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Local existence of the generalized solution for three-dimensional compressible viscous flow of micropolar fluid with cylindrical symmetry

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Abstract

In this work, the three-dimensional model for the compressible micropolar fluid flow is considered, whereby it is assumed that the fluid is viscous, perfect, and heat conducting. The flow between two coaxial thermoinsulated cylinders, which leads to a cylindrically symmetric model with homogeneous boundary data for velocity, microrotation, and heat flux, is analyzed.

The corresponding PDE system is formulated in the Lagrangian setting, and it is proven that this system has a generalized solution locally in time.

MSC: Primary 35D99; 35G61; 35Q35; secondary 76N99

Keywords: Micropolar fluid; Generalized solution; Cylindrical symmetry; Weak and strong convergence

1 Introduction

In this paper, we analyze the compressible flow of an isotropic, viscous, and heat-conducting micropolar fluid, whereby we consider the flow between two coaxial thermoinsulated solid cylinders. We also assume that the fluid is perfect and polytropic in the thermodynamical sense.

The micropolar fluid is a type of fluid which exhibits microrotational effects, as well as microrotational inertia, and it can be perceived as a collection of rigid particles suspended in a viscous medium, which can rotate about the centroid of the volume element. Consequently, it belongs to the class of viscous fluids with a non-symmetric stress tensor, hence the law of conservation of angular momentum must be taken into account. Therefore, in addition to the standard hydrodynamic and thermodynamic variables (mass density, velocity, and temperature), the microrotation vector is introduced to describe the micro phenomena. The micropolar fluid was introduced by Eringen (see [1]) as an extension of the Navier–Stokes model, capable of treating phenomena at the microlevel (see [2]).

As today's science is increasingly engaged with micro and nanotechnology, the need has emerged for models that can handle the impact of the scale. That is why the micropolar continuum has begun to be intensively studied in the last few years. Let us note that the micropolar fluid model has been applied as the model for blood flow (see [3]), water-based nanofluids (see [4]), mimicking physical phenomena of bacteria (see [5]), the behavior of the epididymal material (see [6]), describing lubricants with additives, the motion of the synovial fluid in the joints (see [7]), etc. Recently, the micropolar fluid model has been used to effectively treat some heating problems as well (see [8] and [9]).

The aforementioned model was first analyzed in the one-dimensional case by N. Mujaković in [10] and for the first time in [11] in the three-dimensional case with the assumptions of spherical symmetry of the solution by I. Dražić and N. Mujaković. For the recent progress in the mathematical analysis of these two models, we refer to [12] and [13], and for general theory to [14].

In this paper, assuming that initial functions are cylindrically symmetric and smooth enough, we prove the local existence of the generalized cylindrically symmetric solution to the governing system with homogeneous boundary conditions for velocity, microrotation, and heat flux. In the proof, we follow the ideas from [10] and [11], whereby we apply the Faedo–Galerkin method together with some ideas from [15], where this method was applied for the classical fluid model in the one-dimensional case. Let us mention that the utilization of the Faedo–Galerkin method does not require additional restrictions to initial data, unlike in other approaches. Let us also note that the obtained model is the generalization of the model for the classical fluid considered by Qin in [16] and [17].

The paper is organized as follows. In the next section we state the problem, define the generalized solution to our problem, and present the main result. In the third section, in line with the Faedo–Galerkin method, we introduce the approximate problem and form a series of approximate solutions. In the forth section, we derive some a priori estimates for the obtained approximate solutions, which are the base for the final proof of our main result (the local existence theorem), which is presented in the last section of this paper.

2 Statement of the problem and the main result

The governing initial boundary value problem is derived in [18] and reads as follows:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial}{\partial x}\left(r v^r\right) = 0, \tag{1}$$

$$\frac{\partial v^r}{\partial t} = -Rr\frac{\partial}{\partial x}(\rho\theta) + (\lambda + 2\mu) r \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r v^r\right)\right) + \frac{(v^\varphi)^2}{r}, \tag{2}$$

$$\frac{\partial v^\varphi}{\partial t} = (\mu + \mu_r) r \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r v^\varphi\right)\right) - \frac{v^r v^\varphi}{r} - 2\mu_r r \frac{\partial \omega^z}{\partial x}, \tag{3}$$

$$\frac{\partial v^z}{\partial t} = (\mu + \mu_r) r \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r v^z\right)\right) + (\mu + \mu_r)\frac{v^z}{\rho r^2} + 2\mu_r \frac{\partial}{\partial x}\left(r\omega^\varphi\right), \tag{4}$$

$$j_I \frac{\partial \omega^r}{\partial t} = (c_0 + 2c_d) r \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r \omega^r\right)\right) + j_I \frac{\omega^\varphi v^\varphi}{r} - 4\mu_r \frac{\omega^r}{\rho}, \tag{5}$$

$$j_I \frac{\partial \omega^\varphi}{\partial t} = (c_d + c_a) r \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r \omega^\varphi\right)\right) - j_I \frac{\omega^r v^\varphi}{r} - 2\mu_r r \frac{\partial v^z}{\partial x} - 4\mu_r \frac{\omega^\varphi}{\rho}, \tag{6}$$

$$j_I \frac{\partial \omega^z}{\partial t} = (c_d + c_a) r \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r \omega^z\right)\right) + (c_d + c_a)\frac{\omega^z}{\rho r^2} + 2\mu_r \frac{\partial}{\partial x}\left(r v^\varphi\right) - 4\mu_r \frac{\omega^z}{\rho}, \tag{7}$$

$$
\begin{aligned}
c_v \frac{\partial \theta}{\partial t} = {} & k \frac{\partial}{\partial x}\left(r^2 \rho \frac{\partial \theta}{\partial x}\right) + \rho\left[(\lambda + 2\mu)\frac{\partial}{\partial x}\left(r v^r\right) - R\theta\right]\frac{\partial}{\partial x}\left(r v^r\right) \\
& + (\mu + \mu_r)\rho\left(\frac{\partial}{\partial x}\left(r v^\varphi\right)\right)^2 + (c_d + c_a)\rho\left(\frac{\partial}{\partial x}\left(r\omega^\varphi\right)\right)^2 \\
& + (c_0 + 2c_d)\rho\left(\frac{\partial}{\partial x}\left(r\omega^r\right)\right)^2 + (\mu + \mu_r)\rho r^2\left(\frac{\partial v^z}{\partial x}\right)^2 \\
& + (c_d + c_a)\rho r^2\left(\frac{\partial \omega^z}{\partial x}\right)^2 - 2c_d \frac{\partial}{\partial x}\left(\left(\omega^r\right)^2 + \left(\omega^\varphi\right)^2\right) \\
& - 2\mu \frac{\partial}{\partial x}\left(\left(v^r\right)^2 + \left(v^\varphi\right)^2\right) + 4\mu_r \frac{(\omega^r)^2}{\rho} + 4\mu_r \frac{(\omega^\varphi)^2}{\rho} + 4\mu_r \frac{(\omega^z)^2}{\rho} \\
& + 4\mu_r r\omega^\varphi \frac{\partial v^z}{\partial x} - 4\mu_r \omega^z \frac{\partial}{\partial x}\left(r v^\varphi\right),
\end{aligned}
\tag{8}
$$

$$\rho(x,0) = \rho_0(x), \qquad v^r(x,0) = v_0^r(x), \qquad v^\varphi(x,0) = v_0^\varphi(x), \qquad v^z(x,0) = v_0^z(x), \tag{9}$$

$$\omega^r(x,0) = \omega_0^r(x), \qquad \omega^\varphi(x,0) = \omega_0^\varphi(x), \qquad \omega^z(x,0) = \omega_0^z(x), \qquad \theta(x,0) = \theta_0(x), \tag{10}$$

$$v^r(0,t) = v^r(L,t) = 0, \qquad v^\varphi(0,t) = v^\varphi(L,t) = 0, \qquad v^z(0,t) = v^z(L,t) = 0, \tag{11}$$

$$\omega^r(0,t) = \omega^r(L,t) = 0, \qquad \omega^\varphi(0,t) = \omega^\varphi(L,t) = 0, \qquad \omega^z(0,t) = \omega^z(L,t) = 0, \tag{12}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(L,t) = 0 \tag{13}$$

defined on the domain $Q_T =]0,L[\times]0,T[$, where

$$L = \int_a^b s\rho_0(s)\,ds. \tag{14}$$

Variables in this system are mass density ρ, velocity $\mathbf{v}$, microrotation $\boldsymbol{\omega}$, and temperature θ, and they are defined by

$$\rho(\mathbf{x},t) = \rho(r,t), \qquad \theta(\mathbf{x},t) = \theta(r,t), \tag{15}$$

$$\mathbf{v}(\mathbf{x},t) = v^r(r,t)\mathbf{e}_1 + v^\varphi(r,t)\mathbf{e}_2 + v^z(r,t)\mathbf{e}_3, \tag{16}$$

$$\boldsymbol{\omega}(\mathbf{x},t) = \omega^r(r,t)\mathbf{e}_1 + \omega^\varphi(r,t)\mathbf{e}_2 + \omega^z(r,t)\mathbf{e}_3, \tag{17}$$

where

$$\mathbf{e}_1 = \frac{1}{r}(x_1,x_2,0), \qquad \mathbf{e}_2 = \frac{1}{r}(-x_2,x_1,0), \qquad \mathbf{e}_3 = (0,0,1), \tag{18}$$

$$\mathbf{x} \in \Omega = \left\{(x_1,x_2,x_3) \in \mathbf{R}^3, a < r < b, x_3 \in \mathbf{R}\right\}, \quad a > 0, r = \sqrt{x_1^2 + x_2^2}. \tag{19}$$

Here, the positive constant j_I is microinertia density, λ and μ are the coefficients of viscosity, μ_r, c_0, c_a, and c_d are the coefficients of microviscosity, k ($k \geq 0$) is the heat conduction coefficient, the positive constant R is the specific gas constant, and the positive constant c_v is the specific heat for a constant volume. Coefficients of viscosity and microviscosity have the following properties:

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0, \qquad \mu_r \geq 0, \tag{20}$$

$$c_d \geq 0, \qquad 3c_0 + 2c_d \geq 0, \qquad |c_d - c_a| \leq c_d + c_a. \tag{21}$$

Let us note that equations (1)–(8) are local forms of conservation laws and, with boundary conditions (11)–(13), we describe the acting of the solid thermo-insulated walls.

This problem is written in the Lagrangian description, which is much simpler in comparison to the Eulerian description. Moreover, using the Lagrangian coordinates, we eliminate the hyperbolic part of the system, and the density equation becomes explicitly solvable once the velocity has been determined. At the same time, other equations remain parabolic. Because of this coordinate transform, described in [18], we have the property

$$\frac{\partial r}{\partial x}(x,t) = \frac{1}{\rho(x,t) r(x,t)}, \tag{22}$$

where

$$r(x,t) = r_0(x) + \int_0^t v^r(x,\tau)\,d\tau, \quad (x,t) \in Q_T, \tag{23}$$

$$r_0(x) = \left(a^2 + 2\int_0^x \frac{1}{\rho_0(y)}\,dy\right)^{\frac{1}{2}}, \tag{24}$$

and $a > 0$ is the radius of the smaller boundary cylinder.

The main purpose of this work is to prove that problem (1)–(13) has a generalized (weak) solution in the domain $Q_{T_0} =]0,L[\times]0,T_0[$ for sufficiently small time $T_0 > 0$.

Let us first introduce the vectors

$$\mathbf{V} = \left(v^r, v^\varphi, v^z\right), \qquad \mathbf{W} = \left(\omega^r, \omega^\varphi, \omega^z\right) \tag{25}$$

and the definition of a generalized solution.

