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The local existence of strong solution for the stochastic 3D Boussinesq equations

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Abstract

The stochastic 3D Boussinesq equations with additive noise are considered. We prove the local existence of the strong solution in H^s ($\frac{1}{2} < s \le 1$). We also obtain a new stopping time, which shows that the $H^{\frac{1}{2}^+}$ norm of u controls the breakdown of the strong solution. Furthermore, we give the probability estimate of the lifespan larger than δ ($0 < \delta < 1$).

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Keywords: Stochastic Boussinesq equation; Strong solution; Stopping time estimate

1 Introduction

In this paper, we consider the stochastic 3D Boussinesq equations driven by an additive noise:

$$\begin{cases} du + (u \cdot \nabla u - \nu \Delta u + \nabla p)\,dt = \rho e_3\,dt + \sum_{i=1}^{\infty} \Phi_{1,i}\,dW_i, \\ d\rho + (u \cdot \nabla \rho - \kappa \Delta \rho)\,dt = \sum_{i=1}^{\infty} \Phi_{2,i}\,d\widetilde{W}_i, \quad (t,x) \in (\mathbb{R}^+ \times \mathbb{R}^3), \\ \nabla \cdot u = 0, \\ (u,\rho)|_{t=0} = (u_0, \rho_0), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity field of the flow, ρ is the scalar temperature, and p is the scalar pressure; ν, κ are nonnegative viscosity parameters and e_3 is the vertical unit vector of $\mathbb{R}^3$. $\{W_i\}_{i=1}^{\infty}$, $\{\widetilde{W}_i\}_{i=1}^{\infty}$ are given independent standard Brownian motions defined in the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with $\mathcal{F}_t$ (a set of sub σ-fields of $\mathcal{F}$ with $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ if $0 \le s < t < \infty$), $\Phi_{i,1}$ and $\Phi_{i,2}$ are the components of the random force.

If $\Phi_{i,1} = \Phi_{i,2} \equiv 0$, system (1.1) becomes the deterministic Boussinesq equations, which have been extensively studied for their physical significance as well as mathematical importance in recent years (see, e.g., [1–4]). For 2D Boussinesq equations, the global regularity issue has been settled in the affirmative under various degrees of viscosity: with full viscosity $\nu > 0$ and $\kappa > 0$, partial viscosity $\nu > 0$ and $\kappa = 0$, or $\nu = 0$ and $\kappa > 0$ for anisotropic models [5–10], and with fractional Laplacian dissipation (see [11–15] and the references therein). For the 3D Boussinesq system and the inviscid case in 2D, we can only obtain the local well-posedness result or the global regularity result with respect to small initial data, see [16–21] etc.

In the meantime, researchers are interested in the stochastic Boussinesq equations by considering that a system in reality is usually affected by external perturbations which in many cases are of great uncertainty or random influence. Ferrario [22] first studied the two-dimensional stochastic Boussinesq system with full viscosity, additive noise only on the velocity field equation, and obtained the existence and uniqueness of its solution and invariant measures for the associated semigroup. Following the work of the deterministic case in [6, 7], Pu and Guo [23] studied the stochastic Boussinesq system with partial viscosity with additive noise and obtained the global well-posedness type results. Then, Duan and Millet [24] considered the multiplicative noise case and the large deviations, Brzeźniak and Motyl [25] generalized the existence result (martingale solutions) to the 3D case, and Yamazaki [26] considered the stochastic Boussinesq system with zero dissipation in 2D.

To the best of our knowledge, the problem of the existence and uniqueness of the strong solutions for the 3D stochastic full viscosity Boussinesq equations is still open. In this paper, we prove that the Cauchy problem (1.1) admits a local strong solution under some conditions and obtain a blow-up time. Also, the probability estimate of the lifespan larger than δ $(0 < \delta < 1)$ is given in our paper.

Without loss of generality, we take $\nu = \kappa = 1$. We first state the definition of the local strong solutions for the stochastic Boussinesq equations (1.1).

Definition 1.1 Fix stochastic $\mathcal{S} := (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Let $(u_0, \rho_0) \in H^s_\sigma \times H^s$ be $\mathcal{F}_0$ measurable, $\frac{1}{2} < s \le 1$, $U = (u, \rho)$.

(i) A pair (U, τ) is called a local strong solution of (1.1) if the following conditions are satisfied:

- U is a right continuous progressively measurable process and, for all $0 < T < \infty$,

$$U \in L^2\big(\Omega; L^\infty\big([0,T]; H^s_\sigma \times H^s\big)\big) \cap L^2\big(\Omega; L^2\big([0,T]; H^{s+1}_\sigma \times H^{s+1}\big)\big);$$

- $\tau(\omega)$ is a stopping time with respect to $\mathcal{F}_t$ such that

$$\tau(\omega) = \lim_{N\to\infty} \tau_N(\omega) \quad \text{for almost all } \omega,$$

where, for $N = 1, 2, \ldots$,

$$\tau_N(\omega) = \begin{cases} \inf\{0 \le t < \infty : \|U(t)\|^2_{H^s} + \int_0^t \|\nabla U(s)\|^2_{H^s}\, ds \ge N\}, \\ \infty, \quad \text{if the above set } \{\cdot\} \text{ is empty}; \end{cases} \tag{1.2}$$

- $U(t,x) \in C([0, \tau(\omega)); H^s_\sigma \times H^s)$ for almost all $\omega \in \Omega$, and the following holds $\mathbb{P}$-a.s.:

$$\begin{aligned} \big\langle u(t \wedge \tau_N), \phi\big\rangle - \langle u_0, \phi\rangle = {} & \int_0^{t\wedge\tau_N} \big(\langle \nabla u, \nabla \phi\rangle - \langle u \cdot \nabla u, \phi\rangle\big)\, ds \\ & + \left\langle \sum_{i=1}^\infty \int_0^{t\wedge\tau_N} \Phi_{1,i}\, dW_i(s), \phi \right\rangle, \end{aligned}$$

$$\big\langle \rho(t\wedge\tau_N),\psi\big\rangle - \langle\rho_0,\psi\rangle = \int_0^{t\wedge\tau_N}\big(\langle\nabla\rho,\nabla\psi\rangle - \langle u\cdot\nabla\rho,\psi\rangle\big)\,ds + \left\langle \sum_{i=1}^{\infty}\int_0^{t\wedge\tau_N}\Phi_{2,i}\,d\widetilde{W}_i(s),\psi\right\rangle$$

for all $0\le t<\infty$ and all $(\phi,\psi)\in H^1_\sigma\times H^1$. Here $\langle\cdot,\cdot\rangle$ is the inner product, and the definition of spaces will be given in Sect. 2.

