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The obstacle problem for nonlinear noncoercive elliptic equations with L^1-data

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Abstract

In this paper, we study the obstacle problem governed by nonlinear noncoercive elliptic equations with L^1-data. We prove the existence of an entropy solution and show its continuous dependence on the L^1-data in $W^{1,q}(\Omega)$ with $q > 1$.

Keywords: Obstacle problem; Noncoercive elliptic equation; L^1-data; Entropy solution

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain, $p > 1$, $\theta \geq 0$, $f \in L^1(\Omega)$, and $g, \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. We consider the obstacle problem governed by the noncoercive operator

$$Au = \operatorname{div} \frac{a(x, \nabla u)}{(1 + b(x)|u|)^{\theta(p-1)}} \tag{1}$$

associated with (f, ψ, g), where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying:

$$a(x,\xi) \cdot \xi \geq \alpha|\xi|^p, \tag{2}$$

$$\bigl|a(x,\xi)\bigr| \leq \beta\bigl(j(x) + |\xi|^{p-1}\bigr), \tag{3}$$

$$\bigl(a(x,\zeta) - a(x,\eta)\bigr)(\zeta - \eta) > 0, \tag{4}$$

$$\bigl|a(x,\xi) - a(x,\eta)\bigr| \leq \gamma \begin{cases} |\xi - \eta|^{p-1} & \text{if } 1 < p < 2, \\ (1 + |\xi| + |\eta|)^{p-2}|\xi - \eta| & \text{if } p \geq 2, \end{cases} \tag{5}$$

for almost every $x \in \Omega$ and every $\xi, \zeta, \eta \in \mathbb{R}^N$ with $\zeta \neq \eta$, where α, β, γ are positive constants, j is a nonnegative function in $L^{\frac{p}{p-1}}(\Omega)$, and b is an L^∞-function satisfying, with some $B \geq 0$,

$$0 \leq b(x) \leq B \tag{6}$$

for almost every $x \in \Omega$.

If $f \in W^{-1,p'}(\Omega)$, then the obstacle problem associated with (f, ψ, g) is formulated in terms of the inequality

$$\int_\Omega \frac{a(x, \nabla u)}{(1 + b(x)|u|)^{\theta(p-1)}} \cdot \nabla(v - u)\,dx + \int_\Omega f(v - u)\,dx \geq 0 \quad \forall v \in K_{g,\psi} \cap L^\infty(\Omega) \tag{7}$$

whenever $K_{g,\psi} = \{v \in W^{1,p}(\Omega); v - g \in W_0^{1,p}(\Omega), v \geq \psi \text{ a.e. in } \Omega\} \neq \emptyset$. However, the second integration in (7) is not well defined for $f \in L^1(\Omega)$. Following [1, 3, 5], and so on, we are led to a more general definition of a solution to the obstacle problem with data $f \in L^1(\Omega)$, using the truncation function

$$T_s(r) = \max\{-s, \min\{s, r\}\}, \quad s, r \in \mathbb{R}.$$

Definition 1 An entropy solution to the obstacle problem associated with (f, ψ, g) is a measurable function u such that $u \geq \psi$ a.e. in Ω, $T_s(u) - T_s(g) \in W_0^{1,p}(\Omega)$ for every $s > 0$, and

$$\int_\Omega \frac{a(x, \nabla u)}{(1 + b(x)|u|)^{\theta(p-1)}} \cdot \nabla T_s(v - u)\, dx + \int_\Omega f T_s(v - u)\, dx \geq 0 \quad \forall v \in K_{g,\psi} \cap L^\infty(\Omega). \tag{8}$$

Observe that in the definition a global integrability condition is required neither on u nor on its gradient. As pointed out in [8], if $T_s(u) \in W^{1,p}(\Omega)$ for all $s > 0$, then there exists a unique measurable vector field $U : \Omega \to \mathbb{R}^N$ such that $\nabla(T_s(u)) = \chi_{\{|u|<s\}} U$ a.e. in Ω, $s > 0$, which, in fact, coincides with the standard distributional gradient of ∇u whenever $u \in W^{1,1}(\Omega)$.

The motivation of this paper comes from the study on the Dirichlet boundary value problem

$$\begin{cases} \operatorname{div} \frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Indeed, for the p-Laplacian equation, that is, $\theta = 0$ in (9), the existence and regularity of solutions when f has a fine regularity have been well studied. However, under weaker summability assumptions on f, for example, $f \in L^1(\Omega)$, the gradient of u (and even u itself) may not be in $L^1(\Omega)$. In this case, it is possible to give a meaning to solutions of problem (9) by using the concept of entropy solutions. The works on the theory of entropy solutions for p-Laplacian equations have been applied to unilateral problems in [5, 7, 17], and so on and have been extended in [8, 20] to the obstacle problems with L^1-data in Sobolev spaces with variable exponents and Orlicz–Sobolev spaces, respectively. We remark that the classical obstacle problem for elliptic operators with nonlinear variational energies was considered in [12] and linear elliptic systems involving Radon measures were considered in [19]. Parabolic problems with irregular obstacles and nonstandard $p(x,t)$-growth were considered in [10] and references therein.

If $0 < \theta \leq 1$, then due to the lack of coercivity, the standard Leray–Lions surjectivity theorem cannot be used for the establishment of existence of solutions even in the case $f \in W^{-1,p'}(\Omega)$. To overcome this difficulty, "cutting" the nonlinearity and using the technique of approximation, a pseudomonotone coercive differential operator on $W_0^{1,p}(\Omega)$ can be applied to establish a priori estimates on approximating solutions. Then by the almost everywhere convergence for the gradients of the approximating solutions, the existence and regularity of solutions (or entropy solutions) to problems of the form (9) can be obtained by taking limitation (see, e.g., [1]). For different summability of the data f, Alvino, Boccardo, Ferone, Orsina, Trombetti, et al. have done a lot of work on the existence and regularity of solutions (or entropy solutions) to problems of the form (9) (see [1, 2, 6, 13, 18] and references therein). Particularly, a whole range of existence results have been proven in [6] for $p = 2$ and f regular enough, showing that solutions are in some Sobolev space $W_0^{1,q}(\Omega)$ $(1 < q \leq 2)$ (see also [13, 14, 18]). Nevertheless, little literature has considered the obstacle problem for noncoercive elliptic equations, particularly, for noncoercive elliptic equations with L^1-data.

