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Solvability and coerciveness of multi-point Sturm–Liouville problems with abstract linear functionals

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Abstract

In this paper, we consider a Sturm–Liouville problem with finite discontinuous points inside an interval and with abstract linear functionals in the boundary and transmission conditions. For such a problem, the properties such as isomorphism, Fredholmness, and coerciveness with respect to the spectral parameter are investigated.

Keywords: Sturm–Liouville problems; Transmission conditions; Isomorphism; Coerciveness; Solvability

1 Introduction

We will discuss the following differential equation:

$$M(\lambda)y := -\left(py'\right)'(x) + \lambda^2 y(x) + Ty(x) = f(x), \quad x \in I \tag{1.1}$$

with nonclassical boundary conditions

$$M_k y := a_{01} y^{(n_k)}(\xi_0) + \sum_{h=1}^{n}\left(a_{hk} y^{(n_k)}(\xi_h-) + \tilde{a}_{hk} y^{(n_k)}(\xi_h+)\right)$$

$$+ a_{(n+1)k} y^{(n_k)}(\xi_{n+1}) + \sum_{j=1}^{n_k} \chi_{kj} y^{(n_k)}(x_{kj}) + F_k y = f_k, \tag{1.2}$$

where $k = 1, 2, \ldots, 2(n+1)$, $\xi_0 = -1$, $\xi_h \in (-1, 1)$, $\xi_{n+1} = 1$, $\xi_0 < \xi_1 < \cdots < \xi_{n+1}$; set $I_1 = [\xi_0, \xi_1)$, $I_t = (\xi_{t-1}, \xi_t)$, $I_{n+1} = (\xi_n, \xi_{n+1}]$, $I = \bigcup_{i=1}^{n+1} I_i$, and $J_i = (\xi_{i-1}, \xi_i)$, $J = \bigcup_{i=1}^{n+1} J_i$ ($h = 1, 2, \ldots, n$; $t = 2, 3, \ldots, n$); $p(x)$ is a piecewise constant function, $p(x) = p_i$ for $x \in I_i$ ($i = 1, 2, \ldots, n+1$); T is a linear operator; $a_{0k}, a_{hk}, \tilde{a}_{hk}, a_{(n+1)k}, \chi_{kj}, p_i$ ($j = 1, 2, \ldots, n_k$) are complex coefficients, and assume that $p_i \neq 0$, $|a_{0k}| + \sum_{h=1}^{n}(|a_{hk}| + |\tilde{a}_{hk}|) + |a_{(n+1)k}| \neq 0$; λ is the complex parameter; n_k are integers; $x_{kj} \in J$ are internal points; F_k is a linear function in the space $L_q[-1, 1]$ ($L_q[-1, 1]$ is a set of qth order integrable functions on $[-1, 1]$).

In recent years, the classical Sturm–Liouville problem has been generalized into various types for its new importance in physical sciences and applied mathematics. For example, theoretical investigations have become focused on the discontinuous Sturm–Liouville

problems for their application in physics. The discontinuity of the coefficients of the equations in the Sturm–Liouville problems corresponds to the fact that the heterogeneous media consist of two different materials. Moreover, boundary value problems with discontinuities arise in many physical problems such as heat and mass transfer, electrostatics, and diffraction problem [1, 2]. It should be noted that some works on the spectral properties and coercive solvability of boundary value problems in Sobolev spaces can be found in [3, 4]. Some boundary value problems for differential equations with discontinuous coefficients were investigated by Rasulov [5]. Note that an abstract theory of the boundary value problems with continuous coefficients and an eigenvalue parameter in the boundary conditions have been constructed by Yakubov and Yakubov (see [4] and corresponding bibliography). Many authors have been devoted to the study of discontinuous problems [6–22]. To deal with the discontinuity of the problem, transmission conditions are imposed on the discontinuous points. There are also other terminologies such as point interaction, interface condition, etc. [18, 23]. The properties of isomorphism, Fredholmness, and coerciveness of Sturm–Liouville problems with one discontinuous point were investigated by Mukhtarov and his coauthors in [6, 8, 9, 24].

In this paper, we investigate a Sturm–Liouville problem with discontinuities at finite points and with abstract linear functionals in the boundary-transmission conditions. We obtain the properties such as isomorphism, Fredholmness, and coerciveness of this problem.

2 Boundary value problems with nonhomogeneous transmission conditions

In this section, we consider the homogeneous differential equation

$$M_0(\lambda)y := -\big(py'\big)'(x) + \lambda^2 y(x) = 0, \quad x \in I \tag{2.1}$$

with the nonclassical boundary conditions

$$M_{k0}y := a_{0k}y^{(n_k)}(\xi_0) + \sum_{h=1}^{n}\big(a_{hk}y^{(n_k)}(\xi_h-) + \tilde{a}_{hk}y^{(n_k)}(\xi_h+)\big) + a_{(n+1)k}y^{(n_k)}(\xi_{n+1}) = f_k, \tag{2.2}$$

where $k = 1, 2, \ldots, 2(n + 1)$. We shall use the notations

$$\Gamma_k y := \sum_{j=1}^{n_k} \chi_{kj} y^{(n_k)}(x_{kj}), \quad k = 1, 2, \ldots, 2(n + 1),$$