(ii) We say the strong solutions are unique if, given any pair (U,τ), $(\widetilde{U},\widetilde{\tau})$ of strong solutions,

$$\mathbb{P}\big(1_{U(0)=\widetilde{U}(0)}\big(U(t)-\widetilde{U}(t)\big)=0;\forall t\in[0,\tau\wedge\widetilde{\tau})\big)=1.$$

Let $\Phi_j:=\{\Phi_{j,i}\}_{i=1}^{\infty}$ $(j=1,2)$. Our main results are stated in the following two theorems.

Theorem 1.2 *Fix stochastic* $\mathcal{S}:=(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. *Let* $(u_0,\rho_0)\in L^2(\Omega;H^s_\sigma\times H^s)$ *be* $\mathcal{F}_0$ *measurable,* $(\Phi_1,\Phi_2)\in L^2(\Omega;L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{H}^s\times\mathbb{H}^s))$ *is progressively measurable. Then there exists a unique strong solution* (U,τ) *to system* (1.1).

Moreover, the blow-up time could be weaker. Indeed, we have the following theorem.

Theorem 1.3 *Let* $\frac{1}{2}<s'\le s$, *define*

$$\zeta_K=\begin{cases}\inf\{t\ge 0:\|u(t)\|^2_{H^{s'}}\ge K\},\\ \infty,\quad \text{if the set } \{\cdot\} \text{ is empty},\end{cases}\qquad \zeta=\lim_{K\to\infty}\zeta_K. \tag{1.3}$$

Then $\zeta=\tau$ *a.s., and we have*

$$\mathbb{P}\big(\{\zeta>\delta\}\big)\ge 1-C^*\delta^{\frac{2s'-1}{2s'+1}}\big(\mathcal{A}(\delta)+1\big), \tag{1.4}$$

where

$$\mathcal{A}(\delta)=\mathbb{E}\left(\|u_0\|^2_{H^{s'}}+\int_0^\delta\|\Phi_1\|^2_{\mathbb{H}^{s'}}\,dt\right)+\delta\mathbb{E}\left(\|\rho\|^2_{L^2}+\int_0^\delta\|\Phi_2\|^2_{\mathbb{L}^2}\right).$$

Remark 1.4 In fact, we can consider the more general case that the solution $U\in H^s_0\times H^n$ $(\frac{1}{2}<s,\ \frac{1}{2}<n,\ s-2\le n\le s)$. The proof is similar to ours.

We can construct the strong solution by the contraction mapping principle, cut-off function method, and Cauchy convergence theorem. For the detailed procedure, refer to [27]. Theorem 1.3 can be proved by using the stopping time and energy estimates.

The rest of the paper is organized as follows. We shall introduce some analysis tools in Sobolev spaces and some basic theory of stochastic analysis in Sect. 2. In Sect. 3, we shall prove the local existence of the strong solution, and Theorem 1.3 will be proved in the last section.

2 Preliminaries

Let $H^m = W^{m,2}(\mathbb{R}^3)$ be the usual Sobolev space. We use the same notation H^m for $H^m(\mathbb{R}^3;\mathbb{R}^3)$ also. The symbols $\langle\cdot\rangle_{H^m}$ and $\|\cdot\|_{H^m}$ represent the inner product and the norm of H^m, respectively. We introduce the function space

$$H^m_\sigma = \{f \in H^m; \nabla\cdot f = 0\}.$$

The symbol $\mathbf{P}$ denotes the projection $H^m \to H^m_\sigma$. We recall the following inequality which will be used later: for all $v \in H^{s'}$, $u \in H^{\frac{3}{2}+s-s'}$, and $w \in H^{s+1}$, $\frac{1}{2} < s' \leq s \leq 1$, we have

$$\bigl|\langle v\cdot\nabla u, w\rangle_{H^s}\bigr| \leq C\|v\|_{H^{s'}}\|\nabla u\|_{H^{\frac{1}{2}+s-s'}}\|w\|_{H^{s+1}}, \tag{2.1}$$

since $H^{s'} \subset L^p$, $H^{\frac{1}{2}+s-s'} \subset L^q$, $H^{1-s} \subset L^r$, $1/p + 1/q + 1/r = 1$. By the interpolation inequality, we have

$$\|\nabla u\|_{H^{\frac{1}{2}+s-s'}} \leq C\|u\|_{H^s}^{s'-\frac{1}{2}}\|u\|_{H^{s+1}}^{\frac{3}{2}-s'}. \tag{2.2}$$

We define the space for the white noise. Let $\Phi := \{\Phi_i\}_{i=1}^\infty$, suppose $m \geq 0$, define

$$\mathbb{H}^m := \left\{\Phi \,\middle|\, \forall i, \Phi_i \in H^m \text{ and } \sum_{i=1}^\infty \|\Phi_i\|_{H^m}^2 < \infty\right\},$$

with the norm

$$\|\Phi\|_{\mathbb{H}^m}^2 := \sum_{i=1}^\infty \|\Phi_i\|_{H^m}^2.$$

When $s = 0$, we also denote $\mathbb{L}^2$ by $\mathbb{H}^0$.

Now, we recall some basic theory of stochastic analysis. For details, we refer the reader to [28–31] and the references therein.

Lemma 2.1 *Suppose that $M_t = (M_t^1, M_t^2, \dots, M_t^n)$ is a vector of continuous local martingales, that is, $(M_t^i, \mathcal{F}_t)$ is a local martingale for each $i = 1, 2, \dots, n$ and $t \in \mathbb{R}^+$. Let $A_t = (A_t^1, A_t^2, \dots, A_t^n)$ be a vector of continuous process adapted to the same filtration such that the total variation of A_t^i on each finite interval is bounded almost surely, and $A_0^i = 0$ almost surely. Let $X_t = (X_t^1, X_t^2, \dots, X_t^n)$ be a vector of adapted processes such that $X_t = X_0 + M_t + A_t$, and let $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n)$. Then, for any $t \geq 0$, the equality*

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} f(s,X_s)\,dM_s^i + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} f(s,X_s)\,dA_s^i \\ &+ \frac{1}{2}\sum_{i=1}^n \int_0^t \frac{\partial^2}{\partial_{x_i}\partial_{x_j}} f(s,X_s)\,d\langle M^i, M^j\rangle_s + \int_0^t \frac{\partial}{\partial s} f(s,X_s)\,ds \end{aligned} \tag{2.3}$$

holds almost surely. Here $\langle\cdot,\cdot\rangle_t$ is the cross variation process defined by $\langle X, Y\rangle_t = \frac{1}{4}\{\langle X + Y\rangle_t - \langle X - Y\rangle_t\}$, and $\langle X\rangle_t$ denotes the quadratic variation of X on $[0,t]$.