Motivated by this, we study the obstacle problem governed by (1) and (f, ψ, g) with L^1-data. The main ideas in this paper originate from [1, 8], which can be also applied to the study on a large class of elliptic/parabolic equations [9, 15, 21], potential theory [22], and Schrödinger equations [11, 16]. Throughout this paper, without special statements, we always assume that

$$2 - \frac{1}{N} < p < N, \qquad 0 \leq \theta(p-1) < \frac{N(p-1)}{N-1} - 1, \tag{10}$$

$f \in L^1(\Omega)$, $\psi, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfy $(\psi - g)^+ \in W_0^{1,p}(\Omega)$, and $K_{g,\psi} \neq \emptyset$.

Note that (10) implies that

$$0 \leq \theta < \frac{N}{N-1} - \frac{1}{p-1}, \quad \text{and} \quad \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)} > 1.$$

The main result in this paper is the following:

Theorem 1 *There exists at least one entropy solution u to the obstacle problem associated with (f, ψ, g). In addition, u depends continuously on f, that is, if $f_n \to f$ in $L^1(\Omega)$ and u_n is a solution to the obstacle problem associated with (f_n, ψ, g), then*

$$u_n \to u \quad \text{in } W^{1,q}(\Omega) \text{ for all } q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right).$$

Notations

$\|u\|_p = \|u\|_{L^p(\Omega)}$ is the norm of $L^p(\Omega)$, where $1 \leq p \leq \infty$.
$\|u\|_{1,p} = \|u\|_{W^{1,p}(\Omega)}$ is the norm of $W^{1,p}(\Omega)$, where $1 < p < \infty$.
$p' = \frac{p}{p-1}$ with $1 < p < \infty$.
$u^+ = \max\{u, 0\}$, $u^- = (-u)^+$ for a real-valued function u.
C is a constant, which may be different from line to line.
$\{u > s\} = \{x \in \Omega; u(x) > s\}$.
$\{u \leq s\} = \Omega \setminus \{u > s\}$.
$\{u < s\} = \{x \in \Omega; u(x) < s\}$.
$\{u \geq s\} = \Omega \setminus \{u < s\}$.
$\{u = s\} = \{x \in \Omega; u(x) = s\}$.
$\{t \leq u < s\} = \{u \geq t\} \cap \{u < s\}$.
$\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$.
$|E| = \mathcal{L}^N(E)$ for a measurable set E.

2 Preliminaries on entropy solutions

It is worth noting that, for any function f_n smooth enough, there exists at least one solution to the obstacle problem (7). Indeed, we can proceed exactly as in Theorem 1.1 of [1] to obtain $W^{1,p}$-solutions due to assumptions (2)–(6), which, particularly, are also entropy solutions. In this section, we establish several auxiliary results on convergence of sequences of entropy solutions as $f_n \to f$ in $L^1(\Omega)$. The main techniques used in this section come from [1, 8]. We start with a priori estimate.

Lemma 2 *Let $v_0 \in K_{g,\psi} \cap L^\infty(\Omega)$, and let u be an entropy solution to the obstacle problem associated with (f, ψ, g). Then, we have*

$$\int_{\{|u|<t\}} \frac{|\nabla u|^p}{(1+b(x)|u|)^{\theta(p-1)}}\,dx \le C\big(\|j\|_{p'}^{p'} + \|\nabla v_0\|_p^p + \|f\|_1\big(t + \|v_0\|_\infty\big)\big) \quad \forall t > 0,$$

where C is a positive constant depending only on α, β, and p.

Proof For $t > 0$, taking v_0 as a test function in (8), we compute

$$\begin{aligned}\int_{\{|v_0-u|<t\}} \frac{a(x,\nabla u)\cdot\nabla u}{(1+b(x)|u|)^{\theta(p-1)}}\,dx &\le \int_{\{|v_0-u|<t\}} \frac{a(x,\nabla u)\cdot\nabla v_0}{(1+b(x)|u|)^{\theta(p-1)}}\,dx \\ &\quad + \int_\Omega f\big(T_t(v_0-u)\big)\,dx.\end{aligned}$$

It follows from (2), (3), and Young's inequality with $\varepsilon > 0$ that

$$\begin{aligned}\int_{\{|v_0-u|<t\}} \frac{\alpha|\nabla u|^p}{(1+b(x)|u|)^{\theta(p-1)}}\,dx &\le \int_{\{|v_0-u|<t\}} \frac{\beta(|j| + |\nabla u|^{p-1})\cdot|\nabla v_0|}{(1+b(x)|u|)^{\theta(p-1)}}\,dx + t\|f\|_1 \\ &\le \int_{\{|v_0-u|<t\}} \frac{\beta\varepsilon(|j|^{p'} + |\nabla u|^p)}{(1+b(x)|u|)^{\theta(p-1)}}\,dx \\ &\quad + \int_{\{|v_0-u|<t\}} \frac{\beta C(\varepsilon)|\nabla v_0|^p}{(1+b(x)|u|)^{\theta(p-1)}}\,dx + t\|f\|_1 \\ &\le \varepsilon\int_{\{|v_0-u|<t\}} \frac{|\nabla u|^p}{(1+b(x)|u|)^{\theta(p-1)}}\,dx \\ &\quad + C\big(\|j\|_{p'}^{p'} + \|\nabla v_0\|_p^p\big) + t\|f\|_1.\end{aligned}$$

Thus we have

$$\int_{\{|v_0-u|<t\}} \frac{|\nabla u|^p}{(1+b(x)|u|)^{\theta(p-1)}}\,dx \le C\big(\|j\|_{p'}^{p'} + \|\nabla v_0\|_p^p + t\|f\|_1\big). \tag{11}$$

Replacing t with $t + \|v_0\|_\infty$ in (11) and noting that $\{|u| < t\} \subset \{|v_0 - u| < t + \|v_0\|_\infty\}$, we obtain the desired result. □

In the rest of this section, let $\{u_n\}$ be a sequence of entropy solutions to the obstacle problem associated with (f_n, ψ, g) and assume that

$$f_n \to f \quad \text{in } L^1(\Omega) \quad \text{and} \quad \|f_n\|_1 \le \|f\|_1 + 1.$$

Lemma 3 *For $k > 0$ large enough, there exists a measurable function u such that $u_n \to u$ in measure and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W^{1,p}(\Omega)$. Thus $T_k(u_n) \to T_k(u)$ strongly in $L^p(\Omega)$, and up to a subsequence, $T_k(u_n) \to T_k(u)$ a.e. in Ω.*

