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# Multiplicity of periodic bouncing solutions for generalized impact Hamiltonian systems

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**Abstract**

Applying the Generalized Nonsmooth Saddle Point Theorem, we obtain multiple nontrivial periodic bouncing solutions for systems $\ddot{x} = f(t,x)$ with new conditions. In particular, we generalize the collision axis from $x = 0$ to the axis $x = a$, where $a$ is an arbitrary constant.

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**Keywords:** Generalized Nonsmooth Saddle Point Theorem; Periodic bouncing solution; Impact Hamiltonian systems; Multiplicity

## 1 Introduction

Consider the following Hamiltonian system with an obstacle, that is,

$$\ddot{x} = f(t,x), \quad t \in \mathbf{R} \setminus W, \tag{1.1}$$

associated with the conditions

$$\begin{cases} \dot{x}(t^-) = -\dot{x}(t^+), & t \in W, \\ x(t) \geq a, & \forall t \in \mathbf{R}, \\ x(t) = x(t+T), & \forall t \in \mathbf{R}, \end{cases} \tag{1.2}$$

where $a$ is a constant, $W = \{t \in \mathbf{R} \mid x(t) = a\}$, and $f : \mathbf{R} \times [a,+\infty) \to \mathbf{R}$ is continuous and $T$-periodic in $t$.

**Definition 1.1** (see [11]) Let $x : \mathbf{R} \to \mathbf{R}$ be continuous map. Then $x$ is a nonrivial $T$-periodic bouncing solution of system (1.1) with collision axis at $x = a$ if it satisfies (1.1)–(1.2) and

(1) the set $W$ is nonempty and discrete,

(2) there exists at least one $t_0 \in W$ such that $\dot{x}(t_0^-) \neq 0$.

We call systems, having solutions as in Definition 1.1, *impact Hamiltonian systems*. When $a = 0$, the bouncing periodic solutions of (1.1) have been discussed by some scholars in recent years. To the best of the authors' knowledge, Jiang (see [7]) first proposed a variational method to consider the bouncing periodic solutions of equation (1.1), and obtained the existence of bouncing solutions for impact Hamiltonian systems under a classical superquadratic condition. Then Ding (see [3]) considered the existence of subharmonic bouncing solutions for system (1.1) with sublinear conditions. In 2017, Nie (see [9]) first proved a Generalized Nonsmooth Saddle Point Theorem, which is applied to impact Hamiltonian systems, then obtained nontrivial $kT$-periodic bouncing solutions for system (1.1) with another sublinear condition.

Different from the papers [3, 7] and [9], we focus on the nontrivial $kT$-periodic bouncing solutions for system (1.1) with a new condition separating whether or not $a$ is equal to 0. The idea comes from the papers [7] and [9].

In this paper, we suppose that $\dot{x}(t_i^-) \leq 0$, furthermore, $\dot{x}(t_i^-) < 0$ if $t_i$ is a real impact time. To ensure that there exists at least one real impact time, paper [9] tells us that we must give the following condition:

(B) The inequality $f(t,x) \leq 0$ holds for every $t \in [0,T]$ and $x \geq a$; moreover, $\lim_{x\to+\infty} f(t,x) = -\infty$ or $\limsup_{x\to+\infty} f(t,x) < 0$ holds for every $t \in [0,T]$.

Now, we list our main result of periodic bouncing solution as follows:

Set $\Gamma = \{h \in C([0,+\infty);[0,+\infty)) \mid h$ satisfies (h1)–(h4)$\}$, where

(h1) $h(s) \leq h(t) + C$, for a certain constant $C > 0$ and $s,t \in [0,+\infty)$ with $s \leq t$,

(h2) $h(s+t) \leq C^*(h(s) + h(t))$, for a certain constant $C^* \geq 0$ and $\forall s,t \in [0,+\infty)$,

(h3) $th(t) - 2H(t) \to -\infty$, as $t \to +\infty$,

(h4) $\frac{H(t)}{t^2} \to 0$, as $t \to +\infty$,

and $H(t) := \int_0^t h(s)\,\mathrm{d}s$. Since $h(t) = \sqrt{t}$ can be in $\Gamma$, we get $\Gamma \neq \emptyset$.

We suppose that function $f$ satisfies the following conditions:

(f) There exist $T$-periodic functions $\gamma, g \in L^1([0,T];(0,+\infty))$ and function $h \in \Gamma$ such that

$$\bigl|f\bigl(t,|x|+a\bigr)\bigr| \leq \gamma(t)h\bigl(|x|\bigr) + g(t), \quad \forall x \in \mathbf{R} \text{ and } t \in [0,T]. \tag{1.3}$$

(F1) Function $h \in \Gamma$ in condition (f) satisfies

$$\limsup_{|x|\to+\infty} \frac{1}{H(|x|)} \int_0^T F\bigl(t,|x|+a\bigr)\,\mathrm{d}t < 0,$$

where $F(t,x) = \int_a^x f(t,s)\,\mathrm{d}s$ $(x \geq a)$.

(F2) Function $f(t,x)$ is differentiable for a.e. $t \in [0,T]$ and there exists a constant $\sigma > 0$ such that

$$\left|\frac{\partial F(t,x)}{\partial t}\right| \leq -\sigma F(t,x), \quad \text{a.e. } t \in [0,T] \text{ and } x \in [a,+\infty).$$

**Theorem 1.1** *Suppose function $f$ satisfies conditions* (B), (f), (F1) *and* (F2). *Then system* (1.1) *possesses nontrivial $kT$-periodic bouncing solutions $u_k$ for any sufficiently large integer $k$. Furthermore, $\|u_k\|_{L^\infty} \to +\infty$ as $k \to +\infty$.*

In this paper, we generalize the collision axis from $x = 0$ to the axis $x = a$, so our conditions generalize those in [9]. The main difficulty of this paper is checking whether the corresponding functional is locally Lipschitz and finding the range of the generalized gradients of the above functional. Therefore, we first prove Lemma 2.1 in Sect. 2.

System (1.1) with condition (1.2) is equivalent to the system

$$\ddot{x} = f(t, x + a), \quad t \in \mathbf{R} \setminus W^1, \tag{1.4}$$

associated with the conditions

$$\begin{cases} \dot{x}(t^-) = -\dot{x}(t^+), & t \in W^1, \\ x(t) \geq 0, & \forall t \in \mathbf{R}, \\ x(t) = x(t+T), & \forall t \in \mathbf{R}, \end{cases} \tag{1.5}$$

where $W^1 = \{t \in \mathbf{R} \mid x(t) = 0\}$.

Based on similar proofs in paper [7], we can conclude that if $x : \mathbf{R} \to \mathbf{R}$ is a $T$-periodic solution with isolated zeros of

$$\ddot{x} = f\bigl(t, |x| + a\bigr) \operatorname{sgn}(x), \quad t \in \mathbf{R} \setminus W^1, \tag{1.6}$$

then $|x|$ is a nontrivial $T$-periodic bouncing solution of system (1.4) with collision axis $x = 0$, and vice versa.

