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# Solutions to a phase-field model of sea ice growth

Yangxin Tang[1*]

*Correspondence:
tangyangxin@i.shu.edu.cn
[1]Department of Mathematics, Shanghai University, Shanghai, P.R. China

**Abstract**

We shall apply the phase-field method to investigate the dynamics of sea ice growth. The model consists of two parabolic equations. The existence and uniqueness of weak solutions to an initial-boundary value problem of this model is proved. Then the regularity, large-time behavior of solutions are studied, also the existence of global attractor is proved. The main technique in this article is energy method. Our existence proof is only valid in one space dimension.

**MSC:** 35K51; 74N20

**Keywords:** Phase-field model; Sea ice; Weak solution; Global attractor; Large time behavior

## 1 Introduction

Due to global warming, which leads to significant climate changes and more and more frequently occurring severe weather disasters, the study of global warming seems more important than ever since sea ice has begun to melt (see, e.g., [1–3]) and this makes sea level rise considerably so that some islandish countries may vanish. In this paper we shall employ a phase-field approach to model the growth of sea ice. This approach, though it has been developed since 1980s, thus is very young, is now a very powerful tool for both theoretical and numerical studies in many fields (see, e.g., [4, 5]). To our knowledge, the application of a two-phase field model to investigation of sea ice growth presented in this paper is the first one in phase-field modeling for sea ice evolution.

The evolution of macroscopic sea ice has been studied by means of the classic Stefan's problem (see, e.g., [6]). Fluid flow through sea ice is another point of interest. The permeability of sea ice is important in many physical processes such as the melting and draining from sea ice surface during the melting season (see, e.g., [7]). A new simple model which includes turbulent transport of heat and salt between ice and ocean is introduced and solved analytically (see, e.g., [8]). The mesoscopic numerical simulation of sea ice crystals growth has been studied through Voronoi dynamics during the freezing season (see, e.g., [9]). Overall, sea ice interacts with the climate system of the polar. A one-dimensional enthalpy-based model of sea ice allows for quantitative studies of sea ice and its interaction (see, e.g., [10]). These references need to add appropriate conditions at the interface of tracking movement. Theoretical analysis and numerical simulation are very difficult. In this paper we study a phase-field model for the evolution of the phase interface region in sea water-ice interface phase change problems, which was derived in [5]. The author only simulated dendritic crystal growth without theoretical analysis. We will use the phase-field method in the sea ice growth, more precisely, I do theoretical analysis, regularities, and large time behavior. We formulate this initial-boundary value problem in the one-dimensional case and conclude the introduction by stating our main result.

Let $\Omega \subset \mathbb{R}^3$ be an open set. We introduce a phase-field variable (the order parameter $p \in \mathbb{R}$) to represent the physical state of the system in time and space, that is, to distinguish the liquid phase and solid phase, such as the solid state when the variable is 1. The liquid phase is expressed when the variable is 0. We restrict ourselves to that type of order parameter $p$, which describes the evolution of phase interfacial region.

Now let us establish the free energy function $F$ of the system based on the order parameter $p$, their spatial derivatives $\nabla p$, and the local temperature:

$$F[p,T] = \int_{\Omega} \left( \frac{\varepsilon_{12}^2}{2} |\nabla p|^2 + \widehat{\psi}(p) + e_0 T^2 \right) dV. \tag{1.1}$$

Setting

$$\widehat{\psi}(p) = \frac{1}{4a_{12}} \left( p^2(1-p)^2 - m_{12}^2 \left( \frac{1}{3}p^3 + p^2(1-p) - \frac{1}{3}(1-p)^3 - (1-p)^2 p \right) \right),$$

we choose for $\widehat{\psi} \in C^2(\mathbb{R}, [0,\infty))$ a direct extension of the double well potential with minima at $p = 0$ and $p = 1$. Here, $e_0$, $\varepsilon_{12}$, $a_{12}$, $m_{12}$ are thermophysical data.

$T$ is temperature, it satisfies

$$T = \frac{e}{e_0} + \frac{1}{2}h(p),$$

where $h(p)$ is a non-decreasing smooth function satisfying $h(0) = 0$ near $p = 0$ and $h(1) = 1$ near $p = 1$, $e$ is the local enthalpy, and $e_0$ satisfies

$$e_0 = \frac{L^2}{T_M c_{p^*}},$$

$L$ is the latent heat of fusion for sea-water, $T_M$ is the melting temperature, and $c_{p^*}$ is the specific heat of sea water.

For the two-phase case, we get the following systems:

$$\frac{\partial p}{\partial t} = \frac{1}{\tau_{12}} \left( \varepsilon_{12}^2 \Delta p - \frac{1}{2a_{12}} p(1-p)(1-2p) + \frac{m_{12}}{a_{12}} p(1-p) \right) - \kappa T \frac{\partial h}{\partial p}, \tag{1.2}$$

$$\frac{\partial T}{\partial t} = \nabla \cdot (D \nabla T) + \frac{1}{2} \frac{\partial h}{\partial p} \frac{\partial p}{\partial t} \tag{1.3}$$

for $(t,x) \in (0,\infty) \times \Omega$, $\kappa$, $D$ are constants. The boundary and initial conditions are

$$p(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.4}$$

$$T(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.5}$$

$$p(0,x) = p_0(x), \quad x \in \Omega, \tag{1.6}$$

$$T(0,x) = T_0(x), \quad x \in \Omega. \tag{1.7}$$

*Now we make some assumptions.* We assume that all functions depend on the variables $x_1$ and $t$ and, to simplify the notation, denote $x_1$ by $x$. The set $\Omega$ is a bounded open interval. We write $Q_{T_e} := (0,T_e)\times\Omega$, where $T_e$ is a positive constant, and define

$$(\upsilon,\varphi)_{\mathbb{Z}} = \int_{\mathbb{Z}} \upsilon(y)\varphi(y)\,dy$$

for $\mathbb{Z}=\Omega$ or $\mathbb{Z}=Q_{T_e}$.

Then, under these assumptions, equations (1.2)–(1.3) in the case of one dimension can be rewritten as follows:

$$p_t = \frac{1}{\tau_{12}}\left(\varepsilon_{12}^2 p_{xx} - \frac{1}{2a_{12}}p(1-p)(1-2p) + \frac{m_{12}}{a_{12}}p(1-p)\right) - \kappa T\frac{\partial h}{\partial p}, \quad \text{in } (t,x)\in(0,T_e)\times\Omega, \tag{1.8}$$

$$T_t = DT_{xx} + \frac{1}{2}\frac{\partial h}{\partial p}\frac{\partial p}{\partial t}, \quad \text{in } (t,x)\in(0,T_e)\times\Omega, \tag{1.9}$$

where $\kappa = \frac{2e_0}{\tau_{12}}$. The boundary and initial conditions therefore are

$$p(t,x) = 0, \quad \text{on } (t,x)\in[0,T_e]\times\partial\Omega, \tag{1.10}$$

$$T(t,x) = 0, \quad \text{on } (t,x)\in[0,T_e]\times\partial\Omega, \tag{1.11}$$

$$p(0,x) = p_0(x), \quad \text{in } x\in\Omega, \tag{1.12}$$

$$T(0,x) = T_0(x), \quad \text{in } x\in\Omega. \tag{1.13}$$

**Definition 1.1** Let $p_0\in H_0^1(\Omega)$, $T_0\in L^2(\Omega)$. A function $(p,T)$ with

$$p\in L^\infty\big(0,T_e;H_0^1(\Omega)\big)\cap L^2\big(0,T_e;H^2(\Omega)\big), \tag{1.14}$$