Lemma 2.2 (Burkholder–Davis–Gundy inequality) *Let $T > 0$ and $\{X_t\}_{0\le t\le T}$ be a continuous local martingale such that $X_0 = 0$. For every $0 < p < \infty$, there exist universal constants c_p and C_p, independent of T and X_t, such that*

$$c_p\mathbb{E}\big(\langle X\rangle_T^{\frac{p}{2}}\big) \le \mathbb{E}\Big(\max_{0\le t\le T}|X_s|^p\Big) \le C_p\mathbb{E}\big(\langle X\rangle_T^{\frac{p}{2}}\big). \tag{2.4}$$

3 The local existence of local strong solution

We first give the key estimate to prove the local existence and uniqueness of the strong solution. Fix $N > 0$ to be determined, choose a C^∞-smooth nonincreasing function $\varphi_N : [0,\infty) \to [0,1]$ such that

$$\varphi_N(x) = \begin{cases} 1, & \text{for } |x| < N, \\ 0, & \text{for } |x| > N+1, \end{cases}$$

and

$$\varphi_N^{u,\rho} = \varphi_N\bigg(\|u\|_{H^s} + \|\rho\|_{H^s} + \bigg(\int_0^t \big(\|\nabla u\|_{H^s}^2 + \|\nabla\rho\|_{H^s}^2\big)\,ds\bigg)^{1/2}\bigg).$$

We consider the following Cauchy problem:

$$\begin{cases} du^j + (\varphi_N^{u^j,\rho^j}\mathbf{P}(u^j\cdot\nabla u^j) - \Delta u^j)\,dt = \mathbf{P}(\rho^j e_3)\,dt + \sum_{i=1}^\infty \mathbf{P}\Phi_{1,i}^j\,dW_i, \\ d\rho^j + (\varphi_N^{u^j,\rho^j}(u^j\cdot\nabla\rho^j) - \Delta\rho^j)\,dt = \sum_{i=1}^\infty \Phi_{2,i}^j\,d\widetilde{W}_i, \\ (u^j,\rho^j)|_{t=0} = (u_0^j,\rho_0^j), \end{cases} \tag{3.1}$$

for $j = 1,2$. By Itô's formula in H^s and the equations of (u^j,ρ^j), we have

$$\begin{aligned} &d\|u^1 - u^2\|_{H^s}^2 + 2\|\nabla u^1 - \nabla u^2\|_{H^s}^2\,dt \\ &\quad= -2\big\langle\varphi_N^{u^1,\rho^1}\big(u^1\cdot\nabla u^1\big) - \varphi_N^{u^2,\rho^2}\big(u^2\cdot\nabla u^2\big), u^1 - u^2\big\rangle_{H^s}\,dt \\ &\qquad+ \|\Phi_1^1 - \Phi_1^2\|_{\mathbb{H}^s}^2\,dt + 2\big\langle\Phi_{1,i}^1 - \Phi_{1,i}^2, u^1 - u^2\big\rangle_{H^s}\,dW_i \\ &\qquad+ 2\big\langle\big(\rho^1 - \rho^2\big)e_3, u^1 - u^2\big\rangle_{H^s}\,dt, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &d\|\rho^1 - \rho^2\|_{H^s}^2 + 2\|\nabla\rho^1 - \nabla\rho^2\|_{H^s}^2\,dt \\ &\quad= -2\big\langle\varphi_N^{u^1,\rho^1}\big(u^1\cdot\nabla\rho^1\big) - \varphi_N^{u^2,\rho^2}\big(u^2\cdot\nabla\rho^2\big), \rho^1 - \rho^2\big\rangle_{H^s}\,dt \\ &\qquad+ \|\Phi_1^2 - \Phi_2^2\|_{\mathbb{H}^s}^s\,dt + 2\big\langle\Phi_{i,1}^2 - \Phi_{i,2}^2, \rho^1 - \rho^2\big\rangle_{H^s}\,d\widetilde{W}_i. \end{aligned} \tag{3.3}$$

We have the following proposition.

Proposition 3.1 *For any $T > 0$, we have*

$$\begin{aligned} &\mathbb{E}\bigg(\sup_{0\le t\le T}\big(\|\bar{u}(t)\|_{H^s}^2 + \|\bar{\rho}(t)\|_{H^s}^2\big) + \int_0^T \big(\|\nabla\bar{u}\|_{H^s}^2 + \|\nabla\bar{\rho}\|_{H^s}^2\big)\,dt\bigg) \\ &\quad\le e^{C_{N,T}}\mathbb{E}(\|\bar{u}_0\|_{H^s}^2 + \|\bar{\rho}_0\|_{H^s}^2 + \int_0^T \big(\|\bar{\Phi}_1\|_{\mathbb{H}^s}^2 + \|\bar{\Phi}_2\|_{\mathbb{H}^s}^2\,dt\big), \end{aligned} \tag{3.4}$$

where $\bar{u} = u_1 - u_2$, $\bar{\rho} = \rho_1 - \rho_2$, and $\bar{\Phi}_i = \Phi_i^1 - \Phi_i^2$ $(i = 1,2)$.