$$w_{2i-1} := p_i^{-1/2}, \qquad w_{2i} := -p_i^{-1/2}, \quad i = 1, 2, \ldots, n + 1,$$

$$\alpha := \min_{1 \le i \le n+1}\{\arg p_i\}, \qquad \beta := \max_{1 \le i \le n+1}\{\arg p_i\},$$

$$\omega = \begin{bmatrix} a_{01}w_1^{n_1} & a_{11}w_2^{n_1} & \tilde{a}_{11}w_3^{n_1} & \cdots & a_{(n+1)1}w_{2(n+1)}^{n_1} \\ a_{02}w_1^{n_2} & a_{12}w_2^{n_2} & \tilde{a}_{12}w_3^{n_2} & \cdots & a_{(n+1)2}w_{2(n+1)}^{n_2} \\ a_{03}w_1^{n_3} & a_{13}w_2^{n_3} & \tilde{a}_{13}w_3^{n_3} & \cdots & a_{(n+1)3}w_{2(n+1)}^{n_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{0,2(n+1)}w_1^{n_{2(n+1)}} & a_{1,2(n+1)}w_2^{n_{2(n+1)}} & \tilde{a}_{1,2(n+1)}w_3^{n_{2(n+1)}} & \cdots & a_{(n+1),2(n+1)}w_{2(n+1)}^{n_{2(n+1)}} \end{bmatrix}$$

and

$$\Omega_\varepsilon(\alpha, \beta) := \left\{ \lambda \in \mathbb{C} \,\middle|\, \frac{1}{2}(\pi + \beta + \varepsilon) < \arg \lambda < \frac{1}{2}(3\pi + \alpha - \varepsilon) \right\}$$

for real $\varepsilon > 0$ sufficiently small.

Note that the direct sum of Sobolev spaces $W_q^m = W_q^m(J_1) \oplus W_q^m(J_2) \oplus \cdots \oplus W_q^m(J_n)$ (for an integer $m \geq 0$ and real $q > 1$) is defined as a Banach space of complex-valued function $y = y(x)$ on I which belongs to $W_q^m(J_i)$ ($i = 1, 2, \ldots, (n + 1)$) in intervals J_i respectively, with the norm

$$\|y\|_{q,m} := \sum_{i=1}^{n+1} \|y\|_{W_q^m(J_i)}.$$

Here, as usual, $W_q^m(a, b)$ is the Sobolev space, i.e., the Banach space consisting of all measurable functions that have generalized derivatives up to mth order in the interval (a, b) inclusive with the finite norm

$$\|y\|_{W_q^m(a,b)} := \sum_{v=0}^{m} \left(\int_a^b |y^{(v)}(x)|^q \, dx \right)^{1/q}.$$

Theorem 2.1 *If $\omega \neq 0$, then for any $\varepsilon > 0$ there exists $\mu_\varepsilon > 0$ such that, for all $\lambda \in \Omega_\varepsilon(\alpha, \beta)$ for which $|\lambda| > \mu_\varepsilon$, problem (2.1)–(2.2) has a unique solution $y(x, \lambda) \in W_q^\sigma$ for arbitrary $\sigma \geq \max\{2, \max_{1 \leq k \leq 2(n+1)}\{n_k\} + 1\}$ and λ has the following coercive estimate*:

$$\sum_{\ell=1}^{\sigma} \|y\|_{q,\ell} \leq C(\varepsilon) \sum_{k=1}^{2(n+1)} |\lambda|^{\sigma - n_k - q^{-1}} |f_k|. \tag{2.3}$$

Proof Let $y_k(x, \lambda)$ ($k = 1, 2, \ldots, 2(n+1)$) be the basic solution of equation (2.1), then $y_k(x, \lambda)$ can be represented as

$$y_k(x, \lambda) = \begin{cases} \exp(w_k \lambda(x - \tilde{\xi}_k)), & \text{for } x \in \tilde{I}_k; \\ 0, & \text{for } x \notin \tilde{I}_k, \end{cases} \tag{2.4}$$

where $\tilde{I}_{2i-1} = \tilde{I}_{2i} = I_i$ for every i, $k = 2i - 1, 2i$ ($i = 1, 2, \ldots, n + 1$), $\tilde{\xi}_1 = \xi_0$, $\tilde{\xi}_{2h} = \tilde{\xi}_{2h+1} = \xi_i$, $\tilde{\xi}_{2(n+1)} = \xi_{n+1}$ ($h = 1, 2, \ldots, n$). It is clear that the general solution of (2.1) can be written as

$$y(x, \lambda) = \sum_{k=1}^{2(n+1)} C_k y_k(x, \lambda). \tag{2.5}$$

Substituting equation (2.5) into boundary-transmission conditions (2.2), we obtain a linear system with respect to C_k ($k = 1, 2, \ldots, 2(n + 1)$)

