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Liouville type theorem for a singular elliptic equation with finite Morse index

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Abstract

This paper considers the nonexistence of solutions for the following singular quasilinear elliptic problem:

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = f(|x|)|u|^{r-1}u, & x \in \mathbb{R}^N_+, \\ |x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(|x|)|u|^{q-1}u, & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{0.1}$$

where $\mathbb{R}^N_+ = \{x = (x', x_N) | x' \in \mathbb{R}^{N-1}, x_N > 0\}$ and $\partial\mathbb{R}^N_+ = \{x = (x', x_N) | x' \in \mathbb{R}^{N-1}, x_N = 0\}$. When the weight functions satisfy some suitable assumptions, we prove that problem (0.1) has no nontrivial bounded solutions with finite Morse index.

Keywords: Liouville theorem; Morse index; Singular elliptic equation

1 Introduction and main results

In this paper, we consider the following problem:

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = f(|x|)|u|^{r-1}u, & x \in \mathbb{R}^N_+, \\ |x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(|x|)|u|^{q-1}u, & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1.1}$$

where $a > 0$, $r > 1$, $q > 1$, $p \geq 2$ and $\mathbb{R}^N_+ = \{x = (x', x_N) | x' \in \mathbb{R}^{N-1}, x_N > 0\}$ denotes the upper half-space in $\mathbb{R}^N$.

Liouville type theorems have been widely applied to research the nonexistence of nontrivial solutions for elliptic equations. Liouville theorem was first announced in 1844 by Liouville [1] for the special case of a doubly periodic function. The classical Liouville-type theorem states that a bounded harmonic (or holomorphic) function defined in the entire space $\mathbb{R}^N$ must be constant. Liouville type theorems for solutions with finite Morse indices have been widely studied in the past few decades. The idea of using Morse index of a solution to study a semilinear elliptic equation was first explored by Bahri and Lions in [2], where the following problem was considered on the half-space:

$$\begin{cases} -\Delta u = |u|^{p-1}u, & x \in \mathbb{R}^N_+, \\ u = 0, & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.2}$$

The authors proved that (1.2) has no nontrivial bounded solution with finite Morse index when $1 < p < \frac{N+2}{N-2}$. Later, many authors considered the positive solutions of (1.2) by some delicate methods. In [3], Chen and Li considered the positive solutions of (1.2) by the moving plane method. The authors first proved that the solution is symmetric and constant, then deduced that this constant is just zero. Inspired by the idea in [3], many scholars applied similar methods to research solutions of elliptic equations, see [4–7] and the references therein. Yu [8] studied (1.2) with a Neumann boundary condition. By using an energy estimate and Pohozaev identity, the author gave a result on the nonexistence of a finite Morse index solution.

In [9], Gidas and Spruck considered the elliptic problem

$$-\Delta u = |x|^a |u|^{p-1} u \quad \text{in } \Omega. \tag{1.3}$$

If $a = 0$, the authors proved that (1.3) has no positive solutions if and only if $1 < p < \frac{N+2}{N-2}$ ($= \infty$ if $N = 2$). If $a \neq 0$, problem (1.3) is complicated and less is known. For $a \leq -2$, the authors in [9] established an important result that (1.3) does not possess positive solutions in any domain Ω containing the origin. For $a > -2$, however, problem (1.3) is difficult and there are fewer results since some classical techniques fail for this case. In [10], Phan and Souple studied the positive bounded solution of (1.3) for the special case $a > 0$ and $N = 3$. The authors proved that (1.3) has no positive bounded solution in $\Omega = \mathbb{R}^N$ for $1 < p < p_s(a) = (N+2+2a)/(N-2)$ ($= \infty$ if $N = 2$). In [11], Dancer et al. also studied problem (1.3) with $a > -2$, and classified the existence and behavior at infinity of positive solutions with a finite Morse index. In order to get the results on finite Morse index solutions, a duality method was applied in [11]. It is worth noting that the result on radial solutions of problem (1.3) is complete, see the following proposition in [9, 12].