## 2 Preliminaries

Some concepts and conclusions about the Clarke generalized gradient can be found in [2] and [1], so we omit them.

**Definition 2.1** (see [5]) Function $\varphi$ satisfies the nonsmooth (PS) condition if every sequence $\{x_n\} \subset E$, such that $\{\varphi(x_n)\}$ is bounded and $\lambda(x_n) \to 0$ for $n \to \infty$, has a strongly convergent subsequence, where $\lambda(x) = \inf_{x^* \in \partial\varphi(x)} \|x^*\|_{E^*}$, $E^*$ is the dual space of $E$, and $\partial\varphi(x)$ denotes the Clarke generalized gradient of $\varphi$.

**Theorem 2.1** (Generalized Nonsmooth Saddle Point Theorem, see [9]) *Let $E$ be a real Banach space, and $E = V \oplus X$ with $V \neq \{\mathbf{0}\}$ and $\dim V < +\infty$. Suppose that functional $\varphi$ satisfies the nonsmooth (PS) condition, and for some $x_0 \in X$, there exists a constant $r > 0$ such that $\max_{\nu \in V \cap \partial B_r} \varphi(\nu + x_0) < \inf_{x \in X} \varphi(x)$. If $c$ can be characterized as $c = \inf_{\chi \in \Gamma_1} \max_{\nu \in V \cap \bar{B}_r} \varphi(\chi(\nu + x_0))$, then $c$ is a critical value of $\varphi$, where $\Gamma_1 = \{\chi \in C(V \cap \bar{B}_r + x_0, E) \mid \chi(\nu + x_0) = \nu + x_0, \text{if } \nu \in V \cap \partial B_r\}$ and $B_r = \{x \in E \mid \|x\| < r\}$. Furthermore, we have $c \geq \inf_{x \in X} \varphi(x)$.*

Set

$$H^1_{kT} = \left\{ x : [0, kT] \to \mathbf{R} \;\middle|\; \begin{array}{l} x(t) \text{ is absolutely continuous,} \\ x(0) = x(kT), \dot{x} \in L^2\bigl([0, kT], \mathbf{R}\bigr) \end{array} \right\},$$

in which $k \in \mathbf{N}^*$, then $H^1_{kT}$ is a Hilbert space with the norm defined by $\|x\| = [\int_0^{kT} (|\dot{x}(t)|^2 + |x(t)|^2)\,\mathrm{d}t]^{\frac{1}{2}}$. For $x \in H^1_{kT}$, let $\bar{x} = \frac{1}{kT} \int_0^{kT} x(t)\,\mathrm{d}t$ and $\tilde{x}(t) = x(t) - \bar{x}$. The book [8] tells us the following Wirtinger's inequality:

$$\int_0^{kT} \bigl|\tilde{x}(t)\bigr|^2\,\mathrm{d}t \leq \frac{k^2T^2}{4\pi^2} \int_0^{kT} \bigl|\dot{x}(t)\bigr|^2\,\mathrm{d}t \tag{2.1}$$

and Sobolev's inequality

$$\|\tilde{x}\|_{L^\infty}^2 \le \frac{kT}{12}\int_0^{kT}\bigl|\dot{x}(t)\bigr|^2\,\mathrm{d}t. \tag{2.2}$$

Set $\|x\|_0 = (|\bar{x}|^2 + \int_0^{kT}|\dot{x}(t)|^2\,\mathrm{d}t)^{\frac{1}{2}}$, then the norm $\|\cdot\|_0$ is equivalent to $\|\cdot\|$ (see [8]). Indeed, $H^1_{kT}$ has the decomposition $H^1_{kT} = \mathbf{R} \oplus \tilde{H}^1_{kT}$, where $\tilde{H}^1_{kT} = \{x \in H^1_{kT} \mid \bar{x} = 0\}$. Let $J_k(x) = \int_0^{kT} F(t,|x(t)|+a)\,\mathrm{d}t$, $\forall x \in H^1_{kT}$, and let $\varphi_k(x) = \frac{1}{2}\int_0^{kT}|\dot{x}(t)|^2\,\mathrm{d}t + J_k(x)$, $\forall x \in H^1_{kT}$.

**Lemma 2.1** *If $f$ satisfies condition* (f), *then functional $J_k$ is locally Lipschitz on $H^1_{kT}$ and*

$$\partial J_k(x) \subseteq \bigl[f^-\bigl(t,\bigl|x(t)\bigr|+a\bigr), f^+\bigl(t,\bigl|x(t)\bigr|+a\bigr)\bigr] \quad \text{a.e. } t \in [0,kT], \tag{2.3}$$

*where*

$$f^-\bigl(t,|s|+a\bigr) = \min\Bigl\{\lim_{u\to s^-} f\bigl(t,|s|+a\bigr)\operatorname{sgn}(u), \lim_{u\to s^+} f\bigl(t,|s|+a\bigr)\operatorname{sgn}(u)\Bigr\},$$
$$f^+\bigl(t,|s|+a\bigr) = \max\Bigl\{\lim_{u\to s^-} f\bigl(t,|s|+a\bigr)\operatorname{sgn}(u), \lim_{u\to s^+} f\bigl(t,|s|+a\bigr)\operatorname{sgn}(u)\Bigr\}.$$

*Proof* The main idea comes from [1]. Considering the functional

$$J_k(x) = \int_0^{kT}\mathrm{d}t\int_0^{|x(t)|} f(t,s+a)\,\mathrm{d}s,$$

we have

$$\begin{aligned}
\bigl|J_k(u)-J_k(v)\bigr| &\le \int_0^{kT}\biggl|\int_{|v(t)|}^{|u(t)|}\bigl[\gamma(t)h(s)+g(t)\bigr]\,\mathrm{d}s\biggr|\,\mathrm{d}t \\
&\le \int_0^{kT}\bigl|\gamma(t)\widetilde{h}(t)+g(t)\bigr|\bigl|\bigl|u(t)\bigr|-\bigl|v(t)\bigr|\bigr|\,\mathrm{d}t \\
&\le \biggl(\int_0^{kT}\bigl(\gamma(t)\widetilde{h}(t)+g(t)\bigr)^2\,\mathrm{d}t\biggr)^{\frac{1}{2}}\biggl(\int_0^{kT}\bigl(u(t)-v(t)\bigr)^2\,\mathrm{d}t\biggr)^{\frac{1}{2}} \\
&= K\|u-v\|_{L^2} \le K\cdot C\|u-v\|_{H^1_{kT}},
\end{aligned} \tag{2.4}$$

where $\widetilde{h}(t) = \max\{h(s) \mid s \text{ is between } |v(t)| \text{ and } |u(t)|\}$, $K := (\int_0^{kT}(\gamma(t)\widetilde{h}(t)+g(t))^2\,\mathrm{d}t)^{\frac{1}{2}} > 0$, $C$ is embedding constant, (2.4) means that $J_k$ is locally Lipschitz continuous, so [1] tells us that the generalized gradients of $F$ at $x+a$ do exist. The generalized gradients of $F$ at $x+a$ $(x>0)$ are denoted by $\partial F(t,x+a) = \partial_\nu F^0(t,x+a;\nu)|_{\nu=0}$, where $\partial_\nu F^0(t,x+a;\nu)$ denotes the subdifferential in $\nu$ of $F^0(t,x+a;\nu)$, and $F^0(t,x+a;\nu) = \limsup_{h\to 0,\mu\to 0^+}\frac{1}{\mu}\int_{x+h}^{x+h+\mu\nu} f(t,s+a)\,\mathrm{d}s$. Then we have

$$F^0(t,x+a;\nu) \le \begin{cases} \nu\lim_{\sigma\to 0^+}\min_{s\in[x-\sigma,x+\sigma]} f(t,s+a) = f^-(t,x+a)\nu, & \text{if } \nu \le 0, \\ \nu\lim_{\sigma\to 0^+}\max_{s\in[x-\sigma,x+\sigma]} f(t,s+a) = f^+(t,x+a)\nu, & \text{if } \nu \ge 0. \end{cases} \tag{2.5}$$