Proof Let $s, t, \varepsilon > 0$. We can verify that, for all $m, n \geq 1$,

$$\begin{aligned} \mathcal{L}^N\big(\{|u_n - u_m| > s\}\big) &\leq \mathcal{L}^N\big(\{|u_n| > t\}\big) + \mathcal{L}^N\big(\{|u_m| > t\}\big) \\ &\quad + \mathcal{L}^N\big(\{\big|T_t(u_n) - T_t(u_m)\big| > s\}\big) \end{aligned} \tag{12}$$

and

$$\mathcal{L}^N\big(\{|u_n| > t\}\big) = \frac{1}{t^p}\int_{\{|u_n|>t\}} t^p\,dx \leq \frac{1}{t^p}\int_\Omega \big|T_t(u_n)\big|^p\,dx. \tag{13}$$

Since $v_0 = g + (\psi - g)^+ \in K_{g,\psi} \cap L^\infty(\Omega)$, by Lemma 2 we have

$$\begin{aligned} \int_\Omega \big|\nabla T_t(u_n)\big|^p\,dx &= \int_{\{|u_n|<t\}} |\nabla u_n|^p\,dx \\ &\leq C(1 + Bt)^{\theta(p-1)}\big(\|j\|_{p'}^{p'} + \|\nabla v_0\|_p^p + \|f\|_1\big(t + \|v_0\|_\infty\big)\big). \end{aligned} \tag{14}$$

Note that $T_t(u_n) - T_t(g) \in W_0^{1,p}(\Omega)$. By (13), (14), and Poincaré's inequality, for every $t > \|g\|_\infty$ and for some positive constant C independent of n and t, we have

$$\begin{aligned} \mathcal{L}^N\big(\{|u_n| > t\}\big) &\leq \frac{1}{t^p}\int_\Omega \big|T_t(u_n)\big|^p\,dx \\ &\leq \frac{2^{p-1}}{t^p}\int_\Omega \big|T_t(u_n) - T_t(g)\big|^p\,dx + \frac{2^{p-1}}{t^p}\|g\|_p^p \\ &\leq \frac{C}{t^p}\int_\Omega \big|\nabla T_t(u_n) - \nabla T_t(g)\big|^p\,dx + \frac{2^{p-1}}{t^p}\|g\|_p^p \\ &\leq \frac{C}{t^p}\int_\Omega \big|\nabla T_t(u_n)\big|^p\,dx + \frac{C}{t^p}\|g\|_{1,p}^p \\ &\leq \frac{C(1+t)^{1+\theta(p-1)}}{t^p}. \end{aligned}$$

Since $0 \leq \theta < 1$, there exists $t_\varepsilon > 0$ such that

$$\mathcal{L}^N\big(\{|u_n| > t\}\big) < \frac{\varepsilon}{3} \quad \forall t \geq t_\varepsilon, \forall n \geq 1. \tag{15}$$

Now, as in (13), we have

$$\begin{aligned} \mathcal{L}^N\big(\{\big|T_{t_\varepsilon}(u_n) - T_{t_\varepsilon}(u_m)\big| > s\}\big) &= \frac{1}{s^p}\int_{\{|T_{t_\varepsilon}(u_n)-T_{t_\varepsilon}(u_m)|>s\}} s^p\,dx \\ &\leq \frac{1}{s^p}\int_\Omega \big|T_{t_\varepsilon}(u_n) - T_{t_\varepsilon}(u_m)\big|^p\,dx. \end{aligned} \tag{16}$$

Using (14) and the fact that $T_t(u_n) - T_t(g) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ again, we see that $\{T_{t_\varepsilon}(u_n)\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Thus, up to a subsequence, $\{T_{t_\varepsilon}(u_n)\}$ converges strongly in $L^p(\Omega)$. By (16) there exists $n_0 = n_0(t_\varepsilon, s) \geq 1$ such that

$$\mathcal{L}^N\big(\{\big|T_{t_\varepsilon}(u_n) - T_{t_\varepsilon}(u_m)\big| > s\}\big) < \frac{\varepsilon}{3} \quad \forall n, m \geq n_0. \tag{17}$$

Combining (12), (15), and (17), we obtain

$$\mathcal{L}^N\big(\{|u_n - u_m| > s\}\big) < \varepsilon, \quad \forall n, m \geq n_0.$$

Hence $\{u_n\}$ is a Cauchy sequence in measure, and therefore there exists a measurable function u such that $u_n \to u$ in measure. Note that $T_k(u_n) - T_k(g) \in W_0^{1,p}(\Omega)$. By (14) and Poincaré's inequality we conclude that, for fixed k, $\{T_k(u_n)\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Therefore, $T_k(u_n) \to T_k(u)$ strongly in $L^p(\Omega)$, and, up to a subsequence, $T_k(u_n) \to T_k(u)$ a.e. in Ω. □

Proposition 4 *There exist a subsequence of $\{u_n\}$ and a measurable function u such that, for each $q \in (1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$, we have*

$$u_n \to u \quad \text{strongly in } W^{1,q}(\Omega).$$

Furthermore, if $0 \leq \theta < \min\{\frac{1}{N-p+1}, \frac{N}{N-1} - \frac{1}{p-1}\}$, then

$$\frac{a(x,\nabla u_n)}{(1+b(x)|u_n|)^{\theta(p-1)}} \to \frac{a(x,\nabla u)}{(1+b(x)|u|)^{\theta(p-1)}} \quad \text{strongly in } \big(L^1(\Omega)\big)^N.$$

To prove Proposition 4, we need two preliminary lemmas.

Lemma 5 *There exists a subsequence of $\{u_n\}$ and a measurable function u such that for each $q \in (1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$, we have that $u_n \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$ and $u_n \to u$ strongly in $L^q(\Omega)$.*

Proof Let $k > 0$ and $n \geq 1$. Define $D_k = \{|u_n| \leq k\}$ and $B_k = \{k \leq |u_n| < k+1\}$. Using Lemma 2 with $v_0 = g + (\psi - g)^+$, we get

$$\int_{D_k} \frac{|\nabla u_n|^p}{(1+b(x)|u_n|)^{\theta(p-1)}}\,dx \leq C(1+k), \tag{18}$$

where C is a positive constant depending only on α, β, p, $\|j\|_{p'}$, $\|f\|_1$, $\|\nabla v_0\|_p$, and $\|v_0\|_\infty$.