$$T\in L^\infty\big(0,T_e;L^2(\Omega)\big)\cap L^2\big(0,T_e;H_0^1(\Omega)\big), \tag{1.15}$$

is a weak solution to problem (1.8)–(1.13) if, for all $\varphi\in C_0^\infty((-\infty,T_e)\times\Omega)$, there hold

$$0 = (p,\varphi_t)_{Q_{T_e}} - \frac{1}{\tau_{12}}\varepsilon_{12}^2(p_x,\varphi_x)_{Q_{T_e}} - \frac{1}{\tau_{12}}\frac{1}{2a_{12}}\big(p(1-p)(1-2p),\varphi\big)_{Q_{T_e}} + \frac{1}{\tau_{12}}\frac{m_{12}}{a_{12}}\big(p(1-p),\varphi\big)_{Q_{T_e}} - \kappa T\left(\frac{\partial h}{\partial p},\varphi\right)_{Q_{T_e}} + \big(p_0,\varphi(0)\big)_\Omega, \tag{1.16}$$

$$0 = (T,\varphi_t)_{Q_{T_e}} - (DT_x,\varphi_x)_{Q_{T_e}} + \frac{1}{2}\left(\frac{\partial h}{\partial p},\varphi\right)_{Q_{T_e}} + \big(T_0,\varphi(0)\big)_\Omega. \tag{1.17}$$

The main results of this article are as follows.

**Theorem 1.1** *For all $p_0\in H_0^1(\Omega)$ and $T_0\in L^2(\Omega)$, there exists a unique weak solution $(p,T)$ of problem (1.8)–(1.13), which in addition to (1.14)–(1.15) satisfies*

$$p_t\in L^2(Q_{T_e}), \qquad p\in L^4(Q_{T_e}), \qquad T_t\in L^2\big(0,T_e;H^{-1}(\Omega)\big). \tag{1.18}$$

**Theorem 1.2** *Assume that* $p_0 \in H^2(\Omega)$, $T_0 \in H_0^1(\Omega)$, *then there exists a weak solution* $(p, T)$ *of problem* (1.8)–(1.13), *which in addition to* (1.14)–(1.15) *satisfies*

$$\begin{aligned} & p \in L^\infty\big(0, T_e; H^2(\Omega)\big), \qquad p_t \in L^\infty\big(0, T_e; L^2(\Omega)\big) \cap L^2\big(0, T_e; H_0^1(\Omega)\big), \\ & p_{tt} \in L^2\big(0, T_e; H^{-1}(\Omega)\big), \\ & T \in L^2\big(0, T_e; H^2(\Omega)\big) \cap L^\infty\big(0, T_e; H_0^1(\Omega)\big), \qquad T_t \in L^2\big(0, T_e; H^{-1}(\Omega)\big). \end{aligned} \tag{1.19}$$

*Remark* Assume that $p_0 \in H^k(\Omega)$, $T_0 \in H^{k-1}(\Omega)$, regularity will continue to improve, when $k$ is sufficiently large, then weak solution becomes the classical solution.

**Definition 1.2** Let $X$ be a Banach space. A one-parameter family $S(t)$, $0 \le t < \infty$, of bounded linear operators from $X$ into $X$ is a semigroup bounded linear operator on $X$ if

(i) $S(0) = I$ ($I$ is an identity operator on $X$),
(ii) $S(s + t) = S(s)S(t)$ for every $t, s \ge 0$ (the semigroup property).

**Theorem 1.3** *Let* $\Omega$ *denote an open bounded set of* $\mathbb{R}$ *and* $g_1$ *denote a polynomial. The semigroup* $p(t)$ *associated with the initial-boundary-value problem* (1.8)–(1.13) *possesses a maximal attractor* $\mathcal{A}$ *which is bounded in* $H_0^1(\Omega)$, *compact and connected in* $L^2(\Omega)$. *Its basin of attraction is the whole space* $L^2(\Omega)$, $\mathcal{A}$ *attracts the bounded sets of* $L^2(\Omega)$. *Assume that the coefficient is suitably large. Then* $\|p\|_{L^\infty(\Omega)}$ *and* $\|T\|^2$ *decrease exponentially to* 0 *as* $t \to \infty$.

**Notation** Let $\Omega$ be a domain in $\mathbb{R}^n$, and let $r$ be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable functions $u$ defined on $\Omega$ for which

$$\int_\Omega \big|u(x)\big|^r \, dx < \infty.$$

The Sobolev space $W^{k,r}(\Omega)$ is defined by

$$W^{k,r}(\Omega) = \big\{u \in L^r(\Omega) : D^\alpha u \in L^r(\Omega), 0 \le |\alpha| \le k, 1 \le r \le \infty\big\},$$

where $k$ is any positive integer and $D^\alpha u$ is the weak partial derivative. $\|\cdot\|$, $|\cdot|$, $C$, $\mathcal{A}$, $S(t)$ denote $L^2(\Omega)$-norm, the absolute value, various constants, attractor, semigroup, respectively; $\partial_t$ or $\frac{d}{dt}$ or a subscript $t$ and $\partial_x$ or a subscript $x$ denote the derivative with respect to $t$ and $x$ in the distribution sense, respectively.

The remaining sections are devoted to the proof of Theorem 1.1, Theorem 1.2, and Theorem 1.3. In order to obtain the local solution of the initial-boundary value problem for nonlinear equations (1.8)–(1.13), we construct the approximate sequence

$$\big\{(p, T)_n(t, x) = \big(p^n, T^n\big)(t, x)\big\}_{n=3}^\infty.$$

We prove the existence of weak solutions by iterative method: Choose a known approximate solution $p^{n-1}$, $T^{n-1}$ and determine the next $p^n$, $T^n$ by solving equations to (1.8)–(1.9), proving existence solution by using of Banach's fixed point theorem. When we regard the term $p^n$ in (1.9) as known by use of solution of (1.8), a solution will be obtained if convergence of this procedure can be shown. In Sect. 2 we shall establish some *a priori* estimates for the solution.

We will discuss the regularity of our weak solutions $p$, $T$ for the parabolic systems in Sect. 3.

Section 4 is devoted to investigation of the large time behavior of a solution by using the *a priori* estimates.

## 2 *A priori* estimates

In this section we establish *a priori* estimates for solutions $(p, T)$ to the initial-boundary value problem (1.8)–(1.13).