Definition 1 A generalized solution of problem (1)–(13) in the domain Q_T is the function

$$(x,t) \mapsto (\rho, \mathbf{V}, \mathbf{W}, \theta)(x,t), \quad (x,t) \in Q_T, \tag{26}$$

where

$$\rho \in \mathrm{L}^\infty\left(0,T;\mathrm{H}^1\left(]0,L[\right)\right) \cap \mathrm{H}^1(Q_T), \quad \inf_{Q_T} \rho > 0, \tag{27}$$

$$\mathbf{V}, \mathbf{W} \in \left(\mathrm{L}^\infty\left(0,T;\mathrm{H}^1\left(]0,L[\right)\right)\right)^3 \cap \left(\mathrm{H}^1(Q_T)\right)^3 \cap \left(\mathrm{L}^2\left(0,T;\mathrm{H}^2\left(]0,L[\right)\right)\right)^3, \tag{28}$$

$$\theta \in \mathrm{L}^\infty\left(0,T;\mathrm{H}^1\left(]0,L[\right)\right) \cap \mathrm{H}^1(Q_T) \cap \mathrm{L}^2\left(0,T;\mathrm{H}^2\left(]0,L[\right)\right), \tag{29}$$

that satisfies equations (1)–(8) a.e. in Q_T and conditions (9)–(13) in the sense of traces.

It is important to note that function (26) has the properties of a strong solution. Because of the embedding and interpolation theorems (e.g., in [19] and [20]), we also have:

$$\rho \in \mathrm{L}^\infty\left(0,T;C\left([0,L]\right)\right) \cap C\left([0,T],\mathrm{L}^2\left(]0,L[\right)\right), \tag{30}$$

$$\mathbf{V}, \mathbf{W} \in \left(\mathrm{L}^2\left(0,T;\mathrm{C}^1\left([0,L]\right)\right)\right)^3 \cap \left(\mathrm{C}\left([0,T];\mathrm{H}^1\left(]0,L[\right)\right)\right)^3, \tag{31}$$

$$\theta \in \mathrm{L}^2\big(0,T;\mathrm{C}^1\big([0,L]\big)\big) \cap \mathrm{C}\big([0,T];\mathrm{H}^1\big(]0,L[\big)\big), \tag{32}$$

$$\mathbf{V},\mathbf{W} \in \big(\mathrm{C}(\overline{Q}_T)\big)^3, \tag{33}$$

$$\theta \in \mathrm{C}(\overline{Q}_T). \tag{34}$$

The aim of this paper is to prove the following theorem.

Theorem 1 *Let the functions*

$$\rho_0,\theta_0 \in \mathrm{H}^1\big(]0,L[\big), \quad v_0^r, v_0^\varphi, v_0^z, \omega_0^r, \omega_0^\varphi, \omega_0^z \in \mathrm{H}_0^1\big(]0,L[\big) \tag{35}$$

satisfy the conditions

$$\rho_0(x) \ge m, \qquad \theta_0(x) \ge m \quad \textit{for } x \in\,]0,L[\,, \tag{36}$$

where $m \in \mathbf{R}^+$. Then there exists T_0, $0 < T_0 \le T$, such that problem (1)–(13) has a generalized solution in $Q_0 = Q_{T_0}$, having the property

$$\theta > 0 \quad \textit{in } \overline{Q}_0. \tag{37}$$

For the function r, we have

$$r \in \mathrm{L}^\infty\big(0,T_0;\mathrm{H}^2\big(]0,L[\big)\big) \cap \mathrm{H}^2(Q_0) \cap \mathrm{C}(\overline{Q}_0), \tag{38}$$

$$\frac{a}{2} \le r \le 2M \quad \textit{in } \overline{Q}_0. \tag{39}$$

Remark 1 Because of the embedding $\mathrm{H}^1(]0,L[) \subset \mathrm{C}([0,L])$, we can conclude that there exists $M \in \mathbf{R}^+$ such that

$$\rho_0(x), \big|v_0^r(x)\big|, \big|v_0^\varphi(x)\big|, \big|v_0^z(x)\big|, \big|\omega_0^r(x)\big|, \big|\omega_0^\varphi(x)\big|, \big|\omega_0^z(x)\big|, \theta_0(x) \le M, \tag{40}$$

for $x \in [0,L]$.

The function r_0, introduced by (24), belongs to the space $\mathrm{H}^2(]0,L[) \subset \mathrm{C}^1([0,L])$, and we have

$$0 < a \le r_0(x) \le M, \qquad 0 < a_1 \le r_0'(x) \le M_1, \quad x \in [0,L], \tag{41}$$

where $a_1 = M^{-2}$, $M_1 = (ma)^{-1}$, a is from (24) and m from (36).

The proof of Theorem 1 is essentially based on the Faedo–Galerkin method. We first define the approximate problem (for each $n \in \mathbf{N}$). Then we derive uniform (in n) a priori estimates for approximate solutions, where we utilize the techniques applied in [10, 15], and [11] to similar models. Using the obtained estimates and results of weak compactness, we extract the subsequence of approximate solutions, which has a limit in some weak sense on $]0,L[\,\times\,]0,T_0[$ for sufficiently small $T_0 > 0$.

3 Approximate solutions

In [18], we have already introduced the Faedo–Galerkin approximations to problem (1)–(13), where we used them to find the numerical solution to problem (1)–(13). For the reader's convenience, we will describe it here briefly.

As we have already pointed out, we shall find a local generalized solution to problem (1)–(13) as a limit of approximate solutions

$$\left(\rho^n, v^{rn}, v^{\varphi n}, v^{zn}, \omega^{rn}, \omega^{\varphi n}, \omega^{zn}, \theta^n\right), \quad n \in \mathbf{N}, \tag{42}$$

where ρ^n, v^{rn}, $v^{\varphi n}$, v^{zn}, ω^{rn}, $\omega^{\varphi n}$, ω^{zn}, and θ^n are the approximations of the functions ρ, v^r, v^φ, v^z, ω^r, ω^φ, ω^z, and θ, respectively. We define them by

$$v^{rn}(x,t) = \sum_{i=1}^{n} v_i^{rn}(t) \sin \frac{\pi i x}{L}, \qquad v^{\varphi n}(x,t) = \sum_{i=1}^{n} v_i^{\varphi n}(t) \sin \frac{\pi i x}{L}, \tag{43}$$

$$v^{zn}(x,t) = \sum_{i=1}^{n} v_i^{zn}(t) \sin \frac{\pi i x}{L}, \qquad \omega^{rn}(x,t) = \sum_{j=1}^{n} \omega_j^{rn}(t) \sin \frac{\pi j x}{L}, \tag{44}$$

$$\omega^{\varphi n}(x,t) = \sum_{j=1}^{n} \omega_j^{\varphi n}(t) \sin \frac{\pi j x}{L}, \qquad \omega^{zn}(x,t) = \sum_{j=1}^{n} \omega_j^{zn}(t) \sin \frac{\pi j x}{L}, \tag{45}$$

$$\theta^n(x,t) = \sum_{k=0}^{n} \theta_k^n(t) \cos \frac{\pi k x}{L}. \tag{46}$$

We also define

$$r^n(x,t) = r_0(x) + \int_0^t v^{rn}(x,\tau)\,d\tau = r_0(x) + \sum_{i=1}^{n} \sin \frac{\pi i x}{L} \int_0^t v_i^{rn}(\tau)\,d\tau, \tag{47}$$

where $r_0(x)$ is defined by (24). v_i^{rn}, $v_i^{\varphi n}$, v_i^{zn}, $i = 1, 2, \ldots, n$, ω_j^{rn}, $\omega_j^{\varphi n}$, ω_j^{zn}, $j = 1, \ldots, n$ and θ_k^n, $k = 0, \ldots, n$, are unknown smooth functions defined on the interval $[0, T_n]$, $T_n \le T$.

The approximation ρ^n of the function ρ becomes the solution to the initial problem

$$\frac{\partial \rho^n}{\partial t} + \left(\rho^n\right)^2 \frac{\partial}{\partial x}\left(r^n v^{rn}\right) = 0, \quad \rho^n(x,0) = \rho_0(x), \tag{48}$$

and it can be written in the form

$$\rho^n(x,t) = \frac{\rho_0(x)}{1 + \rho_0(x) \frac{\partial}{\partial x} \int_0^t r^n v^{rn}\,d\tau}. \tag{49}$$

We also have

$$\frac{\partial r^n}{\partial x} = \frac{1}{\rho^n r^n}. \tag{50}$$

Since r^n and v^{rn} are sufficiently smooth functions, we can conclude that the function ρ^n is continuous on the rectangle $[0,L] \times [0,T_n]$ with the property $\rho^n(x,0) = \rho_0(x) \ge m > 0$. Because of the aforementioned, we can conclude that there exists such T_n, $0 < T_n \le T$, that

$$\rho^n(x,t) > 0, \quad \text{for } (x,t) \in [0,L] \times [0,T_n]. \tag{51}$$

Evidently, the boundary conditions

$$v^{rn}(0,t) = v^{rn}(L,t) = v^{\varphi n}(0,t) = v^{\varphi n}(L,t) = v^{zn}(0,t) = v^{zn}(L,t) = 0, \tag{52}$$

$$\omega^{rn}(0,t) = \omega^{rn}(L,t) = \omega^{\varphi n}(0,t) = \omega^{\varphi n}(L,t) = \omega^{zn}(0,t) = \omega^{zn}(L,t) = 0, \tag{53}$$

$$\frac{\partial \theta^n}{\partial x}(0,t) = \frac{\partial \theta^n}{\partial x}(L,t) = 0 \tag{54}$$

are satisfied, which is in accordance with boundary conditions (11)–(13) of the starting problem.

We take the initial conditions for v^{rn}, $v^{\varphi n}$, v^{zn}, ω^{rn}, $\omega^{\varphi n}$, ω^{zn}, and θ^n in the form:

$$v^{rn}(x,0) = v_0^{rn}(x), \qquad v^{\varphi n}(x,0) = v_0^{\varphi n}(x), \qquad v^{zn}(x,0) = v_0^{zn}(x), \tag{55}$$

$$\omega^{rn}(x,0) = \omega_0^{rn}(x), \qquad \omega^{\varphi n}(x,0) = \omega_0^{\varphi n}(x), \qquad \omega^{zn}(x,0) = \omega_0^{zn}(x), \tag{56}$$

$$\theta^n(x,0) = \theta_0^n(x), \quad x \in [0,L], \tag{57}$$

where v_0^{rn}, $v_0^{\varphi n}$, v_0^{zn}, ω_0^{rn}, $\omega_0^{\varphi n}$, ω_0^{zn}, and θ_0^n are defined by

$$v_0^{rn}(x) = \sum_{i=1}^{n} v_{0i}^{r} \sin\frac{\pi i x}{L}, \qquad v_0^{\varphi n}(x) = \sum_{i=1}^{n} v_{0i}^{\varphi} \sin\frac{\pi i x}{L}, \tag{58}$$

$$v_0^{zn}(x) = \sum_{i=1}^{n} v_{0i}^{z} \sin\frac{\pi i x}{L}, \qquad \omega_0^{rn}(x) = \sum_{i=1}^{n} \omega_{0i}^{r} \sin\frac{\pi j x}{L}, \tag{59}$$

$$\omega_0^{\varphi n}(x) = \sum_{i=1}^{n} \omega_{0i}^{\varphi} \sin\frac{\pi j x}{L}, \qquad \omega_0^{zn}(x) = \sum_{i=1}^{n} \omega_{0i}^{z} \sin\frac{\pi j x}{L}, \tag{60}$$

$$\theta_0^n(x) = \sum_{k=0}^{n} \theta_{0k} \cos\frac{\pi k x}{L}, \tag{61}$$

and v_{0i}^r, v_{0i}^φ, v_{0i}^z, ω_{0j}^r, ω_{0j}^φ, ω_{0j}^z, and θ_{0k} are the Fourier coefficients of the functions v_0^r, v_0^φ, v_0^z, ω_0^r, ω_0^φ, ω_0^z, and θ_0, respectively.