Taking the function $T_k(u_n)$ with $k > \{\|g\|_\infty, \|\psi\|_\infty\}$ as a test function for the problem associated with (f_n, ψ, g), we obtain

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot \nabla T_1(u_n - T_k(u_n))}{(1+b(x)|u_n|)^{\theta(p-1)}}\,dx \leq \int_\Omega -f_n T_1\big(u_n - T_k(u_n)\big)\,dx,$$

which, together with (2), gives

$$\int_{B_k} \frac{\alpha|\nabla u_n|^p}{(1+b(x)|u_n|)^{\theta(p-1)}}\,dx \leq \|f_n\|_1 \leq \|f\|_1 + 1. \tag{19}$$

Let $q \in (1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$ and $r = \frac{q\theta(p-1)}{p}$. Noting that $\frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)} < p$, it follows $q < p$. Let $\Theta = \theta(p-1)$. Since $\frac{B-1}{A-1} \leq \frac{B}{A}$ for all $A \geq B > 1$, we have $\frac{p-\Theta-1}{N-\Theta-1} \leq \frac{p-\Theta}{N-\Theta}$, which implies $q < \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)} = \frac{N(p-1)-N\Theta}{N-1-\Theta} < \frac{N(p-\Theta)}{N-\Theta}$. So we get $\frac{\Theta}{p-q} < \frac{N}{N-q}$. It follows that $\frac{pr}{p-q} = \frac{q\theta(p-1)}{p-q} =$

$\frac{q\Theta}{p-q} < \frac{Nq}{N-q} := q^*$. For all $k > 0$, we estimate $\int_{B_k} |\nabla u_n|^q \, dx$:

$$\begin{aligned}
\int_{B_k} |\nabla u_n|^q \, dx &= \int_{B_k} \frac{|\nabla u_n|^q}{(1+b(x)|u_n|)^r} \cdot \bigl(1+b(x)|u_n|\bigr)^r \, dx \\
&\le \left(\int_{B_k} \frac{|\nabla u_n|^p}{(1+b(x)|u_n|)^{\theta(p-1)}} \, dx \right)^{\frac{q}{p}} \left(\int_{B_k} \bigl(1+b(x)|u_n|\bigr)^{\frac{pr}{p-q}} \, dx \right)^{\frac{p-q}{p}} \\
&\le C|B_k|^{\frac{p-q}{p}} + C\left(\int_{B_k} |u_n|^{\frac{pr}{p-q}} \, dx \right)^{\frac{p-q}{p}} \qquad \text{by (19) and (6)} \\
&\le C|B_k|^{\frac{p-q}{p}} + C\left(\int_{B_k} |u_n|^{q^*} \, dx \right)^{\frac{r}{q^*}} \cdot |B_k|^{\frac{p-q}{p} - \frac{r}{q^*}}.
\end{aligned}$$

Since $|B_k| \le \frac{1}{k^{q^*}} \int_{B_k} |u_n|^{q^*} \, dx$, for $k \ge k_0 \ge 1$, we have

$$\begin{aligned}
\int_{B_k} |\nabla u_n|^q \, dx &\le C\left(\frac{1}{k^{q^*}} \int_{B_k} |u_n|^{q^*} \, dx \right)^{\frac{p-q}{p}} + C\frac{1}{k^{q^*(\frac{p-q}{p} - \frac{r}{q^*})}} \left(\int_{B_k} |u_n|^{q^*} \, dx \right)^{\frac{p-q}{p}} \\
&\le \frac{2C}{k^{q^*(\frac{p-q}{p} - \frac{r}{q^*})}} \left(\int_{B_k} |u_n|^{q^*} \, dx \right)^{\frac{p-q}{p}}.
\end{aligned}$$

Summing up from $k = k_0$ to $k = K$ and using Hölder's inequality, we have

$$\sum_{k=k_0}^{K} \int_{B_k} |\nabla u_n|^q \, dx \le C\left(\sum_{k=k_0}^{K} \frac{1}{k^{q^*(\frac{p-q}{p} - \frac{r}{q^*})\frac{p}{q}}} \right)^{\frac{q}{p}} \cdot \left(\sum_{k=k_0}^{K} \int_{B_k} |u_n|^{q^*} \, dx \right)^{\frac{p-q}{p}}. \tag{20}$$

Note that

$$\int_{\{|u_n| \le K\}} |\nabla u_n|^q \, dx = \int_{D_{k_0}} |\nabla u_n|^q \, dx + \sum_{k=k_0}^{K} \int_{B_k} |\nabla u_n|^q \, dx. \tag{21}$$

To estimate the first integral in the right-hand side of (21), using Hölder's inequality, (18), and (6), we obtain

$$\begin{aligned}
\int_{D_{k_0}} |\nabla u_n|^q \, dx &\le \left(\int_{D_{k_0}} \frac{|\nabla u_n|^p}{(1+b(x)|u_n|)^{\theta(p-1)}} \, dx \right)^{\frac{q}{p}} \left(\int_{D_{k_0}} \bigl(1+b(x)|u_n|\bigr)^{\frac{pr}{p-q}} \, dx \right)^{\frac{p-q}{p}} \\
&\le C, \qquad (22)
\end{aligned}$$

where C depends only on α, β, B, p, θ, $\|j\|_{p'}$, $\|f\|_1$, $\|\nabla v_0\|_p$, $\|v_0\|_\infty$, and k_0.