**Lemma 2.1** *There holds, for any* $t \in [0, T_e]$,

$$\int_0^t \|p_t\|^2 \, d\tau + \kappa D \int_0^t \|T_x\|^2 \, d\tau + \|T(t)\|^2 + \|p_x(t)\|^2 + \|p(t)\|_{L^4(\Omega)}^4 \le C. \tag{2.1}$$

*Proof* Multiplying (1.8), (1.9) by $p_t$, $2\kappa T$ and integrating by parts with respect to $x \in \Omega$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|T\|^2 + \frac{\varepsilon_{12}}{\tau_{12}}\frac{d}{dt}\|p_x\|^2 + 2\kappa D\|T_x\|^2 + \frac{1}{2\tau_{12}a_{12}}\int_\Omega p(1-p)(1-2p)p_t \, dx \\ &\quad - \frac{m_{12}}{\tau_{12}a_{12}}\int_\Omega p(1-p)p_t \, dx = 0. \end{aligned} \tag{2.2}$$

Integrating (2.2) in $\tau \in [0, t]$, we have

$$\begin{aligned} &\int_0^t \|p_t\|^2 \, d\tau + 4\kappa D \int_0^t \|T_x\|^2 \, d\tau + 2\|T(t)\|^2 + \frac{\varepsilon_{12}}{\tau_{12}}\|p_x(t)\|^2 + c\|p(t)\|_{L^4(\Omega)}^4 \\ &\quad \le \frac{\varepsilon_{12}}{\tau_{12}}\|p_{0x}\|^2 + 2\|T_0\|^2 + C\big(\|p_0\|_{L^4(\Omega)}^4 + C\big) \le C. \end{aligned} \tag{2.3}$$

Thus we obtain $p_t \in L^2(Q_t)$, $p \in L^\infty(0, t; H_0^1(\Omega)) \cap L^4(Q_t)$, $T \in L^\infty(0, t; L^2(\Omega)) \cap L^2(0, t; H_0^1(\Omega))$. □

**Lemma 2.2** *There holds, for any* $t \in [0, T_e]$,

$$\int_0^t \|p(t)\|_{L^4(\Omega)}^4 \, d\tau \le C, \tag{2.4}$$

$$\int_0^t \|T_t\|_{H^{-1}(\Omega)}^2 \, d\tau \le C. \tag{2.5}$$

*Proof* Noting that $f|_{\partial\Omega} = 0$, we have the Gagliardo–Nirenberg inequality in the form

$$\|f\|_{L^4(\Omega)} \le c\|f_x\|_{L^2(\Omega)}^{\frac{1}{4}}\|f\|_{L^2(\Omega)}^{\frac{3}{4}}.$$

We have

$$\int_0^t \|p(\tau)\|^4 \, d\tau \le C\|p(t)\|_{L^\infty(0,t;L^2(\Omega))}^3 \int_0^t \|p_x(\tau)\| \, d\tau \le Ct^{\frac{1}{2}} \le C_t. \tag{2.6}$$

Using equations (1.8), (1.9), we have

$$\frac{1}{\tau_{12}}\varepsilon_{12}^2 p_{xx} = p_t + \kappa T\frac{\partial h}{\partial p} - \frac{1}{\tau_{12}}\biggl(-\frac{1}{2a_{12}}p(1-p)(1-2p) + \frac{m_{12}}{a_{12}}p(1-p)\biggr)$$
$$= f_1, \tag{2.7}$$
$$T_t = DT_{xx} + \frac{1}{2}\frac{\partial h}{\partial p}\frac{\partial p}{\partial t}. \tag{2.8}$$

Taking the $L^2(\Omega)$-norm on both sides of equation (2.7), squaring and integrating it in $\tau \in (0,t)$, using relation (2.1), we have

$$\frac{1}{\tau_{12}}\varepsilon_{12}^2\int_0^t \|p_{xx}\|^2\,d\tau \le C\biggl(\int_0^t \|p_t\|^2\,d\tau + \int_0^t \|T\|^2\biggl\|\frac{\partial h}{\partial p}\biggr\|^2_{L^\infty(\Omega)} + \frac{1}{\tau_{12}}\biggl(\biggl\|\frac{1}{2a_{12}}p(1-p)(1-2p)\biggr\|^2 + \frac{m_{12}}{a_{12}}\bigl\|p(1-p)\bigr\|^2\biggr)\biggr)\,d\tau \le C.$$

Next we invoke the inequality

$$\beta\|p\|^2_{H^2(\Omega)} \le \|f_1\| + \gamma\|p\|^2 \quad \bigl(p \in H^2(\Omega)\cap H_0^1(\Omega)\bigr) \tag{2.9}$$

for constants $\beta > 0$, $\gamma \ge 0$. Integrating (2.9) in $\tau \in (0,t)$, we have

$$\int_0^t \|p\|^2_{H^2(\Omega)}\,d\tau \le C\int_0^t \bigl(\|f_1\|^2_{L^2(\Omega)} + \|p\|^2_{L^2(\Omega)}\bigr)\,d\tau$$
$$\le C\int_0^t \|f_1\|^2_{L^2(\Omega)}\,d\tau + \|p\|^2_{L^\infty(0,T_e;L^2(\Omega))}\int_0^t d\tau \le C_t.$$

It remains to show that $T_t \in L^2(0,t;H^{-1}(\Omega))$. To do so, (2.8) is changed to

$$\|T_t\|_{H^{-1}(\Omega)} \le C\biggl(\|T_{xx}\|_{H^{-1}(\Omega)} + \biggl\|\frac{\partial h}{\partial p}\frac{\partial p}{\partial t}\biggr\|\biggr).$$

Thus

$$\int_0^t \|T_t\|^2_{H^{-1}(\Omega)}\,d\tau \le C\int_0^t \biggl(\|T_{xx}\|^2_{H^{-1}(\Omega)} + \biggl\|\frac{\partial h}{\partial p}\frac{\partial p}{\partial t}\biggr\|^2\biggr)\,d\tau$$
$$\le C(\|T_{xx}\|^2_{L^2(0,t,H^{-1}(\Omega))} + \biggl\|\frac{\partial h}{\partial p}\biggr\|^2_{L^\infty(0,t;L^\infty(\Omega))}\|p_t\|^2_{L^2(0,t;L^2(\Omega))}$$
$$\le C$$

and $T_t$ is bounded in $L^2(0,t;H^{-1}(\Omega))$.

We have the solution $p$, $T$ of (1.8), (1.9). This allows us to extend the solution $p$, $T$ step-by-step to all of $T_e$. □

**Theorem 2.1** (Uniqueness) *Assume that p and T are the weak solution of* (1.8)–(1.9). *Then the weak solution is unique.*

*Proof* If $\widetilde{p}$, $\widetilde{T}$ are another solution, write $w_1 := p - \widetilde{p}$, $w_2 := T - \widetilde{T}$. Setting $\varphi = w_1$ in (1.16), $\varphi = w_2$ in (1.17) and integrating by parts, by using Young's inequality, we have

$$\begin{aligned} &\int_0^{T_e} \frac{d}{dt}\bigl\|w_1(\tau)\bigr\|^2\,d\tau + \int_0^{T_e} \|w_{1x}\|^2\,d\tau + \int_0^{T_e}\int_\Omega w_1^2\bigl(p^2 + \widetilde{p}^2\bigr)\,dx\,d\tau \\ &\quad\le C\int_0^{T_e} \|w_1\|^2\,d\tau, \end{aligned} \tag{2.10}$$

$$\int_0^{T_e} \frac{d}{dt}\bigl\|w_2(\tau)\bigr\|^2\,d\tau + \int_0^{T_e} \|w_{2x}\|^2\,d\tau \le 0. \tag{2.11}$$

(2.10) is changed to

$$\|w_1\|^2 \le C\int_0^{T_e} \|w_1\|^2\,d\tau.$$

By using of Gronwall's inequality, we obtain $p = \widetilde{p}$ for almost everywhere $Q_{T_e}$.
(2.11) is changed to

$$\|w_2\|^2 \le 0,$$

we obtain $T = \widetilde{T}$ for almost everywhere $Q_{T_e}$. □

## 3 Regularity

In this section we discuss the regularity of the weak solutions $p$, $T$ to the initial-boundary value problem for parabolic-parabolic systems in Sect. 1. Assume that all conditions in Sect. 1 are met and $p_0 \in H^2(\Omega)$, $T_0 \in H_0^1(\Omega)$. For this initial $p_0 \in H^2(\Omega)$, $T_0 \in H_0^1(\Omega)$, solutions $p$, $T$ can be constructed as in Sect. 2. Our eventual goal is to prove that $p$, $T$ is smooth.