Proposition A *Let $N \geq 2$, $a > -2$ and $p > 1$.*

(i) *If $p < p_s(a)$, then (1.3) has no positive radial solution in $\Omega = \mathbb{R}^N$.*

(ii) *If $p \geq p_s(a)$, then (1.3) possesses a bounded, positive radial solution in $\Omega = \mathbb{R}^N$.*

For other manuscripts on Liouville-type theorems for nonlinear elliptic equations, we refer the readers to [13–20].

In recent years, a Liouville-type theorem for a higher order equation was also studied. Hu [21] considered the fourth order elliptic equation

$$-\Delta^2 u = |x|^a |u|^{p-1} u \quad \text{in } \Omega. \tag{1.4}$$

Applying the monotonicity formula and blowing down sequence, the author established a Liouville-type theorem for finite Morse index solutions. In [22], Dávila et al. studied (1.4) for the case $a = 0$ and $\Omega = \mathbb{R}^N$. The authors gave a complete classification of finite Morse index solutions. Theorem 1.3 in [22] generalized a similar result of Farina in [23] for the classical Lane–Emden equation.

Some scholars applied the Liouville-type theorem for elliptical equations with the p-Laplace operator. In [24], the authors considered the following p-Laplace elliptic equations with exponential growth

$$-\Delta_p u = f(x) e^u, \quad x \in \mathbb{R}^N. \tag{1.5}$$

There are few works on the elliptic equation with the p-Laplace operator and exponential growth. By choosing a special test function, the authors gave the result on the nonexistence of positive stable solution for (1.5).

In our paper, we consider solutions of (1.1) in Sobolev space $W^{1,p}(|x|^{-ap},\mathbb{R}^N_+)$. The weight functions $f(|x|)$ and $g(|x|)$ in (1.1) are radial. We are interested in the nonexistence of solutions with a finite Morse index. Our proofs in this paper are partly motivated by [14]. Since $a>0$, problem (1.1) is singular at $x=0$, and we need more a delicate energy estimate and computations. We want to point out that the solutions in our problem (1.1) may change sign, thus the moving plane method mentioned above does not work.

Denote by $J(u)$ the natural functional to problem (1.1), that is,

$$J(u)=\frac{1}{p}\int_{\mathbb{R}^N_+}|x|^{-ap}|\nabla u|^p\,dx-\frac{1}{r+1}\int_{\mathbb{R}^N_+}f\bigl(|x|\bigr)|u|^{r+1}\,dx \\ -\frac{1}{q+1}\int_{\partial\mathbb{R}^N_+}g\bigl(\bigl|x'\bigr|\bigr)|u|^{q+1}\,dx'. \tag{1.6}$$

We define the function

$$Q_u(\varphi)=\int_{\mathbb{R}^N_+}|x|^{-ap}|\nabla u|^p\,dx+(p-2)\int_{\mathbb{R}^N_+}|x|^{-ap}|\nabla u|^{p-4}(\nabla u\cdot\nabla\varphi)^2\,dx. \tag{1.7}$$

It is well known that the Morse index $i(u)$ is defined as the maximal dimension of all subspaces $X\in C_0^1(\mathbb{R}^N)$ such that $Q_u(\varphi)<0$.

In order to get our result, we make the following assumptions:

- (A_1) There exist $b_0>0$, $d_0>0$, $b>\frac{N(2-p)-2p(a+1)}{p}$ and $d>\frac{(2-p)N}{p}-2a-1$ such that $f(|x|)|x|^{-b}\to b_0$ and $g(|x|)|x|^{-d}\to d_0$ as $|x|\to\infty$.
- (A_2) $f(|x|)\in C^1(\mathbb{R}^N_+\backslash\{0\})$ is radial and nonnegative in $\mathbb{R}^N_+$, and $g(|x|)\in C^1(\partial\mathbb{R}^N_+\backslash\{0\})$ is radial and nonnegative in $\partial\mathbb{R}^N_+$.
- (A_3) The functions $f(\tau)$ and $g(\tau)$ satisfy

