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# Schrödinger-type identity to the existence and uniqueness of a solution to the stationary Schrödinger equation

Delin Sun<sup>1\*</sup>

\*Correspondence:  
[dlsun12@foxmail.com](mailto:dlsun12@foxmail.com)

<sup>1</sup>Ulsan Ship and Ocean College,  
Ludong University, Yantai, China

## Abstract

In this article we study applications of the Schrödinger-type identity for obtaining transmutations via the fixed point index for nonlinear integral equations. It is possible to derive a wide range of transmutation operators by this method. Classical Riesz transforms are involved in the Schrödinger-type identity method as basic blocks, among them are Fourier, sine and cosine-Fourier, Hankel, Mellin, Laplace, and some generalized transforms. In this paper, we present a modified Schrödinger-type identity for solutions of a class of linear Schrödinger equations with mixed boundary conditions. The techniques used in our proofs are quite different, and most remarkably some of the proofs become simpler and more straightforward. As an application, we obtain the existence and uniqueness of a solution to the stationary Schrödinger equation in the sense of the Weyl law, which advances the recent results obtained in several articles even in a more general setting.

**Keywords:** Schrödinger-type identity; Neumann boundary condition; Stationary Schrödinger equation

## 1 Introduction

The stationary Schrödinger equations are often used to express real-life problems. Naturally, we focus on finding solutions of linear Schrödinger equations. However, analytical solutions are frequently not possible to find and numerical solutions can be both theoretically and computationally complicated due to the complexity of a modified Schrödinger operator. Frequently, researchers investigate inequalities in weighted Banach spaces; see, for example, [1–6] and the references therein for details. Hence, it is appropriate to derive a generalized stationary Schrödinger inequality equation from modified Schrödinger differential equations (see [4, 5, 7]).

In recent years, under the frame of the auxiliary principle, some references, such as Ding [8], Noor [7], Chen and Zhang [9], Huang [10], and so on, introduced the related properties of weak solutions for generalized stationary Schrödinger equations of fluid dynamics and built the corresponding convergence theorems.

Due to the rapid advancement of computing resource, rigorous derivation of generalized stationary Schrödinger equations was done also for Schrödinger fluids. An asymptotic analysis of the generalized Schrödinger differential equation based on the asymptotic

expansion was presented in [9, 10]. In this paper we investigate a new equation of generalized stationary Schrödinger inequalities and prove an existence and uniqueness theorem of solutions for this kind of equation.

The rest of the paper is organized as follows. In Sect. 2, we present some basic definitions, concepts, and some results that will be used later for our model problem. In Sect. 3, we prove the existence and uniqueness of weak solutions for generalized stationary Schrödinger equations. Finally, the paper is concluded in Sect. 4.

## 2 Preliminaries

In 2013, Ren [11] proposed the following linear Schrödinger model:

$$\begin{aligned} \frac{dS}{dt} &= r(S + I)\left(1 - \frac{S + I}{K}\right) - \beta SI - \eta\gamma_1(S)Y, \\ \frac{dI}{dt} &= \beta SI - \gamma(I)Y - cI, \\ \frac{dY}{dt} &= (\varepsilon\gamma(I) + \eta\varepsilon\gamma_1(S) - d)Y, \end{aligned} \tag{1}$$

where  $S$  is the number of sound prey,  $I$  is the number of infected prey population,  $Y$  is the number of predator population,  $\gamma(I)$  and  $\eta\gamma_1(S)$  are predator functional response functions. Ren analyzed model (1) in terms of positivity, uniqueness, boundedness and studied the existence of the Hopf bifurcation.

Model (1) may be re-written by Wang, Mai, and Wang (see [12]) in a simplified form as follows:

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1 - \frac{S + I}{K}\right) - \beta SI, \\ \frac{dI}{dt} &= -cI + \beta SI - pIY, \\ \frac{dY}{dt} &= -dY + pqIY. \end{aligned} \tag{2}$$

Let  $I = \{1, 2\}$  be an index set,  $H_i$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle_i$  and norm  $\| \cdot \|_i$ , respectively, where  $i = 1, 2$ . Let  $A : H_1 \rightarrow H_1, B : H_2 \rightarrow H_2, F_1 : H_1 \times H_2 \rightarrow H_1$ , and  $\eta_1 : H_1 \times H_1 \rightarrow H_1$  be mappings. Let  $a_i : H_i \times H_i \rightarrow \mathbb{R}$  be a coercive continuous map such that

- (C1)  $a_i(\sigma_i, \sigma_i) \geq c_i \|\sigma_i\|_i^2$ ;
- (C2)  $|a_i(\varrho_i, \sigma_i)| \leq d_i \|\varrho_i\|_i \cdot \|\sigma_i\|_i$

for any  $\varrho_i, \sigma_i \in H_i$ .