By (2.5) and results in [1], we have $\partial F(t,x+a) \subseteq [f^-(t,x+a), f^+(t,x+a)]$. By definition, $\exists h_i \in H^1_{kT}$, $h_i \to 0$ $(i \to +\infty)$ in $H^1_{kT}$ such that

$$J_k^0(x;v) = \limsup_{i\to+\infty,\mu\to 0^+} \frac{1}{\mu} \int_0^{kT} \mathrm{d}t \int_{h_i(t)}^{h_i(t)+\mu v(t)} f\bigl(t, s + |x(t)| + a\bigr)\,\mathrm{d}s,$$

so using the results of [1], we have

$$\begin{aligned} J_k^0(x;v) &\le \int_0^{kT} F^0\bigl(t, |x(t)| + a; v\bigr)\,\mathrm{d}t \\ &= \int_0^{kT} \max\bigl\{\omega \cdot v(t) \mid \omega \in \partial F\bigl(t, |x(t)| + a\bigr)\bigr\}\,\mathrm{d}t \\ &\le \int_{v(t)>0} v(t) f^+\bigl(t, |x(t)| + a\bigr)\,\mathrm{d}t \\ &\quad + \int_{v(t)<0} v(t) f^-\bigl(t, |x(t)| + a\bigr)\,\mathrm{d}t. \end{aligned} \tag{2.6}$$

If $\omega_0 \in \partial J_k(x)$, we are going to prove

$$f^-\bigl(t, |x(t)| + a\bigr) \le \omega_0(t) \le f^+\bigl(t, |x(t)| + a\bigr), \quad \forall t \in \mathbf{R}.$$

Otherwise, there would be a set $E_0$, for example, on which

$$f^-\bigl(t, |x(t)| + a\bigr) > \omega_0(t), \quad \forall t \in E_0. \tag{2.7}$$

Due to results in [1],

$$\partial\varphi(x_0) = \bigl\{w \in E^* \mid \langle w, v\rangle \le \varphi^0(x_0; v), \forall v \in E\bigr\}. \tag{2.8}$$

Let $v_0(t) = -\chi_{E_0}(t)$, the characteristic function of $E_0$. Then, from (2.8), we have

$$J_k^0(x;v_0) \ge \int_0^{kT} \omega_0(t) \cdot v_0(t)\,\mathrm{d}t = -\int_{E_0} \omega_0(t)\,\mathrm{d}t. \tag{2.9}$$

From the definition of $v_0(t)$, we have

$$\begin{aligned} &-\int_{E_0} f^-\bigl(t, |x(t)| + a\bigr)\,\mathrm{d}t \\ &\quad = \int_{v_0(t)>0} v_0(t) \cdot f^+\bigl(t, |x(t)| + a\bigr)\,\mathrm{d}t + \int_{v_0(t)<0} v_0(t) \cdot f^-\bigl(t, |x(t)| + a\bigr)\,\mathrm{d}t. \end{aligned} \tag{2.10}$$

Equations (2.10), (2.6) and (2.9) imply that

$$-\int_{E_0} f^-\bigl(t, |x(t)| + a\bigr)\,\mathrm{d}t \ge -\int_{E_0} \omega_0(t)\,\mathrm{d}t,$$

which contradicts (2.7). Similarly, we can get $\omega_0(t) \le f^+(t, |x(t)| + a)$. □

*Remark* 2.1 For $w \in \partial J_k(x)$, there is a function $\varpi$ with $\varpi(t) \in [f^-(t,|x(t)|+a), f^+(t,|x(t)|+a)]$ a.e. $t \in [0,kT]$ such that $\langle w, v\rangle = \int_0^{kT} \varpi(t)v(t)\,\mathrm{d}t$ holds for all $v \in H^1_{kT}$.

*Remark* 2.2 When $s \neq 0$, we have $f^-(t,|s|+a) = f^+(t,|s|+a) = f(t,|s|+a)\operatorname{sgn}(s)$. If $s = 0$ and $f$ satisfies condition (B), then we have $f^-(t,|s|+a) = f(t,|s|+a)$ and $f^+(t,|s|+a) = -f(t,|s|+a)$.

**Lemma 2.2** *If $x_k \in H^1_{kT}$ is a critical point of $\varphi_k$ on $H^1_{kT}$, then $x_k$ is a periodic solution of equation* (1.6), *and vice versa.*

*Proof* The main idea comes from [7]. Lemma 2.1 means that $\varphi_k$ is locally Lipschitz continuous, so $\partial\varphi_k$ do exist. Let $x$ be a critical point of $\varphi_k$ in $H^1_{kT}$, in other words, $\mathbf{0} \in \partial\varphi_k$. By Remark 2.1, there is a function $g$ with $g(t) \in [f^-(t,|x(t)|+a), f^+(t,|x(t)|+a)]$ such that

$$\int_0^{kT} \bigl(\dot{x}(t)\dot{v}(t) + g(t)v(t)\bigr)\,\mathrm{d}t = 0, \quad \forall v \in H^1_{kT},$$

which shows that the generalized second order derivative $\ddot{x}$ exists and $x$ satisfies

$$\int_0^{kT} \bigl(-\ddot{x}(t) + g(t)\bigr)v(t)\,\mathrm{d}t = 0, \quad \forall v \in H^1_{kT},$$

that is,

$$\ddot{x} = g(t) \quad \text{a.e. } t \in \mathbf{R}. \tag{2.11}$$

We need to prove that $g(t) = f(t,|x|+a)\operatorname{sgn}(x)$. Let $T_0 = \{t \in [0,kT] \mid x(t) = 0\}$ and $T_1 = \{t \in [0,kT] \mid x(t) \neq 0\}$. Remark 2.2 tells us that $g(t) = f^-(t,|x|+a) = f^+(t,|x|+a) = f(t,|x|+a)\operatorname{sgn}(x)$ for $t \in T_1$. Using [6, Lemma 7.7], we obtain $\ddot{x} = \dot{x} = 0$ for a.e. $t \in T_0$, and draw a conclusion from (2.11) that $g(t) = 0 = f(t,|x|+a)\operatorname{sgn}(x)$ holds for a.e. $t \in T_0$. Thus we complete the proof. □