$$\bigl(\tau^\mu f(\tau)\bigr)'>0,\quad \forall\tau=|x|\in\mathbb{R}^N_+\backslash\{0\},\qquad \bigl(\tau^\omega g(\tau)\bigr)'>0,\quad \forall\tau=|x|\in\partial\mathbb{R}^N_+\backslash\{0\},$$

  where $\mu=[Np-(r+1)(1+a-N)]/p$ and $\omega=[(N-1)p-(q+1)(N-p-pa)]/p$.

Our main result on (1.1) is listed below.

Theorem 1 *Assume $N\geq 2$ and suppose functions $f(\tau)$ and $g(\tau)$ satisfy assumptions (A_1)–(A_3). Let $u\in W^{1,p}(|x|^{-ap},\mathbb{R}^N_+)$ be a bounded solution of (1.1). If $i(u)<\infty$, then $u\equiv 0$ in $\mathbb{R}^N_+$.*

This paper is organized as follows. In Sect. 2, we establish several lemmas and estimates. In Sect. 3, we give a Pohozaev identity and then complete the proof of Theorem 1.

2 Preliminary results

In order to study the solutions with a finite Morse index, we will establish several lemmas. We first define a cut-off function $\varphi_{\tau,s}\in[0,1]$ as

$$\varphi_{\tau,s}\bigl(|x|\bigr)=\begin{cases}0, & |x|<\tau\text{ or }|x|>2s,\\ 1, & 2\tau\leq|x|\leq s.\end{cases} \tag{2.1}$$

Furthermore, $|\nabla\varphi_{\tau,s}(|x|)|\leq\frac{2}{\tau}$ for $\tau<|x|\leq 2\tau$ and $|\nabla\varphi_{\tau,s}(|x|)|\leq\frac{2}{s}$ for $s<|x|<2s$.

Lemma 2.1 *Assume $u(x)$ is a solution of* (1.1) *with a finite Morse index, then there exists $\tau_0 > 0$ such that $Q_u(u\varphi_{\tau_0,s}(|x|)) \geq 0$.*

Proof Let $i(u) = k$ and $g(\tau,s) = Q_u(u\varphi_{\tau,s}(|x|))$, where $s > 2\tau > 2\tau_0 > 0$. Assume on the contrary that there exist $s_m \to \infty$, $\tau_m \to \infty$ such that $s_{m+1} > 2\tau_{m+1} > \tau_{m+1} > 2s_m$ and

$$g(r_m,s_m) = Q_u(u\varphi_{\tau_m,s_m}) < 0 \quad \text{for } m = 1,2,\ldots. \tag{2.2}$$

Then, one gets from (2.2) that $u\varphi_{\tau_m,s_m} \not\equiv 0$ for $\forall 1 \leq m \leq k+1$. Note that $\{u\varphi_{\tau_m,s_m}\}_{m=1}^{k+1}$ have disjoint support, which implies that $\{u\varphi_{\tau_m,s_m}\}_{m=1}^{k+1}$ are orthogonal in $L^2(\mathbb{R}^N)$ and linearly independent, so the dimension of the space

$$M_{k+1} = \operatorname{span}\{u\varphi_{\tau_m,s_m}\}_{m=1}^{k+1} \tag{2.3}$$

is $k + 1$. Furthermore, one gets from (2.2) that $Q_u(h) < 0$ for any $h \in M_{k+1}$. Thus, the Morse index of u is at least $k + 1$, which contradicts $i(u) = k$, and we complete the proof of Lemma 2.1. □

Now, we give some estimates.