Proof Set

$$Q_1(t) = \left|\left\langle\varphi_N^{u^1,\rho^1}\left(u^1\cdot\nabla u^1\right) - \varphi_N^{u^2,\rho^2}\left(u^2\cdot\nabla u^2\right), u^1 - u^2\right\rangle_{H^s}\right|,$$
$$Q_2(t) = \left|\left\langle\varphi_N^{u^1,\rho^1}\left(u^1\cdot\nabla\rho^1\right) - \varphi_N^{u^2,\rho^2}\left(u^1\cdot\nabla\rho^2\right), \rho^1 - \rho^2\right\rangle_{H^s}\right|.$$

We consider three cases:

Case 1: $\varphi_N^{u^1,\rho^1} > 0$, $\varphi_N^{u^2,\rho^2} > 0$:

$$\begin{aligned} Q_1(t) &\le \left|\varphi_N^{u^1,\rho^1} - \varphi_N^{u^2,\rho^2}\right|\left|\left\langle u^1\cdot\nabla u^1, u^1 - u^2\right\rangle_{H^s}\right| \\ &\quad + \left|\left\langle\left(u^1 - u^2\right)\cdot\nabla u^1, u^1 - u^2\right\rangle_{H^s}\right| \\ &\quad + \left|\left\langle u^2\cdot\nabla\left(u^1 - u^2\right), u^1 - u^2\right\rangle_{H^s}\right| \\ &:= I_1 + I_2 + I_3, \end{aligned} \tag{3.5}$$

$$\begin{aligned} Q_2(t) &\le \left|\varphi_N^{u^1,\rho^1} - \varphi_N^{u^2,\rho^2}\right|\left|\left\langle u^1\cdot\nabla\rho^1, \rho^1 - \rho^2\right\rangle_{H^s}\right| \\ &\quad + \left|\left\langle\left(u^1 - u^2\right)\cdot\nabla\rho^1, \rho^1 - \rho^2\right\rangle_{H^s}\right| \\ &\quad + \left|\left\langle u^2\cdot\nabla\left(\rho^1 - \rho^2\right), \rho^1 - \rho^2\right\rangle_{H^s}\right| \\ &:= J_1 + J_2 + J_3. \end{aligned} \tag{3.6}$$

Case 2: $\varphi_N^{u^1,\rho^1} > 0$, $\varphi_N^{u^2,\rho^2} = 0$:

$$Q_1(t) \le \left|\varphi_N^{u^1,\rho^1} - \varphi_N^{u^2,\rho^2}\right|\left|\left\langle u^1\cdot\nabla u^1, u^1 - u^2\right\rangle_{H^s}\right|, \tag{3.7}$$
$$Q_2(t) \le \left|\varphi_N^{u^1,\rho^1} - \varphi_N^{u^2,\rho^2}\right|\left|\left\langle u^1\cdot\nabla\rho^1, \rho^1 - \rho^2\right\rangle_{H^s}\right|. \tag{3.8}$$

Case 3: $\varphi_N^{u^1,\rho^1} = 0$, $\varphi_N^{u^2,\rho^2} > 0$:

$$Q_1(t) \le \left|\varphi_N^{u^1,\rho^1} - \varphi_N^{u^2,\rho^2}\right|\left|\left\langle u^2\cdot\nabla u^2, u^1 - u^2\right\rangle_{H^s}\right|, \tag{3.9}$$
$$Q_2(t) \le \left|\varphi_N^{u^1,\rho^1} - \varphi_N^{u^2,\rho^2}\right|\left|\left\langle u^2\cdot\nabla\rho^2, \rho^1 - \rho^2\right\rangle_{H^s}\right|. \tag{3.10}$$

Denote

$$\begin{aligned} Y(t) &= \sup_{0\le t'\le t}\left(\left\|u^1(t') - u^2(t')\right\|_{H^s}^2 + \left\|\rho^1(t') - \rho^2(t')\right\|_{H^s}^2\right) \\ &\quad + \int_0^t\left(\left\|\nabla u^1 - \nabla u^2\right\|_{H^s} + \|\nabla\rho_1 - \nabla\rho_2\|_{H^s}^2\right)dt', \end{aligned}$$

we shall estimate $Q_1(t)$ and $Q_2(t)$. For $Q_1(t)$, by taking $s' = s$ in (2.1) and (2.2), we have

$$\begin{aligned} I_1 &\le CY(t)^{1/2}\left\|u^1 - u^2\right\|_{H^{s+1}}\left\|u^1\right\|_{H^s}\left\|\nabla u^1\right\|_{H^{\frac{1}{2}}} \\ &\le CY(t)^{1/2}\left\|u^1 - u^2\right\|_{H^{s+1}}\left\|u^1\right\|_{H^s}^{s+\frac{1}{2}}\left\|u^1\right\|_{H^{s+1}}^{\frac{3}{2}-s} \\ &\le \frac{1}{4}\left\|u^1 - u^2\right\|_{H^{s+1}}^2 + CY(t)\left\|u^1\right\|_{H^s}^{2s+1}\left\|u^1\right\|_{H^{s+1}}^{3-2s}, \end{aligned} \tag{3.11}$$

$$\begin{aligned}
I_2 &\le C\|u^1-u^2\|_{H^s}\|\nabla u^1\|_{H^{\frac{1}{2}}}\|u^1-u^2\|_{H^{s+1}} \\
&\le C\|u^1-u^2\|_{H^{s+1}}\|u_1\|_{H^s}^{s+\frac{1}{2}}\|u^1\|_{H^{s+1}}^{\frac{3}{2}-s}\|u^1-u^2\|_{H^{s+1}} \\
&\le \frac{1}{4}\|u^1-u^2\|_{H^{s+1}}^2 + C\|u_1\|_{H^s}^{2s+1}\|u^1\|_{H^{s+1}}^{3-2s}\|u_1-u_2\|_{H^s}^2, \\
&\le \frac{1}{4}\|u^1-u^2\|_{H^{s+1}}^2 + CY(t)\|u_1\|_{H^s}^{2s+1}\|u^1\|_{H^{s+1}}^{3-2s},
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
I_3 &\le C\|u^2\|_{H^s}\|\nabla(u^1-u^2)\|_{H^{\frac{1}{2}}}\|u^1-u^2\|_{H^{s+1}} \\
&\le C\|u^2\|_{H^s}\|u^1-u^2\|_{H^s}^{s-\frac{1}{2}}\|u^1-u^2\|_{H^{s+1}}^{\frac{5}{2}-s} \\
&\le \frac{1}{4}\|u^1-u^2\|_{H^{s+1}}^2 + C\|u^1-u^2\|_{H^s}^2\|u^2\|_{H^s}^{\frac{4}{2s-1}} \\
&\le \frac{1}{4}\|u^1-u^2\|_{H^{s+1}}^2 + CY(t)\|u^2\|_{H^s}^{\frac{4}{2s-1}}.
\end{aligned} \tag{3.13}$$

