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Existence and uniqueness of positive solutions for fractional differential equations

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Abstract

By some new integral inequalities of Henry–Gronwall type, we investigate the existence and uniqueness of positive solutions for fractional differential equations.

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1 Introduction

Fractional differential equations have gained considerable importance due to their applications in various sciences such as physics, mechanics, chemistry, engineering, etc. [1–7]. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs [8–10] and the papers in [11–17]. However, there have been few contributions to the existence and uniqueness of the following fractional differential equations:

$$\begin{cases} D_c^{\alpha}x(t) - D_c^{\beta}x(t) = f(t,x(t)), & t \in [0,T), 0 < \beta < \alpha < 1, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

In most of the available literature, fractional integral inequalities play an important role in the qualitative analysis of the solutions for fractional differential equations (see [14–17]). In this paper, by a method introduced by M. Medveď [18], we first study the following Henry–Gronwall integral inequalities:

$$u(t) \le a(t) + b_1(t)\int_0^t (t-s)^{\gamma_1-1} l_1(s)u(s)\,ds + b_2(t)\int_0^t (t-s)^{\gamma_2-1} l_2(s)u(s)\,ds \tag{1.2}$$

and

$$u(t) \le a(t) + b_1(t)\int_0^t (t-s)^{\gamma_1-1} l_1(s)\varphi_1\bigl(u(s)\bigr)\,ds + b_2(t)\int_0^t (t-s)^{\gamma_2-1} l_2(s)\varphi_2\bigl(u(s)\bigr)\,ds, \tag{1.3}$$

where $0 < \gamma_1 < \gamma_2 < 1$, which generalize the famous Henry inequalities [19]. Then using a suitable substitution, we construct an equivalent fractional integral equation of equation (1.1). By the above integral inequalities and fixed point theorem, we present the existence and uniqueness of fractional differential equations (1.1). Finally, some examples are given to illustrate the applications of the obtained results.

2 Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Let $I = [0, a]$ $(0 < a < +\infty)$ be a finite interval. $AC[0, a]$ is the space of functions which are absolutely continuous on I. $L^\infty(0, a)$ is the space of measurable functions $f : I \to \Re$ with the norm $\|f\|_{L^\infty} = \inf\{c > 0, |f(t)| \le c, \text{ a.e. } t \in I\}$. $C^1[0, a]$ is the space of functions which are continuously differentiable on I.

The Riemann–Liouville fractional integral and derivative of order $\alpha \in (0, 1)$ are defined by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > 0$$

and

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(s)}{(t-s)^{\alpha}}\, ds, \quad t > 0.$$

The Caputo fractional derivative of order $\alpha \in (0, 1)$ is defined by

$$D_c^\alpha x(t) = D^\alpha x(t) - \frac{x(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \quad t > 0.$$

In particular, when $x(t) \in AC[0, a]$,

$$D_c^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(s)}{(t-s)^{\alpha}}\, ds, \quad t > 0.$$

Lemma 2.1 ([8]) *Let* $\alpha \in (0, 1)$ *and* $x \in L^\infty(0, a)$ *or* $x \in C[0, a]$, *then*

$$\big(D_c^\alpha I^\alpha x\big)(t) = x(t).$$

Lemma 2.2 ([8]) *Let* $\alpha \in (0, 1)$ *and* $x \in AC[0, a]$ *or* $x \in C^1[0, a]$, *then*

$$\big(I^\alpha D_c^\alpha x\big)(t) = x(t) - x(0).$$

Theorem 2.3 *Let* $0 < \beta < \alpha < 1$ *and* $x \in AC[0, a]$ *or* $x \in C^1[0, a]$, *then*

$$\big(D_c^\alpha I^{\alpha-\beta} x\big)(t) = D_c^\beta x(t) + \frac{x(0)}{\Gamma(1-\beta)} t^{-\beta}.$$