Note that $\sum_{k=k_0}^{K} \frac{1}{k^{q^*(\frac{p-q}{p} - \frac{r}{q^*})\frac{p}{q}}}$ converges since $q^*(\frac{p-q}{p} - \frac{r}{q^*})\frac{p}{q} > 1$. Combining (20)–(22), we get, for k_0 large enough,

$$\int_{\{|u_n| \le K\}} |\nabla u_n|^q \, dx \le C + C\left(\int_{\{|u_n| \le K\}} |u_n|^{q^*} \, dx \right)^{\frac{p-q}{p}}. \tag{23}$$

Since $p > q$, $T_K(u_n) \in W^{1,q}(\Omega)$ and $T_K(g) = g \in W^{1,q}(\Omega)$ for $K > \|g\|_\infty$. Hence $T_K(u_n) - g \in W_0^{1,q}(\Omega)$. Using the Sobolev embedding $W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ and Poincaré's inequality, we obtain

$$\begin{aligned}\|T_K(u_n)\|_{q^*}^q &\le 2^{q-1}\left(\|T_K(u_n)-g\|_{q^*}^q + \|g\|_{q^*}^q\right)\\ &\le C\left(\|\nabla(T_K(u_n)-g)\|_q^q + \|g\|_{q^*}^q\right)\\ &\le C\left(\|\nabla T_K(u_n)\|_q^q + \|\nabla g\|_q^q + \|g\|_{q^*}^q\right)\\ &\le C\left(1+\int_{\{|u_n|\le K\}}|\nabla u_n|^q\,dx\right).\end{aligned} \tag{24}$$

Using the fact that

$$\int_{\{|u_n|\le K\}}|u_n|^{q^*}\,dx \le \int_{\{|u_n|\le K\}}\left|T_K(u_n)\right|^{q^*}\,dx \le \|T_K(u_n)\|_{q^*}^{q^*}, \tag{25}$$

from (23)–(24) we obtain

$$\int_{\{|u_n|\le K\}}|\nabla u_n|^q\,dx \le C + C\left(1+\int_{\{|u_n|\le K\}}|\nabla u_n|^q\,dx\right)^{\frac{q^*}{q}\frac{p-q}{p}}. \tag{26}$$

Note that $p < N \Leftrightarrow \frac{q^*}{q}\frac{p-q}{p} < 1$. It follows from (26) that, for k_0 large enough, $\int_{\{|u_n|\le K\}}|\nabla u_n|^q\,dx$ is bounded independently of n and K. Using (24) and (25), we deduce that $\int_{\{|u_n|\le K\}}|u_n|^{q^*}\,dx$ is also bounded independently of n and K. Letting $K\to\infty$, we deduce that $\|\nabla u_n\|_q$ and $\|u_n\|_{q^*}$ are uniformly bounded independently of n. Particularly, u_n is bounded in $W^{1,q}(\Omega)$. Therefore, there exist a subsequence of $\{u_n\}$ and a function $v\in W^{1,q}(\Omega)$ such that $u_n \rightharpoonup v$ weakly in $W^{1,q}(\Omega)$ and $u_n\to v$ strongly in $L^q(\Omega)$ and a.e. in Ω. By Lemma 3, $u_n\to u$ in measure in Ω, and we conclude that $u=v$ and $u\in W^{1,q}(\Omega)$. □

Lemma 6 *There exist a subsequence of* $\{u_n\}$ *and a measurable function* u *such that* ∇u_n *converges to* ∇u *almost everywhere in* Ω.

Proof The proof is similar to that of [1, Thm. 4.1] and can be also found in [4]. Here we sketch only the main steps due to slight modifications. For $r_2>1$, let $\lambda=\frac{q}{pr_2}<1$, where q is the same as in Lemma 5. Define $A(x,u,\xi)=\frac{a(x,\xi)}{(1+b(x)|u|)^{\theta(p-1)}}$ (for simplicity, we omit the dependence of $A(x,u,\xi)$ on x) and

$$I(n)=\int_\Omega\left(\left(A(u_n,\nabla u_n)-A(u_n,\nabla u)\right)\cdot\nabla(u_n-u)\right)^\lambda dx.$$

We fix $k>0$ and split the integral in $I(n)$ on the sets $\{|u|>k\}$ and $\{|u|\le k\}$, obtaining

$$I_1(n,k)=\int_{\{|u|>k\}}\left(\left(A(u_n,\nabla u_n)-A(u_n,\nabla u)\right)\cdot\nabla(u_n-u)\right)^\lambda dx$$

and

$$I_2(n,k)=\int_{\{|u|\le k\}}\left(\left(A(u_n,\nabla u_n)-A(u_n,\nabla u)\right)\cdot\nabla(u_n-u)\right)^\lambda dx.$$

For $I_2(n,k)$, we have

$$I_2(n,k) \leq I_3(n,k) = \int_\Omega \Big(\big(A_n(u_n,\nabla u_n) - A_n\big(u_n,\nabla T_k(u)\big)\big) \cdot \nabla\big(u_n - T_k(u)\big)\Big)^\lambda dx.$$

Fix $h > 0$ and split $I_3(n,k)$ on the sets $\{|u_n - T_k(u)| > h\}$ and $\{|u_n - T_k(u)| \leq h\}$, obtaining

$$I_4(n,k,h) = \int_{\{|u_n - T_k(u)|>h\}} \Big(\big(A_n(u_n,\nabla u_n) - A_n\big(u_n,\nabla T_k(u)\big)\big) \cdot \nabla\big(u_n - T_k(u)\big)\Big)^\lambda dx$$

and

$$\begin{aligned} I_5(n,k,h) &= \int_{\{|u_n - T_k(u)|\leq h\}} \Big(\big(A_n(u_n,\nabla u_n) - A_n\big(u_n,\nabla T_k(u)\big)\big) \cdot \nabla\big(u_n - T_k(u)\big)\Big)^\lambda dx \\ &= \int_\Omega \Big(\big(A_n(u_n,\nabla u_n) - A_n\big(u_n,\nabla T_k(u)\big)\big) \cdot \nabla T_h\big(u_n - T_k(u)\big)\Big)^\lambda dx \\ &\leq |\Omega|^{1-\lambda}\bigg(\int_\Omega \big(A_n(u_n,\nabla u_n) - A_n\big(u_n,\nabla T_k(u)\big)\big) \cdot \nabla T_h\big(u_n - T_k(u)\big)\, dx\bigg)^\lambda \\ &= |\Omega|^{1-\lambda}\big(I_6(n,k,h)\big)^\lambda. \end{aligned}$$

For $I_6(n,k,h)$, we can split it as the difference $I_7(n,k,h) - I_8(n,k,h)$, where

$$I_7(n,k,h) = \int_\Omega A(u_n,\nabla u_n) \cdot \nabla T_h\big(u_n - T_k(u)\big)\, dx$$

and

$$I_8(n,k,h) = \int_\Omega A\big(u_n,\nabla T_k(u)\big) \cdot \nabla T_h\big(u_n - T_k(u)\big)\, dx.$$

