Yang and Fang *Boundary Value Problems* (2019) 2019:32
https://doi.org/10.1186/s13661-019-1150-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A general decay result for a semilinear heat equation with past and finite history memories

Rui Yang[1*] and Zhong Bo Fang[2]

*Correspondence:
yangr@pusan.ac.kr
[1]Department of Mathematics, Pusan National University, Busan, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the initial-boundary value problem of the following semilinear heat equation with past and finite history memories:

$$u_t - \Delta u + \int_0^t g_1(t-s)\operatorname{div}(a_1(x)\nabla u(s))\,ds + \int_0^{+\infty} g_2(s)\operatorname{div}(a_2(x)\nabla u(t-s))\,ds + f(u) = 0, \quad (x,t) \in \Omega \times [0,+\infty),$$

where Ω is a bounded domain. Under suitable conditions on a_1 and a_2, for a large class of relation functions g_1 and g_2, we establish a general decay estimate, including the usual exponential and polynomial decay cases.

Keywords: Heat equation; Past memory; Finite memory; General decay

1 Introduction

In this paper, we aim at studying a generalized decay result of the following problem:

$$\begin{cases} u_t - \Delta u + \int_0^t g_1(t-s)\operatorname{div}(a_1(x)\nabla u(s))\,ds \\ \quad + \int_0^{+\infty} g_2(s)\operatorname{div}(a_2(x)\nabla u(t-s))\,ds + f(u) = 0, \quad (x,t) \in \Omega \times [0,+\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,+\infty), \\ u(x,-t) = u_0(x,t), \quad (x,t) \in \Omega \times [0,+\infty), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, g_1 and g_2 are positive nonincreasing functions defined on $\mathbb{R}^+ := [0,+\infty)$, a_1 and a_2 are essentially bounded nonnegative functions defined on Ω, f is the nonlinear term, and u_0 is a given initial condition, respectively, satisfying assumptions (H1)–(H5) shown in Sect. 2.

This type of equations describes many mathematical models in engineering and physical sciences; we refer to [1]. For example, in the study of heat conduction in materials with memory, the classical Fourier law of the heat flux is replaced by the following form (see

© The Author(s) 2019. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

[2]):

$$q = -d\nabla u - \int_{-\infty}^{t} g(t-s)\nabla\bigl[a(x)u(x,s)\bigr]\,ds,$$

where q, u, d, and the integral term represent the heat flux, temperature, diffusion coefficient, and memory effect in the material, respectively. From the mathematical point of view, we would expect that the leading term $-d\nabla u$ dominates the integral term in the equation. Hence theory of parabolic equations can be applied to this type of equations.

The study on the global existence, blow-up, and energy decay of solutions for this type of problems involving a finite history memory term has attracted much attention; see [3–12] and the references therein. More precisely, Messaoudi and Tellab [9] studied the quasilinear parabolic system of the form

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(s))\,ds = 0, \quad (x,t) \in \Omega \times [0,+\infty),$$

for $m \geq 2$, where $A(t)$ a bounded and positive definite matrix, and proved a general property of energy decay result, with usual exponential and polynomial decays as particular cases. Fang and Qiu [11] considered the mixed boundary problem of the equation

$$A(x,t)u_t - \Delta u + \int_0^t g(t-s)\operatorname{div}\bigl(a(x)\nabla u(s)\bigr)\,ds = 0, \quad (x,t) \in \Omega \times [0,+\infty),$$

where $A(x,t)$ is a positive function such that $A_t(x,t) \leq 0$. By the technique of Lyapunov functional they proved the existence and uniqueness of a global solution and that the energy functional decays exponentially or polynomially to zero as time tends to infinity. Later, Li et al. [12] were concerned with a mixed boundary value problem of the semilinear parabolic equation

$$u_t - \Delta u + \int_0^t g(t-s)\operatorname{div}\bigl(a(x)\nabla u(s)\bigr)\,ds = 0, \quad (x,t) \in \Omega \times [0,+\infty).$$

Under suitable conditions, a generalized property of energy decay was proved, in which the exponential and polynomial decay results are only particular cases. However, to the best of the authors' knowledge, the decay results for semilinear heat equations with past (infinite) history memory have not been discussed yet. Motivated by this observation, we intend to study the generalized property of energy decay for problem (1.1) in the presence of past and finite history memories, which allows a wide class of memory kernel functions, where the exponential and polynomial decay results are only particular cases (see Example 3.1). Our result is also valid for the problem with past memory or finite memory term case; see Remark 3.2. It is necessary to point out that the argument in [9, 11, 12] cannot be extended to problem (1.1) due to the past memory term. In this paper, we adopt a new approach introduced by Guesmia [13], who investigated a class of hyperbolic problems.

This paper is organized as follows. In Sect. 2, we present preliminaries and some lemmas needed for later work. In Sect. 3, we establish the general decay result and give some examples to illustrate its wide application.

2 Preliminaries and some lemmas

In this section, we give some assumptions, definitions, and lemmas, which will be used to establish our main result.

We first state the following assumptions.

(H1) $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable nonincreasing functions satisfying $g_i(0) > 0$, $i = 1, 2$, and $1 - \|a_1\|_\infty \int_0^{+\infty} g_1(s)\,ds - \|a_2\|_\infty \int_0^{+\infty} g_2(s)\,ds = l > 0$.

(H2) There exists a positive differentiable nonincreasing function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$g_1{}'(t) \le -\xi(t) g_1(t), \quad t \ge 0. \tag{2.1}$$

(H3) There exist a positive constant σ and an increasing strictly convex function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(0, +\infty)$, satisfying $G(0) = G'(0) = 0$ and $\lim_{t\to+\infty} G'(t) = +\infty$, such that

$$g_2{}'(t) \le -\sigma g_2(t), \quad t \ge 0, \tag{2.2}$$

or

$$\int_0^{+\infty} \frac{g_2(t)}{G^{-1}(-g_2{}'(t))}\,dt + \sup_{t\in\mathbb{R}^+} \frac{g_2(t)}{G^{-1}(-g_2{}'(t))} < +\infty. \tag{2.3}$$

(H4) $a_i : \overline{\Omega} \to \mathbb{R}^+$ $(i = 1, 2)$ are in $C^1(\overline{\Omega})$, and there exist two positive constants b_0 and b_1 such that $|\nabla a_1(x)| \le b_1 a_1(x)$ for $x \in \overline{\Omega}$ and

$$a_1(x) + a_2(x) \ge b_0, \quad x \in \overline{\Omega}.$$

(H5) $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and satisfies

$$f(s)s \ge 2F(s) \ge 0, \quad s \in \mathbb{R},$$

where $F(s) = \int_0^s f(z)\,dz$.