## 3 Bouncing solutions for second order Hamiltonian systems

**Lemma 3.1** *Functional $\varphi_k$ satisfies the nonsmooth* (*PS*) *condition iff $f$ satisfies* (f) *and* (F1).

*Proof* The main idea comes from [9]. Let $\{x_n\} \in H^1_{kT}$ be a nonsmooth (PS) sequence, that is, $\{\varphi(x_n)\}$ is bounded and $\lambda(x_n) \to 0$ as $n \to \infty$. According to Remark 2.1 and the definition of $\lambda(x_n)$, for each $n \in \mathbf{N}^*$, there exist functions $\varpi_n \in \partial J_k(x_n)$ and $x_n^* \in \partial\varphi_k(x_n)$ with $\|x_n^*\|_{E^*} \to 0$ as $n \to \infty$ such that

$$\langle x_n^*, v\rangle = \int_0^{kT} \dot{x}_n(t)\dot{v}(t)\,\mathrm{d}t + \int_0^{kT} \varpi_n(t)v(t)\,\mathrm{d}t, \quad \forall v \in H^1_{kT}. \tag{3.1}$$

For $x_n \in H^1_{kT} = \mathbf{R} \oplus \tilde{H}^1_{kT}$, we write $x_n$ as $x_n(t) = \bar{x}_n + \tilde{x}_n(t)$ for all $t \in [0,kT]$, where $\bar{x}_n \in \mathbf{R}$, $\tilde{x}_n(t) \in \tilde{H}^1_{kT}$. It follows from (2.3), (1.3), (h1), and (h2) that

$$\begin{aligned} |\varpi_n(t)| &\le \bigl|f\bigl(t,|x_n(t)|+a\bigr)\bigr| \\ &\le \gamma(t)h\bigl(|x_n(t)|\bigr) + g(t) \end{aligned}$$

$$
\begin{aligned}
&\leq \gamma(t)\big[h\big(|\bar{x}_n| + |\tilde{x}_n(t)|\big) + C\big] + g(t) \\
&\leq C^*\gamma(t)h\big(|\bar{x}_n|\big) + C^*\gamma(t)h\big(|\tilde{x}_n(t)|\big) + C\gamma(t) + g(t).
\end{aligned} \tag{3.2}
$$

Using (3.2), (h1), Young inequality, [9, (1) of Lemma 2.3] and (2.2), we have

$$
\begin{aligned}
&\left|\int_0^{kT} \varpi_n(t)\tilde{x}_n(t)\,\mathrm{d}t\right| \\
&\quad\leq \int_0^{kT} \big|\varpi_n(t)\big|\big|\tilde{x}_n(t)\big|\,\mathrm{d}t \\
&\quad\leq \int_0^{kT} \big[C^*\gamma(t)h\big(|\bar{x}_n|\big) + C^*\gamma(t)h\big(|\tilde{x}_n(t)|\big) + C\gamma(t) + g(t)\big]\big|\tilde{x}_n(t)\big|\,\mathrm{d}t \\
&\quad\leq C^*\|\tilde{x}_n\|_{L^\infty} h\big(|\bar{x}_n|\big)\int_0^{kT} \gamma(t)\,\mathrm{d}t + C^*\big[h\big(\|\tilde{x}_n\|_{L^\infty}\big) + C\big]\|\tilde{x}_n\|_{L^\infty}\int_0^{kT} \gamma(t)\,\mathrm{d}t \\
&\qquad + C\|\tilde{x}_n\|_{L^\infty}\int_0^{kT} \gamma(t)\,\mathrm{d}t + \|\tilde{x}_n\|_{L^\infty}\int_0^{kT} g(t)\,\mathrm{d}t \\
&\quad\leq C^*\left[\frac{3}{C^*kT}\|\tilde{x}_n\|_{L^\infty}^2 + \frac{C^*kT}{3}h^2\big(|\bar{x}_n|\big)\left(\int_0^{kT} \gamma(t)\,\mathrm{d}t\right)^2\right] \\
&\qquad + C^*\big(\varepsilon\|\tilde{x}_n\|_{L^\infty} + C_\varepsilon + C\big)\|\tilde{x}_n\|_{L^\infty}\int_0^{kT} \gamma(t)\,\mathrm{d}t \\
&\qquad + C\|\tilde{x}_n\|_{L^\infty}\int_0^{kT} \gamma(t)\,\mathrm{d}t + \|\tilde{x}_n\|_{L^\infty}\int_0^{kT} g(t)\,\mathrm{d}t \\
&\quad\leq \frac{1}{4}\int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t + \frac{{C^*}^2kT}{3}h^2\big(|\bar{x}_n|\big)\left(\int_0^{kT} \gamma(t)\,\mathrm{d}t\right)^2 \\
&\qquad + \frac{\varepsilon C^*kT}{12}\int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t\int_0^{kT} \gamma(t)\,\mathrm{d}t \\
&\qquad + \big(C^*C_\varepsilon + C^*C + C\big)\left(\frac{kT}{12}\int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\int_0^{kT} \gamma(t)\,\mathrm{d}t \\
&\qquad + \left(\frac{kT}{12}\int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\int_0^{kT} g(t)\,\mathrm{d}t \\
&\quad\leq \left(\frac{1}{4} + \varepsilon C_1\right)\int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t + C_{2,\varepsilon}\left(\int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t\right)^{\frac{1}{2}} + C_3h^2\big(|\bar{x}_n|\big),
\end{aligned} \tag{3.3}
$$

where $C_1 = \frac{C^*kT}{12}\int_0^{kT}\gamma(t)\,\mathrm{d}t > 0$, $C_{2,\varepsilon} = \sqrt{\frac{kT}{12}}(C^*C_\varepsilon + C^*C + C)\int_0^{kT}\gamma(t)\,\mathrm{d}t + \sqrt{\frac{kT}{12}}\int_0^{kT} g(t)\,\mathrm{d}t > 0$ and $C_3 = \frac{{C^*}^2kT}{3}(\int_0^{kT}\gamma(t)\,\mathrm{d}t)^2 > 0$. By (3.1) and (3.3), for $n$ large enough, we have

$$
\begin{aligned}
\|\tilde{x}_n\| &\geq \langle x_n^*, \tilde{x}_n\rangle = \int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t + \int_0^{kT} \varpi_n(t)\tilde{x}_n(t)\,\mathrm{d}t \\
&\geq \left(\frac{3}{4} - \varepsilon C_1\right)\int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t - C_{2,\varepsilon}\left(\int_0^{kT} \big|\dot{x}_n(t)\big|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad - C_3h^2\big(|\bar{x}_n|\big).
\end{aligned} \tag{3.4}
$$

On the other hand, (2.1) implies that

$$\|\tilde{x}_n\| \le \left(1 + \frac{k^2T^2}{4\pi^2}\right)^{\frac{1}{2}} \left(\int_0^{kT} \left|\dot{x}_n(t)\right|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \quad \forall n \in \mathbf{N}^*. \tag{3.5}$$