Let  $b_i : H_i \times H_i \rightarrow \mathbb{R}$  be a map with nondifferentiable terms such that

- (C3)  $b_i$  is a linear function for the first variable;
- (C4)  $b_i$  is a convex function;
- (C5) There exists a positive constant  $\gamma_i$  satisfying

$$\gamma_i \|\varrho_i\|_i \cdot \|\sigma_i\|_i \geq a_i(\varrho_i, \sigma_i)$$

for any  $\varrho_i, \sigma_i \in H_i$ .

$$(C6) \quad b_i(\varrho_i, \sigma_i - w_i) \geq b_i(\varrho_i, \sigma_i) - b_i(\varrho_i, w_i)$$

for any  $\varrho_i, \sigma_i, w_i \in H_i$ .

Based on the above notations, we then define the proposed equation of generalized non-linear stationary Schrödinger inequality problems as follows (see [13, 14]):

$$\begin{aligned} \frac{\partial F_1(x, y)}{\partial x} &= F_1(x, y)[b_1(x, \sigma_1) - \eta_1(x, y)(F_1(x, y) + F_2(x, y)) - a_1(x, y)f_1(x, \sigma_1 - x)], \\ \frac{\partial F_2(x, y)}{\partial x} &= F_2(x, y)[b_1(x, \sigma_1) - \eta_2(x, y)(F_1(x, y) + F_2(x, y)) - a_2(x, y)f_2(x, \sigma_2 - x)], \\ \frac{\partial f_i(x, y)}{\partial x} &= -d(x, y)x_3(x, y) - b(x, y)x_3^2(x, y) + c_1(x, y)x_3(t - \tau)F_1(t - \tau) \\ &\quad + c_2(x, y)x_3(t - \tau)F_2(t - \tau) \quad (i = 1, 2), \end{aligned} \tag{3}$$

where  $F_1, F_2$ , and  $f_i$  are susceptible, infected, and predator population respectively, and the corresponding parameters have the meaning as defined in [15]. Time delay is considered as a gestation period, and a disease can be transmitted by contact and spreads among prey species only.

*Remark 1* There are some special cases for the model problem (3):

- (1) If  $A = B = I, f_i = 0$ , and  $a_i(\varrho_i, \sigma_i) = 0$ , then (3) is equivalent to

$$\langle F_1(x, y), \eta_1(\sigma_1, x) \rangle_1 + b_1(x, \sigma_1) - b_1(x, x) \geq 0$$

for any  $\sigma_1 \in H_1$  and

$$\langle F_2(x, y), \eta_2(\sigma_2, y) \rangle_2 + b_2(y, \sigma_2) - b_2(y, y) \geq 0$$

for any  $\sigma_2 \in H_2$ .

- (2) If  $H_1 = H_2 = H, f_i = f_2 = f, \eta_1 = \eta_2 = \eta, a_1 = a_2 = a, b_1 = b_2 = b$ , then (3) is reduced to

$$\langle F(Ax, Bx) - f, \eta(w, y) \rangle + a(y, w - y) + b(y, w) - b(y, y) \geq 0$$

for any  $v \in H$ .

We then introduce the following definition (see [16, 17]), which will be useful for the proposed method.

**Definition 1** Let  $A : H_1 \rightarrow H_1, B : H_2 \rightarrow H_2, F_1 : H_1 \times H_2 \rightarrow H_1$ , and  $\eta_1 : H_1 \times H_1 \rightarrow H_1$  be mappings. Then

- (1)  $F_1$  is  $\alpha_1$ -strongly monotone if the following inequality holds:

$$\langle F_1(A\varrho_1, \varrho_2) - F_1(A\sigma_1, \varrho_2), \varrho_1 - \sigma_1 \rangle_1 \geq \alpha_1 \|\varrho_1 - \sigma_1\|_1^2,$$

where  $\varrho_1, \sigma_1 \in H_1, \varrho_2 \in H_2$  and  $\alpha_1$  is a positive constant;