Hence, it holds that

$$Q_1(t) \le \frac{3}{4}\|\nabla u^1 - \nabla u^2\|_{H^s}^2 + R_1(t)Y(t), \tag{3.14}$$

where

$$\int_0^T R_1(t)\,dt \le (N+1)^{\frac{4}{2s-1}} + (N+1)T^{s-\frac{1}{2}}$$

according to the definition of φ_N^u. Similarly, we have

$$J_1 \le \frac{1}{4}\|\rho^1-\rho^2\|_{H^{s+1}}^2 + CY(t)\|\rho^1\|_{H^s}^{2s+1}\|\rho^1\|_{H^{s+1}}^{3-2s}, \tag{3.15}$$

$$J_2 \le \frac{1}{4}\|\rho^1-\rho^2\|_{H^{s+1}}^2 + CY(t)\|\rho_1\|_{H^s}^{2s+1}\|u^1\|_{H^{s+1}}^{3-2s}, \tag{3.16}$$

$$J_3 \le \frac{1}{4}\|\rho^1-\rho^2\|_{H^{s+1}}^2 + CY(t)\|u^2\|_{H^s}^{\frac{4}{2s-1}}. \tag{3.17}$$

Therefore

$$Q_2(t) \le \frac{3}{4}\|\nabla\rho^1 - \nabla\rho^2\|_{H^1}^2 + R_2(t)Y(t), \tag{3.18}$$

and

$$\int_0^T R_1(t)\,dt \le (N+1)^{\frac{4}{2s-1}} + (N+1)T^{s-\frac{1}{2}}.$$

In addition,

$$\begin{aligned}
\left|\langle(\rho^1-\rho^2)e_3, u^1-u^2\rangle_{H^s}\right| &\le C\left(\|\rho^1-\rho^2\|_{H^s}^2 + \|u^1-u^2\|_{H^s}^2\right) \\
&\le CY(t).
\end{aligned} \tag{3.19}$$

Introducing a stopping time:

$$\sigma_L = \begin{cases} \inf\{0 \le t \le T : Y(t) > L\}, \\ T, \quad \text{if the set } \{\cdot\} \text{ is empty.} \end{cases}$$

Therefore, combining (3.2)–(3.19) and recalling the classical Grönwall inequality, we get

$$\begin{aligned} Y(T \wedge \sigma_L) \le{}& e^{C_{N,T}} \big(\| u_0^1 - u_0^2 \|_{H^s}^2 + \| \rho_0^1 - \rho_0^2 \|_{H^s}^2 \big) \\ &+ e^{C_{N,T}} \sup_{t \in [0, T\wedge\sigma_L]} \left| \sum_{i=1}^{\infty} \int_0^t \langle u^1 - u^2, \Phi_{i,1}^1 - \Phi_{i,1}^2 \rangle_{H^s} \, dW_i \right| \\ &+ e^{C_{N,T}} \sup_{t \in [0, T\wedge\sigma_L]} \left| \sum_{i=1}^{\infty} \int_0^t \langle \rho^1 - \rho^2, \Phi_{i,2}^1 - \Phi_{i,2}^2 \rangle_{H^s} \, d\widetilde{W}_i \right| \\ &+ e^{C_{N,T}} \int_0^{T\wedge\sigma_L} \big(\| \Phi_1^1 - \Phi_1^2 \|_{\mathbb{H}^s}^2 + \| \Phi_2^1 - \Phi_2^2 \|_{\mathbb{H}^s}^2 \big) \, dt. \end{aligned} \tag{3.20}$$

Furthermore, by the Burkholder–Davis–Gundy inequality (2.4), one has

$$\begin{aligned} & e^{C_{N,T}} \mathbb{E} \left(\sup_{t \in [0, T\wedge\sigma_L]} \left| \sum_{i=1}^{\infty} \int_0^t \langle u^1 - u^2, \Phi_{i,1}^1 - \Phi_{i,1}^2 \rangle_{H^s} \, dW_i \right| \right) \\ &\le e^{C_{N,T}} \mathbb{E} \left(\int_0^{T\wedge\sigma_L} Y(t) \| \Phi_{i,1}^1 - \Phi_{i,1}^2 \|_{\mathbb{H}^s}^2 \, dt \right)^{1/2} \\ &\le \frac{1}{4} \mathbb{E} Y(T \wedge \sigma_L) + e^{C_{N,T}} \mathbb{E} \int_0^T \| \Phi_{i,1}^1 - \Phi_{i,1}^2 \|_{\mathbb{H}^s}^2 \, dt, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} & e^{C_{N,T}} \mathbb{E} \left(\sup_{t \in [0, T\wedge\sigma_L]} \left| \sum_{i=1}^{\infty} \int_0^t \langle \rho^1 - \rho^2, \Phi_{i,2}^1 - \Phi_{i,2}^2 \rangle_{H^s} \, d\widetilde{W}_i \right| \right) \\ &\le e^{C_{N,T}} \mathbb{E} \left(\int_0^{T\wedge\sigma_L} Y(t) \| \Phi_{i,2}^1 - \Phi_{i,2}^2 \|_{\mathbb{H}^s}^2 \, dt \right)^{1/2} \\ &\le \frac{1}{4} \mathbb{E} Y(T \wedge \sigma_L) + e^{C_{N,T}} \mathbb{E} \int_0^T \| \Phi_{i,2}^1 - \Phi_{i,2}^2 \|_{\mathbb{H}^s}^2 \, dt. \end{aligned} \tag{3.22}$$

Taking the mathematical expectation in (3.20), (3.21), and (3.22) implies

$$\begin{aligned} \mathbb{E} Y(T) \le{}& e^{C_{N,T}} \mathbb{E} \big(\| u_0^1 - u_0^2 \|_{H^s}^2 + \| \rho_0^1 - \rho_0^2 \|_{H^s}^2 \big) \\ &+ e^{C_{N,T}} \mathbb{E} \int_0^T \big(\| \Phi_1^1 - \Phi_1^2 \|_{\mathbb{H}^s}^2 + \| \Phi_2^1 - \Phi_2^2 \|_{\mathbb{H}^s}^2 \big) \, dt. \end{aligned} \tag{3.23}$$

□

Hence, we get the key estimate for proving the existence of strong solution. Now, we give the sketch for the proof of the existence and uniqueness of strong solution.