Let $\varepsilon > 0$ be small enough such that $\frac{3}{4} - \varepsilon C_1 > 0$. Using the property of the parabola, then (3.4) and (3.5) mean that there exist two constants $C_4, C_5 > 0$ such that

$$\left(\int_0^{kT} \left|\dot{x}_n(t)\right|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \le C_4 h\big(|\bar{x}_n|\big) + C_5 \quad \text{for } n \text{ large enough.} \tag{3.6}$$

By the mean value theorem for a locally Lipschitz functional (see [5]), there exists $z_n \in \{(1-s)x_n + s\bar{x}_n \mid 0 \le s \le 1\}$ and $z_n^* \in \partial J_k(z_n)$ such that

$$\int_0^{kT} F\big(t, \left|x_n(t)\right| + a\big) \,\mathrm{d}t - \int_0^{kT} F\big(t, |\bar{x}_n| + a\big) \,\mathrm{d}t = \langle z_n^*, x_n - \bar{x}_n \rangle. \tag{3.7}$$

By Lemma 2.1, there exists a function $\varpi_{z_n}(t) \in [f^-(t, |z_n(t)| + a), f^+(t, |z_n(t)| + a)]$ such that

$$\langle z_n^*, x_n - \bar{x}_n \rangle = \int_0^{kT} \big(\varpi_{z_n}(t), x_n(t) - \bar{x}_n\big) \,\mathrm{d}t. \tag{3.8}$$

In the same way as in the computation of (3.3) with $\varepsilon = 1$, together with (3.7) and (3.8), we have

$$\begin{aligned}
&\int_0^{kT} F\big(t, \left|x_n(t)\right| + a\big) \,\mathrm{d}t - \int_0^{kT} F\big(t, |\bar{x}_n| + a\big) \,\mathrm{d}t \\
&\quad = \int_0^{kT} \big(\varpi_{z_n}(t), x_n(t) - \bar{x}_n\big) \,\mathrm{d}t \\
&\quad \le \int_0^{kT} \left|f\big(t, \left|z_n(t)\right| + a\big)\right| \left|x_n(t) - \bar{x}_n\right| \,\mathrm{d}t \\
&\quad = \int_0^{kT} \left|f\big(t, \left|\bar{x}_n + (1-s)\tilde{x}_n(t)\right| + a\big)\right| \left|\tilde{x}_n(t)\right| \,\mathrm{d}t \\
&\quad \le \int_0^{kT} \left[C^*\gamma(t) h\big(|\bar{x}_n|\big) + C^*\gamma(t) h\big(\left|\tilde{x}_n(t)\right|\big) + C\gamma(t) + g(t)\right] \left|\tilde{x}_n(t)\right| \,\mathrm{d}t,
\end{aligned}$$

which is similar to (3.3), and we have

$$\begin{aligned}
&\int_0^{kT} F\big(t, \left|x_n(t)\right| + a\big) \,\mathrm{d}t - \int_0^{kT} F\big(t, |\bar{x}_n| + a\big) \,\mathrm{d}t \\
&\quad \le \left(\frac{1}{4} + C_1\right) \int_0^{kT} \left|\dot{x}_n(t)\right|^2 \,\mathrm{d}t + C_2 \left(\int_0^{kT} \left|\dot{x}_n(t)\right|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} + C_3 h^2\big(|\bar{x}_n|\big).
\end{aligned} \tag{3.9}$$

By (3.9), (3.6), [9, Lemma 2.3] and (F1), one has

$$\begin{aligned}
\varphi_k(x_n) \;=\; & \frac{1}{2} \int_0^{kT} \left|\dot{x}_n(t)\right|^2 \,\mathrm{d}t + \int_0^{kT} \left[F\big(t, \left|x_n(t)\right| + a\big) - F\big(t, |\bar{x}_n| + a\big)\right] \mathrm{d}t \\
& + \int_0^{kT} F\big(t, |\bar{x}_n| + a\big) \,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{3}{4}+C_1\right)\int_0^{kT}\left|\dot{x}_n(t)\right|^2\,\mathrm{d}t + C_2\left(\int_0^{kT}\left|\dot{x}_n(t)\right|^2\,\mathrm{d}t\right)^{\frac{1}{2}} + C_3h^2\left(|\bar{x}_n|\right)\\
&\quad + \int_0^{kT} F\left(t,|\bar{x}_n|+a\right)\mathrm{d}t\\
&\le \left(\frac{3}{4}+C_1\right)\left[C_4h\left(|\bar{x}_n|\right)+C_5\right]^2 + C_2\left[C_4h\left(|\bar{x}_n|\right)+C_5\right] + C_3h^2\left(|\bar{x}_n|\right)\\
&\quad + \int_0^{kT} F\left(t,|\bar{x}_n|+a\right)\mathrm{d}t\\
&\le C_6h^2\left(|\bar{x}_n|\right) + C_7h\left(|\bar{x}_n|\right) + \int_0^{kT} F\left(t,|\bar{x}_n|+a\right)\mathrm{d}t + C_8\\
&= H\left(|\bar{x}_n|\right)\left[C_6\frac{h^2(|\bar{x}_n|)}{H(|\bar{x}_n|)} + C_7\frac{h(|\bar{x}_n|)}{H(|\bar{x}_n|)} + \frac{\int_0^{kT}F(t,|\bar{x}_n|+a)\,\mathrm{d}t}{H(|\bar{x}_n|)}\right] + C_8\\
&\to -\infty, \quad \text{as } |\bar{x}_n| \to +\infty
\end{aligned}
$$

(where constants $C_6 > 0$, $C_7 > 0$, $C_8 > 0$) which contradicts the boundedness of $\{\varphi_k(x_n)\}$, thus $\{\bar{x}_n\}$ is bounded and, together with (3.6), one has that $\{\|x_n\|_0\}$ is bounded. Then, by the equivalence of the two norms, $\{\|x_n\|\}$ is bounded.

Next, we verify that $\{x_n\}$ has a strongly convergent subsequence. The main idea comes from [9] and [10].

Suppose $x_n \rightharpoonup x$ in $H^1_{kT}$, then $x_n \to x$ in $C([0,kT];\mathbf{R})$. The results in [1] imply that $\partial\varphi_k(x_n)$ is weak*-compact, and the set-valued mapping $x \to \partial\varphi_k(x)$ is upper semicontinuous, so, according to [9], we get $x_n \to x$ in $H^1_{kT}$, and hence $\varphi_k$ satisfies the nonsmooth (PS) condition. □

**Lemma 3.2** *For every $k \in \mathbf{N}^*$, functional $\varphi_k(x) \to +\infty$ as $\|x\| \to +\infty$ in $\tilde{H}^1_{kT}$, iff satisfies condition* (f).

*Proof* The main idea comes from [9].