Proof By Lemmas 2.1 and 2.2, we know

$$\begin{aligned}\big(D_c^\alpha I^{\alpha-\beta} x\big)(t) &= D_c^\alpha I^{\alpha-\beta}\big(\big(I^\beta D_c^\beta x\big)(t) + x(0)\big) \\ &= \big(D_c^\alpha I^\alpha D_c^\beta x\big)(t) + D_c^\alpha I^{\alpha-\beta}\big(x(0)\big)\end{aligned}$$

$$= D_c^{\beta}x(t) + \frac{x(0)}{\Gamma(1-\beta)}t^{-\beta}. \tag{2.1}$$

□

Theorem 2.4 *Let* $0 < \beta < \alpha < 1$ *and* $x = I^{\beta}\mu(t)$, *where* $\mu \in C[0,a]$, *then*

$$\bigl(D_c^{\alpha}I^{\alpha-\beta}x\bigr)(t) = D_c^{\beta}x(t).$$

Proof We know

$$\begin{aligned}\bigl(D_c^{\alpha}I^{\alpha-\beta}x\bigr)(t) &= \bigl(D_c^{\alpha}I^{\alpha-\beta}I^{\beta}\mu\bigr)(t)\\ &= \bigl(D_c^{\alpha}I^{\alpha}\mu\bigr)(t)\\ &= \mu(t)\\ &= D_c^{\beta}x(t).\end{aligned} \tag{2.2}$$

□

Theorem 2.5 *Let* $0 < \gamma_1 < \gamma_2 < 1$, $a(t)$, $b_1(t)$, $b_2(t)$, $l_1(t)$, *and* $l_2(t)$ *be continuous, nonnegative functions on* $[0,+\infty)$, *and* $u(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$ *with*

$$u(t) \le a(t) + b_1(t)\int_0^t (t-s)^{\gamma_1-1}l_1(s)u(s)\,ds + b_2(t)\int_0^t (t-s)^{\gamma_2-1}l_2(s)u(s)\,ds. \tag{2.3}$$

Then the following assertions hold:

$$u(t) \le \left(3^{p-1}a^p(t) + 3^{p-1}b^p(t)\left(A(t) + \int_0^t L(s)A(s)\exp\left(\int_s^t L(\tau)\,d\tau\right)ds\right)\right)^{\frac{1}{p}}, \quad t \in [0,+\infty), \tag{2.4}$$

where $b(t) = \max\{\frac{b_1(t)t^{\gamma_1-1+\frac{1}{q}}}{(q(\gamma_1-1)+1)^{\frac{1}{q}}}, \frac{b_2(t)t^{\gamma_2-1+\frac{1}{q}}}{(q(\gamma_2-1)+1)^{\frac{1}{q}}}\}$, $A(t) = \int_0^t 3^{p-1}(l_1^p(s) + l_2^p(s))a^p(s)\,ds$, $L(t) = 3^{p-1}b^p(t)(l_1^p(t) + l_2^p(t))$, *and* $p,q \in (1,+\infty)$ *such that* $\gamma_1 + \frac{1}{q} > 1$ *and* $\frac{1}{q} + \frac{1}{p} = 1$.

Proof Choose nonnegative constants p,q such that $\gamma_1 + \frac{1}{q} > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$. Using the Hölder inequality, we obtain

$$\begin{aligned}u(t) &\le a(t) + b_1(t)\int_0^t (t-s)^{\gamma_1-1}l_1(s)u(s)\,ds\\ &\quad + b_2(t)\int_0^t (t-s)^{\gamma_2-1}l_2(s)u(s)\,ds\\ &\le a(t) + b_1(t)\left(\int_0^t (t-s)^{(\gamma_1-1)q}\,ds\right)^{\frac{1}{q}}\left(\int_0^t \bigl(l_1(s)u(s)\bigr)^p\,ds\right)^{\frac{1}{p}}\\ &\quad + b_2(t)\left(\int_0^t (t-s)^{(\gamma_2-1)q}\,ds\right)^{\frac{1}{q}}\left(\int_0^t \bigl(l_2(s)u(s)\bigr)^p\,ds\right)^{\frac{1}{p}}\\ &\le a(t) + \frac{b_1(t)t^{\gamma_1-1+\frac{1}{q}}}{(q(\gamma_1-1)+1)^{\frac{1}{q}}}\left(\int_0^t l_1^p(s)u^p(s)\,ds\right)^{\frac{1}{p}}\end{aligned}$$

$$+\frac{b_2(t)t^{\gamma_2-1+\frac{1}{q}}}{(q(\gamma_2-1)+1)^{\frac{1}{q}}}\left(\int_0^t l_2^p(s)u^p(s)\,ds\right)^{\frac{1}{p}}. \tag{2.5}$$