Lemma 2.2 *Assume (A_1)–(A_3). If u is a bounded solution of* (1.1) *with a finite Morse index, then*

$$\int_{\mathbb{R}^N_+} f(|x|)|u|^{r+1}\,dx < \infty, \qquad \int_{\partial\mathbb{R}^N_+} g(|x'|)|u|^{q+1}\,dx' < \infty. \tag{2.4}$$

Proof We prove the first claim of (2.4). For this purpose, we will divide our proof into three cases.

(i) $r > p - 1 + \frac{bp}{N}$.

According to Lemma 2.1, there exists $\tau_0 > 0$ such that $Q_u(u\varphi_{\tau_0,s}) \geq 0$ for $s > 2\tau_0$, that is,

$$\begin{aligned}
&q\int_{\partial\mathbb{R}^N_+} g(|x'|)|u|^{q+1}\varphi_{\tau_0,s}\,dx' + r\int_{\mathbb{R}^N_+} f(|x|)\varphi^2_{\tau_0,s}\,dx \\
&\quad\leq \int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}(\nabla u\varphi_{\tau_0,s} + u\nabla\varphi_{\tau_0,s})^2\,dx \\
&\qquad + (p-2)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-4}\bigl(\nabla u\cdot\nabla(u\varphi_{\tau_0,s})\bigr)^2\,dx \\
&\quad= (p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^p\varphi^2_{\tau_0,s}\,dx + 2(p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u\varphi_{\tau_0,s}\nabla u\nabla\varphi_{\tau_0,s}\,dx \\
&\qquad + (p-1)\int_{\mathbb{R}^N_+} |\nabla u|^2|\nabla\varphi|^2_{\tau_0,s}\,dx.
\end{aligned} \tag{2.5}$$

On the other hand, multiplying (1.1) by $u\varphi^2_{\tau_0,s}$ and integrating by parts, one gets

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} f(|x|)|u|^{r+1}\varphi^2_{\tau_0,s}\,dx + \int_{\partial\mathbb{R}^N_+} g(|x'|)|u|^{q+1}\varphi^2_{\tau_0,s}\,dx' \\
&\quad= \int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^p\varphi^2_{\tau_0,s}\,dx \\
&\qquad + 2\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u\varphi_{\tau_0,s}\nabla u\nabla\varphi_{\tau_0,s}\,dx.
\end{aligned} \tag{2.6}$$

It follows from Lemma 2.1 that there exists $\tau_0 > 0$ such that

$$\begin{aligned} &q \int_{\partial\mathbb{R}^N_+} g\big(|x'|\big)|u|^{q+1}\varphi_{\tau_0,s}\,dx' + r\int_{\mathbb{R}^N_+} f\big(|x|\big)|u|^{r+1}\varphi^2_{\tau_0,s}\,dx \\ &\quad \le (p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^p\varphi^2_{\tau_0,s}\,dx \\ &\qquad + 2(p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u\varphi_{\tau_0,s}\nabla u\nabla\varphi_{\tau_0,s}\,dx \\ &\qquad + (p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\varphi_{\tau_0,s}|^2\,dx. \end{aligned} \tag{2.7}$$

Inserting (2.6) into (2.7), one gets

$$\begin{aligned} &q \int_{\partial\mathbb{R}^N_+} g\big(|x'|\big)|u|^{q+1}\varphi_{\tau_0,s}\,dx' + r\int_{\mathbb{R}^N_+} f\big(|x|\big)|u|^{r+1}\varphi^2_{\tau_0,s}\,dx \\ &\quad \le (p-1)\bigg[\int_{\mathbb{R}^N_+} f\big(|x|\big)|u|^{r+1}\varphi^2_{\tau_0,s}\,dx \\ &\qquad + \int_{\partial\mathbb{R}^N_+} g\big(|x|\big)|u|^{q+1}\varphi^2_{\tau_0,s}\,dx'\bigg] + (p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\varphi_{\tau_0,s}|^2\,dx. \end{aligned} \tag{2.8}$$