According to the Faedo–Galerkin method, we take the following approximate conditions:

$$\begin{aligned}&\int_0^L \biggl(\frac{\partial v^{rn}}{\partial t} + Rr^n \frac{\partial}{\partial x}\bigl(\rho^n \theta^n\bigr) - (\lambda + 2\mu) r^n \frac{\partial}{\partial x}\biggl(\rho^n \frac{\partial}{\partial x}\bigl(r^n v^{rn}\bigr)\biggr) \\ &\quad - \frac{(v^{\varphi n})^2}{r^n}\biggr) \sin\frac{\pi i_1 x}{L}\, dx = 0,\end{aligned} \tag{62}$$

$$\begin{aligned}&\int_0^L \biggl(\frac{\partial v^{\varphi n}}{\partial t} - (\mu + \mu_r) r^n \frac{\partial}{\partial x}\biggl(\rho^n \frac{\partial}{\partial x}\bigl(r^n v^{\varphi n}\bigr)\biggr) + \frac{v^{rn} v^{\varphi n}}{r^n} \\ &\quad + 2\mu_r r^n \frac{\partial \omega^{zn}}{\partial x}\biggr) \sin\frac{\pi i_2 x}{L}\, dx = 0,\end{aligned} \tag{63}$$

$$\begin{aligned}
&\int_0^L \left(\frac{\partial v^{zn}}{\partial t} - (\mu + \mu_r) r^n \frac{\partial}{\partial x}\left(\rho^n \frac{\partial}{\partial x}\left(r^n v^{zn}\right)\right) - (\mu + \mu_r)\frac{v^{zn}}{\rho^n (r^n)^2} \right. \\
&\quad \left. - 2\mu_r \frac{\partial}{\partial x}\left(r^n \omega^{\varphi n}\right)\right) \sin \frac{\pi i_3 x}{L}\, dx = 0, \qquad (64)\\
&\int_0^L \left(\frac{\partial \omega^{rn}}{\partial t} - \frac{c_0 + 2c_d}{j_I} r^n \frac{\partial}{\partial x}\left(\rho^n \frac{\partial}{\partial x}\left(r^n \omega^{rn}\right)\right) - \frac{\omega^{\varphi n} v^{\varphi n}}{r^n} \right. \\
&\quad \left. + 4\frac{\mu_r}{j_I}\frac{\omega^{rn}}{\rho^n}\right) \sin \frac{\pi j_1 x}{L}\, dx = 0, \qquad (65)\\
&\int_0^L \left(\frac{\partial \omega^{\varphi n}}{\partial t} - \frac{c_d + c_a}{j_I} r^n \frac{\partial}{\partial x}\left(\rho^n \frac{\partial}{\partial x}\left(r^n \omega^{\varphi n}\right)\right) + \frac{\omega^{rn} v^{\varphi n}}{r^n} + 2\frac{\mu_r}{j_I} r^n \frac{\partial v^{zn}}{\partial x} \right. \\
&\quad \left. + 4\frac{\mu_r}{j_I}\frac{\omega^{\varphi n}}{\rho^n}\right) \sin \frac{\pi j_2 x}{L}\, dx = 0, \qquad (66)\\
&\int_0^L \left(\frac{\partial \omega^{zn}}{\partial t} - \frac{c_d + c_a}{j_I} r^n \frac{\partial}{\partial x}\left(\rho^n \frac{\partial}{\partial x}\left(r^n \omega^{zn}\right)\right) - \frac{c_d + c_a}{j_I}\frac{\omega^{zn}}{\rho^n (r^n)^2} \right. \\
&\quad \left. - 2\frac{\mu_r}{j_I}\frac{\partial}{\partial x}\left(r^n v^{\varphi n}\right) + 4\frac{\mu_r}{j_I}\frac{\omega^{zn}}{\rho^n}\right) \sin \frac{\pi j_3 x}{L}\, dx = 0, \qquad (67)\\
&\int_0^L \left(\frac{\partial \theta^n}{\partial t} - \frac{k}{c_v}\frac{\partial}{\partial x}\left(\left(r^n\right)^2 \rho^n \frac{\partial \theta^n}{\partial x}\right) - \frac{\rho^n}{c_v}\left[(\lambda + 2\mu)\frac{\partial}{\partial x}\left(r^n v^{rn}\right) - R\theta^n\right] \right. \\
&\quad \times \frac{\partial}{\partial x}\left(r^n v^{rn}\right) - \frac{\mu + \mu_r}{c_v}\rho^n \left(\frac{\partial}{\partial x}\left(r^n v^{\varphi n}\right)\right)^2 - \frac{c_d + c_a}{c_v}\rho^n \left(\frac{\partial}{\partial x}\left(r^n \omega^{\varphi n}\right)\right)^2 \\
&\quad - \frac{c_0 + 2c_d}{c_v}\rho^n \left(\frac{\partial}{\partial x}\left(r^n \omega^{rn}\right)\right)^2 - \frac{\mu + \mu_r}{c_v}\rho^n \left(r^n\right)^2 \left(\frac{\partial v^{zn}}{\partial x}\right)^2 \\
&\quad - \frac{c_d + c_a}{c_v}\rho^n \left(r^n\right)^2 \left(\frac{\partial \omega^{zn}}{\partial x}\right)^2 + 2\frac{c_d}{c_v}\frac{\partial}{\partial x}\left(\left(\omega^{rn}\right)^2 + \left(\omega^{\varphi n}\right)^2\right) \\
&\quad + 2\frac{\mu}{c_v}\frac{\partial}{\partial x}\left(\left(v^{rn}\right)^2 + \left(v^{\varphi n}\right)^2\right) - 4\frac{\mu_r}{c_v}\frac{(\omega^{rn})^2}{\rho^n} - 4\frac{\mu_r}{c_v}\frac{(\omega^{\varphi n})^2}{\rho^n} - 4\frac{\mu_r}{c_v}\frac{(\omega^{zn})^2}{\rho^n} \\
&\quad \left. - 4\frac{\mu_r}{c_v} r^n \omega^{\varphi n}\frac{\partial v^{zn}}{\partial x} + 4\frac{\mu_r}{c_v}\omega^{zn}\frac{\partial}{\partial x}\left(r^n v^{\varphi n}\right)\right) \cos \frac{\pi k x}{L}\, dx = 0 \qquad (68)
\end{aligned}$$

for $i_1, i_2, i_3, j_1, j_2, j_3 = 1, \ldots, n$, $k = 0, 1, \ldots, n$.

To simplify the problem, we introduce the functions $z_m^n(t)$ and $\lambda_{pq}^n(t)$ by

$$z_m^n(t) = \int_0^t v_m^{rn}(\tau)\, d\tau, \quad m = 1, \ldots, n, \qquad (69)$$

$$\lambda_{pq}^n(t) = \int_0^t z_p^n(\tau) v_q^{rn}(\tau)\, d\tau, \quad p, q = 1, \ldots, n. \qquad (70)$$

Now, (47) and (49) could be written in the form

$$r^n(x,t) = r_0(x) + \sum_{i=1}^n z_i^n(t) \sin \frac{\pi i x}{L}, \qquad (71)$$

$$
\begin{aligned}
\rho^n(x,t) = \rho_0(x)\Biggl(1 + \rho_0(x)\sum_{j=1}^{n} z_j^n(t)\frac{\partial}{\partial x}\biggl(r_0(x)\sin\frac{\pi jx}{L}\biggr) \\
+ \rho_0(x)\sum_{i,j=1}^{n}\lambda_{ij}^n(t)\frac{\partial}{\partial x}\biggl(\sin\frac{\pi ix}{L}\sin\frac{\pi jx}{L}\biggr)\Biggr)^{-1}.
\end{aligned}
\tag{72}
$$

Taking into account (58)–(72), from (62)–(68) we obtain, for n^2+8n+1-tuple $\mathbf{u}_n$ (with coordinates $v_{i_1}^{rn}$, $v_{i_2}^{\varphi n}$, $v_{i_3}^{zn}$, $\omega_{j_1}^{rn}$, $\omega_{j_2}^{\varphi n}$, $\omega_{j_3}^{zn}$, θ_k^n, z_m^n, λ_{pq}^n, i_1, i_2, i_3, j_1, j_2, j_3, m, p, $q = 1,\dots,n$, $k = 0,1,\dots,n$), the following differential equation:

$$
\dot{\mathbf{u}}_n(t) = \mathbf{F}\bigl(\mathbf{u}_n(t)\bigr)
\tag{73}
$$

with the initial conditions $\mathbf{u}_n(0)$ defined by

$$
v_{i_1}^{rn}(0) = v_{0i_1}^r, \qquad v_{i_2}^{\varphi n}(0) = v_{0i_2}^{\varphi}, \qquad v_{i_3}^{zn}(0) = v_{0i_3}^z(x),
\tag{74}
$$

$$
\omega_{j_1}^{rn}(0) = \omega_{0j_1}^r, \qquad \omega_{j_2}^{\varphi n}(0) = \omega_{0j_2}^{\varphi}, \qquad \omega_{j_3}^{zn}(0) = \omega_{0j_3}^z,
\tag{75}
$$

$$
\theta_k^n(0) = \theta_{0k}, \qquad z_m^n(0) = 0, \qquad \lambda_{pq}^n(0) = 0.
\tag{76}
$$

Let us notice that the function $\mathbf{F}$ on the right-hand side of (73) satisfies the conditions of the Cauchy–Picard theorem, and we can conclude that problem (73)–(76) has a unique smooth solution on a sufficiently small domain $[0,T_n[$, $T_n \le T$. Because of (36), (40), and (41), we easily obtain the following statements.

Lemma 1 *For each $n \in N$, there exists such $T_n, 0 < T_n \le T$, and $Q_n =]0,L[\times]0,T_n[$, that the functions v^{rn}, $v^{\varphi n}$, v^{zn}, ω^{rn}, $\omega^{\varphi n}$, ω^{zn}, and θ^n belong to the class $\mathrm{C}^\infty(\overline{Q}_n)$ and satisfy conditions (55)–(57).*

Moreover, we have $\rho^n \in \mathrm{C}(\overline{Q}_n)$, $r^n \in \mathrm{C}^1(\overline{Q}_n)$, and

$$
\frac{m}{2} \le \rho^n(x,t) \le 2M,
\tag{77}
$$

$$
\frac{a}{2} \le r^n(x,t) \le 2M,
\tag{78}
$$

$$
\frac{a_1}{2} \le \frac{\partial r^n}{\partial x}(x,t) \le 2M_1,
\tag{79}
$$

on $\overline{Q}_n$. The constants m, a, a_1, M, and M_1 are introduced by (36), (40), and (41).

4 Some properties of approximate solutions

In the previous section, we showed that, for each $n \in \mathbf{N}$, there exists $T_n, 0 < T_n \le T$, such that the set Q_n is a domain of the nth approximate solution introduced in Lemma 1. Our first goal is to find such T_0, $0 < T_0 \le T$, that for each $n \in \mathbf{N}$ a solution $\mathbf{u}_n$ to problem (73)–(76) is defined on $[0,T_0]$. Therefore, the approximate functions (43)–(47) and (49) also exist on Q_0. For that we need some interrelationships between the functions ρ^n, v^{rn}, $v^{\varphi n}$, v^{zn}, ω^{rn}, $\omega^{\varphi n}$, ω^{zn}, and θ^n which we, using the ideas adapted from [11], state in the following lemmas.

Hereafter, we denote by $C > 0$ or $C_i > 0$ $(i = 1,2,\dots)$ a generic constant, independent of $n \in \mathbf{N}$, which may have different values in different places.

For simplicity reasons, we use the notations

$$\|f\| = \|f\|_{L^2(]0,L[)}, \tag{80}$$

$$\mathbf{V}^n = \left(v^{rn}, v^{\varphi n}, v^{zn}\right), \qquad \mathbf{V}^n(0) = \left(v_0^{rn}, v_0^{\varphi n}, v_0^{zn}\right), \tag{81}$$

as well as

$$\left\|\frac{\partial^\beta \mathbf{V}^n}{\partial x^\beta}(t)\right\| = \left(\left\|\frac{\partial^\beta v^{rn}}{\partial x^\beta}(t)\right\|^2 + \left\|\frac{\partial^\beta v^{\varphi n}}{\partial x^\beta}(t)\right\|^2 + \left\|\frac{\partial^\beta v^{zn}}{\partial x^\beta}(t)\right\|^2\right)^{\frac{1}{2}}, \tag{82}$$

$$\left\|\frac{\partial^\beta \mathbf{V}^n}{\partial t^\beta}(t)\right\| = \left(\left\|\frac{\partial^\beta v^{rn}}{\partial t^\beta}(t)\right\|^2 + \left\|\frac{\partial^\beta v^{\varphi n}}{\partial t^\beta}(t)\right\|^2 + \left\|\frac{\partial^\beta v^{zn}}{\partial t^\beta}(t)\right\|^2\right)^{\frac{1}{2}} \tag{83}$$

for $\beta = 0, 1, \ldots$. We use the same notations for the vector $\mathbf{W} = (\omega^{rn}, \omega^{\varphi n}, \omega^{zn})$.