Let $b(t)=\max\{\frac{b_1(t)t^{\gamma_1-1+\frac{1}{q}}}{(q(\gamma_1-1)+1)^{\frac{1}{q}}},\frac{b_2(t)t^{\gamma_2-1+\frac{1}{q}}}{(q(\gamma_2-1)+1)^{\frac{1}{q}}}\}$. Then

$$u^p(t)\le 3^{p-1}a^p(t)+3^{p-1}b^p(t)\int_0^t\bigl(l_1^p(s)+l_2^p(s)\bigr)u^p(s)\,ds \tag{2.6}$$

and

$$\begin{aligned}&\int_0^t\bigl(l_1^p(s)+l_2^p(s)\bigr)u^p(s)\,ds\\&\quad\le\int_0^t 3^{p-1}\bigl(l_1^p(s)+l_2^p(s)\bigr)a^p(s)\,ds\\&\qquad+\int_0^t 3^{p-1}b^p(s)\bigl(l_1^p(s)+l_2^p(s)\bigr)\int_0^s\bigl(l_1^p(\tau)+l_2^p(\tau)\bigr)u^p(\tau)\,d\tau\,ds.\end{aligned} \tag{2.7}$$

Let $w(t)=\int_0^t(l_1^p(s)+l_2^p(s))u^p(s)\,ds$, $A(t)=\int_0^t 3^{p-1}(l_1^p(s)+l_2^p(s))a^p(s)\,ds$, and $L(t)=3^{p-1}b^p(t)\times(l_1^p(t)+l_2^p(t))$. Then

$$w(t)\le A(t)+\int_0^t L(s)w(s)\,ds. \tag{2.8}$$

By Gronwall's integral inequality, we have

$$w(t)\le A(t)+\int_0^t L(s)A(s)\exp\left(\int_s^t L(\tau)\,d\tau\right)ds. \tag{2.9}$$

By (2.6) and (2.9) we obtain inequality (2.4) and complete the proof. □

Theorem 2.6 *Let $0<\gamma_1<\gamma_2<1$, $a(t)$, $b_1(t)$, $b_2(t)$, $l_1(t)$, and $l_2(t)$ be nondecreasing, nonnegative, and continuous functions on $[0,T)(0<T\le+\infty)$, $\varphi_1,\varphi_2:[0,+\infty)\to[0,+\infty)$ be continuous, nondecreasing functions, and $u(t)$ be a continuous, nonnegative function on $[0,T)$ with*

$$\begin{aligned}u(t)\le{}&a(t)+b_1(t)\int_0^t(t-s)^{\gamma_1-1}l_1(s)\varphi_1\bigl(u(s)\bigr)\,ds\\&+b_2(t)\int_0^t(t-s)^{\gamma_2-1}l_2(s)\varphi_2\bigl(u(s)\bigr)\,ds.\end{aligned} \tag{2.10}$$