$$(w_1 \lambda)^{n_k} \left(a_{0k} + a_{1k} e^{w_1 \lambda(\xi_1 - \xi_0)} \right) C_1 + (w_2 \lambda)^{n_k} \left(a_{1k} + a_{0k} e^{w_2 \lambda(\xi_0 - \xi_1)} \right) C_2$$
$$+ \sum_{t=2}^{n+1} \left[(w_{2t-1} \lambda)^{n_k} \left(\tilde{a}_{(t-1)k} + a_{tk} e^{w_{2t-1} \lambda(\xi_t - \xi_{t-1})} \right) C_{2t-1} \right.$$
$$\left. + (w_{2t} \lambda)^{n_k} \left(a_{tk} + \tilde{a}_{(t-1)k} e^{w_{2t} \lambda(\xi_{t-1} - \xi_t)} \right) C_{2t} \right] = f_k, \tag{2.6}$$

where $k = 1, 2, \ldots, 2(n + 1)$. It follows from $\lambda \in \Omega_\varepsilon(\alpha, \beta)$ that

$$\frac{1}{2}(\pi + \varepsilon) < \arg(w_k \lambda) < \frac{1}{2}(3\pi - \varepsilon), \quad k = 1, 3, \ldots, 2n + 1;$$

$$-\frac{1}{2}(\pi - \varepsilon) < \arg(w_k \lambda) < \frac{1}{2}(\pi - \varepsilon), \quad k = 2, 4, \ldots, 2(n + 1).$$

Therefore, for this λ and $\varepsilon > 0$ (sufficiently small), we obtain

$$(-1)^{k+1} \operatorname{Re}(w_k \lambda) < -|\lambda||w_k| \sin \frac{\varepsilon}{2}, \quad k = 1, 2, \ldots, 2(n + 1).$$

Thus, the determinant of system (2.6) has the form

$$A(\lambda) = \lambda^{\tilde{n}} \big(\omega + \theta(\lambda) \big),$$

where $\tilde{n} = n_1 + n_2 + \cdots + n_{2(n+1)}$,

$$\theta(\lambda) = e^{\kappa} \begin{bmatrix} a_{11} w_1^{n_1} & a_{01} w_2^{n_1} & \cdots & \tilde{a}_{n1} w_{2(n+1)}^{n_1} \\ a_{12} w_1^{n_2} & a_{02} w_2^{n_2} & \cdots & \tilde{a}_{n2} w_{2(n+1)}^{n_2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,2(n+1)} w_1^{n_{2(n+1)}} & a_{0,2(n+1)} w_2^{n_{2(n+1)}} & \cdots & \tilde{a}_{n,2(n+1)} w_{2(n+1)}^{n_{2(n+1)}} \end{bmatrix},$$

$\kappa = \lambda \sum_{i=1}^{n+1} (\xi_i - \xi_{i-1})(w_{2i-1} - w_{2i})$ and $\theta(\lambda) \to 0$ as $|\lambda| \to \infty$ in the angle $\Omega_\varepsilon(\alpha, \beta)$. Since $\omega \neq 0$, there exists $\mu_\varepsilon > 0$ such that, for all $\lambda \in \Omega_\varepsilon(\alpha, \beta)$ and $|\lambda| > \mu_\varepsilon$, we have $A(\lambda) \neq 0$. So, for these λ, the unique solution for the system of linear homogeneous equations (2.6) has the following representation:

$$C_k(\lambda) = \frac{1}{A(\lambda)} \sum_{\eta=1}^{2(n+1)} A_{\eta k}(\lambda) f_\eta, \quad k = 1, 2, \ldots, 2(n + 1),$$

where $A_{\eta k}(\lambda)$ is an algebraic cofactor of (η, k)th element of the determinant $A(\lambda)$. It is obvious that each of the determinants $A_{\eta k}(\lambda)$ can be represented as

$$A_{\eta k}(\lambda) = \lambda^{\tilde{n} - n_\eta} \big(\omega_{\eta k} + \theta_{\eta k}(\lambda) \big),$$

where $\omega_{\eta k} \in \mathbb{C}$ and $\theta(\lambda) \to 0$ as $|\lambda| \to \infty$ in the angle $\Omega_\varepsilon(\alpha, \beta)$. Hence, we have

$$C_k(\lambda) = \sum_{\eta=1}^{2(n+1)} \lambda^{-n_\eta} \frac{\omega_{\eta k} + \theta_{\eta k}(\lambda)}{\omega + \theta(\lambda)} f_\eta, \quad k = 1, 2, \ldots, 2(n + 1).$$

So, the solution of (2.1)–(2.2) has the following representation:

$$y(x, \lambda) = \sum_{k=1}^{2(n+1)} \sum_{\eta=1}^{2(n+1)} \lambda^{-n_\eta} \frac{\omega_{\eta k} + \theta_{\eta k}(\lambda)}{\omega + \theta(\lambda)} f_\eta y_k(x, \lambda).$$