**Lemma 3.1** *There holds, for any* $t \in [0, T_e]$,

$$\begin{aligned} &\|p\|_{L^\infty(0,T_e;H^2(\Omega))} + \|p_t\|_{L^\infty(0,T_e;L^2(\Omega))} + \|p_t\|_{L^2(0,T_e;H_0^1(\Omega))} + \|p_{tt}\|_{L^2(0,T_e;H^{-1}(\Omega))} \\ &\qquad + \|T\|_{L^\infty(0,T_e;H_0^1(\Omega))} + \|T\|_{L^2(0,T_e;H^2(\Omega))} + \|T\|_{L^2(0,T_e;L^2(\Omega))} \\ &\quad\le \bigl(C + \|p_0\|_{H^2(\Omega)} + \|T_0\|_{H_0^1(\Omega)}\bigr). \end{aligned} \tag{3.1}$$

*Proof* Differentiating (1.8) formally with respect to $t$ yields

$$\begin{aligned} p_{tt} &= \frac{1}{\tau_{12}}\biggl(\varepsilon_{12}^2 p_{xxt} - \frac{1}{2a_{12}}\bigl(1 - 6p + 6p^2\bigr)p_t + \frac{m_{12}}{a_{12}}(1 - 2p)p_t\biggr) \\ &\quad - \kappa T\frac{\partial^2 h}{\partial p^2}p_t - \kappa T_t\frac{\partial h}{\partial p}, \end{aligned} \tag{3.2}$$

$$T_t = DT_{xx} + \frac{1}{2}\frac{\partial h}{\partial p}\frac{\partial p}{\partial t}. \tag{3.3}$$

Multiplying (3.3) by $-T_{xx}$ and integrating by parts with respect to $x$ over $\Omega$, we have

$$\frac{1}{2}\frac{d}{dt}\|T_x\|^2 + D\|T_{xx}\|^2 = -\frac{1}{2}\int_\Omega T_{xx}\frac{\partial h}{\partial p}p_t\,dx \le \varepsilon\|T_{xx}\|^2 + c_\varepsilon\biggl\|\frac{\partial h}{\partial p}\biggr\|_{L^\infty(\Omega)}^2\|p_t\|^2.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|T_x\|^2+(D-\varepsilon)\|T_{xx}\|^2\le c_\varepsilon\left\|\frac{\partial h}{\partial p_1}\right\|^2_{L^\infty(\Omega)}\|p_t\|^2. \tag{3.4}$$

Integrating (3.4) in $\tau\in(0,t)$, we have

$$\|T_x\|^2+2(D-\varepsilon)\int_0^t\|T_{xx}\|^2\,d\tau\le\|T_{0x}\|^2+C\le C. \tag{3.5}$$

(3.3) be changed to

$$DT_{xx}=T_t-\frac{1}{2}\frac{\partial h}{\partial p}\frac{\partial p}{\partial t}=f_3.$$

We use the result of regularity theory of elliptic equations

$$\|T\|_{H^2(\Omega)}\le C\bigl(\|f_3\|+\|T\|\bigr). \tag{3.6}$$

Squaring and integrating (3.6) in $\tau\in(0,t)$, we have

$$\begin{aligned}\int_0^{T_e}\|T\|^2_{H^2(\Omega)}\,d\tau&\le C\int_0^{T_e}\bigl(\|f_3\|^2+\|T\|^2\bigr)\,d\tau\\&\le C\left(C+\|T\|^2_{L^\infty(0,T_e;L^2(\Omega))}\int_o^{T_e}d\tau\right)\le C,\end{aligned} \tag{3.7}$$

we obtain $T\in L^2(0,T_e;H^2(\Omega))$.

Squaring and integrating (3.3) in $\Omega$, we have

$$\|T_t\|^2\le C\|T_{xx}\|^2+C\left\|\frac{\partial h}{\partial p}\right\|^2_{L^\infty(\Omega)}\|p_t\|^2. \tag{3.8}$$

Integrating (3.8) in $\tau\in(0,t)$, we have

$$\int_0^t\|T_t\|^2\,d\tau\le C\int_0^t\|T_{xx}\|^2\,d\tau+C\int_0^t\left\|\frac{\partial h}{\partial p}\right\|^2_{L^\infty(\Omega)}\|p_t\|^2\,d\tau\le C, \tag{3.9}$$

we obtain $T_t\in L^2(Q_{T_e})$.

Multiplying (3.2) by $p_t$ and integrating by parts with respect to $x$ over $\Omega$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|p_t\|^2+\frac{1}{2\tau_{12}a_{12}}\int_\Omega\bigl(1-6p+6p^2\bigr)p_t^2\,dx\\&\quad+\frac{\varepsilon_{12}^2}{\tau_{12}}\|p_{xt}\|^2-\frac{m_{12}}{\tau_{12}a_{12}}\int_\Omega(1-2p)(p_t)^2\,dx\\&\le C\left\|\frac{\partial^2 h}{\partial p^2}\right\|_{L^\infty(\Omega)}\|T\|_{L^\infty(\Omega)}\|p_t\|^2+\frac{1}{2}\left\|\frac{\partial h}{\partial p}\right\|^2_{L^\infty(\Omega)}\|T_t\|^2+\frac{1}{2}\|p_t\|^2\\&\le C\|p_t\|^2+C\|T_t\|^2.\end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|p_t\|^2+\frac{\varepsilon_{12}^2}{2\tau_{12}}\|p_{xt}\|^2+c\int_\Omega p^2(p_t)^2\,dx\le C\|p_t\|^2+C\|T_t\|^2. \tag{3.10}$$

We thereupon conclude from (3.10) that

$$\begin{aligned}&\|p_t(t)\|^2+\int_0^{T_e}\|p_{xt}\|^2\,d\tau+c\int_0^{T_e}\int_\Omega p^2p_{1t}^2\,dx\,d\tau\\&\quad\le e^{Ct}\biggl(\|p_{0t}\|^2+C\int_0^{T_e}\|T_t\|^2\,d\tau\biggr)\le C\bigl(\|p_0\|^2_{H^2(\Omega)}+C\bigr).\end{aligned} \tag{3.11}$$

Multiplying (1.8) by $p_{xx}$ and integrating it in $x\in\Omega$, we have

$$\begin{aligned}\frac{1}{\tau_{12}}\varepsilon_{12}^2(p_{xx},p_{xx})&=\biggl(p_t+\kappa T\frac{\partial h}{\partial p},p_{xx}\biggr)\\&\quad-\frac{1}{\tau_{12}}\biggl(-\frac{1}{2a_{12}}p(1-p)(1-2p)+\frac{m_{12}}{a_{12}}p(1-p),p_{xx}\biggr)\\&=(g_3+p_t,p_{xx}).\end{aligned} \tag{3.12}$$

Next we invoke the inequality

$$\beta\|p\|^2_{H^2(\Omega)}\le(g_3+p_t,p_{xx})+\gamma\|p\|^2,\quad\bigl(p\in H^2(\Omega)\cap H_0^1(\Omega)\bigr)$$

for constants $\beta>0$, $\gamma\ge0$.