To get an equivalent problem of (1.1) for convenience, we define

$$\eta^t(x, s) = u(x, t) - u(x, t - s), \quad \forall x \in \Omega, s, t \ge 0.$$

Obviously, this gives

$$\eta_t^t(x, s) + \eta_s^t(x, s) = u_t(x, t).$$

Together with (1.1), we deduce the problem

$$\begin{cases} u_t - \operatorname{div}[(1 - a_2(x) \int_0^{+\infty} g_2(s)\,ds)\nabla u(t)] + \int_0^t g_1(t - s) \operatorname{div}(a_1(x)\nabla u(s))\,ds \\ \quad - \int_0^{+\infty} g_2(s) \operatorname{div}(a_2(x)\nabla \eta^t(s))\,ds + f(u) = 0, \quad (x, t) \in \Omega \times [0, \infty), \\ \eta_t^t(x, s) + \eta_s^t(x, s) - u_t(x, t) = 0, \quad x \in \Omega, s, t \ge 0, \end{cases} \tag{2.4}$$

and the following initial and boundary conditions:

$$\begin{cases} u(x,t) = \eta^t(x,s) = 0, \quad \forall x \in \partial\Omega, s, t \geq 0, \\ u(x,-t) = u_0(x,t), \quad \eta^t(x,0) = 0, (x,t) \in \Omega \times [0,\infty), \\ \eta^0(x,s) = u_0(x,0) - u_0(x,s), \quad x \in \Omega, s, t \geq 0. \end{cases} \tag{2.5}$$

Now we define the Hilbert space

$$H_0^1(\Omega) = \{u \in H^1(\Omega) | u = 0 \text{ on } \partial\Omega\}$$

and the weight space with respect to g_2

$$\mathrm{M} = \left\{\xi : \mathbb{R}^+ \to H_0^1(\Omega) \middle| \int_0^{+\infty} g_2(s) \left\| \sqrt{a_2(x)} \nabla \xi(s) \right\|_2^2 ds < +\infty \right\}$$

endowed with scalar product and norm

$$\begin{aligned} (\xi, \varsigma)_{\mathrm{M}} &= \int_0^{+\infty} g_2(s) \int_\Omega a_2(x) \nabla \xi(s) \nabla \varsigma(s)\, dx\, ds, \\ \|\xi\|_{\mathrm{M}}^2 &= \int_0^{+\infty} g_2(s) \left\| \sqrt{a_2(x)} \nabla \xi(s) \right\|_2^2 ds < +\infty, \end{aligned}$$

where $\|\cdot\|_q$, $1 \leq q \leq \infty$, is the norm of $L^q(\Omega)$.

We now state the definition of a solution of problem (2.4)–(2.5).

Definition 2.1 A solution of problem (2.4)–(2.5) is a function $u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, $T > 0$, that satisfies

$$\begin{aligned} &\int_\Omega u_t(t)\omega(x,t)\, dx + \int_\Omega \left(1 - a_2(x) \int_0^{+\infty} g_2(s)\, ds\right) \nabla u(t) \nabla \omega(x,t)\, dx \\ &\quad - \int_\Omega \int_0^t g_1(t-s) a_1(x) \nabla u(s) \nabla \omega(x,t)\, ds\, dx + \int_\Omega a_2(x) \int_0^{+\infty} g_2(s) \nabla \eta^t(s) \nabla \omega(x,t)\, ds\, dx \\ &\quad + \int_\Omega f(u)\omega(x,t)\, dx = 0, \\ &\left(\eta_t^t(s), \xi\right)_{\mathrm{M}} = -\left(\eta_s^t(s), \xi\right)_{\mathrm{M}} + \left(u_t(t), \xi\right)_{\mathrm{M}}, \end{aligned}$$

for all $\omega \in C([0,T]; H_0^1(\Omega))$ and $\xi \in \mathrm{M}$.

Remark 2.2 The existence and uniqueness of a global solution for problem (2.4)–(2.5) can be established by using the Galerkin method, the contraction mapping principle, and a continuation argument. The process is similar to that of [11].

Define the modified energy functional of problem (2.4)–(2.5) by

$$\begin{aligned} E(t) &:= E[u](t) \\ &= \frac{1}{2} \int_\Omega k(x,t) \left|\nabla u(t)\right|^2 dx + \frac{1}{2}(g_1 \circ \nabla u)(t) + \frac{1}{2}\left(g_2 \circ \nabla \eta^t\right)(t) + \int_\Omega F\left(u(t)\right) dx, \end{aligned} \tag{2.6}$$

where

$$k(x,t) = 1 - a_1(x)\int_0^t g_1(s)\,ds - a_2(x)\int_0^{+\infty} g_2(s)\,ds,$$
$$(g_1 \circ \nabla u)(t) = \int_0^t g_1(t-s)\left\|\sqrt{a_1(x)}\big(\nabla u(t) - \nabla u(s)\big)\right\|_2^2 ds, \quad \forall u \in H_0^1(\Omega),$$
$$\big(g_2 \circ \nabla \eta^t\big)(t) = \int_0^{+\infty} g_2(s)\left\|\sqrt{a_2(x)}\nabla \eta^t(s)\right\|_2^2 ds.$$

We show the following lemma, which is useful to get $\frac{dE(t)}{dt}$.

Lemma 2.1 *For all $\eta^t(s) \in \mathrm{M}$, we have the equality*

$$\frac{1}{2}\int_\Omega a_2(x)\int_0^{+\infty} g_2{}'(s)\left|\nabla \eta^t(s)\right|^2 ds\,dx = -\int_\Omega a_2(x)\int_0^{+\infty} g_2(s)\nabla \eta^t(s)\nabla \eta_s^t(s)\,ds\,dx.$$

Proof The result can be obtained directly from the calculation

$$\begin{aligned} 0 &= \frac{1}{2}\int_\Omega a_2(x)\frac{d}{ds}\left(\int_0^{+\infty} g_2(s)\left|\nabla \eta^t(s)\right|^2 ds\right) dx \\ &= \frac{1}{2}\int_\Omega a_2(x)\int_0^{+\infty} g_2'(s)\left|\nabla \eta^t(s)\right|^2 ds\,dx \\ &\quad + \int_\Omega a_2(x)\int_0^{+\infty} g_2(s)\nabla \eta^t(s)\nabla \eta_s^t(s)\,ds\,dx. \end{aligned}$$