That is,

$$\begin{aligned} &(q-p-1)\int_{\partial\mathbb{R}^N_+} g\big(|x'|\big)|u|^{q+1}\varphi_{\tau_0,s}\,dx' + (r-p-1)\int_{\mathbb{R}^N_+} f\big(|x|\big)|u|^{r+1}\varphi^2_{\tau_0,s}\,dx \\ &\quad \le (p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\varphi_{\tau_0,s}|^2\,dx. \end{aligned} \tag{2.9}$$

Then, we get that

$$(q-p-1)\int_{\partial\mathbb{R}^N_+} g\big(|x'|\big)|u|^{q+1}\varphi_{\tau_0,s}\,dx' \le (p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\varphi_{\tau_0,s}|^2\,dx \tag{2.10}$$

and

$$(r-p-1)\int_{\mathbb{R}^N_+} f\big(|x|\big)|u|^{r+1}\varphi^2_{\tau_0,s}\,dx \le (p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\varphi_{\tau_0,s}|^2\,dx. \tag{2.11}$$

In the following, we prove that

$$\int_{\mathbb{R}^N_+} f\big(|x|\big)|u|^{r+1}\,dx < \infty. \tag{2.12}$$

By (2.11), we get that

$$\begin{aligned} &(r-p-1)\int_{\mathbb{R}^N_+} f\big(|x|\big)|u|^{r+1}\varphi^2_{\tau_0,s}\,dx \\ &\quad \le (p-1)\int_{\mathbb{R}^N_+} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\varphi_{\tau_0,s}|^2\,dx \end{aligned}$$

$$
\begin{aligned}
&= (p-1)\int_{\tau_0<|x|<2\tau_0} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\varphi_{\tau_0,s}|^2\,dx \\
&\quad + (p-1)\int_{s<|x|<2s} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\varphi_{\tau_0,s}|^2\,dx \\
&\leq (p-1)\biggl(\int_{\tau_0<|x|<2\tau_0} |x|^{-ap}|\nabla u|^p\,dx\biggr)^{\frac{p-2}{p}}\biggl(\int_{\tau_0<|x|<2\tau_0} |x|^{-ap}|u|^p|\nabla u|^p\,dx\biggr)^{\frac{2}{p}} \\
&\quad + (p-1)\biggl(\int_{s\leq|x|\leq 2s} |x|^{-ap}|\nabla u|^p\,dx\biggr)^{\frac{p-2}{p}}\biggl(\int_{s\leq|x|\leq 2s} |x|^{-ap}|u|^p|\nabla\varphi_{r_0,s}|^p\,dx\biggr)^{\frac{2}{p}} \\
&\leq c_1(\tau_0)^{-2a}\int_{\tau_0<|x|<2\tau_0} |x|^{-ap}|u|^p|\nabla u|^p\,dx)^{\frac{2}{p}} + c_2(s)^{-2a-2}\int_{s\leq|x|\leq 2s}|u|^p\,dx.
\end{aligned}
\tag{2.13}
$$

By Hölder inequality, one gets

$$
\int_{s\leq|x|\leq 2s}|u|^p\,dx \leq \biggl(\int_{\Omega_{2s}} f\bigl(|x|\bigr)|u|^{r+1}\,dx\biggr)^{\frac{p}{r+1}}\biggl(\int_{\Omega_{2s}} f\bigl(|x|\bigr)^{\frac{-p}{r+1-p}}\,dx\biggr)^{\frac{r+1-p}{r+1}}, \tag{2.14}
$$

where $\Omega_s = B_s^+\backslash\overline{B_{2\tau_0}^+}$ for $s > 2\tau_0$, and B_s^+ is defined in (3.1). Thus, it follows from (2.13) and (2.14) that

$$
\begin{aligned}
&\int_{\mathbb{R}_+^N} f\bigl(|x|\bigr)|u|^{r+1}\,dx \\
&\quad\leq c_1 + c_2(s)^{-2a-2}\biggl(\int_{\Omega_{2s}} f\bigl(|x|\bigr)|u|^{r+1}\,dx\biggr)^{\frac{2}{r+1}}\biggl(\int_{\Omega_{2s}} f\bigl(|x|\bigr)^{\frac{p}{r+1-p}}\,dx\biggr)^{\frac{2(r+1-p)}{p(r+1)}}.
\end{aligned}
\tag{2.15}
$$