(2) If there exist two positive constants  $\beta_1$  and  $\xi_1$  such that

$$\|F_1(A\varrho_1, B\varrho_2) - F_1(A\sigma_1, B\sigma_2)\|_1 \leq \beta_1 \|\varrho_1 - \sigma_1\|_1 + \xi_1 \|\varrho_2 - \sigma_2\|_2$$

for  $\varrho_1, \sigma_1 \in H_1$  and  $\varrho_2, \sigma_2 \in H_2$ , then  $F_1$  is  $(\beta_1, \xi_1)$ -Lipschitz continuous;

(3) If

$$\|\eta_1(\varrho_1, \sigma_1)\|_1 \leq \delta_1 \|\varrho_1 - \sigma_1\|_1$$

for  $\varrho_1, \sigma_1 \in H_1$ , then  $\eta_1$  is  $\delta_1$ -Lipschitz continuous, where  $\delta_1$  is a positive constant.

(4) If

$$\langle \varrho_1 - \sigma_1, \eta_1(\varrho_1, \sigma_1) \rangle_1 \geq \sigma_1 \|\varrho_1 - \sigma_1\|_1^2$$

for  $\varrho_1, \sigma_1 \in H_1$ , then  $\eta_1$  is  $\sigma_1$ -strongly monotone, where  $\sigma_1$  is a positive constant.

### 3 Existence and uniqueness

In this section, we give an existence and uniqueness theorem of the solution of the auxiliary problem for (3). Under the frame of this theorem, we develop a new linearization iterative algorithm for the proposed model. First of all, let us present the following auxiliary problem for (3).

It follows that there exist  $z_1 \in H_1$  and  $z_2 \in H_2$  such that (see [18])

$$\begin{aligned} & \langle z_1 - \eta_1, \eta_1(\sigma_1, z_1) \rangle_1 + \rho_1 \langle F_1(A\eta_1, Bx_2) - f_1, \eta_1(\sigma_1, z_1) \rangle_1 + \rho_1 [a_1(z_1, \sigma_1 - z_1)] \\ & + \rho_1 [b_1(\eta_1, \sigma_1) - b_1(\eta_1, z_1)] \geq 0 \end{aligned} \tag{4}$$

for any  $\sigma_1 \in H_1$ .

$$\begin{aligned} & \langle z_2 - x_2, \eta_2(\sigma_2, z_2) \rangle_2 + \rho_2 \langle F_2(A\eta_1, Bx_2) - f_2, \eta_2(\sigma_2, z_2) \rangle_2 + \rho_2 [a_2(z_2, \sigma_2 - z_2)] \\ & + \rho_2 [b_2(x_2, \sigma_2) - b_1(x_2, z_2)] \geq 0 \end{aligned} \tag{5}$$

for any  $\sigma_2 \in H_2$  and any given  $(\eta_1, x_2) \in H_1 \times H_2$ .

By applying a fixed point theorem, which is due to Yao et al. (see [19]), we have the following.

**Theorem 1** *Let  $\phi_i$  be a continuous function defined in  $H_1 \times H_2$  and  $S$  be multifunctions defined in  $\mathbb{R}_+$ . If there exists a function  $m \in \mathcal{L}^1_{loc}(\mathbb{R}_+)$  satisfying*

$$m(s) \geq H(F(s, x, y), 0)$$

for any  $t \in \mathbb{R}_+$ , then there exists a solution  $x$  of problem (3) satisfying

$$|\eta_1(s) - \rho_2(s)| \leq \left( |\eta_1(0) - x_0| + \int_0^L \rho_1(s) ds \right) \exp(2\lambda^+ s) \tag{6}$$

for  $s \in [0, L]$ , where  $\lambda^+ = \max\{\lambda, 0\}$ .

*Proof* Define

$$\psi_i(s, \zeta, \beta) = \{x \in F(s, \zeta, \beta) : \langle \dot{x}_1(s) - x, \eta_1(s) - \zeta \rangle \leq \lambda |\eta_1(s) - \zeta|^2\}.$$

It is easy to see that it is nonempty for any  $s \in [0, L]$ . So

$$|\dot{\eta}_1(s) - w| = d\left(\dot{x}_1(s), \phi_i\left(s, \eta_1(s), \max_{s \in S(s)} \eta_1(s)\right)\right) \leq \rho_1(s),$$

which together with (C3) gives that

$$\langle w - x, \eta_1(s) - \zeta \rangle \leq \lambda |\eta_1(s) - \zeta|^2 + |\eta_1(s) - \zeta|.$$