We thereupon conclude from (3.12) that

$$\|p\|_{H^2(\Omega)}\le C\bigl(\|g_3\|+\|p_t\|+\|p\|\bigr)\le C,$$

we obtain $p\in L^\infty(0,T_e;H^2(\Omega))$.

It remains to show that $p_{tt}\in L^2(0,T_e;H^{-1}(\Omega))$. To do so, equation (3.2) is changed to

$$p_{tt}=\frac{1}{\tau_{12}}\biggl(\varepsilon_{12}^2p_{xxt}-\frac{1}{2a_{12}}\bigl(1-6p+6p^2\bigr)p_t+\frac{m_{12}}{a_{12}}(1-2p)p_t\biggr)-\kappa T\frac{\partial^2h}{\partial p^2}p_t-\kappa T_t\frac{\partial h}{\partial p}.$$

Thus

$$\|p_{tt}\|_{H^{-1}(\Omega)}\le C\bigl(\|p_{xxt}\|_{H^{-1}(\Omega)}+\|p_t\|_{L^2(\Omega)}+\|T_t\|_{L^2(\Omega)}\bigr),$$

and so $p_{tt}$ is bounded in $L^2(0,T_e;H^{-1}(\Omega))$. □

## 4 Global attractor

In this section we discuss the existence of a global attractor and the stability of solution to problem (1.8)–(1.13). This amounts to proving that the solutions of the evolution problem remain bounded as $t\to\infty$. Usually, proving the existence of absorbing sets amounts to proving a priori estimates. Once the properties of the semigroup are established, we may apply the general results of the attractor. That theorem produces the existence of an attractor which is maximal among the bounded attractors and among the bounded functional invariant sets; it fully describes the long-time behavior of the solutions of the equations.

### 4.1 Global attractor

Let

$$g_1(p) = \frac{1}{2\tau_{12}a_{12}}p(1-p)(1-2p) - \frac{1}{\tau_{12}m_{12}}p(1-p) \tag{4.1}$$

be a polynomial with a positive leading coefficient. Using Young's inequality, we infer from (4.1) the existence of a constant $c_1 > 0$ such that

$$\left|\frac{1}{2\tau_{12}a_{12}}p\bigl(1-3p^2\bigr) - \frac{1}{\tau_{12}m_{12}}p(1-p)\right| \le \frac{1}{2\tau_{12}a_{12}}p^4 + c_1, \tag{4.2}$$

and hence

$$-\frac{1}{2\tau_{12}a_{12}}p^4 - c_1 \le g_1(p)p \le \frac{1}{2\tau_{12}a_{12}}p^4 + c_1. \tag{4.3}$$

**Lemma 4.1** (The uniform Gronwall Lemma) *Let $g$, $h$, $y$ be three positive locally integral functions on $(t_0,\infty)$ such that $y'$ is locally integrable on $(t_0,\infty)$, and which satisfies*

$$\begin{aligned}
&\frac{dy}{dt} \le gy + h \quad \text{for } t \ge t_0,\\
&\int_t^{t+r} g(s)\,ds \le a_1, \qquad \int_t^{t+r} h(s)\,ds \le a_2,\\
&\int_t^{t+r} y(s)\,ds \le a_3 \quad \text{for } t \ge t_0,
\end{aligned}$$

*where $r$, $a_1$, $a_2$, $a_3$ are positive constants. Then*

$$y(t+r) \le \left(\frac{a_3}{r} + a_2\right)e^{a_1}, \quad \forall t \ge t_0.$$

*Proof of Theorem 1.3* *(a)* Absorbing set in $L^2(\Omega)$ of $p$. Using relation (4.3), we obtain, multiplying (1.8) by $p$ and integrating by parts with respect to $x$ over $\Omega$, where we take the boundary condition (1.10) into account, that for almost all $t$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|p\|^2 + \frac{\varepsilon_{12}^2}{\tau_{12}}\|p_x\|^2 + \frac{1}{2\tau_{12}a_{12}}\|p\|^4\\
&\quad \le \kappa\|T\|\|p\|_{L^2(\Omega)}\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)} + 2c_1|\Omega|\\
&\quad \le c_\varepsilon\|T\|^2\left\|\frac{\partial h}{\partial p}\right\|^2_{L^\infty(\Omega)} + \varepsilon\|p\|^2 + 2c_1|\Omega|,
\end{aligned} \tag{4.4}$$

$|\Omega|$ = the measure (volume) of $\Omega$. Due to Poincaré's inequality, there exists a constant $c_0 = c_0(\Omega)$ such that

$$\|p\| \le c_0\|p_x\|, \quad \forall p \in H_0^1(\Omega),$$

and setting $c_2 = 4c_1|\Omega|$, we infer from (4.4) that

$$\frac{d}{dt}\|p\|^2 + \left(\frac{2\varepsilon_{12}^2}{c_0^2\tau_{12}} - 2\varepsilon\right)\|p\|^2 + \frac{1}{\tau_{12}a_{12}}\|p\|^4 \le c_2 + 2c_\varepsilon\|T\|^2\left\|\frac{\partial h}{\partial p}\right\|^2_{L^\infty(\Omega)},$$

where $\epsilon = \frac{2\varepsilon_{12}^2}{c_0^2\tau_{12}} - 2\varepsilon > 0$.

Using the uniform Gronwall Lemma 4.1 we see that

$$\begin{aligned}\|p(t)\|^2 &\le \|p_0\|^2 e^{-\epsilon t} + e^{-\epsilon t}\int_0^t e^{\epsilon\tau}\left(c_2 + 2c_\varepsilon\|T\|^2\left\|\frac{\partial h}{\partial p}\right\|^2_{L^\infty(\Omega)}\right)d\tau \\ &\le \|p_0\|^2 e^{-\epsilon t} + e^{-\epsilon t}\int_0^t e^{\epsilon\tau}(c_2 + 2c_3c_\varepsilon)\,d\tau \\ &\le \|p_0\|^2 e^{-\epsilon t} + \frac{c_2 + 2c_3c_\varepsilon}{\epsilon}\left(1 - e^{-\epsilon t}\right).\end{aligned} \tag{4.5}$$

Thus

$$\limsup_{t\to\infty}\|p(t)\| \le \rho_0, \quad \rho_0^2 = \frac{c_2 + 2c_3c_\varepsilon}{\epsilon}. \tag{4.6}$$