□

Lemma 2.2 *The energy functional $E(t)$ of problem (2.4)–(2.5) satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) &= -\frac{1}{2}g_1(t)\left\|\sqrt{a_1(x)}\nabla u(t)\right\|_2^2 + \frac{1}{2}\big(g_1{}' \circ \nabla u\big)(t) + \frac{1}{2}\big(g_2{}' \circ \nabla \eta^t\big)(t) - \left\|u_t(t)\right\|_2^2 \\ &\le 0. \end{aligned} \tag{2.7}$$

Proof Differentiating $E(t)$ in (2.6), we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &= -\frac{1}{2}g_1(t)\left\|\sqrt{a_1(x)}\nabla u(t)\right\|_2^2 + \int_\Omega k(x,t)\nabla u(t)\nabla u_t(t)\,dx + \frac{1}{2}\big(g_1' \circ \nabla u\big)(t) \\ &\quad + \frac{1}{2}\big(g_2' \circ \nabla \eta^t\big)(t) + \int_\Omega a_1(x)\int_0^t g_1(t-s)\big(\nabla u(t) - \nabla u(s)\big)\nabla u_t(t)\,ds\,dx \\ &\quad + \int_\Omega a_2(x)\int_0^{+\infty} g_2(s)\nabla \eta^t(s)\nabla \eta_t^t(s)\,ds\,dx + \int_\Omega f(u)u_t(t)\,dx. \end{aligned}$$

Multiplying the first equality in (2.4) by u_t and then integrating the result over Ω, we obtain

$$\begin{aligned} &\int_\Omega u_t^2(t)\,dx + \int_\Omega \nabla u(t)\nabla u_t(t)\,dx - \int_\Omega\int_0^t g_1(t-s)a_1(x)\nabla u(s)\,ds\nabla u_t(t)\,dx \\ &\quad - \int_\Omega\int_0^{+\infty} g_2(s)a_2(x)\nabla u(t-s)\,ds\nabla u_t(t)\,dx + \int_\Omega f(u)u_t(t)\,dx = 0. \end{aligned}$$

Subtracting the above two equalities and using Lemma 2.1, we get (2.7). □

Note that here the positive constant c or C denotes different constants in different places.

3 General decay

In this section, we establish the estimate of general energy decay, which is the main result of this paper. For this purpose, we introduce the perturbed energy functional

$$\Psi(t) = E(t) + \varepsilon_1 \chi(t) + \varepsilon_2 \phi(t),$$

where ε_1 and ε_2 are small positive constants, and

$$\chi(t) = \frac{1}{2}\int_\Omega u^2(t)\,dx,$$
$$\phi(t) = \int_\Omega u(t)\int_0^t g_1(t-s)a_1(x)u(s)\,ds\,dx.$$

We prove the following three lemmas for later use.

Lemma 3.1 *Assume that $u(x,t)$ is a solution of problem* (2.4)–(2.5). *Then there exist two constants $c_1, c_2 > 0$ such that*

$$c_1 E(t) \le \Psi(t) \le c_2 E(t). \tag{3.1}$$

Proof Let c_* be the best constant for the Poincaré inequality $\|u\|_2 \le c_*\|\nabla u\|_2$. It follows directly that

$$\bigl|\chi(t)\bigr| = \frac{1}{2}\left|\int_\Omega u^2(t)\,dx\right| \le \frac{c_*^2}{2}\|\nabla u\|_2^2 \le c_{01}E(t).$$

Applying the Hölder inequality, we obtain

$$\begin{aligned}
\bigl|\phi(t)\bigr| &\le \left|\int_\Omega u(t)\int_0^t g_1(t-s)a_1(x)\bigl(u(t)-u(s)\bigr)\,ds\,dx\right| \\
&\quad + \left|\int_\Omega u(t)\int_0^t g_1(t-s)a_1(x)u(t)\,ds\,dx\right| \\
&\le \left(\frac{\delta_1}{2} + \|a_1\|_\infty\int_0^t g_1(s)\,ds\right)c_*^2\|\nabla u\|_2^2 + \frac{\|a_1\|_\infty\int_0^t g_1(s)\,ds}{2\delta_1}c_*^2(g_1\circ\nabla u)(t) \\
&\le c_{02}E(t).
\end{aligned}$$

Then, selecting $c_1 = 1 - \varepsilon_1 c_{01} - \varepsilon_2 c_{02}$ and $c_2 = 1 + \varepsilon_1 c_{01} + \varepsilon_2 c_{02}$, we get(3.1). □

Lemma 3.2 *Assume that* (H1)–(H5) *hold and $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$. Let $u(x,t)$ be a solution of problem* (2.4)–(2.5). *Then there exists a small constant μ_1 such that*

$$\chi'(t) \le -(l-\mu_1)\|\nabla u\|_2^2 + \frac{1-l}{2\mu_1}\bigl[(g_1\circ\nabla u)(t) + \bigl(g_2\circ\nabla\eta^t\bigr)(t)\bigr] - \int_\Omega uf(u)\,dx. \tag{3.2}$$

Proof Differentiating $\chi(t)$ and using Green's formula and Hölder's inequality, we deduce

$$\begin{aligned}
\chi'(t) &= \int_\Omega u u_t \, dx \\
&= -\int_\Omega \nabla u \left[\left(1 - a_2(x) \int_0^{+\infty} g_2(s)\, ds \right) \nabla u(t) \right] dx \\
&\quad + \int_\Omega \nabla u \int_0^t g_1(t-s) \big(a_1(x) \nabla u(s) \big)\, ds \\
&\quad - \int_\Omega \nabla u \int_0^{+\infty} g_2(s) a_2(x) \nabla \eta^t(s)\, ds\, dx - \int_\Omega u f(u)\, dx \\
&= -\int_\Omega k(x,t) \big| \nabla u(t) \big|^2 dx + \int_\Omega \nabla u \int_0^t g_1(t-s) a_1(x) \big(\nabla u(s) - \nabla u(t) \big)\, ds \\
&\quad - \int_\Omega \nabla u \int_0^{+\infty} g_2(s) a_2(x) \nabla \eta^t(s)\, ds\, dx - \int_\Omega u f(u)\, dx \\
&\le -l \|\nabla u\|_2^2 + \frac{\mu_1 + \mu_2}{2} \|\nabla u\|_2^2 + \frac{\|a_1\|_\infty \int_0^t g_1(s)\, ds}{2\mu_1} (g_1 \circ \nabla u)(t) \\
&\quad + \frac{\|a_2\|_\infty \int_0^\infty g_2(s)\, ds}{2\mu_2} \big(g_2 \circ \nabla \eta^t \big)(t) - \int_\Omega u f(u)\, dx \\
&\le -(l - \mu_1) \|\nabla u\|_2^2 + \frac{1-l}{2\mu_1} \big[(g_1 \circ \nabla u)(t) + \big(g_2 \circ \nabla \eta^t \big)(t) \big] - \int_\Omega u f(u)\, dx.
\end{aligned}$$