Then

$$u(t)\le\left(\Omega^{-1}\left(\Omega\bigl(A(t)\bigr)+B_1(t)\int_0^t l_1^p(s)\,ds+B_2(t)\int_0^t l_2^p(s)\,ds\right)\right)^{\frac{1}{p}},\quad t\in[0,T_1], \tag{2.11}$$

where $A(t)=3^{p-1}a^p(t)$, $B_1(t)=3^{p-1}(\frac{b_1(t)t^{\gamma_1-1+\frac{1}{q}}}{(q(\gamma_1-1)+1)^{\frac{1}{q}}})^p$, $B_2(t)=3^{p-1}(\frac{b_2(t)t^{\gamma_2-1+\frac{1}{q}}}{(q(\gamma_2-1)+1)^{\frac{1}{q}}})^p$, $\Omega(x)=\int_{t_0}^x\frac{1}{\mu_1(t)+\mu_2(t)}\,dt$, $\mu_1(t)=\varphi_1^p(t^{\frac{1}{p}})$, $\mu_2(t)=\varphi_2^p(t^{\frac{1}{p}})$, $t_0>0$, *Ω^{-1} is the inverse of Ω, and* $T_1\in$

$(0,T)$ is such that $\Omega(A(t))+B_1(t)\int_0^t l_1^p(s)\,ds+B_2(t)\int_0^t l_2^p(s)\,ds\in \mathrm{Dom}(\Omega^{-1})$ for all $t\in[0,T_1]$, and $p,q\in(1,+\infty)$ such that $\gamma_1+\frac{1}{q}>1$ and $\frac{1}{q}+\frac{1}{p}=1$.

Proof Choose nonnegative constants p,q such that $\gamma_1+\frac{1}{q}>1$ and $\frac{1}{q}+\frac{1}{p}=1$. Using the Hölder inequality, we obtain

$$\begin{aligned} u(t) &\le a(t)+b_1(t)\int_0^t (t-s)^{\gamma_1-1}l_1(s)\varphi_1\big(u(s)\big)\,ds \\ &\quad + b_2(t)\int_0^t (t-s)^{\gamma_2-1}l_2(s)\varphi_2\big(u(s)\big)\,ds \\ &\le a(t)+\frac{b_1(t)t^{\gamma_1-1+\frac{1}{q}}}{(q(\gamma_1-1)+1)^{\frac{1}{q}}}\left(\int_0^t l_1^p(s)\varphi_1^p\big(u(s)\big)\,ds\right)^{\frac{1}{p}} \\ &\quad + \frac{b_2(t)t^{\gamma_2-1+\frac{1}{q}}}{(q(\gamma_2-1)+1)^{\frac{1}{q}}}\left(\int_0^t l_2^p(s)\varphi_2^p\big(u(s)\big)\,ds\right)^{\frac{1}{p}}. \end{aligned} \tag{2.12}$$

Then

$$\begin{aligned} u^p(t) &\le 3^{p-1}a^p(t)+3^{p-1}\left(\frac{b_1(t)t^{\gamma_1-1+\frac{1}{q}}}{(q(\gamma_1-1)+1)^{\frac{1}{q}}}\right)^p\int_0^t l_1^p(s)\varphi_1^p\big(u(s)\big)\,ds \\ &\quad + 3^{p-1}\left(\frac{b_2(t)t^{\gamma_2-1+\frac{1}{q}}}{(q(\gamma_2-1)+1)^{\frac{1}{q}}}\right)^p\int_0^t l_2^p(s)\varphi_2^p\big(u(s)\big)\,ds. \end{aligned} \tag{2.13}$$

Let $w(t)=u^p(t)$, $A(t)=3^{p-1}a^p(t)$, $B_1(t)=3^{p-1}(\frac{b_1(t)t^{\gamma_1-1+\frac{1}{q}}}{(q(\gamma_1-1)+1)^{\frac{1}{q}}})^p$, and $B_2(t)=3^{p-1}(\frac{b_2(t)t^{\gamma_2-1+\frac{1}{q}}}{(q(\gamma_2-1)+1)^{\frac{1}{q}}})^p$. Fix any $T_0\in[0,T_1]$, then for $t\in[0,T_0]$ and (2.13) we have

$$\begin{aligned} w(t) &\le A(T_0)+B_1(T_0)\int_0^t l_1^p(s)\mu_1\big(w(s)\big)\,ds \\ &\quad + B_2(T_0)\int_0^t l_2^p(s)\mu_2\big(w(s)\big)\,ds. \end{aligned} \tag{2.14}$$