For every $x \in \tilde{H}^1_{kT}$, by (f), (h1), of [9, (1) of Lemma 2.3] and (2.2), we have

$$
\begin{aligned}
\left|\int_0^{kT} F\left(t,|x(t)|+a\right)\mathrm{d}t\right| &\le \int_0^{kT}\mathrm{d}t\int_0^{|x(t)|}\left|f(t,s+a)\right|\mathrm{d}s\\
&\le \int_0^{kT}\mathrm{d}t\int_0^{|x(t)|}\left[\gamma(t)h(s)+g(t)\right]\mathrm{d}s\\
&\le \int_0^{kT}\left[\gamma(t)h\left(\|x\|_{L^\infty}\right)+C\gamma(t)+g(t)\right]\left(\left|x(t)\right|\right)\mathrm{d}t\\
&\le \varepsilon C_9\|x\|^2_{L^\infty} + C_{10,\varepsilon}\|x\|_{L^\infty}\\
&\le \varepsilon C_{11}\int_0^{kT}\left|\dot{x}(t)\right|^2\mathrm{d}t + C_{12,\varepsilon}\left(\int_0^{kT}\left|\dot{x}(t)\right|^2\mathrm{d}t\right)^{\frac{1}{2}},
\end{aligned}
\tag{3.10}
$$

where $C_{10,\varepsilon}, C_{12,\varepsilon}, C_{13,\varepsilon} > 0$ hold for any $\varepsilon > 0$, and $C_9, C_{11} > 0$. Then (3.10) implies

$$
\varphi_k(x) \ge \left(\frac{1}{2}-\varepsilon C_{11}\right)\int_0^{kT}\left|\dot{x}(t)\right|^2\mathrm{d}t - C_{12,\varepsilon}\left(\int_0^{kT}\left|\dot{x}(t)\right|^2\mathrm{d}t\right)^{\frac{1}{2}} \quad \forall x \in \tilde{H}^1_{kT}. \tag{3.11}
$$

Choose $\varepsilon > 0$ small enough such that $\frac{1}{2} - \varepsilon C_{11} > 0$. In $\tilde{H}^1_{kT}$, $\|x\| \to +\infty$ if and only if $(\int_0^{kT} |\dot{x}(t)|^2 \,\mathrm{d}t)^{\frac{1}{2}} \to +\infty$, and then (3.11) implies that $\varphi_k(x) \to +\infty$ as $\|x\| \to +\infty$. □

**Lemma 3.3** *For every $k \in \mathbf{N}^*$, we have $\varphi_k(x + e_k) \to -\infty$ as $|x| \to +\infty$ in $\mathbf{R} \subseteq H^1_{kT}$, where $e_k(t) = k\cos(\frac{2\pi t}{kT}) \in \tilde{H}^1_{kT}$, iff satisfies conditions* (f) *and* (F1).

*Proof* The main idea comes from [9]. Using (3.9), (F1) and [9, Lemma 2.3], we have

$$\begin{aligned}
\varphi_k(x+e_k) &= \frac{2k\pi^2}{T} + \int_0^{kT} \bigl[F\bigl(t, \bigl|x+e_k(t)\bigr| + a\bigr) - F\bigl(t, |x| + a\bigr)\bigr]\,\mathrm{d}t \\
&\quad + \int_0^{kT} F\bigl(t, |x| + a\bigr)\,\mathrm{d}t \\
&\leq \frac{2k\pi^2}{T} + \left(\frac{1}{4} + C_1\right)\int_0^{kT} \bigl|\dot{e}_k(t)\bigr|^2\,\mathrm{d}t + C_2\left(\int_0^{kT} \bigl|\dot{e}_k(t)\bigr|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad + C_3 h^2\bigl(|x|\bigr) + \int_0^{kT} F\bigl(t, |x| + a\bigr)\,\mathrm{d}t \\
&= \frac{2k\pi^2}{T} + \left(\frac{1}{4} + C_1\right)\frac{2k\pi^2}{T} + C_2\pi\sqrt{\frac{2k}{T}} \\
&\quad + H\bigl(|x|\bigr)\left[C_3\frac{h^2(|x|)}{H(|x|)} + \frac{\int_0^{kT} F(t, |x| + a)\,\mathrm{d}t}{H(|x|)}\right] \\
&\to -\infty, \quad \text{as } |x| \to +\infty, x \in \mathbf{R}.
\end{aligned}$$

□

**Proposition 3.1** *If conditions* (f) *and* (F1) *hold, then there exists a constant $r_0 > 0$ large enough such that functional $\varphi_k$ has at least one critical value $c_k$ characterized by*

$$c_k = \inf_{\chi \in \Gamma_2} \max_{x \in [-r_0, r_0]} \varphi_k\bigl(\chi(x + e_k)\bigr),$$

*where $\Gamma_2 = \{\chi \in C([-r_0, r_0] + e_k, E) \mid \chi(e_k \pm r_0) = e_k \pm r_0\}$. Furthermore, for $\forall k \in \mathbf{N}^*$, then*

$$-\infty < \inf_{\tilde{H}^1_{kT}} \varphi_k \leq c_k \leq \sup_{x \in \mathbf{R}} \varphi_k(x + e_k). \tag{3.12}$$

*Proof* Set $V = \mathbf{R}$ and $X = \tilde{H}^1_{kT}$, Lemmas 3.2 and 3.3 tell us that there exists a constant $r_0 > 0$ large enough such that

$$\max_{x \in V \cap \partial B_{r_0}} \varphi_k(x + e_k) < \inf_{x \in X} \varphi_k(x). \tag{3.13}$$

Due to Lemma 3.1 and inequality (3.13), Theorem 2.1 tells us that $c_k$ is a critical value of $\varphi_k$ and $c_k \geq \inf_{\tilde{H}^1_{kT}} \varphi_k$.

Inequality (3.11) tells us that $\inf_{\tilde{H}^1_{kT}} \varphi_k > -\infty$. Moreover, the definition of $c_k$ implies that $c_k \leq \sup_{x \in \mathbf{R}} \varphi_k(x + e_k)$. So, (3.12) holds. □

**Lemma 3.4** *Under condition* (B), *we have*

$$\lim_{|x| \to +\infty} F\bigl(t, |x| + a\bigr) = -\infty \quad \forall t \in [0, T]. \tag{3.14}$$

*Proof* The main idea comes from [9]. For $|x| > 0$, employing the mean value theorem for integrals, we have $F(t, |x| + a) = \int_0^{|x|} f(t, s + a)\,\mathrm{d}s = \int_0^{\frac{|x|}{2}} f(t, s + a)\,\mathrm{d}s + \int_{\frac{|x|}{2}}^{|x|} f(t, s + a)\,\mathrm{d}s \le \int_{\frac{|x|}{2}}^{|x|} f(t, s + a)\,\mathrm{d}s = f(t, \xi + a)\frac{|x|}{2}$, $\xi \in (\frac{|x|}{2}, |x|)$. Then

$$\limsup_{|x|\to+\infty} F\bigl(t, |x| + a\bigr) \le \limsup_{|x|\to+\infty} f(t, \xi + a)\frac{|x|}{2} = -\infty \quad \forall t \in [0, T],$$

which implies that (3.14) holds. □

**Lemma 3.5** *If* (3.14) *holds, then for any constant* $\delta > 0$ *there is a measurable subset* $A_\delta \subset [0, T]$ *with* $\operatorname{meas}([0, T] \setminus A_\delta) < \delta$ *such that*

$$F\bigl(t, |x| + a\bigr) \to -\infty, \quad \textit{uniformly in } t \in A_\delta, \textit{ as } |x| \to +\infty.$$