Setting $\mu_1 = \mu_2$ in the last inequality, we get (3.2). □

Lemma 3.3 *Assume that* (H1)–(H5) *hold and* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. *Let* $u(x,t)$ *be a solution of problem* (2.4)–(2.5). *Then*

$$\begin{aligned}
\big| \phi'(t) \big| &\le \left[1 + \frac{c_*^2}{2} + 2 g_1(0) \|a_1\|_\infty c_*^2 + \big(3 b_1^2 c_*^2 + 3 + c_*^2 \big)(1-l)^2 \right] \|\nabla u\|_2^2 \\
&\quad + \big(3 b_1^2 c_*^2 + 4 + c_*^2 \big)(1-l)(g_1 \circ \nabla u)(t) + (1-l) \big(g_2 \circ \nabla \eta^t \big)(t) \\
&\quad + \frac{1}{2} \int_\Omega \big| f(u) \big|^2 dx - \frac{c_*^2}{2} \|a_1\|_\infty g_1(0) \big(g_1' \circ \nabla u \big)(t).
\end{aligned} \tag{3.3}$$

Proof Differentiating $\phi(t)$ and using Green's formula and Hölder's inequality, we deduce

$$\begin{aligned}
\phi'(t) &= \int_\Omega u_t(t) \int_0^t g_1(t-s) a_1(x) u(s)\, ds\, dx \\
&\quad + \int_\Omega u(t) \int_0^t g_1'(t-s) a_1(x) u(s)\, ds\, dx + g_1(0) \int_\Omega \big| u(t) \big|^2 a_1(x)\, dx \\
&= \int_\Omega \operatorname{div} \left[\left(1 - a_2(x) \int_0^{+\infty} g_2(s)\, ds \right) \nabla u(t) \right] \int_0^t g_1(t-s) a_1(x) u(s)\, ds\, dx \\
&\quad - \int_\Omega \int_0^t g_1(t-s) \operatorname{div} \big(a_1(x) \nabla u(s) \big)\, ds \int_0^t g_1(t-s) a_1(x) u(s)\, ds\, dx \\
&\quad + \int_\Omega \int_0^{+\infty} g_2(s) \operatorname{div} \big(a_2(x) \nabla \eta^t(s) \big)\, ds \int_0^t g_1(t-s) a_1(x) u(s)\, ds\, dx \\
&\quad - \int_\Omega f(u) \int_0^t g_1(t-s) a_1(x) u(s)\, ds\, dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega u(t)\int_0^t g_1'(t-s)a_1(x)u(s)\,ds\,dx+g_1(0)\int_\Omega \bigl|u(t)\bigr|^2 a_1(x)\,dx\\
&=\left.\begin{aligned}&-\int_\Omega k(x,t)\nabla u(t)\int_0^t g_1(t-s)\nabla a_1(x)u(s)\,ds\,dx\\&-\int_\Omega k(x,t)\nabla u(t)\int_0^t g_1(t-s)a_1(x)\nabla u(s)\,ds\,dx\end{aligned}\right\}:=I_1\\
&\left.\begin{aligned}&+\int_\Omega\int_0^t g_1(t-s)a_1(x)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds\int_0^t g_1(t-s)\nabla a_1(x)u(s)\,ds\,dx\\&+\int_\Omega\int_0^t g_1(t-s)a_1(x)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds\int_0^t g_1(t-s)a_1(x)\nabla u(s)\,ds\,dx\end{aligned}\right\}:=I_2\\
&\left.\begin{aligned}&-\int_\Omega\int_0^{+\infty} g_2(s)\bigl(a_2(x)\nabla\eta^t(s)\bigr)\,ds\int_0^t g_1(t-s)\nabla a_1(x)u(s)\,ds\,dx\\&-\int_\Omega\int_0^{+\infty} g_2(s)\bigl(a_2(x)\nabla\eta^t(s)\bigr)\,ds\int_0^t g_1(t-s)a_1(x)\nabla u(s)\,ds\,dx\end{aligned}\right\}:=I_3\\
&-\int_\Omega f(u)\int_0^t g_1(t-s)a_1(x)u(s)\,ds\,dx\\
&+\int_\Omega u(t)\int_0^t g_1'(t-s)a_1(x)u(s)\,ds\,dx+g_1(0)\int_\Omega \bigl|u(t)\bigr|^2 a_1(x)\,dx.
\end{aligned}\tag{3.4}
$$

Using (H4), we estimate the first two terms in the right-hand side of this equality:

$$
\begin{aligned}
|I_1|\le{}&\left|\int_\Omega k(x,t)\nabla u(t)\int_0^t g_1(t-s)\nabla a_1(x)u(s)\,ds\,dx\right|\\
&+\left|\int_\Omega k(x,t)\nabla u(t)\int_0^t g_1(t-s)a_1(x)\nabla u(s)\,ds\,dx\right|\\
\le{}&\frac{\mu_3+\mu_4}{2}\int_\Omega \bigl|k(x,t)\nabla u(t)\bigr|^2\,dx\\
&+\left(\frac{b_1^2c_*^2}{2\mu_3}+\frac{1}{2\mu_4}\right)\int_\Omega\left(\int_0^t g_1(t-s)a_1(x)\nabla u(s)\right)^2 ds\,dx\\
\le{}&\frac{\mu_3+\mu_4}{2}\|\nabla u\|_2^2\\
&+\left(\frac{b_1^2c_*^2}{2\mu_3}+\frac{1}{2\mu_4}\right)\|a_1\|_\infty\int_0^t g_1(s)\,ds\int_\Omega\int_0^t g_1(t-s)a_1(x)\bigl|\nabla u(s)\bigr|^2\,ds\,dx\\
\le{}&\left[\frac{\mu_3+\mu_4}{2}+\left(\frac{b_1^2c_*^2}{2\mu_3}+\frac{1}{2\mu_4}\right)(1+\mu_5)\left(\|a_1\|_\infty\int_0^t g_1(s)\,ds\right)^2\right]\|\nabla u\|_2^2\\
&+\left(\frac{b_1^2c_*^2}{2\mu_3}+\frac{1}{2\mu_4}\right)\left(1+\frac{1}{\mu_5}\right)\|a_1\|_\infty\int_0^t g_1(s)\,ds(g_1\circ\nabla u)(t).
\end{aligned}\tag{3.5}
$$