Note that $|\nabla u_n|$ is bounded in $L^q(\Omega)$ and $\lambda p r_2 = q$. Due to Lemmas 3 and 5, in the same way as Theorem 4.1 in [1], we can get that

$$\lim_{k\to\infty}\limsup_{n\to\infty} I_1(n,k) = 0, \qquad \lim_{h\to\infty}\limsup_{k\to\infty}\limsup_{n\to\infty} I_4(n,k,h) = 0,$$
$$\lim_{n\to\infty} I_8(n,k,h) = 0.$$

For $I_7(n,k,h)$, let $k > \max\{\|g\|_\infty, \|\psi\|_\infty\}$ and $n \geq h + k$. Take $T_k(u)$ as a test function for (8), obtaining

$$I_7(n,k,h) \leq \int_\Omega -f_n T_h\big(u_n - T_k(u)\big)\, dx.$$

Using the strong convergence of f_n in $L^1(\Omega)$, we have

$$\lim_{n\to\infty} I_7(n,k,h) \leq \int_\Omega -f T_h\big(u - T_k(u)\big)\, dx.$$

Note that $h > 0$ and $\lim_{k\to\infty} T_h(u - T_k(u)) = 0$. It follows that

$$\lim_{k\to\infty}\lim_{n\to\infty} I_7(n,k,h) \leq 0.$$

Putting together all the limitations and noting that $I(n) \geq 0$, we have

$$\lim_{n\to\infty} I(n) = 0.$$

The same arguments as in [1] give that, up to a subsequence, $\nabla u_n(x) \to \nabla u(x)$ *a.e.* □

Proof of Proposition 4 We shall prove that ∇u_n converges strongly to ∇u_n in $L^q(\Omega)$ for each $q \in (1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$. To this end, we will apply Vitalli's theorem, using the fact that by Lemma 5, ∇u_n is bounded in $L^q(\Omega)$ for each $q \in (1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$. Let $r \in (q, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$, and let $E \subset \Omega$ be a measurable set. We have, by Hölder's inequality,

$$\int_E |\nabla u_n|^q\,dx \leq \left(\int_E |\nabla u_n|^r\,dx\right)^{\frac{q}{r}} \cdot |E|^{\frac{r-q}{r}} \leq C|E|^{\frac{r-q}{r}} \to 0$$

uniformly in n as $|E| \to 0$. From this and from Lemma 6 we deduce that ∇u_n converges strongly to ∇u in $L^q(\Omega)$.

Now assume that $0 \leq \theta < \min\{\frac{1}{N-p+1}, \frac{N}{N-1} - \frac{1}{p-1}\}$. Note that since ∇u_n converges to ∇u a.e. in Ω, to prove the convergence

$$\frac{a(x,\nabla u_n)}{(1+b(x)|u_n|)^{\theta(p-1)}} \to \frac{a(x,\nabla u)}{(1+b(x)|u|)^{\theta(p-1)}} \quad \text{strongly in } \left(L^1(\Omega)\right)^N,$$

it suffices, thanks to Vitalli's theorem, to show that, for every measurable subset $E \subset \Omega$, $\int_E |\frac{a(x,\nabla u_n)}{(1+b(x)|u_n|)^{\theta(p-1)}}|\,dx$ converges to 0 uniformly in n as $|E| \to 0$. Note that $p-1 < \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}$ by the assumptions. For any $q \in (p-1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$, we deduce by Hölder's inequality that

$$\begin{aligned}\int_E \left|\frac{a(x,\nabla u_n)}{(1+b(x)|u_n|)^{\theta(p-1)}}\right|dx &\leq \beta \int_E \left(j + |\nabla u_n|^{p-1}\right)dx \\ &\leq \beta\|j\|_{p'}|E|^{\frac{1}{p}} + \beta\left(\int_E |\nabla u_n|^q\,dx\right)^{\frac{p-1}{q}} |E|^{\frac{q-p+1}{q}} \\ &\to 0 \quad \text{uniformly in } n \text{ as } |E| \to 0.\end{aligned}$$

□

Lemma 7 *There exists a subsequence of* $\{u_n\}$ *such that, for all* $k > 0$,

$$\frac{a(x,\nabla T_k(u_n))}{(1+b(x)|T_k(u_n)|)^{\theta(p-1)}} \to \frac{a(x,\nabla T_k(u))}{(1+b(x)|T_k(u)|)^{\theta(p-1)}} \quad \textit{strongly in } \left(L^1(\Omega)\right)^N.$$

Proof Let k be a positive number. It is well known that if a sequence of measurable functions $\{u_n\}$ with uniform boundedness in $L^p(\Omega)$ $(p > 1)$ converges in measure to u, then u_n converges strongly to u in $L^1(\Omega)$. First note that the sequence $\{\frac{a(x,\nabla T_k(u_n))}{(1+b(x)|T_k(u_n)|)^{\theta(p-1)}}\}$ is bounded in $L^{p'}(\Omega)$. Indeed, we have by (3) and Lemma 2,

$$\begin{aligned}\int_\Omega \left|\frac{a(x,\nabla T_k(u_n))}{(1+b(x)|T_k(u_n)|)^{\theta(p-1)}}\right|^{p'} dx &\leq \beta\|j\|_{p'}^{p'} + \beta\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+b(x)|T_k(u_n)|)^{\theta p}}\,dx \\ &\leq \beta\|j\|_{p'}^{p'} + \beta\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+b(x)|T_k(u_n)|)^{\theta(p-1)}}\,dx \\ &\leq C.\end{aligned}$$

Next, it suffices to show that there exists a subsequence of $\{u_n\}$ such that

$$\frac{a(x,\nabla T_k(u_n))}{(1+b(x)|T_k(u_n)|)^{\theta(p-1)}} \to \frac{a(x,\nabla T_k(u))}{(1+b(x)|T_k(u)|)^{\theta(p-1)}} \quad \text{in measure.}$$

Note that $u_n, u \in W^{1,q}(\Omega)$, where q is the same as in Proposition 4. The a.e. convergence of u_n to u and the fact that $\nabla u_n \to \nabla u$ in measure imply that

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{in measure.}$$

Let s, t be positive numbers and write $\nabla_A u = \frac{a(x,\nabla u)}{(1+b(x)|u|)^{\theta(p-1)}}$. Define

$$\begin{aligned}
&E_n = \bigl\{\bigl|\nabla_A T_k(u_n) - \nabla_A T_k(u)\bigr| > s\bigr\},\\
&E_n^1 = \bigl\{\bigl|\nabla T_k(u_n)\bigr| > t\bigr\},\\
&E_n^2 = \bigl\{\bigl|\nabla T_k(u)\bigr| > t\bigr\},\\
&E_n^3 = E_n \cap \bigl\{\bigl|\nabla T_k(u_n)\bigr| \le t\bigr\} \cap \bigl\{\bigl|\nabla T_k(u)\bigr| \le t\bigr\}.
\end{aligned}$$

