - B_*^{k+1}\|u\|_{1,\Omega}^{k-1}\bigr) \\ &= \|u\|_{1,\Omega}^2\bigl(1 - \phi\bigl(\|u\|_{1,\Omega}\bigr)\bigr) > 0. \end{aligned}$$

(ii) Condition $I(u) < 0$ gives

$$\begin{aligned} \phi(\lambda_0)\|u\|_{1,\Omega}^2 &= \|u\|_{1,\Omega}^2 < \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1} \\ &\le \bigl(c_*^{p+1}\|u\|_{1,\Omega}^{p-1} + B_*^{k+1}\|u\|_{1,\Omega}^{k-1}\bigr)\|u\|_{1,\Omega}^2 = \phi\bigl(\|u\|_{1,\Omega}\bigr)\|u\|_{1,\Omega}^2, \end{aligned}$$

which implies $\|u\|_{1,\Omega} \neq 0$ and $\|u\|_{1,\Omega} > \lambda_0$ by the monotonicity of ϕ.

(iii) If $I(u) = 0$ and $\|u\|_{1,\Omega} \neq 0$, then

$$\phi(\lambda_0)\|u\|_{1,\Omega}^2 = \|u\|_{1,\Omega}^2 = \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1} \le \phi\bigl(\|u\|_{1,\Omega}\bigr)\|u\|_{1,\Omega}^2,$$

and from the monotonicity of ϕ we get $\|u\|_{1,\Omega} > \lambda_0$. □

Lemma 2.4 *Let $I(u)$ be the Nehari functional defined in* (2.4), *then* (i) $d \ge d_0 = (\frac{1}{2} - \max\{\frac{1}{p+1}, \frac{1}{k+1}\})\lambda_0^2$; (ii) *if $u \in H^1$ and $I(u) < 0$, then $I(u) < \min\{p+1, k+1\}(J(u) - d)$.*

Proof (i) For $u \in N$ (or $I(u) = 0$ and $\|u\|_{1,\Omega} \neq 0$), by Lemma 2.3, we have $\|u\|_{1,\Omega} > \lambda_0$. Hence

$$\begin{aligned} J(u) &\ge \frac{1}{2}\|u\|_{1,\Omega}^2 - \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\bigl(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1}\bigr) \\ &= \left(\frac{1}{2} - \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\right)\|u\|_{1,\Omega}^2 + \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}I(u) \\ &= \left(\frac{1}{2} - \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\right)\|u\|_{1,\Omega}^2 \ge \left(\frac{1}{2} - \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\right)\lambda_0^2, \end{aligned}$$

which gives $d \ge d_0$.

(ii) By Lemma 2.2, there exists $\lambda_0 \in (0,1)$ such that $I(\lambda_0 u) = 0$ since $I(1u) = I(u) < 0$. Combining this $I(\lambda_0 u) = 0$ with the definition of d and the fact that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_{1,\Omega}^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{k+1}\|u\|_{k+1,\Gamma}^{k+1} \\ &= \left(\frac{1}{2} - \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\right)\|u\|_{1,\Omega}^2 + \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} I(u) \\ &\quad + \left[\max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} - \frac{1}{p+1}\right]\|u\|_{p+1}^{p+1} \\ &\quad + \left[\max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} - \frac{1}{k+1}\right]\|u\|_{k+1,\Gamma}^{k+1}, \end{aligned}$$

we obtain

$$\begin{aligned} d &\le J(\lambda_0 u) \\ &= \left(\frac{1}{2} - \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\right)\lambda_0^2\|u\|_{1,\Omega}^2 \\ &\quad + \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} I(\lambda_0 u) + \left[\max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} - \frac{1}{p+1}\right]\lambda_0^{(p+1)}\|u\|_{p+1}^{p+1} \\ &\quad + \left[\max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} - \frac{1}{k+1}\right]\lambda_0^{(k+1)}\|u\|_{k+1,\Gamma}^{k+1} \\ &\le \left(\frac{1}{2} - \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\right)\|u\|_{1,\Omega}^2 + \left[\max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} - \frac{1}{p+1}\right]\|u\|_{p+1}^{p+1} \\ &\quad + \left[\max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} - \frac{1}{k+1}\right]\|u\|_{k+1,\Gamma}^{k+1} \\ &= J(u) - \max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\} I(u), \end{aligned}$$

which gives the result since $\max\{\frac{1}{p+1}, \frac{1}{k+1}\} = \min\{p+1, k+1\}$ for $p, k > 1$. □