1. Let ν_ε be the Friedrichs mollifier. By the contraction mapping principle, we can show that the approximate mollified equation

$$
\begin{aligned}
&du_\varepsilon + \bigl(\varphi_N^{u_{\varepsilon_i},\rho_{\varepsilon_i}} \mathbf{P}(u_\varepsilon \cdot \nabla u_\varepsilon) - \Delta u_\varepsilon\bigr)\,dt = \mathbf{P}\rho_\varepsilon e_3\,dt + \sum_{i=1}^{\infty} \mathbf{P}\Phi_{1,i} * \nu_\varepsilon\,dW_i,\\
&d\rho_\varepsilon + \bigl(\varphi_N^{u_{\varepsilon_i},\rho_{\varepsilon_i}} u_\varepsilon \cdot \nabla \rho_\varepsilon - \Delta \rho_\varepsilon\bigr)\,dt = \sum_{i=1}^{\infty} \Phi_{2,i} * \nu_\varepsilon\,d\widetilde{W}_i,\\
&(u_\varepsilon, \rho_\varepsilon)|_{t=0} = (u_0 * \nu_\varepsilon, \rho_0 * \nu_\varepsilon),
\end{aligned}
$$

admits a unique global strong solution.

2. It follows from (3.23) that $\{U_{\varepsilon_i}\}$ is a Cauchy sequence in

$$
L^2\bigl(\Omega; C\bigl([0,T]; H^s_\sigma \times H^s\bigr)\bigr) \cap L^2\bigl(\Omega; L^2\bigl(0,T; H^{s+1}_\sigma \times H^{s+1}\bigr)\bigr).
$$

Let U be the limit. We can extract a subsequence still denoted by U_{ε_i} such that

$$
U_{\varepsilon_i}(\omega) \to U(\omega) \quad \text{in } C\bigl([0,T]; H^s_\sigma \times H^s\bigr) \cap L^2\bigl(0,T; H^{s+1}_\sigma \times H^{s+1}\bigr) \quad \mathbb{P}\text{-a.s.}
$$

Then $\varphi_N^{u_{\varepsilon_i},\rho_{\varepsilon_i}} \to \varphi_N^{u,\rho}$. Hence U is the unique solution of the modified equation

$$
\begin{aligned}
&du + \bigl(\varphi_N^{u,\rho} \mathbf{P}(u \cdot \nabla u) - \Delta u\bigr)\,dt = \mathbf{P}\rho e_3\,dt + \sum_{i=1}^{\infty} \mathbf{P}\Phi_{1,i}\,dW_i,\\
&d\rho + \bigl(\varphi_N^{u,\rho} u \cdot \nabla \rho - \Delta \rho\bigr)\,dt = \sum_{i=1}^{\infty} \Phi_{2,i}\,d\tilde{W}_i,\\
&(u, \rho)|_{t=0} = (u_0, \rho_0).
\end{aligned}
$$

3. Define the stopping time, and drop the cut-off function by the uniqueness of U, we can get the local strong solution of (1.1). If there are two solutions U defined for $[0,\tau)$ and $\widetilde{U}$ defined for $[0,\widetilde{\tau})$, then by inequality (3.4), for any $N \ge 1$,

$$
U = \widetilde{U} \quad \text{on } [0, \tau_N \wedge \widetilde{\tau}_N].
$$

If $\widetilde{\tau}_N < \tau_N$, then $\|U(\tau_N)\|^2_{H^s} \ge N$, which contradicts the definition of τ_N in (1.2). Hence $\tau_N \le \widetilde{\tau}_N$. Similarly, we have $\widetilde{\tau}_N \le \tau_N$, thus $\widetilde{\tau}_N = \tau_N$. Therefore, we obtain the uniqueness of solutions.

4 Proof of Theorem 1.3

4.1 A new blow-up time

We introduce another stopping time as follows:

$$
\widetilde{\zeta}_K = \begin{cases} \inf\{t \ge 0 : \|u(t \wedge \tau)\|^2_{H^{s'}} \ge K\}, \\ \tau, \quad \text{if the set } \{\cdot\} \text{ is empty.} \end{cases} \tag{4.1}
$$

We have the following proposition.

Proposition 4.1 *For any $K > 0$, $\widetilde{\zeta}_K < \tau$ almost surely on the set $\{\tau < \infty\}$.*

Proof Define $S_k^N = \{\tau_N \le \widetilde{\zeta}_K\} \cap \{\tau_N \le k\}$ for $k \ge 1$ and $N \ge 1$. According to Itô's formula and the equation of (u, ρ), we have

$$\begin{aligned} &d\big(\|u\|_{H^s}^2 + \|\rho\|_{H^s}^2\big) + 2\big(\|\nabla u\|_{H^s}^2 + \|\nabla \rho\|_{H^s}^2\big)\,dt \\ &\quad = \big(-2\langle u\cdot\nabla u, u\rangle_{H^s} - 2\langle u\cdot\nabla\rho,\rho\rangle_{H^s} + \langle \rho e_3, u\rangle_{H^s}\big)\,dt \\ &\qquad + 2\sum_{i=1}^{\infty}\big(\langle \Phi_{1,i}, u\rangle_{H^s}\,dW_i + \langle \Phi_{2,i},\rho\rangle_{H^s}\,d\widetilde{W}_i\big) + \big(\|\mathbf{P}\Phi_1\|_{\mathbb{H}^s}^2 + \|\Phi_2\|_{\mathbb{H}^s}^2\big)\,dt. \end{aligned} \tag{4.2}$$

By (2.1) and (2.2), Young's inequality, and Hölder's inequality, we bound

$$\begin{aligned} \big|\langle u\cdot\nabla u, u\rangle_{H^s}\big| &\le \|u\|_{H^{s'}}\|u\|_{H^{\frac{3}{2}+s-s'}}\|u\|_{H^{s+1}} \\ &\le \|u\|_{H^{s'}}\|u\|_{H^s}^{s'-\frac{1}{2}}\|u\|_{H^{s+1}}^{\frac{5}{2}-s'} \\ &\le \frac{1}{2}\|\nabla u\|_{H^s}^2 + C\|u\|_{H^s}^2\big(1+\|u\|_{H^{s'}}^{\frac{4}{2s'-1}}\big), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \big|\langle u\cdot\nabla \rho, \rho\rangle_{H^s}\big| &\le \|u\|_{H^{s'}}\|\rho\|_{H^{\frac{3}{2}+s-s'}}\|\rho\|_{H^{s+1}} \\ &\le \|u\|_{H^{s'}}\|\rho\|_{H^s}^{s'-\frac{1}{2}}\|\rho\|_{H^{s+1}}^{\frac{5}{2}-s'} \\ &\le \frac{1}{2}\|\nabla \rho\|_{H^s}^2 + C\|\rho\|_{H^s}^2\big(1+\|u\|_{H^{s'}}^{\frac{4}{2s'-1}}\big), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \big|\langle \rho e_3, u\rangle_{H^s}\big| &\le C\|\rho\|_{H^s}\|u\|_{H^s} \\ &\le C\big(\|\rho\|_{H^s}^2 + \|u\|_{H^s}^2\big). \end{aligned} \tag{4.5}$$