Note that $E_n \subset E_n^1 \cup E_n^2 \cup E_n^3$ for each $n \ge 1$. Since by Lemma 5 the sequence $\{u_n\}$ and the function u are uniformly bounded in $W^{1,q}(\Omega)$, we obtain

$$\begin{aligned}
&\mathcal{L}^N\bigl(E_n^1\bigr) \le \frac{1}{t^q}\int_\Omega \bigl|\nabla T_k(u_n)\bigr|^q\,dx \le \frac{1}{t^q}\int_\Omega |\nabla u_n|^q\,dx \le \frac{C}{t^q},\\
&\mathcal{L}^N\bigl(E_n^2\bigr) \le \frac{1}{t^q}\int_\Omega \bigl|\nabla T_k(u)\bigr|^q\,dx \le \frac{1}{t^q}\int_\Omega |\nabla u|^q\,dx \le \frac{C}{t^q}.
\end{aligned}$$

We deduce that, for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\mathcal{L}^N\bigl(E_n^1\bigr) + \mathcal{L}^N\bigl(E_n^2\bigr) < \frac{\varepsilon}{3} \quad \forall t \ge t_\varepsilon, \forall n \ge 1. \tag{27}$$

Note that, for $a \ge b \ge 0$ and $\tau \ge 0$, we have the inequality

$$\left|\frac{1}{(1+a)^\tau} - \frac{1}{(1+b)^\tau}\right| = \left|\frac{\tau(b-a)}{(1+c)^{1+\tau}}\right| \le \tau|b-a| \quad \text{for some } c \in [b,a].$$

From (3), (5), and (6) we deduce that, in E_n^3,

$$\begin{aligned}
s &< \bigl|\nabla_A T_k(u_n) - \nabla_A T_k(u)\bigr|\\
&= \left|\frac{a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))}{(1+b(x)|T_k(u_n)|)^{\theta(p-1)}}\right.\\
&\quad \left. + \left(\frac{1}{(1+b(x)|T_k(u_n)|)^{\theta(p-1)}} - \frac{1}{(1+b(x)|T_k(u)|)^{\theta(p-1)}}\right) a\bigl(x,\nabla T_k(u)\bigr)\right|\\
&\le \theta(p-1)B\bigl|T_k(u_n) - T_k(u)\bigr| \cdot \beta\bigl(j + \bigl|\nabla T_k(u)\bigr|^{p-1}\bigr)\\
&\quad + \gamma \begin{cases} |\nabla T_k(u_n) - \nabla T_k(u)|^{p-1} & \text{if } 1 < p < 2,\\ |\nabla T_k(u_n) - \nabla T_k(u)|(1 + |\nabla T_k(u_n)| + |\nabla T_k(u)|)^{p-2} & \text{if } p \ge 2 \end{cases}
\end{aligned}$$

$$\begin{aligned}&\le C_0 j\big|T_k(u_n)-T_k(u)\big|\\&\quad + C_0\big(1+t^{p-1}+t^{p-2}\big)\big(\big|T_k(u_n)-T_k(u)\big| + \big|\nabla T_k(u_n)-\nabla T_k(u)\big|\big),\end{aligned}$$

which leads to $E_n^3 \subset F_1 \cup F_2$ with

$$\begin{aligned}F_1 &= \left\{ j\big|T_k(u_n)-T_k(u)\big| > \frac{s}{2C_0}\right\},\\ F_2 &= \left\{ \big|T_k(u_n)-T_k(u)\big| + \big|\nabla T_k(u_n)-\nabla T_k(u)\big| > \frac{s}{2C_0(1+t^{p-1}+t^{p-2})}\right\}.\end{aligned}$$

In F_1, we have

$$\mathcal{L}^N(F_1) = \frac{2C_0}{s}\int_{F_1}\frac{s}{2C_0}\,dx < \frac{2C_0}{s}\int_{F_1} j\big|T_k(u_n)-T_k(u)\big|\,dx.$$

By Lemma 3 we deduce that there exists $n_0 = n_0(S, C_0, \varepsilon)$ such that

$$\mathcal{L}^N(F_1) \le \frac{\varepsilon}{3} \quad \forall n \ge n_0. \tag{28}$$

Note that $F_2 \subset F_3 \cup F_4$ with

$$\begin{aligned}F_3 &= \left\{ \big|T_k(u_n)-T_k(u)\big| > \frac{s}{4C_0(1+t^{p-1}+t^{p-2})}\right\},\\ F_4 &= \left\{ \big|\nabla T_k(u_n)-\nabla T_k(u)\big| > \frac{s}{4C_0(1+t^{p-1}+t^{p-2})}\right\}.\end{aligned}$$

Using the convergence in measure of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ and of $T_k(u_n)$ to $T_k(u)$, for $t = t_\varepsilon$, we obtain the existence of $n_1 = n_1(s,\varepsilon) \ge 1$ such that

$$\mathcal{L}^N(F_2) \le \mathcal{L}^N(F_3) + \mathcal{L}^N(F_4) < \frac{\varepsilon}{3} \quad \forall n \ge n_1. \tag{29}$$

Combining (27), (28), and (29), we obtain

$$\mathcal{L}^N\big(\big\{\big|\nabla_A T_k(u_n) - \nabla_A T_k(u)\big| > s\big\}\big) < \varepsilon \quad \forall n \ge \max\{n_0, n_1\}.$$

Hence the sequence $\{\nabla_A T_k(u_n)\}$ converges in measure to $\nabla_A T_k(u)$, and the lemma follows. □

3 Proof of the main result

Now we have gathered all the lemmas needed to prove the existence of an entropy solution to the obstacle problem associated with (f, ψ, g). In this section, let f_n be a sequence of smooth functions converging strongly to f in $L^1(\Omega)$ with $\|f_n\|_1 \le \|f\|_1 + 1$. We consider the sequence of approximated obstacle problems associated with (f_n, ψ, g). The proof originates from [8]. We provide details for readers' convenience.