From the expression of $y(x, \lambda)$ we obtain that, for each integer $\tau \geq 0$ and $\lambda \in \Omega_\varepsilon(\alpha, \beta)$, $|\lambda| \to \infty$, the estimate

$$\left\| y^{(\tau)} \right\|_{L_q(-1,1)} \leq C \sum_{\eta=1}^{2(n+1)} \left(|\lambda|^{\tau - n_\eta} |f_\eta| \sum_{k=1}^{2(n+1)} \left\| y_k(x, \lambda) \right\|_{L_q(\tilde{I}_k)} \right) \tag{2.7}$$

is valid. Further, the inequalities

$$\begin{aligned}
\left\| y_{2i-1}(x, \lambda) \right\|_{L_q(\tilde{I}_{2i-1})}^q &= \int_{\xi_{i-1}}^{\xi_i} e^{q \operatorname{Re}(w_{2i-1}\lambda)(x - \xi_{i-1})} \, dx \\
&\leq \int_{\xi_{i-1}}^{\xi_i} e^{-q|w_{2i-1}||\lambda|(x - \xi_{i-1}) \sin \frac{\varepsilon}{2}} \, dx \\
&\overset{t=|\lambda|(x-\xi_{i-1})}{=} \int_0^{|\lambda|(\xi_i - \xi_{i-1})} e^{-q|w_{2i-1}|(\sin \frac{\varepsilon}{2})t} d\left(\frac{t}{|\lambda|} + \xi_{i-1} \right) \\
&\leq |\lambda|^{-1} \int_0^{+\infty} e^{-q|w_{2i-1}|(\sin \frac{\varepsilon}{2})t} dt \\
&= |\lambda|^{-1} \left(q|w_{2i-1}| \sin \frac{\varepsilon}{2} \right)^{-1} \\
&= C_{2i-1}(\varepsilon)|\lambda|^{-1},
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\left\| y_{2i}(x, \lambda) \right\|_{L_q(\tilde{I}_{2i})}^q &= \int_{\xi_{i-1}}^{\xi_i} e^{q \operatorname{Re}(w_{2i}\lambda)(x - \xi_i)} \, dx \\
&\leq \int_{\xi_{i-1}}^{\xi_i} e^{-q|w_{2i}||\lambda|(\xi_i - x) \sin \frac{\varepsilon}{2}} \, dx \\
&\overset{t=|\lambda|(\xi_i - x)}{=} \int_{|\lambda|(\xi_i - \xi_{i-1})}^0 e^{-q|w_{2i}|(\sin \frac{\varepsilon}{2})t} d\left(\xi_i - \frac{t}{|\lambda|} \right) \\
&\leq |\lambda|^{-1} \int_0^{+\infty} e^{-q|w_{2i}|(\sin \frac{\varepsilon}{2})t} dt \\
&= |\lambda|^{-1} \left(q|w_{2i}| \sin \frac{\varepsilon}{2} \right)^{-1} \\
&= C_{2i}(\varepsilon)|\lambda|^{-1},
\end{aligned} \tag{2.9}$$

where $i = 1, 2, \ldots, (n+1)$, hold by (2.4). Substituting (2.8), (2.9) into (2.7) yields

$$\left\| y^{(\tau)}(x) \right\|_{L_q(-1,1)} \leq C(\varepsilon) \sum_{\eta=1}^{2(n+1)} |\lambda|^{\tau - n_\eta - q^{-1}} |f_\eta|,$$

which in turn gives us the needed estimation (2.3). The proof is completed. □

3 Fredholm property for multi-point boundary value problem with functional conditions

Let $\widetilde{M}$ be the linear operator corresponding to problem (1.1)–(1.2). Suppose that $\sigma \geq \max\{2, \max_{1 \leq k \leq 2(n+1)}\{n_k\} + 1\}$ and define $\widetilde{M}$ from W_q^σ into $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$ by the rule

$$\widetilde{M}y = \left(M(\lambda), M_1 y, M_2 y, \ldots, M_{2(n+1)} y \right).$$

Theorem 3.1 *Assume that the following conditions hold*:

(1) *For $x \in I_i$, $p_i \neq 0$*;

(2) *F_k ($k = 1, 2, \ldots, 2(n + 1)$) are continuous functionals in W_q^σ*;

(3) *The operator T from W_q^σ into $W_q^{\sigma-2}$ is compact.*

Then $\widetilde{M}$ is bounded and Fredholm operator.

Proof The operator $\widetilde{M}$ can be represented as

$$\widetilde{M}_0 y = \left(M_0(\lambda)y, M_{10}y, M_{20}y, \ldots, M_{2(n+1)0}y\right),$$

$$\widetilde{M}_1 y = (Ty, \Gamma_1 y + F_1 y, \Gamma_2 y + F_2 y, \ldots, \Gamma_{2(n+1)}y + F_{2(n+1)}y).$$

The operator $\widetilde{M}_0$ is an isomorphism from W_q^σ onto $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$ by Theorem 2.1. Furthermore, it follows from (2) and (3) that the operator $\widetilde{M}_1$ acts compactly from W_q^σ onto $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$.