So

$$\begin{aligned} & \langle \dot{x}_1(s) - x, \eta_1(s) - \zeta \rangle \\ & \leq \langle w - x, \eta_1(s) - \zeta \rangle + |\dot{x}_1(s) - w| |\eta_1(s) - \zeta| \\ & \leq \lambda |\eta_1(s) - \zeta|^2 + |\eta_1(s) - \zeta| \left( \lambda \left| \max_{s \in S(s)} \eta_1(s) - \beta \right| + \rho_1(s) \right), \end{aligned}$$

which yields that there exists a solution  $x$  of problem

$$\begin{aligned} \dot{x}(s) & \in G\left(s, \rho_2(s), \max_{s \in S(s)} x(s)\right), \\ x(0) & = x_0 \end{aligned} \tag{7}$$

satisfying

$$\begin{aligned} & \langle \dot{x}_1(s) - \dot{x}(s), \eta_1(s) - \rho_2(s) \rangle \\ & \leq \lambda |\eta_1(s) - \rho_2(s)|^2 + |\eta_1(s) - \rho_2(s)| \left( \lambda \left| \max_{s \in S(s)} \eta_1(s) - \max_{s \in S(s)} x(s) \right| + \rho_1(s) \right) \\ & \leq \lambda |\eta_1(s) - \rho_2(s)|^2 + |\eta_1(s) - \rho_2(s)| \left( \lambda \max_{s \in S(s)} |\eta_1(s) - x(s)| + \rho_1(s) \right) \end{aligned} \tag{8}$$

for any  $s \in [0, L]$ .

Define

$$r(s) = \max_{s \in S(s)} |\eta_1(s) - \rho_2(s)|,$$

which is absolutely continuous and differentiable. It follows that

$$r(s)\dot{r}(s) = \frac{d}{dt} r^2(s) \leq \lambda r(s) \left( r(s) + \max_{s \in S(s)} r(s) \right) \tag{9}$$

from (8).

And define

$$T = \{s \in [0, L] : r(s) = 0\}.$$

It follows that

$$\dot{r}(s) \leq \lambda^+ \left( r(s) + \max_{s \in S(s)} r(s) \right) + \rho_1(s) \tag{10}$$

for any  $t \notin T$  from (9), which yields that  $\bar{r}$  is the solution of

$$\begin{aligned} \dot{\bar{r}}(s) &= \lambda^+ \left( \bar{r}(s) + \max_{s \in S(s)} \bar{r}(s) \right) + \rho_1(s), \quad s \in [0, L] \\ \bar{r}(0) &= r(0). \end{aligned}$$

So

$$\bar{r}(s) \leq r(0) + \int_0^s (2\lambda^+ \bar{r}(\tau) + \rho_1(\tau)) \, d\tau,$$

which implies the required conclusion. □

**Theorem 2** *Let the following conditions hold:*

- (1)  $a_i : H_i \times H_i \rightarrow \mathbb{R}$  satisfy (C1) and (C2),  $b_i : H_i \times H_i \rightarrow \mathbb{R}$  with (C3)–(C6);
- (2)  $\eta_i$  is  $\sigma_i$ -strongly monotone;
- (3)  $\langle F_i(Ax, By), \eta_i(\sigma_i, \cdot) \rangle$  is concave and upper semicontinuous.

*Then the auxiliary problems (4) and (5) are solvable.*

*Proof* Let mappings  $\phi_i, \psi_i : H_i \times H_i \rightarrow \mathbb{R}$  be defined as follows:

$$\phi_i(\sigma_i, z_i) = \langle \sigma_i - x_i, \eta_i(\sigma_i, z_i) \rangle_i + \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(\sigma_i, z_i) \rangle_i$$

and

$$\psi_i(\sigma_i, z_i) = \langle z_i - x_i, \eta_i(\sigma_i, z_i) \rangle_i + \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(\sigma_i, z_i) \rangle_i,$$

respectively.

So

$$\begin{aligned} \phi_i(\sigma_i, z_i) - \psi_i(\sigma_i, z_i) &= \langle \sigma_i - z_i, \eta_i(\sigma_i, z_i) \rangle_i + \rho_i [a_i(\sigma_i - z_i, \sigma_i - z_i)] \\ &\geq (\sigma_i + \rho_i c_i) \|\sigma_i - z_i\|_i^2 \geq 0 \end{aligned}$$

for any  $\sigma_i, z_i \in H_i$ , which yields that 1 (1) holds. And  $a_i$  is coercive continuous, then  $a_i(\sigma_i, \sigma_i - z_i)$  is weakly upper semicontinuous.