Now we define the subsets of H^1 related to problem (1.1)–(1.3). Set

$$W = \{u \in H^1 | J(u) < d, I(u) > 0\}, \qquad V = \{u \in H^1 | J(u) < d, I(u) < 0\}. \tag{2.8}$$

Lemma 2.5 *If* $u_0, u_1 \in H^1$ *and* $0 < E(0) < d$, u *is a weak solution of problem* (1.1)–(1.3), *then* (i) $u \in W$ *if* $I(u_0) > 0$ *or* $\|u\|_{1,\Omega} = 0$; (ii) $u \in V$ *if* $I(u_0) < 0$.

Proof We only prove (i), the proof for (ii) is similar. Let T_m be maximal existence time of a weak solution of $u(t)$. We are going to prove that $u \in W$ for $0 < t < T_m$. From the energy identity (2.5), we have

$$E(t) = \frac{1}{2}\|u_t\|_{1,\Omega}^2 + J(u) = E(0) < d \quad \text{for any } t \in [0, T),$$

which implies $J(u(t)) < d$. To prove that $u \in W$ for $0 < t < T_m$, we argue by contradiction. Indeed, if it is not the case, there would exist $t_0 \in (0, T_m)$ such that $u(t_0) \in N$. By the definition of $d = \inf_{u \in N} J(u)$, one has $d < J(u(t_0)) < E(t_0) \le d$, we reach a contradiction. □

3 Global existence and blow-up of solutions

In this section, we prove the global existence and blow-up of solutions for problem (1.1)–(1.3).

Theorem 3.1 *Let* $u_0, u_1 \in H^1$, $0 < E(0) < d$, $I(u_0) > 0$ *or* $\|u\|_{1,\Omega} = 0$, *and* p, k *satisfy* (2.1), *then the weak solution* u *of problem* (1.1)–(1.3) *in Theorem* 2.1 *can be extended to* $(0, \infty)$.

Proof By Lemma 2.5, we have $u \in W$, then $I(u) > 0$ and $J(u) < d$ for all $t \in (0, T)$. Therefore

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_{p+1}^{p+1} + \left(\frac{1}{2} - \frac{1}{k+1}\right)\|u\|_{k+1,\Gamma}^{k+1} \\
&= \frac{1}{2}\left(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1}\right) - \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{k+1}\|u\|_{k+1,\Gamma}^{k+1} \\
&= \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}I(u) - \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{k+1}\|u\|_{k+1,\Gamma}^{k+1} \\
&\le J(u) < d
\end{aligned} \tag{3.1}$$

for all $t \in (0, T)$. Define $\alpha = \min\{\frac{1}{2} - \frac{1}{p+1}, \frac{1}{2} - \frac{1}{k+1}\} > 0$, then (3.1) implies

$$\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1} < \frac{d}{\alpha} \tag{3.2}$$

for all $t \in (0, T)$. By the energy identity, the definition of $J(u)$ and (3.2), we have

$$\begin{aligned}
\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_{1,\Omega}^2 &= E(0) + \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{1}{k+1}\|u\|_{k+1,\Gamma}^{k+1} \\
&< d\left(1 + \frac{1}{\alpha}\max\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\right)
\end{aligned} \tag{3.3}$$

for all $t \in (0, T)$. It follows from (3.3) and from a standard continuous argument that the local weak solution u furnished by Theorem 2.1 can be extended to the whole interval $[0, \infty)$, that is, u is a global solution. □