Therefore, by the definition of isomorphism and Theorem 1.2.8 in [3] (or [25, p. 238]), the operator $\widetilde{M} = \widetilde{M}_0 + \widetilde{M}_1$ is Fredholm. Moreover, it is obvious that the operator $\widetilde{M}$ is bounded. So, the desired results are obtained. $\square$

4 Isomorphism and coerciveness of the principal part of the problem

We consider the principle part of main problem (1.1)–(1.2) without internal points, that is,

$$M_0(\lambda)y := -\left(py'\right)'(x) + \lambda^2 y(x) = f(x), \quad x \in I, \tag{4.1}$$

$$M_{k0}y := a_{0k}y^{(n_k)}(\xi_0) + \sum_{h=1}^{n}\left(a_{hk}y^{(n_k)}(\xi_h-) + \tilde{a}_{hk}y^{(n_k)}(\xi_h+)\right)$$

$$+ a_{(n+1)k}y^{(n_k)}(\xi_{n+1}) = f_k \tag{4.2}$$

for $k = 1, 2, \ldots, 2(n + 1)$. The corresponding operator is

$$\widehat{M}_0 y := \left(M_0(\lambda)y, M_{10}y, M_{20}y, \ldots, M_{2(n+1)0}y\right).$$

Theorem 4.1 *Let the condition $\omega \neq 0$ and $\sigma \geq \max\{2, \max_{1\leq k\leq 2(n+1)}\{n_k\} + 1\}$ be satisfied. Then, for each $\varepsilon > 0$, there exists $\mu_\varepsilon > 0$ such that, for all complex numbers $\lambda \in \Omega_\varepsilon(\alpha, \beta)$, $|\lambda| > \mu_\varepsilon$, the operator $\widehat{M}_0$ from W_q^σ onto $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$ is an isomorphism, and for these λ the inequality*

$$\sum_{\ell=0}^{\sigma}|\lambda|^{\sigma-\ell}\|y\|_{q,\ell} \leq C(\varepsilon)\left(\|f\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|f\|_{q,0} + \sum_{k=1}^{2(n+1)}|\lambda|^{\sigma-n_k-q^{-1}}|f_k|\right) \tag{4.3}$$

holds for the solution of (4.1)–(4.2).

Proof Obviously, the linear operator $\widehat{M}_0$ acts continuously from the space W_q^σ into $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$. Let us prove that, for any $(f(x), f_1, f_2, \ldots, f_{2(n+1)}) \in W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$ and f_i, problem (4.1)–(4.2) has a unique solution belonging to W_q^σ. Denote by $f_i(x)$ the restriction of $f(x)$ on the interval J_i. Let $\tilde{f}_i(x) \in W_q^{\sigma-2}(\mathbb{R})$ be an extension of $f_i(x) \in W_q^{\sigma-2}(J_i)$. By

Lemma 1.7.6 in [3] there exists an extension operator $T_i f_i := \tilde{f}_i$ from $W_q^{\sigma-2}$ into $W_q^{\sigma-2}(\mathbb{R})$ is bounded for $i = 1, 2, \ldots, n+1$, where as usual $\mathbb{R} = (-\infty, +\infty)$. We shall find the solution $y(x, \lambda)$ of problem (4.1)–(4.2) in the form of $y(x, \lambda) = y_1(x, \lambda) + y_2(x, \lambda)$, where $y_1(x, \lambda) = (y_{1i}(x, \lambda))$, the function $y_{1i}(x, \lambda)$ is the restriction of the solution $\tilde{y}_{1i}(x, \lambda)$ on J_i of the following equation:

$$-\left(p_i \tilde{y}'\right)'(x) + \lambda^2 \tilde{y}(x) = \tilde{f}_i(x), \quad x \in \mathbb{R}$$

for $i = 1, 2, \ldots, n+1$.

By virtue of Theorem 3.2.1 in [3], we get that this equation has a unique solution $\tilde{y}_{1i} = \tilde{y}_{1i}(x, \lambda) \in W_q^{\sigma}(\mathbb{R})$, and for y_{1i}, the estimate

$$\sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell} \|y_{1i}\|_{W_q^\ell(J_i)} \leq C(\varepsilon)\left(\|f\|_{W_q^{\sigma-2}(J_i)} + |\lambda|^{\sigma-2}\|f\|_{L_q(J_i)}\right), \tag{4.4}$$

where $i = 1, 2, \ldots, n+1$, holds for all $\lambda \in \Omega_\varepsilon(\alpha, \beta)$ sufficiently large in modulus.

Hence, the function

$$y_1(x, \lambda) = \begin{cases} y_{11}(x, \lambda), & \text{for } x \in J_1; \\ y_{12}(x, \lambda), & \text{for } x \in J_2; \\ \quad \vdots \\ y_{1,n+1}(x, \lambda), & \text{for } x \in J_{n+1}, \end{cases} \tag{4.5}$$

satisfies equation (4.1), and from (4.4) the following estimate

$$\sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell} \|y_1\|_{q,\ell} \leq C(\varepsilon)\left(\|f\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|f\|_{q,0}\right) \tag{4.6}$$

holds for all $\lambda \in \Omega_\varepsilon(\alpha, \beta)$ sufficiently large in modulus. In light of solution (4.5), consider the following boundary value problem:

$$-\left(py'\right)'(x) - \lambda^2 y(x) = 0, \quad x \in J,$$
$$M_{k0}y = f_k - M_{k0}y_1(x, \lambda), \quad k = 1, 2, \ldots, 2(n+1).$$