Note that $s > 2\tau_0 > 1$, then if $r > p-1+\frac{bp}{N}$, we get

$$
\int_{\Omega_{2s}} f\bigl(|x|\bigr)^{\frac{p}{r+1-p}}\,dx \leq c_3\int_{\Omega_{2s}} |x|^{\frac{-bp}{r+1-p}}\,dx \leq c_3|r|^{N-\frac{bp}{r+1-p}}. \tag{2.16}
$$

Combining (2.15) with (2.16), we obtain

$$
\begin{aligned}
\int_{\Omega_{2s}} f\bigl(|x|\bigr)^{\frac{p}{r+1-p}}\,dx &\leq c_0 + c_1|2s|^{\frac{2N(r+1-p)-2bp}{p(r+1)}-2(a+1)}\biggl(\int_{\Omega_{2s}} f\bigl(|x|\bigr)|u|^{r+1}\,dx\biggr)^{\frac{2}{r+1}} \\
&= c_0 + c_1|s|^{\theta}2^{\theta}\biggl(\int_{\Omega_{2s}} f\bigl(|x|\bigr)|u|^{r+1}\,dx\biggr)^{\frac{2}{r+1}},
\end{aligned}
\tag{2.17}
$$

where

$$
\theta = \frac{2N(r+1-p)-2bp}{p(r+1)} - 2(a+1) < 0. \tag{2.18}
$$

In the following, we will prove the first part of (2.4) by contradiction.

Suppose $\int_{\mathbb{R}_+^N} f(|x|)|u|^{r+1}\,dx = \infty$. Then one gets by (2.17) that there exists a constant $\alpha > 0$ such that

$$
G(s) = \int_{\Omega_s} f\bigl(|x|\bigr)|u|^{r+1}\,dx \leq \alpha\biggl(\int_{\Omega_{2s}} f\bigl(|x|\bigr)|u|^{r+1}\,dx\biggr)^{\frac{2}{r+1}}2^{\theta}s^{\theta}. \tag{2.19}
$$

Integrating (2.19), we get

$$G(s) \le \alpha^{\gamma_m} 2^{\gamma_m \theta} s^{\gamma_m \theta} \bigl(G(2^{m+1}s)\bigr)^{(\frac{2}{r+1})^{m+1}}, \tag{2.20}$$

where

$$\beta = \frac{2}{r+1} < 1, \qquad \gamma_m = 1 + \beta + \beta^2 + \cdots + \beta^m = \frac{1-\beta^m}{1-\beta} \to \frac{1}{1-\beta} \quad \text{as } m \to \infty.$$

On the other hand, by our assumption, the solution is bounded. So, there exists $M > 0$ such that $|u(x)| \le M$ in $\partial\mathbb{R}^{\mathbb{N}}_+$ and

$$\begin{aligned} G(2^{m+1}s) &= \int_{\Omega_{2^{m+1}s}} f(|x|)|u|^{r+1}\,dx \le \frac{3}{2} b_0 M^{r+1} \int_{\Omega_{2^{m+1}s}} |x|^b\,dx \\ &= \frac{3}{2} b_0 M^{r+1} \frac{\omega_N (2^{m+1}s)^{N+b}}{N+b}. \end{aligned} \tag{2.21}$$

Then we get from (2.20) and (2.21) that

$$\begin{aligned} G(s) &\le \alpha^{\gamma_m} 2^{\gamma_m\theta} s^{\gamma_m\theta} \left(\frac{3}{2} b_0 \omega_N \frac{(2^{m+1}s)^{N+b}}{N+b}\right)^{(\frac{2}{r+1})^{m+1}} \\ &= c_0 \alpha^{\gamma_m} 2^{\gamma_m\theta + (m+1)(N+b)\beta^{m+1}} s^{\gamma_m\theta + (N+b)\beta^{m+1}}, \end{aligned} \tag{2.22}$$

where θ is defined as (2.18).