In what follows, we use the inequalities

$$|f|^2 \le 2\|f\|\left\|\frac{\partial f}{\partial x}\right\|, \qquad \|f\| \le 2\left\|\frac{\partial f}{\partial x}\right\|, \qquad |f| \le 2\left\|\frac{\partial f}{\partial x}\right\|, \tag{84}$$

$$\left|\frac{\partial f}{\partial x}\right|^2 \le 2\left\|\frac{\partial f}{\partial x}\right\|\left\|\frac{\partial^2 f}{\partial x^2}\right\|, \qquad \left\|\frac{\partial f}{\partial x}\right\| \le 2\left\|\frac{\partial^2 f}{\partial x^2}\right\|, \qquad \left|\frac{\partial f}{\partial x}\right| \le 2\left\|\frac{\partial^2 f}{\partial x^2}\right\| \tag{85}$$

which are valid for the function f vanishing at $x = 0$ and $x = L$, and with the first derivative vanishing at some point $x \in [0, L]$. These inequalities satisfy the functions v^{rn}, $v^{\varphi n}$, v^{zn}, ω^{rn}, $\omega^{\varphi n}$, and ω^{zn}. The function θ^n satisfies only (85). Let us note that inequalities (84) and (85) follow from the Gagliardo–Ladyzhenskaya, the Friedrichs, and the Poincaré inequalities, adapted in accordance with the spaces of functions used in this work.

Hereafter, we use T_n, $0 < T_n \le T$, from Lemma 1.

Lemma 2 *For* $t \in [0, T_n]$, *we have*

$$\left\|\frac{\partial^2 r^n}{\partial x^2}(t)\right\|^2 \le C\left(1 + \int_0^t \left\|\frac{\partial^2 \mathbf{V}^n}{\partial x^2}(\tau)\right\|^2 d\tau\right). \tag{86}$$

Proof From (47) we have

$$\frac{\partial^2 r^n}{\partial x^2} = r_0'' + \int_0^t \frac{\partial^2 v^{rn}}{\partial x^2}\, d\tau \tag{87}$$

and by using Remark 1 we immediately obtain (86). □

Lemma 3 *For* $t \in [0, T_n]$, *we have*

$$\begin{aligned}&\|\mathbf{W}^n(t)\|^2 + \int_0^t \|\mathbf{W}^n(\tau)\|^2\, d\tau + \int_0^t \left\|\frac{\partial \mathbf{W}^n}{\partial x}(\tau)\right\|^2 d\tau \\ &\quad \le C\left(1 + \int_0^t \|\mathbf{V}^n(\tau)\|^2 d\tau\right).\end{aligned} \tag{88}$$

Proof After multiplying (65), (66), and (67), respectively, by $j_I\omega^{rn}_{j_1}$, $j_I\omega^{\varphi n}_{j_2}$, and $j_I\omega^{zn}_{j_3}$, summing over $j_1, j_2, j_3 = 1, \dots, n$, and by using formula (50), integration by parts, and boundary conditions, we obtain

$$\begin{aligned}
&\frac{j_I}{2}\frac{d}{dt}\left\|\mathbf{W}^n(t)\right\|^2 + (c_0 + 2c_d)\left(\int_0^L \frac{(\omega^{rn})^2}{(r^n)^2\rho^n}\,dx + \int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial \omega^{rn}}{\partial x}\right)^2 dx\right)\\
&\quad + (c_d + c_a)\left(\int_0^L \frac{(\omega^{\varphi n})^2}{(r^n)^2\rho^n}\,dx + \int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial \omega^{\varphi n}}{\partial x}\right)^2 dx\right.\\
&\quad \left. + \int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial \omega^{zn}}{\partial x}\right)^2 dx\right)\\
&\quad + 4\mu_r\left(\int_0^L \frac{(\omega^{rn})^2}{\rho^n}\,dx + \int_0^L \frac{(\omega^{\varphi n})^2}{\rho^n}\,dx + \int_0^L \frac{(\omega^{zn})^2}{\rho^n}\,dx\right)\\
&= 2\mu_r\left(\int_0^L \frac{v^{zn}}{r^n\rho^n}\omega^{\varphi n}\,dx + \int_0^L r^n v^{zn}\frac{\partial \omega^{\varphi n}}{\partial x}\,dx - \int_0^L r^n v^{\varphi n}\frac{\partial \omega^{zn}}{\partial x}\,dx\right). \qquad (89)
\end{aligned}$$

Now, we will use (77) and (78) as well as the Young inequality with the parameter $\varepsilon > 0$ applied to the integrals on the right-hand side of (89). We obtain

$$\begin{aligned}
&\frac{j_I}{2}\frac{d}{dt}\left\|\mathbf{W}^n(t)\right\|^2 + C_1\left(\left\|\mathbf{W}^n(t)\right\|^2 + \left\|\frac{\partial \mathbf{W}^n}{\partial x}(t)\right\|^2\right)\\
&\leq \varepsilon\left\|\mathbf{W}^n(t)\right\|^2 + \varepsilon\left\|\frac{\partial \mathbf{W}^n}{\partial x}(t)\right\|^2 + C_2\left\|\mathbf{V}^n(t)\right\|^2. \qquad (90)
\end{aligned}$$

Integrating (90) over $[0,t], 0 < t \leq T_n$, and taking into account that

$$\left\|\mathbf{W}^n(0)\right\|^2 = \left\|\omega_0^{rn}\right\|^2 + \left\|\omega_0^{\varphi n}\right\|^2 + \left\|\omega_0^{zn}\right\|^2 \leq C \qquad (91)$$

from (90), for sufficiently small $\varepsilon > 0$, we obtain (88). □

Lemma 4 *For $t \in [0, T_n]$, we have*

$$\left|\int_0^L \theta^n(x,t)\,dx\right| \leq C\left(1 + \left\|\frac{\partial \mathbf{V}^n}{\partial x}(t)\right\|^2 + \int_0^t \left\|\frac{\partial \mathbf{V}^n}{\partial x}(\tau)\right\|^2 d\tau\right). \qquad (92)$$

Proof First, we multiply (62) and (63), respectively, by $v^{rn}_{i_1}$ and $v^{\varphi n}_{i_2}$, sum over $i_1, i_2 = 1, \dots, n$, and add (68) for $k = 0$ (multiplied by c_v). After we integrate the resulting equality over $[0, L]$ and employ integration by parts, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\left\|v^{rn}(t)\right\|^2 + \frac{1}{2}\left\|v^{\varphi n}(t)\right\|^2 + c_v\int_0^L \theta^n(x,t)\,dx\right)\\
&= (c_d + c_a)\int_0^L \rho^n\left(\frac{\partial}{\partial x}\left(r^n\omega^{\varphi n}\right)\right)^2 dx + (c_0 + 2c_d)\int_0^L \rho^n\left(\frac{\partial}{\partial x}\left(r^n\omega^{rn}\right)\right)^2 dx\\
&\quad + (\mu + \mu_r)\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial v^{zn}}{\partial x}\right)^2 dx + (c_d + c_a)\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial \omega^{zn}}{\partial x}\right)^2 dx
\end{aligned}$$

$$+ 4\mu_r\left(\int_0^L \frac{(\omega^{rn})^2}{\rho^n}\,dx + \int_0^L \frac{(\omega^{\varphi n})^2}{\rho^n}\,dx + \int_0^L \frac{(\omega^{zn})^2}{\rho^n}\,dx\right)$$
$$+ 4\mu_r \int_0^L r^n\omega^{\varphi n}\left(\frac{\partial v^{zn}}{\partial x}\right)dx + 2\mu_r \int_0^L r^n v^{\varphi n}\left(\frac{\partial \omega^{zn}}{\partial x}\right)dx. \tag{93}$$

Integrating (93) over $[0,t]$, $0 < t \le T_n$, using

$$\|\mathbf{V}^n(0)\|^2 = \|v_0^{rn}\|^2 + \|v_0^{\varphi n}\|^2 + \|v_0^{zn}\|^2 \le C, \qquad \int_0^L \theta_0^n(x,t)\,dx \le \|\theta_0^n\| \le C, \tag{94}$$

the Young inequality, and properties (77)–(79), from (93) we have

$$\left|\int_0^L \theta^n(x,t)\,dx\right| \le C\biggl(1 + \|v^{rn}(t)\|^2 + \|v^{\varphi n}(t)\|^2 + \|v^{zn}(t)\|^2$$
$$+ \int_0^t \biggl(\|\omega^{rn}(\tau)\|^2 + \|\omega^{\varphi n}(\tau)\|^2 + \|\omega^{zn}(\tau)\|^2 + \left\|\frac{\partial v^{zn}}{\partial x}(\tau)\right\|^2$$
$$+ \left\|\frac{\partial \omega^{rn}}{\partial x}(\tau)\right\|^2 + \left\|\frac{\partial \omega^{\varphi n}}{\partial x}(\tau)\right\|^2 + \left\|\frac{\partial \omega^{zn}}{\partial x}(\tau)\right\|^2\biggr)d\tau\biggr). \tag{95}$$

Since, because of (84), we have

$$\|\mathbf{V}^n\|^2 \le 2\left\|\frac{\partial \mathbf{V}^n}{\partial x}\right\|^2, \tag{96}$$

taking into account (88), from (95) we easily obtain (92). □

Lemma 5 *For* $(x,t) \in \overline{Q}_n$, *we have*

$$|\theta^n(x,t)| \le C\left(1 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\| + \left\|\frac{\partial \mathbf{V}^n}{\partial x}(t)\right\|^2 + \int_0^t \left\|\frac{\partial \mathbf{V}^n}{\partial x}(\tau)\right\|^2 d\tau\right). \tag{97}$$

Proof Let $t \in [0,T_n]$ be fixed, but arbitrary. As the function θ^n is continuous with respect to the variable $x \in [0,L]$ (see Lemma 1), there exist such $x_1(t), x_2(t) \in [0,L]$ that we have

$$m_n(t) = \min_{x\in[0,L]} \theta^n(x,t) = \theta^n\bigl(x_1(t),t\bigr), \tag{98}$$
$$M_n(t) = \max_{x\in[0,L]} \theta^n(x,t) = \theta^n\bigl(x_2(t),t\bigr). \tag{99}$$

Now, using the Hölder inequality, we obtain

$$\theta^n(x,t) - m_n(t) = \int_{x_1}^x \frac{\partial \theta^n}{\partial x}(y,t)\,dy \le \sqrt{L}\left\|\frac{\partial \theta^n}{\partial x}(t)\right\|, \tag{100}$$

which implies

$$\theta^n(x,t) \le \sqrt{L}\left\|\frac{\partial \theta^n}{\partial x}(t)\right\| + \frac{1}{L}\left|\int_0^L \theta^n(x,t)\,dx\right|. \tag{101}$$

Analogously, using the function $M_n(t)$, we obtain

$$\theta^n(x,t) \geq -\sqrt{L}\left\| \frac{\partial \theta^n}{\partial x}(t) \right\| - \frac{1}{L}\left| \int_0^L \theta^n(x,t)\,dx \right|, \tag{102}$$

which together with (101) implies

$$\left|\theta^n(x,t)\right| \leq C\left(\left\| \frac{\partial \theta^n}{\partial x}(t) \right\| + \left| \int_0^L \theta^n(x,t)\,dx \right| \right). \tag{103}$$

Now, using (92), from (102) we immediately obtain (97). □

Lemma 6 *For* $t \in [0, T_n]$, *we have*

$$\left\| \frac{\partial \rho^n}{\partial x}(t) \right\|^2 \leq C\left(1 + \left(\int_0^t \left\| \frac{\partial^2 \mathbf{V}^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^2 \right). \tag{104}$$