We estimate the second two terms in the right-hand side of (3.4):

$$
\begin{aligned}
|I_2|\le{}&\left|\int_\Omega\int_0^t g_1(t-s)a_1(x)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds\int_0^t g_1(t-s)\nabla a_1(x)u(s)\,ds\,dx\right|\\
&+\left|\int_\Omega\int_0^t g_1(t-s)a_1(x)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds\int_0^t g_1(t-s)a_1(x)\nabla u(s)\,ds\,dx\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\mu_6+\mu_7}{2}\int_\Omega\left(\int_0^t g_1(t-s)a_1(x)\big(\nabla u(s)-\nabla u(t)\big)\,ds\right)^2 dx \\
&\quad+\left(\frac{b_1^2c_*^2}{2\mu_6}+\frac{1}{2\mu_7}\right)\int_\Omega\left(\int_0^t g_1(t-s)a_1(x)\nabla u(s)\,ds\right)^2 dx \\
&\le \left(\frac{b_1^2c_*^2}{2\mu_6}+\frac{1}{2\mu_7}\right)(1+\mu_5)\left(\|a_1\|_\infty\int_0^t g_1(s)\,ds\right)^2\|\nabla u\|_2^2 \\
&\quad+\left[\frac{\mu_6+\mu_7}{2}+\left(\frac{b_1^2c_*^2}{2\mu_6}+\frac{1}{2\mu_7}\right)\left(1+\frac{1}{\mu_5}\right)\right]\|a_1\|_\infty\int_0^t g_1(s)\,ds(g_1\circ\nabla u)(t).
\end{aligned}
\tag{3.6}
$$

Similarly, we estimate the third two terms in the right-hand side of (3.4):

$$
\begin{aligned}
|I_3| &\le \left|\int_\Omega\int_0^{+\infty} g_2(s)\big(a_2(x)\nabla\eta^t(s)\big)\,ds\int_0^t g_1(t-s)\nabla a_1(x)u(s)\,ds\,dx\right| \\
&\quad+\left|\int_\Omega\int_0^{+\infty} g_2(s)\big(a_2(x)\nabla\eta^t(s)\big)\,ds\int_0^t g_1(t-s)a_1(x)\nabla u(s)\,ds\,dx\right| \\
&\le \frac{\mu_8+\mu_9}{2}\|a_2\|_\infty\int_0^{+\infty} g_2(s)\,ds\big(g_2\circ\nabla\eta^t\big) \\
&\quad+\left(\frac{b_1^2c_*^2}{2\mu_8}+\frac{1}{2\mu_9}\right)(1+\mu_5)\left(\|a_1\|_\infty\int_0^t g_1(s)\,ds\right)^2\|\nabla u\|_2^2 \\
&\quad+\left(\frac{b_1^2c_*^2}{2\mu_8}+\frac{1}{2\mu_9}\right)\left(1+\frac{1}{\mu_5}\right)\|a_1\|_\infty\int_0^t g_1(s)\,ds(g_1\circ\nabla u)(t).
\end{aligned}
\tag{3.7}
$$

The seventh term in the right-hand side of (3.4) gives

$$
\begin{aligned}
&\left|\int_\Omega f(u)\int_0^t g_1(t-s)a_1(x)u(s)\,ds\,dx\right| \\
&\quad\le\left|\int_\Omega f(u)\int_0^t g_1(t-s)a_1(x)\big(u(s)-u(t)\big)\,ds\,dx\right| \\
&\qquad+\left|\int_\Omega f(u)\int_0^t g_1(t-s)a_1(x)u(t)\,ds\,dx\right| \\
&\quad\le\frac{\mu_{10}}{2}\int_\Omega\big|f(u)\big|^2dx+\frac{1}{2\mu_{10}}\int_\Omega\left(\int_0^t g_1(t-s)a_1(x)u(s)\,ds\right)^2dx \\
&\quad\le\frac{\mu_{10}}{2}\int_\Omega\big|f(u)\big|^2dx+\frac{c_*^2}{2\mu_{10}}(1+\mu_5)\left(\|a_1\|_\infty\int_0^t g_1(s)\,ds\right)^2\|\nabla u\|_2^2 \\
&\qquad+\frac{c_*^2}{2\mu_{10}}\left(1+\frac{1}{\mu_5}\right)\|a_1\|_\infty\int_0^t g_1(s)\,ds(g_1\circ\nabla u)(t).
\end{aligned}
\tag{3.8}
$$

From the last two terms in the right-hand side of (3.4) we get

$$
\begin{aligned}
&\left|\int_\Omega u(t)\int_0^t g_1'(t-s)a_1(x)u(s)\,ds\,dx\right| \\
&\quad\le\left|\int_\Omega u(t)\int_0^t g_1'(t-s)a_1(x)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\qquad+\left|\int_\Omega\big(u(t)\big)^2\int_0^t g_1'(t-s)a_1(x)\,ds\,dx\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\mu_{11}c_*^2}{2}\|\nabla u\|_2^2 + \frac{1}{2\mu_{11}}\int_\Omega \left(\int_0^t g_1'(t-s)a_1(x)\big(u(t)-u(s)\big)\,ds\right)^2 dx \\
&\quad + \|a_1\|_\infty \big|g_1(t)-g_1(0)\big| c_*^2\|\nabla u\|_2^2 \\
&\le \left(\frac{\mu_{11}c_*^2}{2} + \|a_1\|_\infty \big|g_1(t)-g_1(0)\big| c_*^2\right)\|\nabla u\|_2^2 \\
&\quad - \frac{c_*^2}{2\mu_{11}}\|a_1\|_\infty \big|g_1(t)-g_1(0)\big|\big(g_1'\circ\nabla u\big)(t)
\end{aligned}
\tag{3.9}
$$

and

$$
g_1(0)\int_\Omega \big|u(t)\big|^2 a_1(x)\,dx \le g_1(0)\|a_1\|_\infty c_*^2\|\nabla u\|_2^2. \tag{3.10}
$$