Note that, when $m \to \infty$,

$$\begin{aligned} &\gamma_m\theta + (m+1)(N+b)\beta^{m+1} \to \beta_0 = \frac{p+1}{p-1}\theta, \\ &\gamma_m\theta + (N+b)\beta^{m+1} \to \beta_0 = \frac{p+1}{p-1}\theta. \end{aligned} \tag{2.23}$$

Then, there exists $c_2 > 0$ such that

$$G(s) \le c_2 s^{\frac{\beta_0}{2}}, \quad s > 2\tau_0, m > 1. \tag{2.24}$$

Since $\beta_0 < 0$, (2.24) yields $G(\infty) = 0$, which contradicts $\int_{\mathbb{R}^N_+} f(|x|)|u|^{r+1}\,dx = \infty$. Thus, we complete the proof of (i).

(ii) $1 < r < p - 1 + \frac{bp}{N}$.

For a large $\tau_0 > 0$, we get

$$\int_{\Omega_{2s}} f(|x|)^{\frac{p}{r+1-p}}\,dx \le c \int_{\Omega_{2s}} |x|^{\frac{p}{r+1-p}}\,dx \le c(2\tau_0)^{N - \frac{bp}{r+1-p}} < 1. \tag{2.25}$$

Then, we get from (2.15) and (2.25) that

$$\int_{\Omega_s} f(|x|)|u|^{r+1}\,dx \le c_1 + c_2(s)^{-2a-2} \left(\int_{\Omega_{2s}} f(|x|)|u|^{r+1}\,dx\right)^{\frac{2}{r+1}}. \tag{2.26}$$

Suppose $\int_{\mathbb{R}_+^N} f(|x|)|u|^{r+1}\,dx = \infty$. Then there exists $\alpha > 0$ such that

$$\begin{aligned} G(s) &= \int_{\Omega_s} f\big(|x|\big)|u|^{p+1}\,dx \le \alpha s^{-(2\alpha+2)} \biggl(\int_{\Omega_{2s}} f\big(|x|\big)|u|^{r+1}\,dx\biggr)^{\frac{2}{r+1}} \\ &\le \alpha^{\gamma_m} s^{-(2\alpha+2)\gamma_m} G\big(2^{m+1}s\big)^{\beta^{m+1}}, \quad m = 0,1,2,\ldots. \end{aligned} \tag{2.27}$$

Thus, one gets from (2.21) and (2.27) that

$$G(s) \le c_0 \alpha^{\gamma_m} s^{-(2\alpha+2)\gamma_m + (N+b)\beta^{m+1}} 2^{(m+1)(N+b)\beta^{m+1}}. \tag{2.28}$$

Since $\gamma_m \to \gamma_0 = \frac{p+1}{p-1}$ as $m \to \infty$ and $\beta < 1$, we obtain $G(\infty) = 0$, which contradicts $\int_{\mathbb{R}_+^N} f(|x|)|u|^{r+1}\,dx = \infty$. The proof of (ii) is completed.

(iii) $r = p - 1 + \frac{bp}{N}$.

For this case, there exists a constant $c_0 > 0$ such that

$$\int_{\Omega_{2s}} f\big(|x|\big)^{-\frac{p}{r+1-p}}\,dx \le \int_{\Omega_{2s}} |x|^{\frac{-bp}{r+1-p}}\,dx \le c_0 \int_{2\tau_0}^{2s} \rho^{-1}\,d\rho \le c_0 s. \tag{2.29}$$