Proof Taking the derivative of the function ρ^n represented by (49) with respect to x and by using (36) and (77)–(79), we obtain

$$\left| \frac{\partial \rho^n}{\partial x} \right| \leq C\left(\left|\rho_0'\right| + \int_0^t \left(\left|v^{rn}\right| \left| \frac{\partial^2 r^n}{\partial x^2} \right| + \left| \frac{\partial v^{rn}}{\partial x} \right| + \left| \frac{\partial^2 v^{rn}}{\partial x^2} \right| \right) d\tau \right).$$

With the help of (84) and (85) applied to the function v^{rn}, the Hölder, and the Young inequalities as well as (86), we obtain (104). □

Lemma 7 *For* $t \in [0, T_n]$, *we have*

$$\begin{aligned}
&\frac{d}{dt}\left(\left\| \frac{\partial \mathbf{V}^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \mathbf{W}^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) \\
&\quad + C_1\left(\left\| \frac{\partial^2 \mathbf{V}^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 \mathbf{W}^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 \right) \\
&\leq C\left(1 + \left\| \frac{\partial \mathbf{V}^n}{\partial x}(t) \right\|^{16} + \left\| \frac{\partial \mathbf{W}^n}{\partial x}(t) \right\|^{16} + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^{16} \right. \\
&\quad \left. + \left(\int_0^t \left\| \frac{\partial^2 \mathbf{V}^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^8 \right).
\end{aligned} \tag{105}$$

Proof We apply a similar procedure as in [11], Lemma 4.6, and in [10], Lemma 5.6. Multiplying (62)–(68), respectively, by $-\frac{(\pi i_1)^2}{L^2} v_{i_1}^{rn}$, $-\frac{(\pi i_2)^2}{L^2} v_{i_2}^{\varphi n}$, $-\frac{(\pi i_3)^2}{L^2} v_{i_3}^{zn}$, $-\frac{(\pi j_1)^2}{L^2} \omega_{j_1}^{rn}$, $-\frac{(\pi j_2)^2}{L^2} \omega_{j_2}^{\varphi n}$, $-\frac{(\pi j_3)^2}{L^2} \omega_{j_3}^{zn}$, and $-\frac{(\pi k)^2}{L^2} \theta_k^n$, after summation over $i_1, i_2, i_3, j_1, j_2, j_3, k = 1, 2, \ldots, n$, using (50) and addition of the obtained equations, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\| \frac{\partial \mathbf{V}^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \mathbf{W}^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) \\
&\quad + (\lambda + 2\mu) \int_0^L \rho^n \left(r^n\right)^2 \left(\frac{\partial^2 v^{rn}}{\partial x^2} \right)^2 dx + (\mu + \mu_r) \int_0^L \rho^n \left(r^n\right)^2 \left(\frac{\partial^2 v^{\varphi n}}{\partial x^2} \right)^2 dx \\
&\quad + (\mu + \mu_r) \int_0^L \rho^n \left(r^n\right)^2 \left(\frac{\partial^2 v^{zn}}{\partial x^2} \right)^2 dx + \frac{c_0 + 2c_d}{j_I} \int_0^L \rho^n \left(r^n\right)^2 \left(\frac{\partial^2 \omega^{rn}}{\partial x^2} \right)^2 dx
\end{aligned}$$

$$+\frac{c_d+c_a}{j_I}\int_0^L \rho^n\big(r^n\big)^2\left(\frac{\partial^2\omega^{\varphi n}}{\partial x^2}\right)^2 dx+\frac{c_d+c_a}{j_I}\int_0^L \rho^n\big(r^n\big)^2\left(\frac{\partial^2\omega^{zn}}{\partial x^2}\right)^2 dx$$
$$+\frac{k}{c_v}\int_0^1 \rho^n\big(r^n\big)^2\left(\frac{\partial^2\theta^n}{\partial x^2}\right)^2 dx=\sum_{p=1}^{61} I_p, \tag{106}$$

where

$$I_1=(\lambda+2\mu)\int_0^L \frac{v^{rn}}{(r^n)^2\rho^n}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx,\qquad I_2=-2(\lambda+2\mu)\int_0^L \frac{\partial v^{rn}}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx,$$
$$I_3=-(\lambda+2\mu)\int_0^L \big(r^n\big)^2\frac{\partial\rho^n}{\partial x}\frac{\partial v^{rn}}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx,\qquad I_4=R\int_0^L r^n\theta^n\frac{\partial\rho^n}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx,$$
$$I_5=R\int_0^L r^n\rho^n\frac{\partial\theta^n}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx,\qquad I_6=-\int_0^L \frac{(v^{\varphi n})^2}{r^n}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx,$$
$$I_7=(\mu+\mu_r)\int_0^L \frac{v^{\varphi n}}{(r^n)^2\rho^n}\frac{\partial^2 v^{\varphi n}}{\partial x^2}\,dx,\qquad I_8=-2(\mu+\mu_r)\int_0^L \frac{\partial v^{\varphi n}}{\partial x}\frac{\partial^2 v^{\varphi n}}{\partial x^2}\,dx,$$
$$I_9=-(\mu+\mu_r)\int_0^L \big(r^n\big)^2\frac{\partial\rho^n}{\partial x}\frac{\partial v^{\varphi n}}{\partial x}\frac{\partial^2 v^{\varphi n}}{\partial x^2}\,dx,\qquad I_{10}=2\mu_r\int_0^L r^n\frac{\partial\omega^{zn}}{\partial x}\frac{\partial^2 v^{\varphi n}}{\partial x^2}\,dx,$$
$$I_{11}=\int_0^L \frac{v^{rn}v^{\varphi n}}{r^n}\frac{\partial^2 v^{\varphi n}}{\partial x^2}\,dx,\qquad I_{12}=2(\mu+\mu_r)\int_0^L \frac{v^{zn}}{(r^n)^2\rho^n}\frac{\partial^2 v^{zn}}{\partial x^2}\,dx,$$
$$I_{13}=-2(\mu+\mu_r)\int_0^L \frac{\partial v^{zn}}{\partial x}\frac{\partial^2 v^{zn}}{\partial x^2}\,dx,\qquad I_{14}=-(\mu+\mu_r)\int_0^L \big(r^n\big)^2\frac{\partial\rho^n}{\partial x}\frac{\partial v^{zn}}{\partial x}\frac{\partial^2 v^{zn}}{\partial x^2}\,dx,$$
$$I_{15}=-2\mu_r\int_0^L \frac{\omega^{\varphi n}}{r^n\rho^n}\frac{\partial^2 v^{zn}}{\partial x^2}\,dx,\qquad I_{16}=-2\mu_r\int_0^L r^n\frac{\partial\omega^{\varphi n}}{\partial x}\frac{\partial^2 v^{zn}}{\partial x^2}\,dx,$$
$$I_{17}=\frac{c_0+2c_d}{j_I}\int_0^L \frac{\omega^{rn}}{(r^n)^2\rho^n}\frac{\partial^2\omega^{rn}}{\partial x^2}\,dx,\qquad I_{18}=-\frac{2(c_0+2c_d)}{j_I}\int_0^L \frac{\partial\omega^{rn}}{\partial x}\frac{\partial^2\omega^{rn}}{\partial x^2}\,dx,$$
$$I_{19}=-\frac{c_0+2c_d}{j_I}\int_0^L \big(r^n\big)^2\frac{\partial\rho^n}{\partial x}\frac{\partial\omega^{rn}}{\partial x}\frac{\partial^2\omega^{rn}}{\partial x^2}\,dx,\qquad I_{20}=-\int_0^L \frac{\omega^{\varphi n}v^{\varphi n}}{r^n}\frac{\partial^2\omega^{rn}}{\partial x^2}\,dx,$$
$$I_{21}=\frac{4\mu_r}{j_I}\int_0^L \frac{\omega^{rn}}{\rho^n}\frac{\partial^2\omega^{rn}}{\partial x^2}\,dx,\qquad I_{22}=\frac{c_d+c_a}{j_I}\int_0^L \frac{\omega^{\varphi n}}{(r^n)^2\rho^n}\frac{\partial^2\omega^{\varphi n}}{\partial x^2}\,dx,$$
$$I_{23}=-\frac{2(c_d+c_a)}{j_I}\int_0^L \frac{\partial\omega^{\varphi n}}{\partial x}\frac{\partial^2\omega^{\varphi n}}{\partial x^2}\,dx,$$
$$I_{24}=-\frac{c_d+c_a}{j_I}\int_0^L \big(r^n\big)^2\frac{\partial\rho^n}{\partial x}\frac{\partial\omega^{\varphi n}}{\partial x}\frac{\partial^2\omega^{\varphi n}}{\partial x^2}\,dx,$$
$$I_{25}=\int_0^L \frac{\omega^{rn}v^{\varphi n}}{r^n}\frac{\partial^2\omega^{\varphi n}}{\partial x^2}\,dx,\qquad I_{26}=\frac{4\mu_r}{j_I}\int_0^L \frac{\omega^{\varphi n}}{\rho^n}\frac{\partial^2\omega^{\varphi n}}{\partial x^2}\,dx,$$
$$I_{27}=\frac{2\mu_r}{j_I}\int_0^L r^n\frac{\partial v^{zn}}{\partial x}\frac{\partial^2\omega^{\varphi n}}{\partial x^2}\,dx,\qquad I_{28}=\frac{2(c_d+c_a)}{j_I}\int_0^L \frac{\omega^{zn}}{(r^n)^2\rho^n}\frac{\partial^2\omega^{zn}}{\partial x^2}\,dx,$$
$$I_{29}=-\frac{2(c_d+c_a)}{j_I}\int_0^L \frac{\partial\omega^{zn}}{\partial x}\frac{\partial^2\omega^{zn}}{\partial x^2}\,dx,$$
$$I_{30}=-\frac{c_d+c_a}{j_I}\int_0^L \big(r^n\big)^2\frac{\partial\rho^n}{\partial x}\frac{\partial\omega^{zn}}{\partial x}\frac{\partial^2\omega^{zn}}{\partial x^2}\,dx,$$
$$I_{31}=\frac{4\mu_r}{j_I}\int_0^L \frac{\omega^{zn}}{\rho^n}\frac{\partial^2\omega^{zn}}{\partial x^2}\,dx,\qquad I_{32}=-\frac{2\mu_r}{j_I}\int_0^L r^n\frac{\partial v^{\varphi n}}{\partial x}\frac{\partial^2\omega^{zn}}{\partial x^2}\,dx,$$