Let $V(t)=A(T_0)+B_1(T_0)\int_0^t l_1^p(s)\mu_1(w(s))\,ds+B_2(T_0)\int_0^t l_2^p(s)\mu_2(w(s))\,ds$, then we get

$$\begin{aligned} V'(t) &= B_1(T_0)l_1^p(t)\mu_1\big(w(t)\big)+B_2(T_0)l_2^p(t)\mu_2\big(w(t)\big) \\ &\le B_1(T_0)l_1^p(t)\mu_1\big(V(t)\big)+B_2(T_0)l_2^p(t)\mu_2\big(V(t)\big). \end{aligned} \tag{2.15}$$

This yields

$$\frac{V'(t)}{\mu_1(V(t))+\mu_2(V(t))}\le B_1(T_0)l_1^p(t)+B_2(T_0)l_2^p(t) \tag{2.16}$$

or

$$\frac{d}{dt}\Omega\big(V(t)\big)\le B_1(T_0)l_1^p(t)+B_2(T_0)l_2^p(t). \tag{2.17}$$

Integrating this inequality from 0 to $t \in [0, T_0]$, we obtain

$$\Omega\bigl(V(t)\bigr) \le \Omega\bigl(A(T_0)\bigr) + \int_0^t B_1(T_0)l_1^p(s) + B_2(T_0)l_2^p(s)\,ds, \tag{2.18}$$

then

$$V(t) \le \Omega^{-1}\biggl(\Omega\bigl(A(T_0)\bigr) + \int_0^t B_1(T_0)l_1^p(s) + B_2(T_0)l_2^p(s)\,ds\biggr), \quad t \in [0, T_0] \tag{2.19}$$

and

$$u(t) \le \biggl(\Omega^{-1}\biggl(\Omega\bigl(A(T_0)\bigr) + \int_0^t B_1(T_0)l_1^p(s) + B_2(T_0)l_2^p(s)\,ds\biggr)\biggr)^{\frac{1}{p}}, \quad t \in [0, T_0]. \tag{2.20}$$

So

$$u(T_0) \le \biggl(\Omega^{-1}\biggl(\Omega\bigl(A(T_0)\bigr) + B_1(T_0)\int_0^{T_0} l_1^p(s)\,ds + B_2(T_0)\int_0^{T_0} l_2^p(s)\,ds\biggr)\biggr)^{\frac{1}{p}}. \tag{2.21}$$

Now replacing T_0 by t in inequality (2.21), we obtain the result (2.11) valid for $t \in [0, T_1]$ provided

$$\Omega\bigl(A(t)\bigr) + B_1(t)\int_0^t l_1^p(s)\,ds + B_2(t)\int_0^t l_2^p(s)\,ds \in \operatorname{Dom}\bigl(\Omega^{-1}\bigr)$$

for all $t \in [0, T_1]$. □

Lemma 2.7 ([20, 21]) *Let E be a Banach space X, C be a closed, convex subset of E, U be an open subset of C, and $P \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact map. Then either*

(a) *F has a fixed point in $\overline{U}$; or*

(b) *there are $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1 - \lambda)P$.*

Lemma 2.8 ([20, 21]) *Let E be a Hausdorff locally convex linear topological space, C be a convex subset of E, U be an open subset of C, and $P \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact map. Then either*

(a) *F has a fixed point in $\overline{U}$; or*

(b) *there are $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1 - \lambda)P$.*

3 Main results

In this section, we give the existence and uniqueness results of the fractional differential equations (1.1).

Theorem 3.1 *$f : \Re^+ \times \Re \to \Re$ is a continuous function. If $x(\cdot) \in C[0, a]$ is the solution of the following integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha - \beta)}\int_0^t (t-s)^{\alpha-\beta-1}\bigl(x(s) - x_0\bigr)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\bigl(s, x(s)\bigr)\,ds, \tag{3.1}$$

then $x(t)$ is the solution of the fractional integral equation (1.1).