Theorem 3.2 *Let* $u_0, u_1 \in H^1$, $0 < E(0) < d$, $I(u_0) < 0$, *and* p, k *satisfy* (2.1), *then the weak solution* u *of problem* (1.1)–(1.3) *blows up in finite time, that is, the maximum existence time* T_m *of* u *is finite and*

$$\lim_{t \to T_m} \sup\left[\|u\|_{1,\Omega}^2 + \|u_t\|_{1,\Omega}^2\right] = +\infty.$$

Proof Arguing by contradiction, we assume that $T_m = +\infty$. Set

$$H(t) = \|u\|_{1,\Omega}^2, \tag{3.4}$$

then by taking the time derivative of the function $H(t)$, performing integration by parts, and using equations (1.1) and (1.2), we get

$$H'(t) = 2(u, u_t) + 2(\nabla u, \nabla u_t), \tag{3.5}$$

$$\begin{aligned} H''(t) &= 2\|u_t\|^2 + 2\|\nabla u_t\|^2 + 2(u,u_{tt}) + 2(\nabla u,\nabla u_{tt}) \\ &= 2\|u_t\|^2 + 2\|\nabla u_t\|^2 - 2I(u). \end{aligned} \tag{3.6}$$

By virtue of the Schwarz inequality, we have

$$H'^2(t) \le 4H(t)\big(\|u_t\|^2 + \|\nabla u_t\|^2\big).$$

Hence

$$\begin{aligned} H(t)H''(t) - \frac{\rho+2}{4}H'^2(t) &\ge H(t)\big[H''(t) - (\rho+2)\big(\|u_t\|^2 + \|\nabla u_t\|^2\big)\big] \\ &= H(t)\big[-\rho\big(\|u_t\|^2 + \|\nabla u_t\|^2\big) - 2I(u)\big], \end{aligned} \tag{3.7}$$

where we denote $\rho = \min\{k,p\}$. Next, we treat the part $-\rho(\|u_t\|^2 + \|\nabla u_t\|^2)$ in the above estimate. By the energy identity $E(t) = E(0)$, we have

$$-\rho\big(\|u_t\|^2 + \|\nabla u_t\|^2\big) = 2\rho\big(J(u) - E(0)\big).$$

Substituting this into (3.7), we find

$$\begin{aligned} H(t)H''(t) - \frac{\rho+2}{4}H'^2(t) &\ge 2H(t)\big[\rho\big(J(u) - E(0)\big) - I(u)\big] \\ &\ge 2H(t)\big[\rho\big(J(u) - d\big) - I(u)\big]. \end{aligned} \tag{3.8}$$

From $u_0 \in V$ and Lemma 2.5, we have $u(t) \in V$, that is, $I(u) < 0$ for all $0 < t < \infty$, then Lemma 2.4(ii) holds, by (3.8), which leads to

$$H(t)H''(t) - \frac{\rho+2}{4}H'^2(t) \ge 0.$$

So,

$$\big(H^{-\beta}(t)\big)'' = \frac{-\beta}{H^{\beta+2}(t)}\big(H(t)H''(t) - (\beta+1)H'^2(t)\big) < 0, \quad \beta = \frac{\rho}{4}. \tag{3.9}$$

From (3.6), Lemma 2.4(ii), and $E(0) < d$, we see that

$$H''(t) \ge -2I(u) \ge 2\rho\big(d - J(u)\big) \ge 2\rho\big(d - E(0)\big),$$

and then

$$H'(t) \ge 2\rho\big(d - E(0)\big) + H'(0).$$

Hence there exists $t_0 \ge 0$ such that $H'(t) > H'(0) > 0$ for $t_0 < +\infty$ and

$$H(t) > H'(0)(t - t_0) + H(t_0) \ge H'(0)(t - t_0).$$

Consequently, there exists t_1 such that $H(t_1) > 0$ and $H'(t_1) > 0$. From this and (3.9), one can find $T_1 > 0$ such that $\lim_{t\to T_1} H^{-\beta}(t) = 0$, therefore $\lim_{t\to T_1} H(t) = +\infty$, which contradicts $T_m = +\infty$. Finally, from $T_m < +\infty$ and Theorem 2.1, we get the result. □

Remark Combining this method with the method of [18, 19, 27], we can also consider equations (1.7), (1.8).

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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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