$$I_{33} = -\frac{2\mu_r}{j_I}\int_0^L \frac{v^{\varphi n}}{r^n\rho^n}\frac{\partial^2\omega^{zn}}{\partial x^2}\,dx, \qquad I_{34} = -\frac{2k}{c_v}\int_0^L \frac{\partial\theta^n}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{35} = -\frac{k}{c_v}\int_0^L \left(r^n\right)^2\frac{\partial\rho^n}{\partial x}\frac{\partial\theta^n}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx, \qquad I_{36} = \frac{R}{c_v}\int_0^L \frac{v^{rn}}{r^n}\theta^n\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{37} = \frac{R}{c_v}\int_0^L \rho^n r^n\theta^n\frac{\partial v^{rn}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx, \qquad I_{38} = -\frac{2(\lambda+2\mu)}{c_v}\int_0^L v^{rn}\frac{\partial v^{rn}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{39} = -\frac{\lambda+2\mu}{c_v}\int_0^L \frac{(v^{rn})^2}{(r^n)^2\rho^n}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{40} = -\frac{\lambda+2\mu}{c_v}\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial v^{rn}}{\partial x}\right)^2\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{41} = -\frac{2(\mu+\mu_r)}{c_v}\int_0^L v^{\varphi n}\frac{\partial v^{\varphi n}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx, \qquad I_{42} = -\frac{\mu+\mu_r}{c_v}\int_0^L \frac{(v^{\varphi n})^2}{(r^n)^2\rho^n}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{43} = -\frac{\mu+\mu_r}{c_v}\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial v^{\varphi n}}{\partial x}\right)^2\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{44} = -\frac{2(c_d+c_a)}{c_v}\int_0^L \omega^{\varphi n}\frac{\partial\omega^{\varphi n}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{45} = -\frac{c_d+c_a}{c_v}\int_0^L \frac{(\omega^{\varphi n})^2}{(r^n)^2\rho^n}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{46} = -\frac{c_d+c_a}{c_v}\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial\omega^{\varphi n}}{\partial x}\right)^2\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{47} = -\frac{2(c_0+2c_d)}{c_v}\int_0^L \omega^{rn}\frac{\partial\omega^{rn}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{48} = -\frac{c_0+2c_d}{c_v}\int_0^L \frac{(\omega^{rn})^2}{(r^n)^2\rho^n}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{49} = -\frac{c_0+2c_d}{c_v}\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial\omega^{rn}}{\partial x}\right)^2\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{50} = -\frac{\mu+\mu_r}{c_v}\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial v^{zn}}{\partial x}\right)^2\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{51} = -\frac{c_d+c_a}{c_v}\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial\omega^{zn}}{\partial x}\right)^2\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{52} = \frac{4\mu}{c_v}\int_0^L v^{rn}\frac{\partial v^{rn}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{53} = \frac{4\mu}{c_v}\int_0^L v^{\varphi n}\frac{\partial v^{\varphi n}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx, \qquad I_{54} = \frac{4c_d}{c_v}\int_0^L \omega^{rn}\frac{\partial\omega^{rn}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{55} = \frac{4c_d}{c_v}\int_0^L \omega^{\varphi n}\frac{\partial\omega^{\varphi n}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx, \qquad I_{56} = -\frac{4\mu_r}{c_v}\int_0^L \frac{(\omega^{rn})^2}{\rho^n}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{57} = -\frac{4\mu_r}{c_v}\int_0^L \frac{(\omega^{\varphi n})^2}{\rho^n}\frac{\partial^2\theta^n}{\partial x^2}\,dx, \qquad I_{58} = -\frac{4\mu_r}{c_v}\int_0^L \frac{(\omega^{zn})^2}{\rho^n}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{59} = -\frac{4\mu_r}{c_v}\int_0^L r^n\omega^{\varphi n}\frac{\partial v^{zn}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx, \qquad I_{60} = \frac{4\mu_r}{c_v}\int_0^L \frac{\omega^{zn}v^{\varphi n}}{r^n\rho^n}\frac{\partial^2\theta^n}{\partial x^2}\,dx,$$

$$I_{61} = \frac{4\mu_r}{c_v}\int_0^L r^n\omega^{zn}\frac{\partial v^{\varphi n}}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}\,dx.$$

Now, we will estimate integrals I_1–I_{61}. Taking into account (77)–(79) as well as (84) and the Young inequality, we obtain

$$\begin{aligned} |I_1| &= (\lambda+2\mu)\left|\int_0^L \frac{v^{rn}}{(r^n)^2\rho^n}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx\right| \le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 + C\left\|v^{rn}(t)\right\|^2 \\ &\le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 + C\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^2 \le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 + C\left(\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^{16}+1\right). \end{aligned} \tag{107}$$

In the same way, we estimate integrals I_7, I_{12}, I_{15}, I_{17}, I_{21}, I_{22}, I_{26}, I_{28}, I_{31}, and I_{33}.

We estimate integrals I_2, I_5, I_8, I_{10}, I_{13}, I_{16}, I_{18}, I_{23}, I_{27}, I_{29}, I_{32}, and I_{34} by using the Young inequality, and in some cases by using (77) and (78). For instance, we have

$$\begin{aligned} |I_2| &= 2(\lambda+2\mu)\left|\int_0^L \frac{\partial v^{rn}}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx\right| \\ &\le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 + C\left(\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^{16}+1\right). \end{aligned} \tag{108}$$

Now, we estimate integral I_3. To do it, we need (84), the Hölder, and the Young inequalities, as well as (78) and (104). We have

$$\begin{aligned} |I_3| &= (\lambda+2\mu)\left|\int_0^L \left(r^n\right)^2\frac{\partial\rho^n}{\partial x}\frac{\partial v^{rn}}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx\right| \\ &\le C\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^{\frac12}\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^{\frac12}\int_0^L\left|\frac{\partial\rho^n}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\right|dx \\ &\le C\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^{\frac12}\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^{\frac32}\left\|\frac{\partial\rho^n}{\partial x}(t)\right\| \\ &\le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 + C\left(\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^4+\left\|\frac{\partial\rho^n}{\partial x}(t)\right\|^8\right) \\ &\le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^{16}+\left(\int_0^t\left\|\frac{\partial^2\mathbf{V}^n}{\partial x^2}(\tau)\right\|^2d\tau\right)^8\right). \end{aligned} \tag{109}$$

We use the same approach in the estimates of integrals I_9, I_{14}, I_{19}, I_{24}, I_{30}, and I_{35}.

To estimate integral I_4, we need the Hölder and the Young inequalities, as well as (77), (78), (84), (97), and (104). We have

$$\begin{aligned} |I_4| &= R\left|\int_0^L r^n\theta^n\frac{\partial\rho^n}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx\right| \le C\max_{x\in[0,L]}\left|\theta^n\right|\int_0^L\left|\frac{\partial\rho^n}{\partial x}\frac{\partial^2 v^{rn}}{\partial x^2}\right|dx \\ &\le C\max_{x\in[0,L]}\left|\theta^n\right|\left\|\frac{\partial\rho^n}{\partial x}(t)\right\|\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\| \le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 \\ &\quad + C\left(\max_{x\in[0,L]}\left|\theta^n\right|^2+\left\|\frac{\partial\rho^n}{\partial x}(t)\right\|^2\right) \le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^{16} + C\left(1+\left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^{16}\right. \\ &\quad \left. +\left\|\frac{\partial\mathbf{V}^n}{\partial x}(t)\right\|^{16}+\left(\int_0^t\left\|\frac{\partial\mathbf{V}^n}{\partial x}(\tau)\right\|^2d\tau\right)^8+\left(\int_0^t\left\|\frac{\partial^2\mathbf{V}^n}{\partial x^2}(\tau)\right\|^2d\tau\right)^8\right). \end{aligned} \tag{110}$$

In the same way we estimate integrals I_{36} and I_{37}.

We base the estimate of integral I_6 on (77), (78), (84), and (85) as well as on the Young inequality. We obtain

$$|I_6| = \left|\int_0^L \frac{(v^{\varphi n})^2}{r^n}\frac{\partial^2 v^{rn}}{\partial x^2}\,dx\right| \le \varepsilon\left\|\frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial v^{\varphi n}}{\partial x}(t)\right\|^{16}\right). \tag{111}$$

We perform the estimates of integrals I_{39}, I_{40}, I_{42}, I_{43}, I_{45}, I_{46}, I_{48}, I_{49}, I_{50}, I_{51}, and I_{56}–I_{58} analogously.

We still have to estimate integrals I_{11}, I_{20}, I_{25}, I_{38}, I_{41}, I_{44}, I_{47}, I_{52}–I_{55}, and I_{59}–I_{61}. To do this, we use (77), (85), and the Young and the Hölder inequalities. For instance, we have

$$\begin{aligned}
|I_{11}| &= \left|\int_0^L \frac{v^{rn}v^{\varphi n}}{r^n}\frac{\partial^2 v^{\varphi n}}{\partial x^2}\,dx\right| \le C\left\|\frac{\partial v^{\varphi n}}{\partial x}(t)\right\|\int_0^L\left|v^{rn}\frac{\partial^2 v^{\varphi n}}{\partial x^2}\right|dx \\
&\le C\left\|\frac{\partial v^{\varphi n}}{\partial x}(t)\right\|\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|\left\|\frac{\partial^2 v^{\varphi n}}{\partial x^2}(t)\right\| \\
&\le \varepsilon\left\|\frac{\partial^2 v^{\varphi n}}{\partial x^2}(t)\right\|^2 + C\left(\left\|\frac{\partial v^{\varphi n}}{\partial x}(t)\right\|^4+\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^4\right) \\
&\le \varepsilon\left\|\frac{\partial^2 v^{\varphi n}}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial v^{\varphi n}}{\partial x}(t)\right\|^{16}+\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^{16}\right).
\end{aligned} \tag{112}$$

Using the obtained estimates with a sufficiently small ε together with (77)–(79), from (106) we obtain (105). □

Lemma 8 *There exists such T_0 $(0 < T_0 \le T)$ that, for each $n \in \mathbf{N}$, the Cauchy problem* (73)–(76) *has a unique solution defined on* $[0, T_0]$. *Moreover, for the functions $\mathbf{V}^n$, $\mathbf{W}^n$, θ^n, ρ^n, and r^n, we have*

$$\begin{aligned}
&\max_{t\in[0,T_0]}\left(\left\|\frac{\partial \mathbf{V}^n}{\partial x}(t)\right\|^2+\left\|\frac{\partial \mathbf{W}^n}{\partial x}(t)\right\|^2+\left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^2\right) \\
&\quad + C_1\int_0^{T_0}\left(\left\|\frac{\partial^2 \mathbf{V}^n}{\partial x^2}(t)\right\|^2+\left\|\frac{\partial^2 \mathbf{W}^n}{\partial x^2}(t)\right\|^2+\left\|\frac{\partial^2 \theta^n}{\partial x^2}(\tau)\right\|^2\right)dt \le C_2,
\end{aligned} \tag{113}$$

$$\frac{a}{2} \le r^n(x,t) \le 2M, \qquad \frac{a_1}{2} \le \frac{\partial r^n}{\partial x}(x,t) \le 2M_1, \tag{114}$$

$$\frac{m}{2} \le \rho^n(x,t) \le 2M, \quad (x,t)\in \overline{Q}_0, \tag{115}$$

$$\max_{t\in[0,T_0]}\left\|\frac{\partial \rho^n}{\partial x}(t)\right\| \le C, \tag{116}$$

$$\max_{(x,t)\in\overline{Q}_0}\left|\theta^n(x,t)\right| \le C, \tag{117}$$

$$\max_{t\in[0,T_0]}\left\|\frac{\partial^2 r^n}{\partial x^2}(t)\right\| \le C, \tag{118}$$

$$\max_{t\in[0,T_0]}\left(\left\|\mathbf{V}^n(t)\right\|^2+\left\|\mathbf{W}^n(t)\right\|^2+\left\|\theta^n(t)\right\|^2\right) \le C, \tag{119}$$

where a, a_1, m, and M are defined by (36) *and* (40)–(41).

Proof To obtain estimate (113), we use a similar approach as in [11], Lemma 4.7, [10], Lemma 5.6, and [15] pp. 64–67. First, we introduce the function

$$y_n(t) = \left\| \frac{\partial \mathbf{V}^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \mathbf{W}^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 + C_1 \int_0^t \left\| \frac{\partial^2 \mathbf{V}^n}{\partial x^2}(\tau) \right\|^2 d\tau, \tag{120}$$

where C_1 is the constant introduced in (105). Using Lemma 7, we find that function y_n satisfies the differential inequality

$$\dot{y}_n(t) \le C\bigl(1 + y_n^8(t)\bigr). \tag{121}$$

Let $\overline{C}$ be a constant defined by

$$\overline{C} = \left\| \frac{d\mathbf{V}_0}{dx} \right\|^2 + \left\| \frac{d\mathbf{W}_0}{dx} \right\|^2 + \left\| \frac{d\theta_0}{dx} \right\|^2, \tag{122}$$

where $\mathbf{V}_0 = (v_0^r, v_0^\varphi, v_0^z)$ and $\mathbf{W}_0 = (\omega_0^r, \omega_0^\varphi, \omega_0^z)$. It is easy to see that we have

$$y_n(0) \le \overline{C}. \tag{123}$$

Now, we compare the solution of problem (121)–(123) with the solution of the Cauchy problem

$$\dot{y}(t) = C\bigl(1 + y^8(t)\bigr), \tag{124}$$

$$y(0) = \overline{C}. \tag{125}$$

Let $[0, T'[$, $0 < T' \le T$ be an existence interval of the solution to problem (124)–(125). Because of the property of the maximal solution for problem (121)–(123), we conclude that

$$y_n(t) \le y(t), \quad t \in [0, T'[. \tag{126}$$

Let T_0 be such that $0 < T_0 \le T'$. From (126) we have

$$\max_{t\in[0,T_0]} y_n(t) \le \max_{t\in[0,T_0]} y(t) = C_3, \tag{127}$$

which together with (120) and (105) implies

$$\begin{aligned} &\frac{d}{dt}\left(\left\| \frac{\partial \mathbf{V}^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \mathbf{W}^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) \\ &\quad + C_1 \left(\left\| \frac{\partial^2 \mathbf{V}^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 \mathbf{W}^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 \right) \le C_4. \end{aligned} \tag{128}$$

Integrating (128) over $[0, t]$, $0 < t \le T_0$ and using (123), we immediately obtain (113).