It follows that  $b_i$  is convex and lower semicontinuous, which means that  $\langle F_i(Ax, By), \eta_i(\sigma_i, \cdot) \rangle$  is concave and upper semicontinuous. Hence,  $\phi_i(\sigma_i, \cdot)$  and  $\{\sigma_i \in H_i : \psi_i(\sigma_i, z_i)\}$  are convex. That is to say, conditions (2) and (3) hold.

Furthermore,

$$w_i = [\sigma_i + \rho_i c_i]^{-1} \left\{ \rho_i \|F_i(A\eta_1, Bx_2) - f_i\|_i + (\rho_{1i} + \rho_i \gamma_i) \|x_i\|_i \right\}$$

and

$$K_i = \{z_i \in H_i : \|z_i\|_i \leq w_i\}$$

for each  $i \in I$ .

So

$$\begin{aligned} \psi_i(\sigma_{0_i}, z_i) &= \psi_i(0, z_i) \\ &= \langle z_i - x_i, \eta_i(0, z_i) \rangle_i + \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(0, z_i) \rangle_i \\ &\quad + \rho_i [a_i(z_i, 0 - z_i)] \\ &= -\langle 0 - z_i, \eta_i(0, z_i) \rangle_i + \langle x_i, \eta_i(z_i, 0) \rangle_i + \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(0, z_i) \rangle_i \\ &\quad - \rho_i [a_i(z_i, z_i)] \\ &\leq -\sigma_i \|z_i\|_i^2 + \rho_{1_i} \|x_i\|_i \|z_i\|_i + \rho_i \|F_i(A\eta_1, Bx_2) - f_i\|_i \|z_i\|_i \\ &\quad - \rho_i c_i \|z_i\|_i^2 + \rho_i \gamma_i \|x_i\|_i \|z_i\|_i \\ &= -\|z_i\|_i \{ (\sigma_i + \rho_i c_i) \|z_i\|_i - \rho_i \|F_i(A\eta_1, Bx_2) - f_i\|_i - (\rho_{1_i} + \rho_i \gamma_i) \|x_i\|_i \} \\ &\leq 0. \end{aligned}$$

There exists  $z_i^* \in H_i$  such that

$$\langle \sigma_i - x_i, \eta_i(\sigma_i, z_i^*) \rangle_i + \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(\sigma_i, z_i^*) \rangle_i + \rho_i [a_i(\sigma_i, \sigma_i - z_i^*)] \geq 0 \tag{11}$$

for any  $s \in (0, 1]$  and  $\sigma_i \in H_i$  from Theorem 1.

If  $\sigma_i$  is replaced by  $y_{i,t}$  in (11), then we obtain that

$$\begin{aligned} 0 &\leq \langle y_{i,t} - x_i, \eta_i(y_{i,t}, z_i^*) \rangle_i + \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(y_{i,t}, z_i^*) \rangle_i \\ &\quad + \rho_i [a_i(y_{i,t}, y_{i,t} - z_i^*)] + \rho_i [b_i(x_i, y_{i,t})] \\ &= -\langle y_{i,t} - x_i, \eta_i(z_i^*, y_{i,t}) \rangle_i - \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(z_i^*, y_{i,t}) \rangle_i \\ &\quad + \rho_i [a_i(y_{i,t}, y_{i,t} - z_i^*)] + \rho_i [b_i(x_i, y_{i,t})] \\ &\leq -t \langle y_{i,t} - x_i, \eta_i(z_i^*, \sigma_i) \rangle_i - \rho_i t \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(z_i^*, \sigma_i) \rangle_i, \end{aligned}$$

where

$$y_{i,t} := t\sigma_i + (1 - t)z_i^*.$$

In fact, in the above formulation, the second inequality comes from that  $\eta_i$  is affine in the second variable,  $\eta_i(z_i^*, z_i^*) = 0$  and  $b_i$  satisfies (C3)–(C6).