By Theorem 2.1, this problem has a unique solution $y_2 = y_2(x, \lambda) \in W_q^\sigma$ for all complex numbers $\lambda \in \Omega_\varepsilon(\alpha, \beta)$ sufficiently large in modulus, and for these λ the estimate

$$\sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell} \|y_2\|_{q,\ell} \leq C(\varepsilon) \sum_{k=1}^{2(n+1)} |\lambda|^{\sigma-n_k-q^{-1}}\left(|f_k| + |M_{k0}y_1|\right) \tag{4.7}$$

is valid. Applying Theorem 1.7.7/2 in [3] and (2.3), one has that, for all $\lambda \in \Omega_\varepsilon(\alpha, \beta)$ and $\sigma \geq \max\{2, \max_{1 \leq k \leq 2(n+1)}\{n_k\} + 1\}$, the following estimates hold:

$$|\lambda|^{\sigma-n_k-q^{-1}}|M_{k0}y_1| \leq C|\lambda|^{\sigma-n_k-q^{-1}} \sum_{i=1}^{n+1} \|y_1\|_{C^{n_k}(I_i)}$$

$$\leq C \sum_{i=1}^{n+1}\left(|\lambda|^\sigma \|y_{1i}\|_{L_q(J_i)} + \|y_{1i}\|_{W_q^\sigma(J_i)}\right)$$

$$\leq C\big(|\lambda|^{\sigma} \|y_1\|_{q,0} + \|y_1\|_{q,\sigma}\big)$$

$$\leq C(\varepsilon)\big(|\lambda|_{\sigma-2}\|f\|_{q,0} + \|f\|_{q,\sigma-2}\big). \tag{4.8}$$

Via (4.7) and (4.8) we have the inequality

$$\sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell} \|y_2\|_{q,\ell} \leq C(\varepsilon)\left(\|f\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|f\|_{q,0} + \sum_{k=1}^{2(n+1)} |\lambda|^{\sigma-n_k-q^{-1}}|f_k|\right). \tag{4.9}$$

It is easy to see that $y(x,\lambda) = y_1(x,\lambda) + y_2(x,\lambda)$ is the solution of problem (4.1)–(4.2). Taking into account estimates (4.6) and (4.9), we see that for this solution the needed estimation (4.3) is valid. Moreover, from estimate (4.3) the uniqueness of the solution follows. On the other hand, by Theorem 3.1 the operator $\widehat{M}_0$ is Fredholm from W_q^{σ} onto $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$. Now, an isomorphism of this operator follows from the fact that it is a Fredholm and one-to-one operator. So, the proof of the theorem is completed. □

5 Solvability and coerciveness of the main problem with nonclassical boundary conditions

Now, we will study the main problem (1.1)–(1.2).

Theorem 5.1 *Assume that the following conditions hold*:

(1) $\omega \neq 0$ *and* $\sigma \geq \max\{2, \max_{1 \leq k \leq 2(n+1)}\{n_k\} + 1\}$;

(2) *The operator T from W_q^{σ} into $W_q^{\sigma-2}$ is compact, and for all $\varepsilon > 0$*

$$\|Ty\|_{q,0} \leq \varepsilon\|y\|_{q,2} + C(\varepsilon)\|y\|_{q,0}, \quad y \in W_q^2;$$

$$\|Ty\|_{q,\sigma-2} \leq \varepsilon\|y\|_{q,\sigma} + C(\varepsilon)\|y\|_{q,0}, \quad y \in W_q^{\sigma};$$

(3) *Functionals F_k in $W_q^{n_k}$ $(k = 1, 2, \ldots, 2(n+1))$ are continuous.*

Then, for each $\varepsilon > 0$, there exists $\mu_\varepsilon > 0$ such that, for all $\lambda \in \Omega_\varepsilon(\alpha,\beta)$ and $|\lambda| > \mu_\varepsilon$, the operator

$$\widehat{M}y = \big(M(\lambda)y, M_1y, M_2y, \ldots, M_{2(n+1)}y\big)$$

is an isomorphism from W_q^{σ} onto $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$, and for these λ we have the following coercive estimate for the solution of problem (1.1)–(1.2):

$$\sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell} \|y\|_{q,\ell} \leq C(\varepsilon)\left(\|f\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|f\|_{q,0} + \sum_{k=1}^{2(n+1)} |\lambda|^{\sigma-n_k-q^{-1}}|f_k|\right), \tag{5.1}$$

where $C(\varepsilon)$ is a constant which depends only on ε.

Proof Let $(f(x), f_1, f_2, \ldots, f_{2(n+1)})$ be any element of $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$. Suppose that there exists a solution $y = y(x,\lambda)$ of problem (1.1)–(1.2) corresponding to this element. Then this solution satisfies the equalities

$$M_0(\lambda)y := -\big(py'\big)' + \lambda^2 y = M(\lambda)y - Ty, \tag{5.2}$$

$$M_{k0}y := a_{0k}y^{(n_k)}(\xi_0) + \sum_{h=1}^{n}\left(a_{hk}y^{n_k}(\xi_h-) + \tilde{a}_{hk}y^{(n_k)}(\xi_h+)\right)$$

$$+ a_{(n+1)k}y^{(n_k)}(\xi_{n+1}) = M_k y - \Gamma_k y - F_k y, \tag{5.3}$$

where $k = 1, 2, \ldots, 2(n+1)$. By Theorem 4.1 we have that for this solution the following a priori estimates hold:

$$\sum_{\ell=0}^{\sigma}|\lambda|^{\sigma-\ell}\|y\|_{q,\ell} \leq C(\varepsilon)\left(\|f\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|f\|_{q,0} + \sum_{k=1}^{2(n+1)}|\lambda|^{\sigma-n_k-q^{-1}}|f_k|\right)$$

$$= C(\varepsilon)\left(\left\|M(\lambda)y - Ty\right\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\left\|M(\lambda)y - Ty\right\|_{q,0}\right.$$

$$\left. + \sum_{k=1}^{2(n+1)}|\lambda|^{\sigma-n_k-q^{-1}}|M_k y - \Gamma_k y - F_k y|\right)$$

$$\leq C(\varepsilon)\left(\|f\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|f\|_{q,0} + \|Ty\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|Ty\|_{q,0}\right.$$

$$\left. + \sum_{k=1}^{2(n+1)}|\lambda|^{\sigma-n_k-q^{-1}}\left(|f_k| + |\Gamma_k y| + |F_k y|\right)\right). \tag{5.4}$$

Let ζ be any real number satisfying

$$0 < \zeta$$

$$< \min\left\{\frac{\xi_i - \xi_{i-1}}{2}, |\xi_i - x_{kj}|, |\xi_0 - x_{kj}|:\right.$$

$$\left. i = 1, 2, \ldots, n+1; k = 1, 2, \ldots, 2(n+1); j = 1, 2, \ldots, n_k\right\}.$$

Using the same method in [15, Sect. 2.8.3], it is not difficult to construct a function $\phi(x) \in \mathbb{C}_0^\infty(\mathbb{R})$ such that

$$\phi(x) = \begin{cases} 1, & x \in \bigcup_{i=1}^{n+1}[\xi_{i-1} + \zeta, \xi_i - \zeta]; \\ 0, & x \in \bigcup_{i=1}^{n+1}[\xi_0, \xi_0 + \frac{\zeta}{2}] \cup (\bigcup_{i=1}^{n}[\xi_i - \frac{\zeta}{2}, \xi_i + \frac{\zeta}{2}]) \cup [\xi_{n+1} - \frac{\zeta}{2}, \xi_{n+1}] \end{cases}$$

and $0 \leq \phi(x) \leq 1$ for all $x \in [-1, 1]$. It is obvious that

$$|\Gamma_k y| \leq C\left\|y^{(n_k)}\right\|_{C(\bigcup_{i=1}^{n+1}[\xi_{i-1}+\zeta,\xi_i-\zeta])} \leq C\left\|(\phi y)^{(n_k)}\right\|_{C[-1,1]}. \tag{5.5}$$

By Theorem 3.10.4 in [15], for $y \in W_q^\sigma$, the estimate

$$|\lambda|^{\sigma-n_k-q^{-1}}\left\|y^{(n_k)}\right\|_{C[-1,1]} \leq C\left(\|y\|_{q,\sigma} + |\lambda|^\sigma\|y\|_{q,0}\right) \tag{5.6}$$

holds. By Theorem 4.1, from (5.4), (5.5) it follows that, for all $\lambda \in \Omega_\varepsilon(\alpha, \beta)$ sufficiently large in modulus, the following estimates hold:

$$
\begin{aligned}
|\lambda|^{\sigma - n_k - q^{-1}} |\Gamma_k y| &\leq C |\lambda|^{\sigma - n_k - q^{-1}} \left\| (\phi y)^{(n_k)} \right\|_{C[-1,1]} \\
&\leq C \big(\|\phi y\|_{q,\sigma} + |\lambda|^\sigma \|\phi y\|_{q,0} \big) \\
&\leq C(\varepsilon) \big(\left\| M_0(\lambda)(\phi y) \right\|_{q,\sigma-2} + |\lambda|^{\sigma-2} \left\| M_0(\lambda)(\phi y) \right\|_{q,0} \big) \\
&\leq C(\varepsilon) \left(\left\| M_0(\lambda) y \right\|_{q,\sigma-2} + |\lambda|^{\sigma-2} \left\| M_0(\lambda) y \right\|_{q,0} + \sum_{\ell=0}^{\sigma-1} |\lambda|^{\sigma-1-\ell} \|y\|_{q,\ell} \right) \\
&\leq C(\varepsilon) \Bigg(\left\| M(\lambda) y \right\|_{q,\sigma-2} + |\lambda|^{\sigma-2} \left\| M(\lambda) y \right\|_{q,0} + \| Ty \|_{q,\sigma-2} \\
&\qquad + |\lambda|^{\sigma-2} \| Ty \|_{q,0} + \sum_{\ell=0}^{\sigma-1} |\lambda|^{\sigma-1-\ell} \|y\|_{q,\ell} \Bigg) \\
&\leq C(\varepsilon) \Bigg(\| f \|_{q,\sigma-2} + |\lambda|^{\sigma-2} \| f \|_{q,0} + \| Ty \|_{q,\sigma-2} \\
&\qquad + |\lambda|^{\sigma-2} \| Ty \|_{q,0} + \sum_{\ell=0}^{\sigma-1} |\lambda|^{\sigma-1-\ell} \|y\|_{q,\ell} \Bigg).
\end{aligned}
\tag{5.7}
$$