Proof If $x(t) \in C[0,a]$ is the solution of the integral equation (3.1), we know $x(0) = x_0$ and

$$x(t) - x_0 = I^{\alpha-\beta}\big(x(t) - x_0\big) + I^{\alpha} f\big(t, x(t)\big) = I^{\alpha-\beta}\mu(t), \tag{3.2}$$

where $\mu(t) = x(t) - x_0 + I^{\beta} f(t, x(t))$. By (3.1) and (3.2), we obtain

$$x(t) - x_0 = I^{\alpha-\beta}\big(x(t) - x_0\big) + I^{\alpha} f\big(t, x(t)\big) = I^{2(\alpha-\beta)}\mu(t) + I^{\alpha} f\big(t, x(t)\big). \tag{3.3}$$

If $2(\alpha - \beta) < \alpha$, then $x(t) - x_0 = I^{2(\alpha-\beta)}\mu_1(t)$, where $\mu_1(t) = \mu(t) + I^{2\beta-\alpha} f(t, x(t))$. By the same step, we obtain $x(t) - x_0 \in I^{\alpha}\phi_1(t)$ and $x(t) - x_0 \in I^{\beta}\phi_2(t)$, where $\phi_1(t), \phi_2(t) \in C[0,a]$.

By Lemma 2.1 and Theorem 2.4, we get

$$\begin{aligned} D_c^{\alpha} x(t) &= D_c^{\alpha} I^{\alpha-\beta}\big(x(t) - x_0\big) + D_c^{\alpha} I^{\alpha} f\big(t, x(t)\big) \\ &= D_c^{\beta}\big(x(t) - x_0\big) + f\big(t, x(t)\big) \\ &= D_c^{\beta} x(t) + f\big(t, x(t)\big). \end{aligned} \tag{3.4}$$

□

Theorem 3.2 *Let $x_0 > 0, f : \Re^+ \times \Re^+ \to \Re^+$ be a continuous function, and there exist non-negative continuous functions $l(t)$ and $k(t)$ such that*

$$\big|f(t,x)\big| \leq l(t)|x| + k(t)$$

for all $x \in \Re^+$, $t \in [0,\infty)$. Then equation (1.1) has at least one positive solution on $[0,\infty)$.

Proof Consider the operator $G : W \to W$ defined by

$$\begin{aligned} (Gx)(t) = x_0 &+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}\big(x(s) - x_0\big)\,ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s)\big)\,ds, \end{aligned} \tag{3.5}$$

where $W = \{x(t) \in C[0,+\infty) | x(t) \geq x_0\}$.

By Theorem 3.1, we know that the fixed points of operator G are solutions of equation (1.1). We can show that $G : W \to W$ is continuous and compact by the usual techniques (see [12, 13]).

Let $U = \{x \in W : |x(t)| < (3^{p-1}a^p(t) + 3^{p-1}b^p(t)(A(t) + \int_0^t L(s)A(s)\exp(\int_s^t L(\tau)\,d\tau)\,ds))^{\frac{1}{p}} + 1, t \in [0,\infty)\}$, where $a(t) = |x_0| + |\frac{x_0 t^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha-\beta)}| + \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha-1+\frac{1}{q}}}{((\alpha-1)q+1)^{\frac{1}{q}}} (\int_0^t k^p(s)\,ds)^{\frac{1}{p}}$, $b(t) = \max\{\frac{\frac{1}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1+\frac{1}{q}}}{(q(\alpha-\beta-1)+1)^{\frac{1}{q}}}, \frac{\frac{1}{\Gamma(\alpha)} t^{\alpha-1+\frac{1}{q}}}{(q(\alpha-1)+1)^{\frac{1}{q}}}\}$, $A(t) = \int_0^t 3^{p-1}(1 + l^p(s))a^p(s)\,ds$, $L(t) = 3^{p-1}b^p(t)(1 + l^p(t))$, and $p, q \in (1,+\infty)$ such that $\alpha - \beta + \frac{1}{q} > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$.

If $x \in W$ is any solution of

$$\begin{aligned} x(t) = (1-\lambda)x_0 + \lambda\bigg(&\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}\big(x(s) - x_0\big)\,ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s)\big)\,ds\bigg) \end{aligned} \tag{3.6}$$

for $\lambda \in (0,1)$.