Substituting inequalities (3.5)–(3.10) into (3.4), we obtain

$$
\begin{aligned}
\big|\phi'(t)\big| \le{}& \left[\frac{\mu_3+\mu_4}{2} + \frac{\mu_{11}c_*^2}{2} + \|a_1\|_\infty\big|g_1(t)-g_1(0)\big|c_*^2 + g_1(0)\|a_1\|_\infty c_*^2\right. \\
&+ \left(\frac{b_1^2c_*^2}{2\mu_3} + \frac{1}{2\mu_4} + \frac{b_1^2c_*^2}{2\mu_6} + \frac{1}{2\mu_7} + \frac{b_1^2c_*^2}{2\mu_8} + \frac{1}{2\mu_9} + \frac{c_*^2}{2\mu_{10}}\right) \\
&\left.\times (1+\mu_5)\left(\|a_1\|_\infty\int_0^t g_1(s)\,ds\right)^2\right]\|\nabla u\|_2^2 \\
&+ \left[\frac{\mu_6+\mu_7}{2} + \left(\frac{b_1^2c_*^2}{2\mu_3} + \frac{1}{2\mu_4} + \frac{b_1^2c_*^2}{2\mu_6} + \frac{1}{2\mu_7} + \frac{b_1^2c_*^2}{2\mu_8} + \frac{1}{2\mu_9} + \frac{c_*^2}{2\mu_{10}}\right)\left(1+\frac{1}{\mu_5}\right)\right] \\
&\times \|a_1\|_\infty\int_0^t g_1(s)\,ds(g_1\circ\nabla u)(t) \\
&+ \frac{\mu_8+\mu_9}{2}\|a_2\|_\infty\int_0^{+\infty} g_2(s)\,ds\big(g_2\circ\nabla\eta^t\big) \\
&+ \frac{\mu_{10}}{2}\int_\Omega \big|f(u)\big|^2\,dx \\
&- \frac{c_*^2}{2\mu_{11}}\|a_1\|_\infty\big|g_1(t)-g_1(0)\big|\big(g_1'\circ\nabla u\big)(t).
\end{aligned}
$$

Set

$$
\mu_3=\mu_4=\mu_5=\mu_6=\mu_7=\mu_8=\mu_9=\mu_{10}=\mu_{11}=1.
$$

Then we obtain

$$
\begin{aligned}
\big|\phi'(t)\big| \le{}& \left[1 + \frac{c_*^2}{2} + 2g_1(0)\|a_1\|_\infty c_*^2 + \big(3b_1^2c_*^2+3+c_*^2\big)(1-l)^2\right]\|\nabla u\|_2^2 \\
&+ \big(3b_1^2c_*^2+4+c_*^2\big)(1-l)(g_1\circ\nabla u)(t) + (1-l)\big(g_2\circ\nabla\eta^t\big)(t) \\
&+ \frac{1}{2}\int_\Omega \big|f(u)\big|^2\,dx - \frac{c_*^2}{2}\|a_1\|_\infty g_1(0)\big(g_1'\circ\nabla u\big)(t).
\end{aligned}
$$

□

We now state and prove our main theorem.

Theorem 3.1 *Assume that* (H1)–(H5) *hold and* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. *Assume further that in case of* (2.3), *there exists* $M_0 > 0$ *satisfying*

$$\int_\Omega \left|\nabla u_0(x,t)\right|^2 dx \le M_0, \quad t > 0. \tag{3.11}$$

Then there exist positive constants ε_0, k_1, *and* k_2 *such that the energy functional of problem* (2.4)–(2.5) *satisfies*

$$E(t) \le k_1 G_1^{-1}\left(k_2 \int_0^t \xi(s)\,ds\right), \quad t > 0, \tag{3.12}$$

where $G_1(t) = \int_t^1 \frac{1}{G_0(s)}\,ds$, *and*

$$G_0(t) = \begin{cases} t & \text{if (2.2) holds,} \\ tG'(\varepsilon_0 t) & \text{if (2.3) holds.} \end{cases}$$

Remark 3.2 This result contains two particular cases: (i) $a_1 = 0$, where we can take $\xi \equiv 1$, and (ii) $a_2 = 0$, where we can take $G_0(t) = t$. Hence our result is more general; for example, case (ii) contains Theorem 1 in [12].

Proof Applying (2.7), (3.2), and (3.3), for some $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we obtain

$$\begin{aligned}
\Psi'(t) &= E'(t) + \varepsilon_1 \chi'(t) + \varepsilon_2 \phi'(t) \\
&\le -\frac{1}{2} g_1(t) \left\|\sqrt{a_1(x)}\nabla u(t)\right\|_2^2 + \frac{1}{2}\left(g_2{}' \circ \nabla \eta^t\right)(t) \\
&\quad - \left\|u_t(t)\right\|_2^2 + \frac{1-l}{2\mu_1}\varepsilon_1\left[(g_1 \circ \nabla u)(t) + \left(g_2 \circ \nabla \eta^t\right)(t)\right] - \varepsilon_1 \int_\Omega uf(u)\,dx \\
&\quad + \left\{\left[1 + \frac{c_*^2}{2} + 2g_1(0)\|a_1\|_\infty c_*^2 + \left(3b_1^2 c_*^2 + 3 + c_*^2\right)(1-l)^2\right]\varepsilon_2 - (l-\mu_1)\varepsilon_1\right\}\|\nabla u\|_2^2 \\
&\quad + \left(3b_1^2 c_*^2 + 4 + c_*^2\right)(1-l)\varepsilon_2 (g_1 \circ \nabla u)(t) + (1-l)\varepsilon_2\left(g_2 \circ \nabla \eta^t\right)(t) \\
&\quad + \frac{1}{2}\varepsilon_2 \int_\Omega \left|f(u)\right|^2 dx + \frac{1}{2}\left[1 - c_*^2 \varepsilon_2 \|a_1\|_\infty g_1(0)\right]\left(g_1' \circ \nabla u\right)(t).
\end{aligned}$$

Choosing ε_1 and ε_2 such that

$$\left[1 + \frac{c_*^2}{2} + 2g_1(0)\|a_1\|_\infty c_*^2 + \left(3b_1^2 c_*^2 + 3 + c_*^2\right)(1-l)^2\right]\varepsilon_2 - (l-\mu_1)\varepsilon_1 < 0$$

and

$$1 - c_*^2 \varepsilon_2 \|a_1\|_\infty g_1(0) > 0,$$

we deduce

$$\Psi'(t) \le -cE(t) + C\left(g_1 \circ \nabla u + g_2 \circ \nabla \eta^t\right), \quad t > 0. \tag{3.13}$$