*Proof* Set $f_n(t) = \inf_{|x|>n} -F(t, |x| + a)$. Then (3.14) implies that $f_n(t) \to +\infty$, as $n \to \infty$, $\forall t \in [0, T]$. By the continuity of $F(t, x)$ in $x$ and the measurability of $F(t, x)$ in $t$, for any $\delta > 0$, by [12, Lemma 1], there exists a measurable subset $A_\delta \subset [0, T]$ with $\operatorname{meas}([0, T] \setminus A_\delta) < \delta$ such that $f_n(t) \to +\infty$ as $n \to \infty$ uniformly for every $t \in A_\delta$. Then we obtain that $F(t, |x| + a) \to -\infty$, uniformly in $t \in A_\delta$, as $|x| \to +\infty$. □

**Lemma 3.6** *For every* $k \in \mathbf{N}^*$, *let* $x_k$ *be a critical point of functional* $\varphi_k$, *then* $\|x_k\|_{L^\infty} \to +\infty$ *as* $k \to +\infty$, *if condition* (f) *holds.*

*Proof* From conditions (f), (h1) and [9, (1) of Lemma 2.3], we have

$$\begin{aligned}
\frac{c_k}{k} &= \frac{\varphi_k(x_k)}{k} \\
&\ge \frac{1}{k}\int_0^{kT} F\bigl(t, \bigl|x_k(t)\bigr| + a\bigr)\,\mathrm{d}t \\
&= \frac{1}{k}\int_0^{kT} \mathrm{d}t \int_0^{|x_k(t)|} f(t, s + a)\,\mathrm{d}s \\
&\ge -\frac{1}{k}\int_0^{kT} \mathrm{d}t \int_0^{|x_k(t)|} \bigl[\gamma(t)h(s) + g(t)\bigr]\,\mathrm{d}s \\
&\ge -\frac{1}{k}\int_0^{kT} \bigl[\gamma(t)h\bigl(\|x_k\|_{L^\infty}\bigr) + C\gamma(t) + g(t)\bigr]\bigl(\|x_k\|_{L^\infty}\bigr)\,\mathrm{d}t \\
&\ge -\bigl[\varepsilon\|\gamma\|_{L^1}\|x_k\|_{L^\infty}^2 + (C_\varepsilon + C)\|\gamma\|_{L^1}\|x_k\|_{L^\infty} + \|g\|_{L^1}\|x_k\|_{L^\infty}\bigr] \\
&\ge -\bigl[\varepsilon C_{13}^2\|\gamma\|_{L^1} + C_{13}(C_\varepsilon + C)\|\gamma\|_{L^1} + C_{13}\|g\|_{L^1}\bigr] := L.
\end{aligned} \tag{3.15}$$

By (3.12), (3.15) and Lemma 3.5, the results in [9] and [12] imply that $\|x_k\|_{L^\infty} \to +\infty$ as $k \to +\infty$. □

**Proposition 3.2** *Suppose* $f$ *satisfies conditions* (B), (F2) *and* $x_k$ *is a* $kT$*-periodic solution of* (1.6), *where* $k \in \mathbf{N}^*$ *is large enough, then* $W_k^1 = \{t \in \mathbf{R} \mid x_k(t) = 0\}$ *is nonempty and its points are isolated. Moreover, there exists at least one* $t_0 \in W_k^1$ *such that* $\dot{x}_k(t_0^-) \neq 0$.

*Proof* The idea comes from [7, 9] and [11].

For every $k \in \mathbf{N}^*$, let $G_k(t) = \frac{1}{2}|\dot{x}_k(t)|^2 - F(t, |x_k(t)| + a)$, then $G_k(t)$ is well-defined for all $t \in \mathbf{R}$ via (1.5), $G_k(t) \geq 0$ holds for all $t \in \mathbf{R}$ via the nonpositivity of $F$ and $G(t)$ is $kT$-periodic, continuous, differentiable for $t \in \mathbf{R} \setminus W_k^1$. By the definition of $G_k$, we get

$$\begin{aligned} G_k'(t) &= \dot{x}_k(t)\bigl[\ddot{x}_k(t) - f\bigl(t, \bigl|x_k(t)\bigr| + a\bigr)\operatorname{sgn}\bigl(x_k(t)\bigr)\bigr] - \frac{\partial F(t, |x_k(t)| + a)}{\partial t} \\ &= -\frac{\partial F(t, |x_k(t)| + a)}{\partial t}, \quad t \in \mathbf{R} \setminus W_k^1, \end{aligned}$$

then by (F2) we have

$$\bigl|G_k'(t)\bigr| \leq -\sigma F\bigl(t, \bigl|x_k(t)\bigr| + a\bigr) \leq \sigma G_k(t), \quad t \in \mathbf{R} \setminus W_k^1. \tag{3.16}$$

If $G_k(0) = 0$, then by Gronwall's inequality in [4], we have $G_k(t) \equiv 0$ on $\mathbf{R}$. Then $\dot{x}_k(t) \equiv 0$ and $F(t, |x_k(t)| + a) \equiv 0$ on $\mathbf{R}$, which means that $x_k(t) \equiv a_k \in \mathbf{R}$ and $f(t, |a_k| + a) = \ddot{x}_k(t) \equiv 0$, and so we get $\lim_{k\to+\infty} f(t, |a_k| + a) = 0$. Lemma 3.6 implies that $\lim_{k\to+\infty} |a_k| = +\infty$, so we have $\lim_{|a_k|\to+\infty} f(t, |a_k| + a) = 0$, which contradicts condition (B). Therefore we have $G_k(0) > 0$.

Using (3.16), we get

$$\bigl[G_k(t)\mathrm{e}^{\sigma t}\bigr]' = \bigl(G_k'(t) + \sigma G_k(t)\bigr)\mathrm{e}^{\sigma t} \geq 0, \quad t \in \mathbf{R} \setminus W_k^1. \tag{3.17}$$

Note that $G_k(t)\mathrm{e}^{\sigma t}$ is continuous and (3.17) implies that $G_k(t)\mathrm{e}^{\sigma t} \geq G_k(0)$ holds for all $t \in [0, kT]$, that is,

$$G_k(t) \geq G_k(0)\mathrm{e}^{-\sigma t} > 0, \quad t \in [0, kT]. \tag{3.18}$$

To prove that $W_k^1$ is nonempty and discrete, it is sufficient to prove that $\widehat{W}_k^1 = \{t \in [0, kT] \mid x_k(t) = 0\}$ is nonempty and finite.