Now, using inequalities (84)–(85) for the function v^{rn} and (113), we derive the following estimate:

$$\int_0^{T_0} \left|v^{rn}(x,t)\right| d\tau \le 4\int_0^{T_0} \left\| \frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\| d\tau$$
$$\le 4\left(\int_0^{T_0} \left\| \frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\|^2 d\tau\right)^{\frac{1}{2}} T_0^{\frac{1}{2}} \le 4\left(C_2 C_1^{-1}\right)^{\frac{1}{2}} T_0^{\frac{1}{2}}. \tag{129}$$

In the same way we get

$$\int_0^{T_0} \left| \frac{\partial v^{rn}}{\partial x}(t)\right| d\tau \le 2\int_0^{T_0} \left\| \frac{\partial^2 v^{rn}}{\partial x^2}(t)\right\| d\tau \le 2\left(C_2 C_1^{-1}\right)^{\frac{1}{2}} T_0^{\frac{1}{2}}, \tag{130}$$

where C_1 and C_2 are taken from (113). Estimates (129) and (130) are also valid for the functions $v^{\varphi n}$, v^{zn}, ω^{rn}, $\omega^{\varphi n}$, and ω^{zn}.

With the help of (41), (129), and (130), we can easily conclude that for

$$T_0 = \min\left\{ T', \frac{a^2 C_1}{64 C_2}, \frac{a_1^2 C_1}{16 C_2}, \frac{C_1}{64 M^2 (2M_1 + M)} \right\} \tag{131}$$

from (47) and (49) we obtain (114) and (115).

Because of (96) and the same inequality for the function $\mathbf{W}$, from (113) for $t \in [0, T_0]$, we obtain

$$\left\|\mathbf{V}^n\right\|^2 + \left\|\mathbf{W}^n\right\|^2 \le C. \tag{132}$$

From (113) and (46) we obtain

$$\left\| \frac{\partial \theta^n}{\partial x}(t)\right\| = \sum_{k=1}^n \left(\theta_k^n(t)\right)^2 \left(\frac{\pi k}{L}\right)^2 \int_0^L \sin^2 \frac{\pi k x}{L}\, dx = \sum_{k=1}^n \left(\theta_k^n(t)\right)^2 \frac{(\pi k)^2}{2L} \le C_2. \tag{133}$$

Also, taking into account (113) from (93), for $t \in [0, T_0]$, we have

$$\left|\theta_0^n(t) L\right| = \left| \int_0^L \theta^n(x,t)\, dx \right| \le C. \tag{134}$$

Therefore, we obtain

$$\max_{t\in[0,T_0]} \left\|\theta^n(t)\right\|^2 \le C. \tag{135}$$

Estimates (132) and (135) give (119). Finally, from (104), (97), and (86), we immediately get (116), (117), and (118), respectively. □

Lemma 9 *Let T_0 be defined by Lemma 8. Then, for each $n \in \mathbf{N}$, we have*

$$\int_0^{T_0} \left(\left\| \frac{\partial \mathbf{V}^n}{\partial t}(\tau)\right\|^2 + \left\| \frac{\partial \mathbf{W}^n}{\partial t}(\tau)\right\|^2 + \left\| \frac{\partial \theta^n}{\partial t}(\tau)\right\|^2 \right) d\tau \le C, \tag{136}$$

$$\max_{t\in[0,T_0]} \left\| \frac{\partial \rho^n}{\partial t}(t)\right\| \le C, \tag{137}$$

$$\begin{aligned} &\max_{t\in[0,T_0]}\left\|\frac{\partial r^n}{\partial t}(t)\right\|\leq C, \qquad \max_{t\in[0,T_0]}\left\|\frac{\partial^2 r^n}{\partial x\partial t}(t)\right\|\leq C, \\ &\int_0^{T_0}\left\|\frac{\partial^2 r^n}{\partial x^2}(\tau)\right\|^2 d\tau\leq C. \end{aligned} \tag{138}$$

Proof First, we multiply (62)–(68) respectively by $\frac{dv_i^{rn}}{dt}$, $\frac{dv_i^{\varphi n}}{dt}$, $\frac{dv_i^{zn}}{dt}$, $\frac{\omega_j^{rn}}{dt}$, $\frac{\omega_j^{\varphi n}}{dt}$, $\frac{\omega_j^{zn}}{dt}$, $\frac{\theta_k^n}{dt}$, summarize over $i_1,i_2,i_3,j_1,j_2,j_3=1,2,\ldots,n$, $k=0,1,2,\ldots,n$. After we integrate the obtained equality over $[0,L]$, we get the equality which is very similar to (106). We estimate it in the same way as in the proof of Lemma 7 and obtain (136).

From (48), after we use the Hölder inequality as well as (77) and (78), we get

$$\left\|\frac{\partial\rho^n}{\partial t}(t)\right\|^2\leq C\left(\left\|v^{rn}\right\|^2+\left\|v^{rn}\right\|\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|+\left\|\frac{\partial v^{rn}}{\partial x}(t)\right\|^2\right). \tag{139}$$

Using (113) and (119) from (139), by integrating over $[0,T_0]$, we easily obtain (137). Estimates (138) follow directly from (47) and (49). □

Using the results from Lemmas 8 and 9, we easily derive the following statements.

Proposition 1 *Let T_0 be defined by Lemma 8. Then, for the sequence*

$$\left\{\left(r^n,\rho^n,\mathbf{V}^n,\mathbf{W}^n,\theta^n\right):n\in\mathbf{N}\right\}, \tag{140}$$

we have:

(i) $\{r^n\}$ *is bounded in* $\mathrm{L}^\infty(Q_0)$, $\mathrm{L}^\infty(0,T_0;\mathrm{H}^2(]0,L[))$ *and* $\mathrm{H}^2(Q_0)$;
(ii) $\{\frac{\partial r^n}{\partial x}\}$ *is bounded in* $\mathrm{L}^\infty(Q_0)$;
(iii) $\{\rho^n\}$ *is bounded in* $\mathrm{L}^\infty(Q_0)$, $\mathrm{L}^\infty(0,T_0;\mathrm{H}^1(]0,L[))$, *and* $\mathrm{H}^1(Q_0)$;
(iv) $\{\mathbf{V}^n\}$ *and* $\{\mathbf{W}^n\}$ *are bounded in* $(\mathrm{L}^\infty(0,T_0;\mathrm{H}^1(]0,L[)))^3$, $(\mathrm{H}^1(Q_0))^3$, *and* $(\mathrm{L}^2(0,T_0;\mathrm{H}^2(]0,L[)))^3$;
(v) $\{\theta^n\}$ *is bounded in* $\mathrm{L}^\infty(0,T_0;\mathrm{H}^1(]0,L[))$, $\mathrm{H}^1(Q_0)$, *and* $\mathrm{L}^2(0,T_0;\mathrm{H}^2(]0,L[))$.

5 The proof of Theorem 1

To prove Theorem 1, we extract the convergent subsequence of sequence (140) and show that the limit of this subsequence is a solution to our problem. The proof is very similar to the proof of Theorem 2.1 in [11]; therefore, we omit the details of some proofs hereafter and refer to the corresponding results from [11].

Let $T_0\in\mathbf{R}^+$ be defined by Lemma 8. Theorem 1 follows directly from the following lemmas.

Lemma 10 (in [11], Lemma 5.1) *There exist a function*

$$r\in\mathrm{L}^\infty\left(0,T_0;\mathrm{H}^2\left(]0,L[\right)\right)\cap\mathrm{H}^2(Q_0)\cap\mathrm{C}(\overline{Q}_0) \tag{141}$$

and a subsequence (*for simplicity reasons denoted again as* $\{r^n\}$) *of* $\{r^n\}$ *with the properties*

$$r^n\overset{*}{\rightharpoonup}r\quad in\ \mathrm{L}^\infty\left(0,T_0;\mathrm{H}^2\left(]0,L[\right)\right), \tag{142}$$

$$r^n\rightharpoonup r\quad in\ \mathrm{H}^2(Q_0), \tag{143}$$

$$r^n \to r \quad in\ \mathrm{C}(\overline{Q}_0), \tag{144}$$

$$\frac{\partial r^n}{\partial x} \to \frac{\partial r}{\partial x} \quad in\ \mathrm{C}(\overline{Q}_0). \tag{145}$$

The function r satisfies the conditions

$$\frac{a}{2} \le r \le 2M \quad in\ \overline{Q}_0, \tag{146}$$

$$r(x,0) = r_0(x), \quad x \in [0,L], \tag{147}$$

where r_0 is defined by (24).

Lemma 11 (in [11], Lemma 5.2) *There exists a function*

$$\rho \in \mathrm{L}^\infty\bigl(0,T_0;\mathrm{H}^1(]0,L[)\bigr) \cap \mathrm{H}^1(Q_0) \cap \mathrm{C}(\overline{Q}_0) \tag{148}$$

and a subsequence (*for simplicity reasons denoted again as* $\{\rho^n\}$) *of* $\{\rho^n\}$ *with the properties*

$$\rho^n \overset{*}{\rightharpoonup} \rho \quad in\ \mathrm{L}^\infty\bigl(0,T_0;\mathrm{H}^1(]0,L[)\bigr), \tag{149}$$

$$\rho^n \rightharpoonup \rho \quad in\ \mathrm{H}^1(Q_0), \tag{150}$$

$$\rho^n \to \rho \quad in\ \mathrm{C}(\overline{Q}_0). \tag{151}$$

The function ρ satisfies the conditions

$$\frac{m}{2} \le \rho(x,t) \le 2M \quad in\ \overline{Q}_0, \tag{152}$$

$$\rho(x,0) = \rho_0(x), \quad x \in [0,L]. \tag{153}$$

Lemma 12 *There exist functions* $\mathbf{V} = (v^r, v^\varphi, v^z, \omega^r)$, $\mathbf{W} = (\omega^r, \omega^\varphi, \omega^z)$, *and θ such that*

$$\mathbf{V},\mathbf{W} \in \bigl(\mathrm{L}^\infty\bigl(0,T_0;\mathrm{H}^1(]0,L[)\bigr)\bigr)^3 \cap \bigl(\mathrm{H}^1(Q_0)\bigr)^3 \cap \bigl(\mathrm{L}^2\bigl(0,T_0;\mathrm{H}^2(]0,L[)\bigr)\bigr)^3, \tag{154}$$

$$\theta \in \mathrm{L}^\infty\bigl(0,T_0;\mathrm{H}^1(]0,L[)\bigr) \cap \mathrm{H}^1(Q_0) \cap \mathrm{L}^2\bigl(0,T_0;\mathrm{H}^2(]0,L[)\bigr) \tag{155}$$

and a subsequence of $\{(\mathbf{V}^n,\mathbf{W}^n,\theta^n)\}$ (*for simplicity reasons denoted again as* $\{(\mathbf{V}^n,\mathbf{W}^n,\theta^n)\}$) *of* $\{(\mathbf{V}^n,\mathbf{W}^n,\theta^n)\}$ *with the properties*:

$$\bigl(\mathbf{V}^n,\mathbf{W}^n,\theta^n\bigr) \overset{*}{\rightharpoonup} (\mathbf{V},\mathbf{W},\theta) \quad in\ \bigl(\mathrm{L}^\infty\bigl(0,T_0;\mathrm{H}^1(]0,L[)\bigr)\bigr)^7, \tag{156}$$

$$\bigl(\mathbf{V}^n,\mathbf{W}^n,\theta^n\bigr) \rightharpoonup (\mathbf{V},\mathbf{W},\theta) \quad in\ \bigl(\mathrm{H}^1(Q_0)\bigr)^7, \tag{157}$$

$$\bigl(\mathbf{V}^n,\mathbf{W}^n,\theta^n\bigr) \rightharpoonup (\mathbf{V},\mathbf{W},\theta) \quad in\ \bigl(\mathrm{L}^2\bigl(0,T_0;\mathrm{H}^2(]0,L[)\bigr)\bigr)^7, \tag{158}$$

$$\bigl(\mathbf{V}^n,\mathbf{W}^n,\theta^n\bigr) \to (\mathbf{V},\mathbf{W},\theta) \quad in\ \bigl(\mathrm{L}^2(Q_0)\bigr)^7. \tag{159}$$

The functions $\mathbf{V}$, $\mathbf{W}$, *and θ satisfy the conditions*

$$\mathbf{V}(0,t) = \mathbf{V}(L,t) = \mathbf{W}(0,t) = \mathbf{W}(0,t) = \mathbf{0}, \quad t \in [0,T_0], \tag{160}$$

$$\frac{\partial\theta}{\partial x}(0,t) = \frac{\partial\theta}{\partial x}(L,t) = 0, \quad \textit{a.e. in }]0,T_0[, \tag{161}$$

$$\mathbf{V}(x,0) = \mathbf{V}_0(x), \qquad \mathbf{W}(x,0) = \mathbf{W}_0(x), \qquad \theta(x,0) = \theta_0(x), \quad x \in [0,L], \tag{162}$$

where $\mathbf{V}_0 = (v_0^r, v_0^\varphi, v_0^z)$, $\mathbf{W}_0 = (\omega_0^r, \omega_0^\varphi, \omega_0^z)$, *and* θ_0 *are defined by* (35).