To get our conclusion, we deal with two cases to estimate $(g_2 \circ \nabla\eta^t)(t)$:

Case 1. Condition (2.2) holds. Using (2.7), we get

$$\big(g_2 \circ \nabla\eta^t\big)(t) \le -\frac{1}{\sigma}\big(g_2{}' \circ \nabla\eta^t\big)(t) \le -\frac{2}{\sigma}E'(t), \quad t > 0. \tag{3.14}$$

Case 2. Condition (2.3) holds. We follow the idea of [13]. Let $G^*(t) = \sup_{s\ge 0}\{ts - G(s)\}$ be the dual function of the convex function G and set $K(s) = \frac{s}{G^{-1}(s)}$, $s \in \mathbb{R}^+$. Noting that G^{-1} is a positive concave function and $G^{-1}(0) = 0$, for any $0 \le s_1 < s_2$, we derive

$$\begin{aligned} K(s_1) &= \frac{s_1}{G^{-1}(\frac{s_1}{s_2}s_2 + (1 - \frac{s_1}{s_2})0)} \\ &\le \frac{s_1}{\frac{s_1}{s_2}G^{-1}(s_2) + (1 - \frac{s_1}{s_2})G^{-1}(0)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2), \end{aligned}$$

which implies that $K(s)$ is nondecreasing.

From (2.6) and (3.11) we arrive at

$$\big\|\sqrt{a_2(x)}\nabla\eta^t\big\|_2^2 \le 2\big\|\sqrt{a_2(x)}\nabla u(t)\big\|_2^2 + 2\big\|\sqrt{a_2(x)}\nabla u(t-s)\big\|_2^2 \le cE(t) \le c.$$

Choosing $\tau_1, \tau_2 > 0$ and applying the Young inequality $ts \le G(t) + G^*(s)$ and the fact that $K(s)$ and G^* are nondecreasing, we have

$$\begin{aligned} \big(g_2 \circ \nabla\eta^t\big)(t) &= \frac{1}{\tau_1 G'(\varepsilon_0 E(t))}\int_0^{+\infty} G^{-1}\big(-\tau_2 g_2{}'(s)\big\|\sqrt{a_2(x)}\nabla\eta^t\big\|_2^2\big) \\ &\quad \times \frac{\tau_1 G'(\varepsilon_0 E(t))g_2(s)\|\sqrt{a_2(x)}\nabla\eta^t\|_2^2}{G^{-1}(-\tau_2 g_2{}'(s)\|\sqrt{a_2(x)}\nabla\eta^t\|_2^2)}\,ds \\ &\le -\frac{\tau_2}{\tau_1 G'(\varepsilon_0 E(t))}\big(g_2{}' \circ \nabla\eta^t\big)(t) \\ &\quad + \frac{1}{\tau_1 G'(\varepsilon_0 E(t))}\int_0^{+\infty} G^*\left(\frac{c\tau_1 G'(\varepsilon_0 E(t))g_2(s)}{G^{-1}(-c\tau_2 g_2{}'(s))}\right)ds. \end{aligned}$$

Using (2.7) and $G^*(s) = s(G')^{-1}(s) - G((G')^{-1}(s)) \le s(G')^{-1}(s)$, we get

$$\begin{aligned} \big(g_2 \circ \nabla\eta^t\big)(t) &\le -\frac{2\tau_2}{\tau_1 G'(\varepsilon_0 E(t))}E'(t) \\ &\quad + c\int_0^{+\infty} \frac{g_2(s)}{G^{-1}(-c\tau_2 g_2{}'(s))}\big(G'\big)^{-1}\left(\frac{c\tau_1 G'(\varepsilon_0 E(t))g_2(s)}{G^{-1}(-c\tau_2 g_2{}'(s))}\right)ds. \end{aligned}$$

Setting $\tau_2 = \frac{1}{c}$, by (2.3) we know that

$$\big(g_2 \circ \nabla\eta^t\big)(t) \le -\frac{2}{c\tau_1 G'(\varepsilon_0 E(t))}E'(t) + c\big(G'\big)^{-1}\big(c\tau_1 G'\big(\varepsilon_0 E(t)\big)\big)\int_0^{+\infty} \frac{g_2(s)}{G^{-1}(-g_2{}'(s))}\,ds.$$

Then, setting $\tau_1 = \frac{1}{c}$ and using (2.3) again, we obtain

$$G'\big(\varepsilon_0 E(t)\big)\big(g_2 \circ \nabla\eta^t\big)(t) \le -cE'(t) + c\varepsilon_0 E(t)G'\big(\varepsilon_0 E(t)\big), \quad t > 0. \tag{3.15}$$

From (3.14) and (3.15) it is easy to see that

$$\frac{G_0(E(t))}{E(t)}\big(g_2 \circ \nabla \eta^t\big)(t) \le -cE'(t) + c\varepsilon_0 G_0\big(E(t)\big), \quad t > 0, \tag{3.16}$$

with G_0 given in Theorem 3.1.

Now, multiplying (3.13) by $\frac{G_0(E(t))}{E(t)}$, then using (3.16), and selecting ε_0 sufficiently small, we get

$$\frac{G_0(E(t))}{E(t)}\Psi'(t) + cE'(t) \le -cG_0\big(E(t)\big) + C\frac{G_0(E(t))}{E(t)}(g_1 \circ \nabla u)(t), \quad t > 0. \tag{3.17}$$

Define

$$L(t) = \frac{G_0(E(t))}{E(t)}\Psi(t) + cE(t), \quad t > 0.$$

From the definition of G_0 we have that $t \mapsto \frac{G_0(E(t))}{E(t)}$ is nonincreasing and nonnegative. Applying Lemma 3.1, we have $L(t) \sim E(t)$ for $t > 0$.