There exists an absorbing set $\mathcal{B}_0$ in $L^2(\Omega)$, namely, any ball of $L^2(\Omega)$ centered at 0 of radius $\rho_0' > \rho_0$. If $\mathcal{B}$ is a bounded set of $L^2(\Omega)$, included in a ball $B(0,R)$ of $L^2(\Omega)$ centered at 0 of radius $R$, then $S(t)\mathcal{B} \subset B(0,\rho_0')$ for $t \ge t_0(\mathcal{B};\rho_0')$

$$t_0 = \frac{1}{\epsilon}\log\frac{R^2}{(\rho_0')^2 - \rho_0^2}. \tag{4.7}$$

We also infer from (4.4), after integration in $t$, that

$$\begin{aligned}&\int_t^{t+r}\frac{2\varepsilon_{12}^2}{\tau_{12}}\|p_x\|^2\,d\tau + \frac{1}{\tau_{12}a_{12}}\int_t^{t+r}\|p\|^4\,d\tau \\ &\quad\le c_2 r + 2\varepsilon\int_t^{t+r}\|p(\tau)\|^2\,d\tau + rc_3c_\varepsilon + \|p(t)\|^2, \quad r > 0.\end{aligned} \tag{4.8}$$

With (4.6) we conclude that

$$\begin{aligned}&\limsup_{t\to\infty}\left(2\epsilon\int_t^{t+r}\|p_x\|^2\,d\tau + \frac{1}{\tau_{12}a_{12}}\int_t^{t+r}\|p\|^4\,d\tau\right) \\ &\quad\le (c_2 + c_3c_\varepsilon)r + (2\varepsilon r + 1)\rho_0^2, \quad r > 0,\end{aligned} \tag{4.9}$$

and if $p_0 \in \mathcal{B} \subset B(0,R)$ and $t \ge t_0(\mathcal{B},\rho_0')$, then

$$\begin{aligned}&2\epsilon\int_t^{t+r}\|p_x\|^2\,d\tau + \frac{2}{2\tau_{12}a_{12}}\int_t^{t+r}\|p\|^4\,d\tau \\ &\quad\le (c_2 + c_3c_\varepsilon)r + (2\varepsilon r + 1)\left(\rho_0'\right)^2, \quad r > 0.\end{aligned} \tag{4.10}$$

*(b) Absorbing set in* $H_0^1(\Omega)$ *of* $p$. We now prove the existence of an absorbing set in $H_0^1(\Omega)$ and the uniform compactness of $S(t)$. For that purpose we need another energy-type equality; it is obtained by multiplying (1.8) by $-p_{xx}$ and integrating by parts with

respect to $x$ over $\Omega$, where we take the boundary condition (1.10) into account, that for almost all $t$

$$\frac{1}{2}\frac{d}{dt}\|p_x\|^2+\frac{\varepsilon_{12}^2}{\tau_{12}}\|p_{xx}\|^2+c\int_\Omega\left(p^2-1\right)(p_x)^2\,dx=\kappa\int_\Omega Tp_{xx}\frac{\partial h}{\partial p}\,dx. \tag{4.11}$$

As in (4.3), we can prove with repeated applications of Young's inequality that there exists $c_2'>0$ such that

$$\frac{3}{4\tau_{12}a_{12}}p^2-c_2'\le\frac{dg_1}{dp}\le\frac{9}{4\tau_{12}a_{12}}p^2+c_2',\quad\forall p\in R. \tag{4.12}$$

We also infer from general results on the Dirichlet problem in $\Omega$ that $|\triangle p|$ is, on $H_0^1(\Omega)\cap H^2(\Omega)$, a norm equivalent to that induced by $H^2(\Omega)$.

Setting $c_3'=\frac{3}{4\tau_{12}a_{12}}>0$, we then deduce from (4.12)

$$\frac{d}{dt}\|p_x\|^2+\epsilon'\|p_x\|^2+c_3'\int_\Omega(p)^2(p_x)^2\,dx\le 2c_\varepsilon\|T\|^2\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}^2, \tag{4.13}$$

where $\epsilon'=\frac{\frac{2\varepsilon_{12}^2}{\tau_{12}}-2\varepsilon}{c_0'^2}-2c_2'>0$. If $p_0\in H_0^1(\Omega)$, then the uniform Gronwall Lemma 4.1 shows that

$$\left\|p_x(t)\right\|^2\le\|p_0\|_{H_0^1(\Omega)}^2e^{-\epsilon't}+\frac{2c_3c_\varepsilon}{\epsilon'}\left(1-e^{-\epsilon't}\right),\quad t>0. \tag{4.14}$$

A bound valid for all $t\in R^+$ is obtained by application of the uniform Gronwall lemma; for arbitrary fixed $r>0$, we write (4.13)

$$\frac{d}{dt}\|p_x\|^2\le 2c_2'\|p_x\|^2+2c_\varepsilon\|T\|^2\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}^2.$$

Multiplying by $e^{-2c_2't}$, we obtain the relation

$$\frac{d}{dt}\left(e^{-2c_2't}\|p_x\|^2\right)\le 2e^{-2c_2't}c_\varepsilon\|T\|^2\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}^2\le 2c_\varepsilon\|T\|^2\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}^2.$$

Then, by integration between $t$ and $t+r$, we have

$$\left\|p_x(t+r)\right\|^2\le\left(\frac{\int_t^{t+r}\|p_x(t)\|^2\,d\tau}{r}+a_2\right)e^{a_1}\le\left(\frac{a_3}{r}+a_2\right)e^{a_1},\quad t\ge t_*, \tag{4.15}$$

provided

$$\int_t^{t+r}\left\|p_x(t)\right\|^2\,d\tau\le a_3,\qquad a_1=2c_2'r,\qquad a_2=2c_\varepsilon c_3r,\quad t\le t_*.$$

An explicit value of $a_3$ can be derived from (4.4) and the computation above when $t_*=0$. Hence (4.15) provides a uniform bound for $p_x$, $t>r$, while (4.13) provides a uniform bound for $p_x$ for $0 < t < r$. For our purpose, it is simpler and sufficient to set $t_* = t_0$, in which case, the value of $a_3$ is given by (4.10),

$$a_3 = \frac{(c_2 + c_3 c_\varepsilon)r + (2\varepsilon r + 1)(\rho_0')^2}{2\epsilon}. \tag{4.16}$$

It follows that the ball of $H_0^1(\Omega)$ centered at 0 of radius $\rho_1$ is absorbing in $H_0^1(\Omega)$, when

$$\rho_1^2 = \left(\frac{a_3}{r} + a_2\right)e^{a_1},$$

and if $p_0$ belongs to the ball $B(0, R')$ of $H_0^1(\Omega)$ centered at 0 of radius $r$, then $p(t)$ enters this absorbing set denoted by $\mathcal{B}_1$ at a time $t \leq t_0 + r$ and remains in it for $t \geq t_0 + r$. At the same time, this result provides the uniform compactness of $S(t)$: any bounded set $\mathcal{B}$ of $L^2(\Omega)$ is included in such a ball $B(0, R')$, and for $p_0 \in \mathcal{B}$ and $\geq t_0 + r$, $t_0$, $r$ as above, $p(t)$ belongs to $\mathcal{B}_1$ which is bounded in $H_0^1(\Omega)$ and relatively compact in $L^2(\Omega)$.