Proof of Theorem 1 Let $v \in K_{g,\psi} \cap L^\infty(\Omega)$. Taking v as a test function in (8) associated with (f_n, ψ, g), we get

$$\int_\Omega \frac{a(x,\nabla u_n)}{(1+b(x)|u_n|)^{\theta(p-1)}}\cdot \nabla T_t(v-u_n)\,dx \ge \int_\Omega -f_n T_t(v-u_n)\,dx.$$

Since $\{|v-u_n|<t\}\subset\{|u_n|<s\}$ with $s=t+\|v\|_\infty$, the previous inequality can be written as

$$\int_\Omega \chi_n \nabla_A T_s(u_n)\cdot\nabla v\,dx \geq \int_\Omega -f_n T_t(v-u_n)\,dx + \int_\Omega \chi_n \nabla_A T_s(u_n)\cdot\nabla T_s(u_n)\,dx, \tag{30}$$

where $\chi_n=\chi_{\{|v-u_n|<t\}}$ and $\nabla_A u=\frac{a(x,\nabla u)}{(1+b(x)|u|)^{\theta(p-1)}}$. It is clear that $\chi_n \rightharpoonup \chi$ weakly* in $L^\infty(\Omega)$. Moreover, χ_n converges a.e. to $\chi_{\{|v-u|<t\}}$ in $\Omega\setminus\{|v-u|=t\}$. It follows that

$$\chi=\begin{cases}1 & \text{in } \{|v-u|<t\},\\ 0 & \text{in } \{|v-u|>t\}.\end{cases}$$

Note that we have $\mathcal{L}^N(\{|v-u|=t\})=0$ for a.e. $t\in(0,\infty)$. So there exists a measurable set $\mathcal{O}\subset(0,\infty)$ such that $\mathcal{L}^N(\{|v-u|=t\})=0$ for all $t\in(0,\infty)\setminus\mathcal{O}$. Assume that $t\in(0,\infty)\setminus\mathcal{O}$. Then χ_n converges weakly* in $L^\infty(\Omega)$ and a.e. in Ω to $\chi=\chi_{\{|v-u|<t\}}$. Since $\nabla T_s(u_n)$ converges a.e. to $\nabla T_s(u)$ in Ω (Proposition 4), by Fatou's lemma we obtain

$$\liminf_{n\to\infty}\int_\Omega \chi_n\nabla_A T_s(u_n)\cdot\nabla T_s(u_n)\,dx \geq \int_\Omega \chi\nabla_A T_s(u)\cdot\nabla T_s(u)\,dx. \tag{31}$$

Using the strong convergence of $\nabla_A T_s(u_n)$ to $\nabla_A T_s(u)$ in $L^1(\Omega)$ (Lemma 7) and the weak* convergence of χ_n to χ in $L^\infty(\Omega)$, we obtain

$$\lim_{n\to\infty}\int_\Omega \chi_n\nabla_A T_s(u_n)\cdot\nabla v\,dx = \int_\Omega \chi\nabla_A T_s(u)\cdot\nabla v\,dx. \tag{32}$$

Moreover, since f_n converges to f in $L^1(\Omega)$ and $T_t(v-u_n)$ converges to $T_t(v-u)$ weakly* in $L^\infty(\Omega)$, by passing to the limit in (30) and taking into account (31)–(32) we obtain

$$\int_\Omega \chi\nabla_A T_s(u)\cdot\nabla v\,dx - \int_\Omega \chi\nabla_A T_s(u)\cdot\nabla T_s(u)\,dx \geq \int_\Omega -fT_t(v-u)\,dx,$$

which can be written as

$$\int_{\{|v-u|\leq t\}} \chi\nabla_A T_s(u)\cdot(\nabla v-\nabla u)\,dx \geq \int_\Omega -fT_t(v-u)\,dx$$

or, since $\chi=\chi_{\{|v-u|<t\}}$ and $\nabla(T_t(v-u))=\chi_{\{|v-u|<t\}}\nabla(v-u)$,

$$\int_\Omega \nabla_A u\cdot\nabla T_t(v-u)\,dx \geq \int_\Omega -fT_t(v-u)\,dx, \quad \forall t\in(0,\infty)\setminus\mathcal{O}.$$

For $t\in\mathcal{O}$, we know that there exists a sequence $\{t_k\}$ in $(0,\infty)\setminus\mathcal{O}$ such that $t_k\to t$ due to $|\mathcal{O}|=0$. Therefore we have

$$\int_\Omega \nabla_A u\cdot\nabla T_{t_k}(v-u)\,dx \geq \int_\Omega -fT_{t_k}(v-u)\,dx \quad \forall k\geq 1. \tag{33}$$

Since $\nabla(v-u)=0$ a.e. in $\{|v-u|=t\}$, the left-hand side of (33) can be written as

$$\int_\Omega \nabla_A u\cdot\nabla T_{t_k}(v-u)\,dx = \int_{\Omega\setminus\{|v-u|=t\}} \chi_{\{|v-u|<t_k\}}\nabla_A u\cdot\nabla(v-u)\,dx.$$

The sequence $\chi_{\{|v-u|<t_k\}}$ converges to $\chi_{\{|v-u|<t\}}$ a.e. in $\Omega \setminus \{|v-u|=t\}$ and therefore converges weakly* in $L^\infty(\Omega \setminus \{|v-u|=t\})$. We obtain

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \nabla_A u\cdot\nabla T_{t_k}(v-u)\,dx &= \int_{\Omega\setminus\{|v-u|=t\}} \chi_{\{|v-u|<t\}}\nabla_A u\cdot\nabla(v-u)\,dx \\ &= \int_\Omega \chi_{\{|v-u|<t\}}\nabla_A u\cdot\nabla(v-u)\,dx \\ &= \int_\Omega \nabla_A u\cdot\nabla T_t(v-u)\,dx. \end{aligned} \tag{34}$$

For the right-hand side of (33), we have

$$\left|\int_\Omega -fT_{t_k}(v-u)\,dx - \int_\Omega -fT_t(v-u)\,dx\right| \le |t_k-t|\|f\|_1 \to 0 \quad \text{as } k\to\infty. \tag{35}$$

It follows from (33)–(35) that we have the inequality

$$\int_\Omega \nabla_A u\cdot\nabla T_t(v-u)\,dx \ge \int_\Omega -fT_t(v-u)\,dx \quad \forall t\in(0,\infty).$$

Hence, u is an entropy solution of the obstacle problem associated with (f,ψ,g). The regularity of the entropy solution u is guaranteed by Proposition 4. □

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