Differentiating $L(t)$, we get

$$\begin{aligned} L'(t) &= \left(\frac{G_0(E(t))}{E(t)}\right)'\Psi(t) + \frac{G_0(E(t))}{E(t)}\Psi'(t) + cE'(t) \\ &\le -cG_0\big(E(t)\big) + C\frac{G_0(E(t))}{E(t)}(g_1 \circ \nabla u)(t), \quad t > 0. \end{aligned} \tag{3.18}$$

To handle the last term in (3.18), multiplying it by $\xi(t)$ in both sides, and using (H2), (2.7), the fact that $t \mapsto \frac{G_0(E(t))}{E(t)}$ is nonnegative, we obtain

$$\begin{aligned} \xi(t)L'(t) &\le -c\xi(t)G_0\big(E(t)\big) + C\frac{G_0(E(t))}{E(t)}\xi(t)(g_1 \circ \nabla u)(t) \\ &\le -c\xi(t)G_0\big(E(t)\big) - C\big({g_1}' \circ \nabla u\big)(t) \\ &\le -c\xi(t)G_0\big(E(t)\big) - CE'(t), \quad t > 0. \end{aligned} \tag{3.19}$$

Finally, set $I(t) = \xi(t)L(t) + CE(t)$. Obviously, $I(t) \sim E(t)$. Since $\xi(t)$ is nonincreasing, it follows form (3.19) that

$$I'(t) \le -c\xi(t)G_0\big(E(t)\big), \quad t > 0. \tag{3.20}$$

Since $G_0(s) > 0$ for $s > 0$ and $I(t) \sim E(t)$, we can see that

$$-\frac{I'(t)}{G_0(I(t))} \ge c\xi(t), \quad t > 0,$$

and thus

$$\big(G_1\big(I(t)\big)\big)' \ge c\xi(t), \quad t > 0.$$

Integrating this inequality over $(0,t)$, we obtain

$$G_1\big(I(t)\big) \geq c\int_0^t \xi(s)\,ds + G_1\big(I(0)\big) \geq c\int_0^t \xi(s)\,ds, \quad t>0.$$

Since G_1 is nonincreasing, it is easy to get that

$$I(t) \leq {G_1}^{-1}\left(c\int_0^t \xi(s)\,ds\right), \quad t>0.$$

Since $I(t)\sim E(t)$, we obtain (3.12). The proof is completed. □

Now we give some examples to illustrate the result in Theorem 3.1, in which the exponential and polynomial decay estimates are only particular cases.

Example 3.1

(i) Let $g_1(t)=\lambda e^{-(t+1)^p}$, $\lambda>0$, $p\geq 1$, and $g_2(t)=e^{-(t+1)^q}$, $q\geq 1$. We can see that (2.1) and (2.2) hold for $\xi=p$ and $\sigma=q$, respectively. Then (3.12) gives the exponential decay estimate

$$E(t)\leq k_1 e^{-k_2 pt}.$$

(ii) Let $g_1(t)=(1+t)^\nu$, $\nu<-1$, and $g_2(t)=\lambda e^{-(t+1)^p}$, $\lambda>0$, $p\geq 1$. Similarly, we can check that (2.1) and (2.2) hold for $\xi(t)=\frac{-\nu}{1+t}$ and $\sigma=p$, respectively. Then (3.12) gives the polynomial decay estimate

$$E(t)\leq k_1(1+t)^{k_2\nu}.$$

(iii) Let $g_1(t)=\frac{1}{(2+t)^\nu(\ln(2+t))^\lambda}$, $\nu,\lambda>0$, and $g_2(t)=\frac{1}{(1+t)^q}$, $q>1$. We can see that (2.1) holds for $\xi(t)=\frac{\nu}{2+t}+\frac{\lambda}{\ln(2+t)}$ and (2.3) holds for $G(t)=t^{\frac{1}{p}+1}$, $p\in(0,\frac{q-1}{2})$. Then we obtain the decay estimate

$$E(t)\leq \frac{k_1}{(k_2\int_0^t \xi(s)\,ds+1)^p} = \frac{k_1}{(k_2\ln[(2+t)^\nu(\ln(2+t))^\lambda]-k_2\ln[2^\nu(\ln 2)^\lambda]+1)^p}.$$

Acknowledgements
The authors are sincerely grateful to the editors and anonymous referees for careful reading of the original manuscript and useful comments.

Funding
To be appeared.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the manuscript. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Pusan National University, Busan, Republic of Korea. [2]School of Mathematical Sciences, Ocean University of China, Qingdao, People's Republic of China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 13 September 2018 Accepted: 4 February 2019 Published online: 11 February 2019

References

1. Pao, C.: Solution of a nonlinear integrodifferential system arising in nuclear reactor dynamics. J. Math. Anal. Appl. **48**, 470–492 (1974)
2. Nohel, J.: Nonlinear Volterra equations for heat flow in materials with memory. In: Integral and Functional Differential Equations. Lecture Notes in Pure and Appl. Math. Dekker, New York (1981)
3. Yamada, Y.: Asymptotic stability for some systems of semilinear Volterra diffusion equations. J. Differ. Equ. **52**, 295–326 (1984)
4. Bellout, H.: Blow-up of solutions of parabolic equation with nonlinear memory. J. Differ. Equ. **70**, 42–68 (1987)
5. Cannon, J.R., Lin, Y.: A priori l^2 error estimates for finite-element methods for nonlinear diffusion equations with memory. SIAM J. Numer. Anal. **27**, 595–607 (1990)
6. Yin, H.M.: Weak and classical solutions of some nonlinear Volterra integro-differential equations. Commun. Partial Differ. Equ. **17**, 1369–1385 (1992)
7. Messaoudi, S.A.: Blow-up of solutions of a semilinear heat equation with a visco-elastic term. Prog. Nonlinear Differ. Equ. Appl. **64**, 351–356 (2005)
8. Messaoudi, S.A.: Blow-up of solutions of a semilinear heat equation with a memory term. Abstr. Appl. Anal. **2005**, 87–94 (2005)
9. Messaoudi, S.A., Tellab, B.: A general decay result in a quasilinear parabolic system with viscoelastic term. Appl. Math. Lett. **25**, 443–447 (2012)
10. Liu, G., Chen, H.: Global and blow-up of solutions for a quasilinear parabolic system with viscoelastic and source terms. Math. Methods Appl. Sci. **37**, 148–156 (2014)
11. Fang, Z.B., Qiu, L.R.: Global existence and uniform energy decay rates for the semilinear parabolic equation with a memory term and mixed boundary condition. Abstr. Appl. Anal. **2013**, Article ID 532935 (2013). https://doi.org/10.1155/2013/532935
12. Li, C.J., Qiu, L.R., Fang, Z.B.: General decay rate estimates for a semilinear parabolic equation with memory term and mixed boundary condition. Bound. Value Probl. **2014**, 197 (2014). https://doi.org/10.1186/s13661-014-0197-0
13. Guesmia, A.: Asymptotic stability of abstract dissipative systems with infinite memory. J. Math. Anal. Appl. **382**, 748–760 (2011)

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com