Proof As in [11], Lemma 5.3, conclusions (156)–(159) follow from Proposition 1.

To verify the boundary and initial conditions (160)–(162), we use the Green formula in the same way as in [11]. Here, we demonstrate the proof for the boundary condition (161) at $x = 0$.

Let φ be a function from $\mathrm{C}^\infty([0,L])$, which is equal to zero in a neighborhood of L, with $\varphi(0) \neq 0$ and $u \in \mathrm{L}^\infty(]0,T_0[)$. Using the integration by parts for $\frac{\partial\theta}{\partial x}$ and $\frac{\partial\theta^n}{\partial x}$, we obtain

$$\begin{aligned}
&\int_0^{T_0}\int_0^L \frac{\partial^2\theta}{\partial x}(x,t)u(t)\varphi(x)\,dx\,dt + \int_0^{T_0}\int_0^L \frac{\partial\theta}{\partial x}(x,t)u(t)\frac{d\varphi}{dx}(x)\,dx\,dt \\
&\quad = -\varphi(0)\int_0^{T_0} \frac{\partial\theta}{\partial x}(0,t)u(t)\,dt,
\end{aligned} \tag{163}$$

$$\begin{aligned}
&\int_0^{T_0}\int_0^L \frac{\partial^2\theta^n}{\partial x}(x,t)u(t)\varphi(x)\,dx\,dt + \int_0^{T_0}\int_0^L \frac{\partial\theta^n}{\partial x}(x,t)u(t)\frac{d\varphi}{dx}(x)\,dx\,dt \\
&\quad = -\varphi(0)\int_0^{T_0} \frac{\partial\theta^n}{\partial x}(0,t)u(t)\,dt = 0.
\end{aligned} \tag{164}$$

Passing to the limit when $n \to \infty$ in (164), comparing (164) and (163), and by using the convergence (158), we obtain

$$\frac{\partial\theta}{\partial x}(0,t) = 0 \quad \text{a.e. in } t \in]0,T_0[. \tag{165}$$

In a similar way, we obtain all the remaining equalities in (160)–(162). □

Lemma 13 *The functions* r, ρ, v, ω, θ, *defined by Lemmas* 10, 11, *and* 12 *satisfy equations* (1)–(8) *a.e. in* Q_0.

Proof The proof of this lemma is based on strong and weak convergences from Proposition 1. As the procedure is the same as in [11], Lemma 5.4, we will demonstrate here the idea of the proof just for equation (8), which is the most complex.

Let $\{(r^n, \rho^n, \mathbf{V}^n, \mathbf{W}^n, \theta^n) : n \in \mathbf{N}\}$ be the subsequence defined by Lemmas 10, 11, and 12, and let $\varphi \in \mathcal{D}(]0,T_0[)$, where $\mathcal{D}$ denotes the space of test functions. We first rewrite equation (68) in the following form:

$$\begin{aligned}
&\int_0^{T_0}\int_0^L \frac{\partial\theta^n}{\partial t}\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt - \frac{k}{c_v}\int_0^{T_0}\int_0^L \frac{\partial}{\partial x}\left(\left(r^n\right)^2\rho^n\frac{\partial\theta^n}{\partial x}\right)\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt \\
&\quad - \frac{1}{c_v}\int_0^{T_0}\int_0^L \rho^n\left[(\lambda+2\mu)\frac{\partial}{\partial x}\left(r^n v^{rn}\right) - R\theta^n\right]\frac{\partial}{\partial x}\left(r^n v^{rn}\right)\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt \\
&\quad - \frac{\mu+\mu_r}{c_v}\int_0^{T_0}\int_0^L \rho^n\left(\frac{\partial}{\partial x}\left(r^n v^{\varphi n}\right)\right)^2\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt \\
&\quad - \frac{c_d+c_a}{c_v}\int_0^{T_0}\int_0^L \rho^n\left(\frac{\partial}{\partial x}\left(r^n \omega^{\varphi n}\right)\right)^2\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\frac{c_0+2c_d}{c_v}\int_0^{T_0}\int_0^L \rho^n\left(\frac{\partial}{\partial x}\left(r^n\omega^{rn}\right)\right)^2\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt\\
&-\frac{\mu+\mu_r}{c_v}\int_0^{T_0}\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial v^{zn}}{\partial x}\right)^2\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt\\
&-\frac{c_d+c_a}{c_v}\int_0^{T_0}\int_0^L \rho^n\left(r^n\right)^2\left(\frac{\partial \omega^{zn}}{\partial x}\right)^2\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt\\
&+2\frac{c_d}{c_v}\int_0^{T_0}\int_0^L \frac{\partial}{\partial x}\left(\left(\omega^{rn}\right)^2+\left(\omega^{\varphi n}\right)^2\right)\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt\\
&+2\frac{\mu}{c_v}\int_0^{T_0}\int_0^L \frac{\partial}{\partial x}\left(\left(v^{rn}\right)^2+\left(v^{\varphi n}\right)^2\right)\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt\\
&-4\frac{\mu_r}{c_v}\int_0^{T_0}\int_0^L \left(\frac{(\omega^{rn})^2}{\rho^n}+\frac{(\omega^{\varphi n})^2}{\rho^n}+\frac{(\omega^{zn})^2}{\rho^n}\right)\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt=0.
\end{aligned}\tag{166}
$$

Now, we should show the convergence for each integrand on the left-hand side of (166). Here we will demonstrate the following convergence:

$$
\begin{aligned}
&\int_0^{T_0}\int_0^L \rho^n\left(\frac{\partial}{\partial x}\left(r^n\omega^{rn}\right)\right)^2\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt\\
&\quad\to\int_0^{T_0}\int_0^L \rho\left(\frac{\partial}{\partial x}\left(r\omega^{r}\right)\right)^2\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt,
\end{aligned}\tag{167}
$$

when $n\to\infty$. Using integration by parts, as well as the Hölder inequality, we have

$$
\begin{aligned}
&\left|\int_0^{T_0}\int_0^L\left[\rho^n\left(\frac{\partial}{\partial x}\left(r^n\omega^{rn}\right)\right)^2-\rho\left(\frac{\partial}{\partial x}\left(r\omega^{r}\right)\right)^2\right]\cos\frac{\pi kx}{L}\varphi(t)\,dx\,dt\right|\\
&\quad\le C\max_{\overline{Q_0}}\left|\rho^n-\rho\right|\left(\left\|\omega^{rn}\right\|^2+\left\|\frac{\partial\omega^{rn}}{\partial x}\right\|^2\right)\\
&\qquad+C\int_0^{T_0}\varphi(t)\int_0^L\left(\frac{\partial}{\partial x}\left(r^n\omega^{rn}\right)-\frac{\partial}{\partial x}\left(r\omega^{r}\right)\right)\\
&\qquad\times\left(\frac{\partial}{\partial x}\left(r^n\omega^{rn}\right)+\frac{\partial}{\partial x}\left(r\omega^{r}\right)\right)\rho\cos\frac{\pi kx}{L}\,dx\,dt\\
&\quad=C\max_{\overline{Q_0}}\left|\rho^n-\rho\right|\left(\left\|\omega^{rn}\right\|^2+\left\|\frac{\partial\omega^{rn}}{\partial x}\right\|^2\right)-C\int_0^{T_0}\varphi(t)\int_0^L\left(r^n\omega^{rn}-r\omega^{r}\right)\\
&\qquad\times\left(\frac{\partial}{\partial x}\left(r^n\omega^{rn}\right)+\frac{\partial}{\partial x}\left(r\omega^{r}\right)\right)\frac{\partial}{\partial x}\left(\rho\cos\frac{\pi kx}{L}\right)dx\,dt\\
&\qquad-C\int_0^{T_0}\varphi(t)\int_0^L\left(r^n\omega^{rn}-r\omega^{r}\right)\cdot\left(\frac{\partial^2}{\partial x^2}\left(r^n\omega^{rn}\right)+\frac{\partial^2}{\partial x^2}\left(r\omega^{r}\right)\right)\rho\cos\frac{\pi kx}{L}\,dx\,dt\\
&\quad\le C\max_{\overline{Q_0}}\left|\rho^n-\rho\right|\left(\left\|\omega^{rn}\right\|^2+\left\|\frac{\partial\omega^{rn}}{\partial x}\right\|^2\right)\\
&\qquad+C\left(\int_0^{T_0}\left\|r^n\omega^{rn}-r\omega^{r}\right\|^2dt\right)^{\frac12}\left(\int_0^{T_0}\left\|\frac{\partial}{\partial x}\left(r^n\omega^{rn}\right)+\frac{\partial}{\partial x}\left(r\omega^{r}\right)\right\|^2dt\right)^{\frac12}\\
&\qquad+C\left(\int_0^{T_0}\left\|r^n\omega^{rn}-r\omega^{r}\right\|^2dt\right)^{\frac12}\left(\int_0^{T_0}\left\|\frac{\partial^2}{\partial x^2}\left(r^n\omega^{rn}\right)+\frac{\partial^2}{\partial x^2}\left(r\omega^{r}\right)\right\|^2dt\right)^{\frac12}.
\end{aligned}\tag{168}
$$

Taking into account the strong convergences $\rho^n \to \rho$ and $r^n\omega^{rn} \to r\omega^r$, from (168) we easily obtain (167).

In the same way, we can derive the convergences of other integrals in (166). $\square$

Let us note that from (47) we have

$$\int_0^{T_0}\int_0^L r^n(x,t)\varphi(x,t)\,dx\,dt = \int_0^{T_0}\int_0^L \left(r_0(x) + \int_0^t v^{rn}(x,\tau)\,d\tau\right)\varphi(x,t)\,dx\,dt \tag{169}$$

for all $\varphi \in \mathrm{L}^2(Q_0)$, which together with (144) and (159) implies

$$r(x,t) = r_0(x) + \int_0^t v^r(x,t)\,d\tau, \quad (x,t). \tag{170}$$

For the function θ, we have the following property.

Lemma 14 (in [11], Lemma 5.5) *There exists T_0, $0 < T_0 \le T$, such that the function θ defined by Lemma 12 satisfies the condition*

$$\theta > 0 \quad in\ \overline{Q}_0. \tag{171}$$

The conclusions of Theorem 1 are an immediate consequence of the above lemmas.

6 Conclusion

The initial boundary problem for the 3-D flow of a compressible viscous micropolar fluid with cylindrical symmetry and homogeneous boundary conditions for velocity, microrotation, and heat flux has a generalized solution locally in time.

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