*(c)* Absorbing set in $L^2(\Omega)$ of $T$. Making use of relations (4.10) and (4.15), we obtain

$$\begin{aligned}
&\|p_x(t+r)\|^2 + \left(\frac{\varepsilon_{12}^2}{\tau_{12}} - \varepsilon\right)\int_t^{t+r} \|p_{xx}\|^2\,d\tau + c\int_t^{t+r}\int_\Omega p^2(p_x)^2\,dx\,d\tau \\
&\leq 2c_\varepsilon \int_t^{t+r} \|T\|^2 \left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}^2 d\tau + c_3' \int_t^{t+r} \|p_x\|^2\,d\tau + \|p_x(t)\|^2 \\
&\leq 2c_3 c_\varepsilon r + c_3' a_3 + \rho_0^2.
\end{aligned}$$

Squaring and integrating (1.8) over $\Omega$, we have

$$\begin{aligned}
\|p_t\|^2 \leq{}& \frac{2\varepsilon_{12}^2}{\tau_{12}}\|p_{xx}\|^2 + 4\kappa^2\|T\|^2\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}^2 \\
&+ C\big(\|p(1-p)(1-2p)\|^2 + \|p(1-p)\|^2\big).
\end{aligned} \tag{4.17}$$

Integrating (4.17) in $\tau \in (t, t+r)$, we have

$$\int_t^{t+r} \|p_t\|^2\,d\tau \leq \frac{\varepsilon_{12}^2}{\tau_{12}} \frac{2c_3 c_\varepsilon r + c_3' a_3 + \rho_0^2}{\frac{\varepsilon_{12}^2}{\tau_{12}} - \varepsilon} + (c_2 + 2c_3 c_\varepsilon)r + 2(2\varepsilon r + 1)\rho_0^2 + 2c_1 r + 4c_3 c_\varepsilon r.$$

We multiply (1.9) by $T$ and integrate by parts with respect to $x$ over $\Omega$, where we take the boundary condition (1.11) into account, that for almost all $t$

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + D\|T_x\|^2 = \frac{1}{2}\int_\Omega T\frac{\partial h}{\partial p}p_t\,dx \leq \varepsilon\|T\|^2 + c_\varepsilon\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}\|p_t\|^2. \tag{4.18}$$

Due to Poincaré's inequality, there exists a constant $c_0 = c_0(\Omega)$ such that

$$\|T\| \leq c_0\|T_x\|, \quad \forall T \in L^2(\Omega).$$

We infer from (4.18) that

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + \left(\frac{D}{c_0^2} - \varepsilon\right)\|T\|^2 \leq c_\varepsilon\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}\|p_t\|^2.$$

Using the uniform Gronwall lemma, we see that

$$\|T(t)\|^2 \le e^{-2(\frac{D}{c_0^2}-\varepsilon)}\|T(0)\|^2 + 2C_1c_\varepsilon.$$

Thus

$$\limsup_{t\to\infty}\|T\| \le \rho_2, \qquad \rho_2^2 = 2C_1c_\varepsilon. \tag{4.19}$$

There exists an absorbing set $\mathcal{B}_2$ in $L^2(\Omega)$, namely, any ball of $L^2(\Omega)$ centered at 0 of radius $\rho_2' > \rho_2$. If $\mathcal{B}$ is a bounded set of $L^2(\Omega)$, included in a ball $B(0,R)$ of $L^2(\Omega)$ centered at 0 of radius $R$, then $S(t)\mathcal{B} \subset B(0,\rho_2')$ for $t \ge t_0(\mathcal{B};\rho_2')$

$$t_1 = \frac{1}{2(\frac{D}{c_0^2}-\varepsilon)}\log\frac{R^2}{(\rho_2')^2-\rho_2^2}.$$

We also infer from (4.18), after integration in $t$, that

$$\begin{aligned}2D\int_t^{t+r}\|T_x\|^2\,d\tau \le{}& \|T(t)\|^2 + 2\varepsilon\int_t^{t+r}\|T(t)\|^2\,d\tau \\ &+ 2c_\varepsilon\int_t^{t+r}\left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}\|p_t\|^2\,d\tau, \quad r>0.\end{aligned}$$

With (4.19) we conclude that

$$\begin{aligned}&\limsup_{t\to\infty}2D\int_t^{t+r}\|T_x\|^2\,d\tau \\ &\quad\le (1+2r\varepsilon)\rho_2^2 + 2c_\varepsilon\frac{\varepsilon_{12}^2}{\tau_{12}}\frac{2c_3c_\varepsilon r + c_3'a_3 + \rho_0^2}{\frac{\varepsilon_{12}^2}{\tau_{12}}-\varepsilon} \\ &\qquad + 2c_\varepsilon\big((c_2+2c_3c_\varepsilon)r + 2(2\varepsilon r+1)\rho_0^2 + 2c_1r + 4c_3c_\varepsilon r\big), \quad r>0,\end{aligned}$$

and if $T_0 \in \mathcal{B} \subset B(0,R)$ and $t \ge t_1(\mathcal{B},\rho_2')$, then

$$\begin{aligned}&2D\int_t^{t+r}\|T_x\|^2\,d\tau \\ &\quad\le (1+2r\varepsilon)\big(\rho_2'\big)^2 + 2c_\varepsilon\frac{\varepsilon_{12}^2}{\tau_{12}}\frac{2c_3c_\varepsilon r + c_3'a_3 + \rho_0^2}{\frac{\varepsilon_{12}^2}{\tau_{12}}-\varepsilon} \\ &\qquad + 2c_\varepsilon\big((c_2+2c_3c_\varepsilon)r + 2(2\varepsilon r+1)\big(\rho_0'\big)^2 + 2c_1r + 4c_3c_\varepsilon r\big), \quad r>0.\end{aligned}$$

□

### 4.2 Large-time behavior of the solutions

**Lemma 4.2** *Let $f = f(t)$, which satisfies $f \in L^1(\mathbb{R}^+)$, $\frac{df}{dt} \in L^1(\mathbb{R}^+)$, and*

$$\int_0^\infty |f(t)|\,dt \le C,$$

$$\int_0^\infty \left|\frac{df}{dt}\right|\,d\tau \le C.$$

*Then*

$$\lim_{t\to\infty} f(t) = 0. \tag{4.20}$$

**Theorem 4.1** *Let $p = p(t,x)$, $T = T(t,x)$, $\|p\|^2 \in L^1(\mathbb{R}^+)$, $\frac{d}{dt}\|p\|^2 \in L^1(\mathbb{R}^+)$, $\|T\|^2 \in L^1(\mathbb{R}^+)$, $\frac{d}{dt}\|T\|^2 \in L^1(\mathbb{R}^+)$, and*

$$\int_0^\infty \|p_x\|^2\,dt \le C, \qquad \int_0^\infty \left|\frac{d}{dt}\|p_x\|^2\right| d\tau \le C,$$
$$\int_0^\infty \|T\|^2\,dt \le C, \qquad \int_0^\infty \left|\frac{d}{dt}\|T\|^2\right| d\tau \le C.$$

*Then*

$$\lim_{t\to\infty} \|p\|_{L^\infty(\Omega)} = 0, \qquad \lim_{t\to\infty} \|T\|^2 = 0. \tag{4.21}$$