Then

$$\begin{aligned}
|x(t)| &\le |x_0| + \left|\frac{x_0 t^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha-\beta)}\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)\,ds\right| + \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds\right| \\
&\le |x_0| + \left|\frac{x_0 t^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha-\beta)}\right| + \frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-1+\frac{1}{q}}}{((\alpha-1)q+1)^{\frac{1}{q}}}\left(\int_0^t k^p(s)\,ds\right)^{\frac{1}{p}} \\
&\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|x(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)|x(s)|\,ds. \qquad (3.7)
\end{aligned}$$

Consequently, by Theorem 2.5, we can get

$$|x(t)| \le \left(3^{p-1}a^p(t) + 3^{p-1}b^p(t)\left(A(t) + \int_0^t L(s)A(s)\exp\left(\int_s^t L(\tau)\,d\tau\right)ds\right)\right)^{\frac{1}{p}},$$
$$t \in [0,\infty). \qquad (3.8)$$

Applying Lemma 2.8, we can obtain that G has at least one fixed point in W. Thus, the proof is completed. □

Theorem 3.3 *If $f : \Re^+ \times \Re^+ \to \Re^+$ is a continuous function and*

$$|f(t,x) - f(t,y)| \le l(t)|x-y|$$

for all $x, y \in \Re^+$ and $t \in [0,+\infty)$, where nonnegative function $l(t) \in C[0,+\infty)$, then equation (1.1) has a unique positive solution on $[0,+\infty)$.

Proof By Theorem 3.2, we suppose that $x_1(t)$, $x_2(t)$ are two positive solutions of equation (1.1). Then

$$\begin{aligned}
|x_1(t) - x_2(t)| &\le \left|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\bigl(x_1(s)-x_2(s)\bigr)\,ds\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl(f\bigl(s,x_1(s)\bigr) - f\bigl(s,x_2(s)\bigr)\bigr)\,ds\right| \\
&\le \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|x_1(s)-x_2(s)|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)|x_1(s)) - x_2(s)|\,ds. \qquad (3.9)
\end{aligned}$$

By Theorem 2.5, we can get $x_1(t) = x_2(t)$. □

Theorem 3.4 *Let $x_0 > 0$, $f : [0,T] \times \Re^+ \to \Re^+$ be a continuous function, and there exist a nonnegative function $l(t) \in C[0,T]$ and a nonnegative nondecreasing function $\omega \in C[0,+\infty)$ such that*

$$|f(t,x)| \le l(t)\omega(|x|).$$

Then the initial value problem (1.1) *has at least a continuous positive solution on* $[0,T]$ *provided that*

$$\Omega\bigl(A(t)\bigr)+tB_1(t)+B_2(t)\int_0^t l^p(s)\,ds\in \operatorname{Dom}\bigl(\Omega^{-1}\bigr)$$

for all $t\in[0,T]$, *where* $A(t)=3^{p-1}(|x_0|+\frac{|x_0t^{\alpha-\beta}|}{(\alpha-\beta)\Gamma(\alpha-\beta)})^p$, $B_1(t)=3^{p-1}(\frac{\frac{1}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1+\frac{1}{q}}}{(q(\alpha-\beta-1)+1)^{\frac{1}{q}}})^p$, $B_2(t)=3^{p-1}(\frac{\frac{1}{\Gamma(\alpha)}t^{\alpha-1+\frac{1}{q}}}{(q(\alpha-1)+1)^{\frac{1}{q}}})^p$, $\Omega(x)=\int_{t_0}^x\frac{1}{\mu_1(t)+\mu_2(t)}\,dt$, $\mu_1(t)=t$, $\mu_2(t)=\omega^p(t^{\frac{1}{p}})$, $t_0>0$, Ω^{-1} *is the inverse of* Ω, *and* $p,q\in(1,+\infty)$ *such that* $\alpha-\beta+\frac{1}{q}>1$ *and* $\frac{1}{q}+\frac{1}{p}=1$.