Hence, we derive

$$\langle y_{i,t} - x_i, \eta_i(\sigma_i, z_i^*) \rangle_i + \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(\sigma_i, z_i^*) \rangle_i + \rho_i [a_i(y_{i,t}, \sigma_i - z_i^*)] \geq 0,$$

which yields that

$$\begin{aligned} & \langle z_i^* - x_i, \eta_i(\sigma_i, z_i^*) \rangle_i + \rho_i \langle F_i(A\eta_1, Bx_2) - f_i, \eta_i(\sigma_i, z_i^*) \rangle_i + \rho_i [a_i(z_i^*, \sigma_i - z_i^*)] \\ & + \rho_i [b_i(x_i, \sigma_i) - b_i(x_i, z_i^*)] \geq 0 \end{aligned}$$

in the case  $t \rightarrow 0^+$ .

Therefore,  $z_1^* \in H_1$  and  $z_2^* \in H_2$  are the solutions of auxiliary problems (4) and (5), respectively.

In the following, we prove the uniqueness of problems (4) and (5). For any two solutions  $z_1, z'_1 \in H_1$  of (4), we have

$$\begin{aligned} & \langle z_1 - \eta_1, \eta_1(\sigma_1, z_1) \rangle_1 + \rho_1 \langle F_1(A\eta_1, Bx_2) - f_1, \eta_1(\sigma_1, z_1) \rangle_1 + \rho_1 [a_1(z_1, \sigma_1 - z_1)] \\ & + \rho_1 [b_1(\eta_1, \sigma_1) - b_1(\eta_1, z_1)] \geq 0 \quad \forall \sigma_1 \in H_1, \end{aligned} \tag{12}$$

$$\begin{aligned} & \langle z'_1 - \eta_1, \eta_1(\sigma_1, z'_1) \rangle_1 + \rho_1 \langle F_1(A\eta_1, Bx_2) - f_1, \eta_1(\sigma_1, z'_1) \rangle_1 + \rho_1 [a_1(z'_1, \sigma_1 - z'_1)] \\ & + \rho_1 [b_1(\eta_1, \sigma_1) - b_1(\eta_1, z'_1)] \geq 0 \quad \forall \sigma_1 \in H_1. \end{aligned} \tag{13}$$

Taking  $\sigma_1 = z'_1$  in (12) and  $\sigma_1 = z_1$  in (13), we obtain that

$$\begin{aligned} & \langle z_1 - \eta_1, \eta_1(z'_1, z_1) \rangle_1 + \rho_1 \langle F_1(A\eta_1, Bx_2) - f_1, \eta_1(z'_1, z_1) \rangle_1 + \rho_1 [a_1(z_1, z'_1 - z_1)] \\ & + \rho_1 [b_1(\eta_1, z'_1) - b_1(\eta_1, z_1)] \geq 0, \end{aligned} \tag{14}$$

$$\begin{aligned} & \langle z'_1 - \eta_1, \eta_1(z_1, z'_1) \rangle_1 + \rho_1 \langle F_1(A\eta_1, Bx_2) - f_1, \eta_1(z_1, z'_1) \rangle_1 + \rho_1 [a_1(z'_1, z_1 - z'_1)] \\ & + \rho_1 [b_1(\eta_1, z_1) - b_1(\eta_1, z'_1)] \geq 0. \end{aligned} \tag{15}$$

Adding (14) and (15), we deduce that

$$\begin{aligned} \sigma_1 \|z_1 - z'_1\|_1^2 & \leq \langle z_1 - z'_1, \eta_1(z_1, z'_1) \rangle_1 \\ & \leq -\rho_1 [a(z_1 - z'_1, z_1 - z'_1)] \\ & \leq -\rho_1 c_1 \|z_1 - z'_1\|_1^2, \end{aligned}$$

which yields  $z_1 = z'_1$ . That is to say,  $z_1^* \in H_1$  is the unique solution of (4). In a similar fashion,  $z_2^* \in H_2$  is the unique solution of (5). □

### 4 Conclusions

In this work, we have studied applications of the Schrödinger-type identity for obtaining transmutations via the fixed point index for nonlinear integral equations. It was possible to derive a wide range of transmutation operators by this method. Classical Riesz transforms were involved in the Schrödinger-type identity method as basic blocks, among them are Fourier, sine and cosine-Fourier, Hankel, Mellin, Laplace, and some generalized transforms. In this paper, we presented a modified Schrödinger-type identity for solutions of a class of linear Schrödinger equations with mixed boundary conditions. The techniques used in our proofs were quite different, and most remarkably some of the proofs became simpler and more straightforward. As an application, we obtained the existence



and uniqueness of a solution to the stationary Schrödinger equation in the sense of the Weyl law, which advanced the recent results obtained in several articles even in a more general setting.

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