By Grönwall's inequality, we obtain

$$\begin{aligned} &\sup_{t\in[0,k\wedge\tau_N\wedge\widetilde{\zeta}_K]}\big(\|u\|_{H^s}^2 + \|\rho\|_{H^s}^2\big) + \int_0^{k\wedge\tau_N\wedge\widetilde{\zeta}_K}\big(\|\nabla u\|_{H^s}^2 + \|\nabla\rho\|_{H^s}^2\big)\,ds \\ &\quad \le C\Bigg(\|u_0\|_{H^s}^2 + \|\rho_0\|_{H^s}^2 + \int_0^{k\wedge\tau_N\wedge\widetilde{\zeta}_K}\big(\|\Phi_1\|_{\mathbb{H}^s}^2 + \|\Phi_2\|_{\mathbb{H}^s}^2\big)\,dt \\ &\qquad + \sup_{t\in[0,k\wedge\tau_N\wedge\widetilde{\zeta}_K]}\sum_{i=1}^{\infty}\bigg|\int_0^t \langle\Phi_{1,i},u\rangle_{H^s}\,dW_i + \int_0^t\langle\Phi_{2,i},\rho\rangle_{H^s}\,d\widetilde{W}_i\bigg|\Bigg) \\ &\qquad \times \exp\bigg(C\int_0^{k\wedge\tau_N\wedge\widetilde{\zeta}_K}\big(1+\|u\|_{H^{s'}}^{\frac{4}{2s'-1}}\big)\,ds\bigg). \end{aligned} \tag{4.6}$$

According to the definition of $\widetilde{\zeta}_K$, we have

$$\int_0^{k\wedge\tau_N\wedge\widetilde{\zeta}_K}\big(1+\|u\|_{H^{s'}}^{\frac{4}{2s'-1}}\big)\,ds \le \big(1+K^{\frac{2}{2s'-1}}\big)k \quad \text{a.s.} \tag{4.7}$$

Furthermore, by the Burkholder–Davis–Gundy inequality (2.4), one has

$$\begin{aligned}
&e^{C_{K,k}}\mathbb{E}\sup_{t\in[0,k\wedge\tau_N\wedge\widetilde{\zeta}_K]}\sum_{i=1}^{\infty}\left|\int_0^t\langle\Phi_{1,i},u\rangle_{H^s}\,dW_i+\int_0^t\langle\Phi_{2,i},\rho\rangle_{H^s}\,d\widetilde{W}_i\right| \\
&\quad\le e^{C_{K,k}}\mathbb{E}\left(\int_0^{k\wedge\tau_N\wedge\widetilde{\zeta}_K}\left(\|u\|_{H^s}^2\|\Phi_1\|_{\mathbb{H}^s}^2+\|\rho\|_{H^s}^2\|\Phi_2\|_{\mathbb{H}^s}^2\right)ds\right)^{1/2} \\
&\quad\le\frac{1}{2}\mathbb{E}\sup_{t\in[0,k\wedge\tau_N\wedge\widetilde{\zeta}_K]}\left(\|u\|_{H^s}^2+\|\rho\|_{H^s}^2\right)+e^{C_{K,k}}\mathbb{E}\int_0^k\left(\|\Phi_1\|_{\mathbb{H}^s}^2+\|\Phi_2\|_{\mathbb{H}^s}^2\right)dt.
\end{aligned}\tag{4.8}$$

Taking the mathematical expectation in (4.6), we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,k\wedge\tau_N\wedge\widetilde{\zeta}_K]}\left(\|u\|_{H^s}^2+\|\rho\|_{H^s}^2\right)+\mathbb{E}\int_0^{k\wedge\tau_N\wedge\widetilde{\zeta}_K}\left(\|\nabla u\|_{H^s}^2+\|\nabla\rho\|_{H^s}^2\,ds\right) \\
&\quad\le e^{C_{K,k}}\mathbb{E}\left(\|u_0\|_{H^s}^2+\|\rho_0\|_{H^s}^2\right)+e^{C_{K,k}}\mathbb{E}\int_0^k\left(\|\Phi_1\|_{\mathbb{H}^s}^2+\|\Phi_2\|_{\mathbb{H}^s}^2\right)dt.
\end{aligned}\tag{4.9}$$

By the definition of A_N^k, one has

$$A_N^k\subset\left\{\sup_{t\in[0,k\wedge\tau_N\wedge\widetilde{\zeta}_K]}\left(\|u\|_{H^s}^2+\|\rho\|_{H^s}^2\right)+\int_0^{k\wedge\tau_N\wedge\widetilde{\zeta}_K}\left(\|\nabla u\|_{H^s}^2+\|\nabla\rho\|_{H^s}^2\,ds\right)\ge N\right\}.$$

By Chebyshev's inequality, it follows from (4.9) that

$$\mathbb{P}\left(A_N^k\right)\le\frac{\exp(C_{K,k})}{N}\mathbb{E}\left(\|u_0\|_{H^s}^2+\|\rho_0\|_{H^s}^2+\int_0^k\left(\|\Phi_1\|_{\mathbb{H}^s}^2+\|\Phi_2\|_{\mathbb{H}^s}^2\right)dt\right).\tag{4.10}$$

Define $A^k=\{\tau\le\zeta_K\}\cap\{\tau\le k\}$, then $A^k\subset A_N^k$ for all N, so $\mathbb{P}(A^k)=0$. Since $\{\tau\le\zeta_K\}\cap\{\tau<\infty\}=\bigcup_{k=1}^{\infty}A^k$, then $\mathbb{P}(\{\tau\le\zeta_K\}\cap\{\tau<\infty\})=0$. The proof of Proposition 4.1 is thus complete. □

Then, by the definition of ζ_K (1.3), we have $\widetilde{\zeta}_K=\zeta_K$ and $\zeta\le\tau$ a.s. On the other hand, $\|u\|_{H^{s'}}\le\|u\|_{H^s}$, then $\|u(\tau_N)\|_{H^{s'}}\le N$. Due to the definition of ζ_K, we have $\tau_N\le\zeta_N$. Therefore,

$$\tau=\zeta\quad\text{a.s.},$$

and ζ is another blow-up time.