By virtue of Theorem 1.7.7/2(b) and Remark 1.1.7/5 in [3], the following inequality

$$
\|y\|_{q,\ell} \leq \delta \|y\|_{q,\ell+1} + C(\delta) \|y\|_{q,0}
$$

holds for any $\delta > 0$. Then, by virtue of (5.7),

$$
\begin{aligned}
|\lambda|^{\sigma - n_k - q^{-1}} \| \Gamma_k y \| &\leq C(\varepsilon) \Bigg(\left\| M(\lambda) y \right\|_{q,\sigma-2} + |\lambda|^{\sigma-2} \left\| M(\lambda) y \right\|_{q,0} + \| Ty \|_{q,\sigma-2} \\
&\qquad + |\lambda|^{\sigma-2} \| Ty \|_{q,0} + \sum_{\ell=0}^{\sigma-1} |\lambda|^{\sigma-1-\ell} \big(\delta \|y\|_{q,\ell+1} + C(\delta) \|y\|_{q,0} \big) \Bigg) \\
&\leq C(\varepsilon) \big(\left\| M(\lambda) y \right\|_{q,\sigma-2} + |\lambda|^{\sigma-2} \left\| M(\lambda) y \right\|_{q,0} + \| Ty \|_{q,\sigma-2} \big) \\
&\qquad + |\lambda|^{\sigma-2} \| Ty \|_{q,0} + C(\varepsilon) \big(\delta + C(\delta) |\lambda|^{-1} \big) \sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell} \|y\|_{q,\ell}.
\end{aligned}
\tag{5.8}
$$

From conditions (2), (3), inequality (5.8), and Theorem 1.7.7/2 in [3], for any $\delta > 0$, we have

$$
\begin{aligned}
\| Ty \|_{q,\sigma-2} &+ |\lambda|^{\sigma-2} \| Ty \|_{q,0} + \sum_{k=1}^{2(n+1)} |\lambda|^{\sigma - n_k - q^{-1}} \big(|\Gamma_k y| + |F_k y| \big) \\
&\leq C(\varepsilon) \big(\left\| M(\lambda) y \right\|_{q,\sigma-2} + |\lambda|^{\sigma-2} \left\| M(\lambda) y \right\|_{q,0} \big) + \delta \big(\|y\|_{q,\sigma} + |\lambda|^{\sigma-2} \|y\|_{q,2} \big) \\
&\quad + C(\delta) |\lambda|^{\sigma-2} \|y\|_{q,0} + C(\varepsilon) \big(\delta + C(\delta) |\lambda|^{-1} \big) \sum_{\ell=0}^{\sigma} |\lambda|_{\sigma-\ell} \|y\|_{q,\ell} \\
&\quad + \sum_{k=1}^{2(n+1)} |\lambda|^{\sigma - n_k - q^{-1}} \|y\|_{q,n_k}
\end{aligned}
$$

$$
\leq C(\varepsilon)\big(\|f\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|f\|_{q,0}\big)
$$

$$
+ C(\varepsilon)\big(\delta + C(\delta)|\lambda|^{-q^{-1}}\big) \sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell}\|y\|_{q,\ell}. \tag{5.9}
$$

Substituting (5.9) into (5.4) yields

$$
\sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell}\|y\|_{q,\ell} \leq C(\varepsilon)\Bigg(\|f\|_{q,\sigma-2} + |\lambda|^{\sigma-2}\|f\|_{q,0} + \sum_{k=1}^{2(n+1)} |\lambda|^{\sigma-n_k-q^{-1}}|f_k|
$$

$$
+ \big(\delta + C(\delta)|\lambda|^{-q^{-1}}\big) \sum_{\ell=0}^{\sigma} |\lambda|^{\sigma-\ell}\|y\|_{q,\ell} \Bigg).
$$

It is obvious that for fixed $\varepsilon > 0$ it is possible to choose $\delta > 0$ so small and $|\lambda|$ so large that $C(\varepsilon)(\delta + C(\delta)|\lambda|^{-q^{-1}}) < 1$. Hence, for $\lambda \in \Omega_{\varepsilon}(\underline{w}, \overline{w})$ sufficiently large in modulus, we obtain a priori estimate (5.1).

It follows from estimate (5.1) that we can obtain the uniqueness property of solution of problem (1.1)–(1.2), i.e., the operator $\widehat{M}$ is a one-to-one operator. Moreover, by Theorem 3.1 the operator $\widehat{M}$ from W_q^{σ} onto $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$ is Fredholm.

In view of condition (2), the operator T from W_q^{σ} into $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$ is compact. From above, we get that the operator $\widehat{M}$ is an isomorphism from W_q^{σ} onto $W_q^{\sigma-2} \oplus \mathbb{C}^{2(n+1)}$. The proof is completed. $\qquad\square$

6 Conclusion

Sturm–Liouville problem with discontinuous points inside an interval has attracted extensive attention for its wide application in physical and mathematical fields. In this paper, we go into Sturm–Liouville problem with finite discontinuous points, and for such a problem, we establish the properties such as isomorphism, Fredholmness, and coerciveness with respect to the spectral parameter. These results are of both theoretical and practical significance.

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