Proof Consider the operator $G:W\to W$ defined by

$$\begin{aligned}(Gx)(t)={}&x_0+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\bigl(x(s)-x_0\bigr)\,ds\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds,\end{aligned}\tag{3.10}$$

where $W=\{x\in C[0,T]|x(t)\geq x_0\}$.

Similarly with the proof of Theorem 3.2, we can show that $G:W\to W$ is continuous and compact.

Let $U=\{x\in W:|x(t)|<(\Omega^{-1}(\Omega(A(t))+tB_1(t)+B_2(t)\int_0^t l^p(s)\,ds))^{\frac{1}{p}}+1,t\in[0,T]\}$.

If $x\in W$ is any solution of

$$\begin{aligned}x(t)={}&(1-\lambda)x_0+\lambda\biggl(\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\bigl(x(s)-x_0\bigr)\,ds\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds\biggr)\end{aligned}$$

for $\lambda\in(0,1)$.

Then

$$\begin{aligned}\bigl|x(t)\bigr|\leq{}&|x_0|+\frac{|x_0t^{\alpha-\beta}|}{(\alpha-\beta)\Gamma(\alpha-\beta)}\\&+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\bigl|x(s)\bigr|\,ds\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}l(s)\omega\bigl(\bigl|x(s)\bigr|\bigr)\,ds.\end{aligned}\tag{3.11}$$

By Theorem 2.6, we can get

$$\bigl|x(t)\bigr|\leq\biggl(\Omega^{-1}\biggl(\Omega\bigl(A(t)\bigr)+tB_1(t)+B_2(t)\int_0^t l^p(s)\,ds\biggr)\biggr)^{\frac{1}{p}},\quad t\in[0,T].\tag{3.12}$$

By Lemma 2.7, G has at least one fixed point in W. Thus, the proof is completed. □

4 Examples

Example 4.1

$$\begin{cases} D_c^{\frac{1}{2}} x(t) - D_c^{\frac{1}{4}} x(t) = t^2 x^{\frac{1}{2}}(t), \\ x(0) = 1. \end{cases} \tag{4.1}$$

We know $|t^2 x^{\frac{1}{2}}(t)| \leq \frac{t^2(|x(t)|+1)}{2}$, all assumptions of Theorem 3.2 are satisfied. Hence equation (4.1) has at least one positive solution on $[0, +\infty)$.

Example 4.2

$$\begin{cases} D_c^{\frac{1}{2}} x(t) - D_c^{\frac{1}{4}} x(t) = e^t \ln(1 + x(t)), \\ x(0) = 1. \end{cases} \tag{4.2}$$

We know $|\ln(1+x) - \ln(1+y)| \leq |x-y|$ for all $x, y \in (0, +\infty)$. From Theorem 3.3, equation (4.2) has a unique positive solution on $[0, +\infty)$.

Example 4.3

$$\begin{cases} D_c^{\frac{1}{2}} x(t) - D_c^{\frac{1}{4}} x(t) = tx^2(t), \\ x(0) = 1. \end{cases} \tag{4.3}$$

Let $q = \frac{5}{4}$ and $p = 5$, from Theorem 3.4, equation (4.3) has at least one positive solution on $[0, T]$ provided that

$$\begin{aligned} &\ln\left(3^4\left(1 + \frac{4T^{\frac{1}{4}}}{\Gamma(\frac{1}{4})}\right)^5\right) + 3^4\left(\frac{16^{\frac{4}{5}} T^{\frac{1}{20}}}{\Gamma(\frac{1}{4})}\right)^5 T + 3^4\left(\frac{8^{\frac{4}{5}} T^{\frac{3}{10}}}{3^{\frac{4}{5}} \Gamma(\frac{1}{2})}\right)^5 \frac{T^6}{6} \\ &\quad < \ln\left(1 + 3^4\left(1 + \frac{4T^{\frac{1}{4}}}{\Gamma(\frac{1}{4})}\right)^5\right). \end{aligned}$$

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