4.2 The proof of (1.4)

By Itô's formula for $\|\rho\|_{L^2}^2$, we have

$$d\|\rho\|_{L^2}^2+2\|\nabla\rho\|_{L^2}^2\,dt=2\sum_{i=1}^{\infty}\langle\rho,\Phi_{2,i}\rangle_{L^2}\,d\widetilde{W}_i+\|\Phi_2\|_{\mathbb{L}^2}^2\,dt.\tag{4.11}$$

By the Burkholder–Davis–Gundy inequality (2.4), we have

$$\mathbb{E}\sup_{t'\in[0,\zeta_K\wedge t]}2\sum_{i=1}^{\infty}\int_0^{t'}\langle\rho,\Phi_{2,i}\rangle_{L^2}\,d\widetilde{W}_i$$

$$
\begin{aligned}
&\le C\mathbb{E}\left(\int_0^{t\wedge\zeta_K}\|\rho\|_{L^2}^2\|\Phi_2\|_{\mathbb{L}^2}^2\,dt\right)^{1/2}\\
&\le \frac{1}{4}\mathbb{E}\sup_{t'\in[0,\zeta_K\wedge t]}\|\rho\|_{L^2}^2+\mathbb{E}\int_0^t\|\Phi_2\|_{\mathbb{L}^2}^2\,dt. \qquad (4.12)
\end{aligned}
$$

Therefore, we obtain

$$
\mathbb{E}\sup_{t'\in[0,\zeta_K\wedge t]}\|\rho\|_{L^2}^2\le C\mathbb{E}\|\rho_0\|_{L^2}^2+\mathbb{E}\int_0^t\|\Phi_2\|_{\mathbb{L}^2}^2\,dt. \qquad (4.13)
$$

Applying Itô's formula for $\|u\|_{H^{s'}}^2$, we have

$$
\begin{aligned}
d\|u\|_{H^{s'}}^2+2\|\nabla u\|_{H^{s'}}^2\,dt &= 2\langle u\cdot\nabla u,u\rangle_{H^{s'}}\,dt+2\langle\rho e_3,u\rangle_{H^{s'}}\,dt\\
&\quad+2\sum_{i=1}^{\infty}\langle\Phi_{1,i},u\rangle_{H^{s'}}\,dW_i+\|\mathbf{P}\Phi_1\|_{\mathbb{H}^{s'}}^2\,dt. \qquad (4.14)
\end{aligned}
$$

By (2.1), we bound

$$
\begin{aligned}
2\left|\langle u\cdot\nabla u,u\rangle_{H^{s'}}\right| &\le C\|u\|_{H^{s'}}\|\nabla u\|_{H^{\frac{1}{2}}}\|u\|_{H^{s'+1}}\\
&\le C\|u\|_{H^{s'}}^{s'+\frac{1}{2}}\|u\|_{H^{s'+1}}^{\frac{5}{2}-s'}\\
&\le \frac{1}{4}\|u\|_{H^{s'+1}}^2+C\left(\|u\|_{H^{s'}}^{\frac{4s'+2}{2s'-1}}+\|u\|_{H^{s'}}^2\right), \qquad (4.15)
\end{aligned}
$$

and

$$
\begin{aligned}
2\left|\langle\rho e_3,u\rangle_{H^{s'}}\right| &\le \|\rho\|_{L^2}\|u\|_{H^{2s'}}\\
&\le \frac{1}{4}\|\nabla u\|_{H^{s'}}^2+C\left(\|\rho\|_{L^2}^2+\|u\|_{H^{s'}}^2\right). \qquad (4.16)
\end{aligned}
$$

Similar to (4.8), we have

$$
\begin{aligned}
&\sup_{t'\in[0,t\wedge\zeta_K]}\sum_{i=1}^{\infty}\left|\int_0^{t'}\langle\Phi_{1,i},u\rangle_{H^{s'}}\,dW_i\right|\\
&\le \frac{1}{2}\mathbb{E}\sup_{t'\in[0,t\wedge\zeta_K]}\|u\|_{H^{s'}}^2+C\mathbb{E}\int_0^t\|\Phi_1\|_{\mathbb{H}^{s'}}^2\,dt. \qquad (4.17)
\end{aligned}
$$

Combining (4.13)–(4.17) and recalling the definition of ζ_K, we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\delta\wedge\zeta_K]}\|u\|_{H^{s'}}^2+\mathbb{E}\int_0^{\delta\wedge\zeta_K}\|\nabla u\|_{H^{s'}}^2\,dt\\
&\le C\left(\mathbb{E}\left(\|u_0\|_{H^{s'}}^2+\int_0^{\delta}\|\Phi_1\|_{\mathbb{H}^{s'}}^2\,dt\right)+K^{1+\frac{2}{2s'-1}}\delta+K\delta\right.\\
&\quad\left.+\delta\mathbb{E}\left(\|\rho_0\|_{L^2}^2+\int_0^{\delta}\|\Phi_2\|_{\mathbb{L}^2}^2\right)\right)\\
&\le C\left(K^{1+\frac{1}{2s'-1}}\delta+K\delta+\mathcal{A}(\delta)\right), \qquad (4.18)
\end{aligned}
$$

where we define

$$\mathcal{A}(\delta)=\mathbb{E}\left(\|u_0\|_{H^{s'}}^2+\int_0^\delta\|\Phi_1\|_{\mathbb{H}^{s'}}^2\,dt\right)+\delta\mathbb{E}\left(\|\rho_0\|_{L^2}^2+\int_0^\delta\|\Phi_2\|_{\mathbb{L}^2}^2\right).$$

By the definition of ζ_K, one has

$$\{\omega|\zeta_K\le\delta\}\subset\left\{\omega\Big|\sup_{t\in[0,\delta\wedge\zeta_K]}\|u\|_{H^{s'}}^2\ge K\right\}.$$

By Chebyshev's inequality, we obtain

$$\mathbb{P}\big(\{\zeta_K\le\delta\}\big)\le\frac{C(K^{1+\frac{2}{2s'-1}}\delta+K\delta+\mathcal{A}(\delta))}{K}.$$

Let δ be given such that $0<\delta<1$. Choose an integer $K>0$ such that

$$\frac{1}{K+1}\le\delta^{\frac{2s'-1}{2s'+1}}<\frac{1}{K},$$

then

$$\mathbb{P}\big(\{\zeta>\delta\}\big)\ge\mathbb{P}\big(\{\zeta_K>\delta\}\big)\ge 1-C^*\delta^{\frac{2s'-1}{2s'+1}}\big(\mathcal{A}(\delta)+1\big).$$

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