*Proof* Multiplying (1.8) by $p$ and integrating by parts with respect to $x$ over $\Omega$, we have

$$\frac{1}{2}\frac{d}{dt}\|p\|^2 + \left(\frac{\varepsilon_{12}^2}{\tau_{12}} - \varepsilon - cc_0\right)\|p_x\|^2 + \frac{c}{\tau_{12}a_{12}}\|p\|^4_{L^4(\Omega)} \le c_\varepsilon \|T_x\|^2 \left\|\frac{\partial h}{\partial p}\right\|^2_{L^\infty(\Omega)}, \tag{4.22}$$

where $\alpha = \frac{\varepsilon_{12}^2}{\tau_{12}} - \varepsilon - cc_0 > 0$. Integrating (4.22) in $\tau \in (0,t)$, we have

$$\|p(t)\|^2 + \alpha\int_0^t \|p_x\|^2\,d\tau + \frac{c}{\tau_{12}a_{12}}\int_0^t \|p\|^4_{L^4(\Omega)}\,d\tau \le \|p(0)\|^2 + C_\varepsilon\int_0^t \|T_x\|^2\,d\tau \le C.$$

Thus we obtain

$$\int_0^t \|p_x\|^2\,d\tau \le C, \tag{4.23}$$

$$\int_0^t \|p\|^4_{L^4(\Omega)}\,d\tau \le C. \tag{4.24}$$

Multiplying (1.8) by $-p_{xx}$ and integrating by parts with respect to $x$ over $\Omega$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|p_x\|^2 + \left(\frac{\varepsilon_{12}^2}{\tau_{12}} - \varepsilon\right)\|p_{xx}\|^2 + \frac{1}{\tau_{12}a_{12}}\int_\Omega p^2p_x^2\,dx - c_2'\int_\Omega (p_x)^2\,dx \\ &\quad\le c_\varepsilon\left\|\frac{\partial h}{\partial p}\right\|^2_{L^\infty(\Omega)} + \|T_x\|^2,\end{aligned} \tag{4.25}$$

where $\beta = \frac{\varepsilon_{12}^2}{\tau_{12}} - \varepsilon > 0$. Integrating (4.25) in $\tau \in (0,t)$, using relations (4.23) and (4.24), we have

$$\begin{aligned}&\|p_x\|^2 + \beta\int_0^t \|p_{xx}\|^2\,d\tau + \frac{1}{\tau_{12}a_{12}}\int_0^t\int_\Omega p^2p_x^2\,dx\,d\tau \\ &\quad\le \|p_x(0)\|^2 + c_2'\int_0^t \|p_x\|^2\,dx\,d\tau + C\int_0^t \|T_x\|^2\,d\tau \le C.\end{aligned}$$

Thus we obtain

$$\int_0^t \|p_{xx}\|^2 \, d\tau \le C, \tag{4.26}$$

$$\int_0^t \int_\Omega p^2 (p_x)^2 \, dx \, d\tau \le C. \tag{4.27}$$

Multiplying (1.8) by $-p_{xx}$ and integrating by parts with respect to $x$ over $\Omega$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|p_x\|^2 = {} & -\frac{\varepsilon_{12}^2}{\tau_{12}}\|p_{xx}\|^2 + \frac{1}{2a_{12}\tau_{12}}\int_\Omega p(1-p)(1-2p)p_{xx}\, dx \\ & + \kappa \int_\Omega T p_{xx} \frac{\partial h}{\partial p}\, dx. \end{aligned} \tag{4.28}$$

Taking the absolute value on both sides of equation (4.28) and integrating it in $\tau \in (0,t)$, using relations (4.26), (4.27), we have

$$\begin{aligned} \frac{1}{2}\int_0^t \left|\frac{d}{dt}\|p_x\|^2\right| d\tau \le {} & \frac{1}{2a_{12}\tau_{12}}\int_0^t \left|\int_\Omega p(1-p)(1-2p)p_{xx}\, dx\right| d\tau \\ & + \frac{\varepsilon_{12}^2}{\tau_{12}}\int_0^t \|p_{xx}\|^2 \, d\tau + \kappa \int_0^t \left|\int_\Omega T p_{xx}\frac{\partial h}{\partial p}\, dx\right| d\tau \\ \le {} & \frac{\varepsilon_{12}^2}{\tau_{12}}\int_0^t \|p_{xx}\|^2 \, d\tau + C\int_0^t \bigl(p^2(p_x)^2 + p_x^2\bigr)\, d\tau + c_\varepsilon \int_0^t \|p_x\|^2 \, d\tau \\ & + \varepsilon \int_0^t \|T_x\|^2 \left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)}^2 d\tau \le C. \end{aligned} \tag{4.29}$$

Thus we obtain

$$\frac{1}{2}\int_0^t \left|\frac{d}{dt}\|p_x\|^2\right| d\tau \le C. \tag{4.30}$$

By use of Lemma 4.2 and relations (4.23), (4.30), we obtain

$$\lim_{t\to\infty} \|p_x\|^2 = 0. \tag{4.31}$$

By use of Poincaré's inequality, since $p \in H_0^1(\Omega)$, we have

$$\|p\|_{L^\infty(\Omega)} \le C\left(\int_\Omega |p_x|^2 \, dx\right)^{\frac{1}{2}} \to 0. \tag{4.32}$$

Multiplying (1.9) by $T$ and integrating by parts with respect to $x$ over $\Omega$, we have

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + (D-\varepsilon)\|T_x\|^2 \le c_\varepsilon \left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)} \|p_t\|^2, \tag{4.33}$$

where $D - \varepsilon > 0$. Integrating (4.33) in $\tau \in (0,t)$, we get

$$\|T\|^2 + (D-\varepsilon)\int_0^t \|T_x\|^2 \le C. \tag{4.34}$$

Multiplying (1.9) by $T$ and integrating by parts with respect to $x$ over $\Omega$, we have

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + \left(\frac{D}{c_0^2} - \varepsilon\right)\|T\|^2 \le c_\varepsilon \left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)} \|p_t\|^2, \tag{4.35}$$

where $\frac{D}{c_0^2} - \varepsilon > 0$. Integrating (4.35) in $\tau \in (0,t)$, we have

$$\|T\|^2 + \left(\frac{D}{c_0^2} - \varepsilon\right)\int_0^t \|T\|^2 \le C. \tag{4.36}$$

Taking the absolute value on both sides of equation (4.33) and integrating it in $\tau \in (0,t)$, using relation (4.34), we have

$$\int_0^t \left|\frac{1}{2}\frac{d}{dt}\|T\|^2\right| d\tau \le C\int_0^t \|T_x\|^2\, d\tau + C\int_0^t \left\|\frac{\partial h}{\partial p}\right\|_{L^\infty(\Omega)} \|p_t\|^2\, d\tau \le C. \tag{4.37}$$

Since $T|_{\partial\Omega} = 0$, we have

$$\|T\|^2 = 0, \qquad t \to \infty. \tag{4.38}$$

□

## 5 Conclusion

With the help of Banach's fixed point theorem, we prove the existence of weak solutions and study the regularity of weak solutions for the phase-field model. Also we study the existence of a global attractor for this simplified model and investigate the large time behavior of weak solutions.

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**Availability of data and materials**

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

**Competing interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**Authors' contributions**

The author carried out the paper and drafted the manuscript. The author read and approved the final manuscript.

## Publisher's Note



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