First, we consider whether $\widehat{W}_k^1$ is nonempty. If not, we have $x_k \in C^2([0, kT], \mathbf{R})$, and, without loss of generality, we suppose that $x_k(t) > 0$ holds for all $t \in [0, kT]$. From (1.6), we get

$$\ddot{x}_k(t) = f\bigl(t, x_k(t) + a\bigr) \quad \forall t \in [0, kT],$$

and $x_k(t + kT) = x_k(t)$. By integrating the above equation on $[0, kT]$, we get

$$0 = \int_0^{kT} \ddot{x}_k(t)\,\mathrm{d}t = \int_0^{kT} f\bigl(t, x_k(t) + a\bigr)\,\mathrm{d}t,$$

which implies that $\ddot{x}_k(t) = f(t, x_k(t) + a) \equiv 0$ holds for any $t \in [0, kT]$ due to the nonpositivity of $f$. Since $x_k(t)$ is $kT$-periodic and continuous on $\mathbf{R}$, we have $x_k(t) \equiv b_k > 0$ for all $t \in [0, kT]$. Lemma 3.6 implies that $\lim_{k\to+\infty} b_k = +\infty$. Then $\lim_{b_k\to+\infty} f(t, b_k + a) = 0$, which contradicts condition (B). So $\widehat{W}_k^1$ is nonempty.

Next, we prove that the set $\widehat{W}_k^1$ is finite. Otherwise, since $\widehat{W}_k^1$ is compact, there would be a sequence $\{t_j\}_1^\infty \subset \widehat{W}_k^1$ with $0 \leq t_1 < t_2 < \cdots < t_j < \cdots \leq kT$, which, passing to a subsequence, still denoted by $\{t_j\}$, if necessary, we can take to be such that $t_j \to \beta \in \widehat{W}_k^1$ as

$j \to +\infty$. Then

$$\dot{x}_k(\beta^-) = \lim_{j\to+\infty} \frac{x_k(t_j) - x_k(\beta)}{t_j - \beta} = 0. \tag{3.19}$$

On the other hand, since $\widehat{W}_k^1$ is nonempty, by (3.18), one has that $\dot{x}_k(t^-) \neq 0$ holds for all $t \in \widehat{W}_k^1$, which contradicts (3.19), so $\widehat{W}_k^1$ is finite. □

*Proof of Theorem* 1.1 Proposition 3.1 tells us that $\varphi_k$ has a critical point $x_k$, and Lemma 2.2 implies that $x_k$ is a solution of (1.6) for every $k \in \mathbf{N}^*$. Proposition 3.2 implies that $u_k := |x_k|$ satisfies Definition 1.1 for $k \in \mathbf{N}^*$ large enough, so $u_k$ is a nontrivial $kT$-periodic bouncing solution of system (1.4). Furthermore, Lemma 3.6 implies that $\|u_k\|_{L^\infty} \to +\infty$ as $k \to +\infty$. Thus we complete the proof. □

## 4 Example

In this section, we present an example to demonstrate our Theorem 1.1.

*Example* 4.1 We define a $T$-periodic function $\theta \in C(\mathbf{R}, (-\infty, 0))$ with

$$\theta(t) = \begin{cases} -1, & t \in [0, \frac{T}{2}], \\ \sin \frac{2\pi t}{T} - 1, & t \in (\frac{T}{2}, T]. \end{cases}$$

Function $f : \mathbf{R} \times [a, +\infty) \to \mathbf{R}$ is defined as

$$f(t,x) = \theta(t) \frac{2(x-a)\ln(100 + (x-a)^2) - \frac{2(x-a)^3}{100+(x-a)^2}}{\ln^2(100 + (x-a)^2)} - 1.$$

Then $f(t,x)$ is $T$-periodic in $t$, continuous for any $t$ and $x$, differentiable at every $t \in \mathbf{R}$ except at $t = \frac{mT}{2}$ ($m \in \mathbf{N}^*$) and

$$F(t,x) := \int_a^x f(t,s)\,ds = \theta(t) \frac{(x-a)^2}{\ln(100 + (x-a)^2)} - (x-a).$$

The definition of function $f$ implies

$$\begin{aligned} \bigl|f\bigl(t, |x| + a\bigr)\bigr| &\le \bigl|\theta(t)\bigr| \left[ \frac{2|x|}{\ln(100 + x^2)} + \frac{2|x|}{\ln^2(100 + x^2)} \right] + 1 \\ &\le 4\bigl|\theta(t)\bigr| \frac{|x|}{\ln(100 + x^2)} + 1, \quad \forall x \in \mathbf{R} \text{ and } t \in [0, T]. \end{aligned} \tag{4.1}$$

Let $h(t) = \frac{t}{\ln(100+t^2)}$, $\gamma(t) = 4|\theta(t)|$ and $g(t) \equiv 1$. Obviously, $h$ satisfies conditions (h1), (h2) and (h4). The results of [9] imply that $h$ satisfies condition (h3). So (4.1) implies that assumption (f) holds.

Note that $f(t, x + a)$ is the same as $f(t,x)$ in [9]. Similarly to paper [9], assumptions (F1), (F2) and (B) hold.

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**Authors' contributions**
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**References**
1. Chang, K.C.: Variational methods for non-differentiable functionals and their applications to partial differential equations. J. Math. Anal. Appl. **80**, 102–129 (1981). https://doi.org/10.1016/0022-247X(81)90095-0
2. Clarke, F.H.: A new approach to Lagrange multipliers. Math. Oper. Res. **1**, 165–174 (1976). https://doi.org/10.1287/moor.1.2.165
3. Ding, W.: Subharmonic solutions of sublinear second order systems with impacts. J. Math. Anal. Appl. **379**, 538–548 (2011). https://doi.org/10.1016/j.jmaa.2011.01.036
4. Evans, L.C.: Partial Differential Equations. Grad. Stud. Math., vol. 19, pp. 624–625. Am. Math. Soc., Rhode Island (1998)
5. Gasinski, L., Papageorgiou, N.S.: Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems. Ser. Math. Anal. Appl., pp. 181–183. Chapman & Hall/CRC, Boca Raton (2005)
6. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
7. Jiang, M.Y.: Periodic solutions of second order differential equations with an obstacle. Nonlinearity **19**, 1165–1183 (2006) http://iopscience.iop.org/0951-7715/19/5/007
8. Mawhin, J., Willem, M.: Critical Point Theory and Hamiltonian Systems. Appl. Math. Sci., pp. 8–10. Springer, New York (1989)
9. Nie, Q.Q., Guo, F., Wang, M.W.: Generalized nonsmooth saddle point theorem and its applications on second order Hamiltonian systems. Chaos Solitons Fractals **104**, 741–747 (2017). https://doi.org/10.1016/j.chaos.2017.09.032
10. Ning, Y., An, T.Q.: Periodic solutions of a class of nonautonomous second-order Hamiltonian systems with nonsmooth potentials. Bound. Value Probl. **2015**, 34 (2015). http://link.springer.com/article/10.1186/s13661-015-0292-x
11. Qian, D.B., Torres, P.J.: Periodic motions of linear impact oscillators via the successor map. SIAM J. Math. Anal. **36**, 1707–1725 (2005). https://doi.org/10.1137/S003614100343771X
12. Tang, C.L., Wu, X.P.: Periodic solutions for second order systems with not uniformly coercive potential. J. Math. Anal. Appl. **259**, 386–397 (2001). https://doi.org/10.1006/jmaa.2000.7401