Moreover, similar to case (i) and (ii), we can obtain

$$G(s) \le c_1 + c_2 s^{-(2\alpha+1)} \biggl(\int_{\Omega_{2s}} f\big(|x|\big)|u|^{r+1}\,dx\biggr)^{\frac{2}{r+1}}, \tag{2.30}$$

and there exists a constant $\alpha > 0$ such that

$$\begin{aligned} G(s) &\le \alpha^{\gamma_m} s^{-(2\alpha+1)\gamma_m} \big(G\big(2^{m+1}s\big)\big)^{(\frac{2}{r+1})^{(m+1)}} \\ &\le c_0 \alpha^{\gamma_m} s^{-(2a+1)\gamma_m + (N+a)\beta^{m+1}} 2^{(m+1)(N+a)\beta^{m+1}}. \end{aligned} \tag{2.31}$$

If $\int_{\mathbb{R}_+^N} f(|x|)|u|^{r+1}\,dx = \infty$, we can similarly get from (2.31) that $G(\infty) = 0$, which is a contraction. As a result, we complete the proof of (iii).

Next, we will prove $\int_{\partial\mathbb{R}_+^N} g(|x'|)|u|^{q+1}\,dx' < \infty$.

It follows from (2.10) that

$$\begin{aligned} \int_{B_s^0 \setminus B_{\tau_0}^0} g\big(|x'|\big)|u|^{q+1}\,dx' &\le \int_{\partial\mathbb{R}_+^N} g\big(|x'|\big)|u|^{q+1}\phi^2\,dx' \le \int_{\mathbb{R}_+^N} |x|^{-ap}|\nabla u|^{p-2}u^2|\nabla\phi|^2\,dx \\ &\le \frac{2}{s^2}\biggl(\int_{\Omega_{2s}} |x|^{-ap}|\nabla u|^p\,dx\biggr)^{\frac{p-2}{p}} \biggl(\int_{\Omega_{2s}} |x|^{-ap}|u|^p\,dx\biggr)^{\frac{2}{p}} \\ &\le cs^{-2-2a}\biggl(\int_{\Omega_{2s}} f\big(|x|\big)|u|^{r+1}\,dx\biggr)^{\frac{2}{r+1}} \biggl(\int_{\Omega_{2s}} f\big(|x|\big)^{\frac{-p}{r+1-p}}\,dx\biggr)^{\frac{2(r+1-p)}{p(r+1)}} \\ &\le cs^{\theta}. \end{aligned} \tag{2.32}$$

Noting that $\theta < 0$, we get from (2.32) that

$$\int_{\mathbb{R}^{N-1} \setminus B_{\tau_0}} g\big(|x'|\big)|u|^{q+1}\,dx' = 0. \tag{2.33}$$

Furthermore, we get the second claim in (2.4), and the proof of this lemma is completed. □

3 The proof of Theorem 1

In this part, we will complete the proof of Theorem 1. To make the proof clear, we give the following symbols;

$$\begin{aligned} S_s^+ &= \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}^+ : |x| = s\}, \\ B_s^0 &= \{x = (x', 0), x' \in \mathbb{R}^{N-1} : |x'| < s\}, \\ \partial B_s^0 &= \{x = (x', 0), x' \in \mathbb{R}^{N-1} : |x'| = s\}, \\ B_s^+ &= \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}^+ : |x| < s\}. \end{aligned} \tag{3.1}$$

It is obvious that $\partial B_R^+ = S_R^+ \cup B_R^0$, where $R \in \mathbb{R}^+$. In order to prove the nonexistence of solutions, we need to establish the following Pohozaev identity for problem (1.1).

Lemma 3.1 *Let u be a solution of* (1.1), *then for any $R > 0$ the following equality holds*:

$$\begin{aligned} &\left(\frac{N}{p} - 1 - a\right) \int_{B_R^+} |x|^{-ap} |\nabla u|^p \,dx - \frac{1}{r+1} \int_{B_R^+} \big[Nf(|x|) + |x| f'(|x|)\big] |u|^{r+1} \,dx \\ &\quad - \frac{1}{q+1} \int_{B_R^0} |u|^{q+1} \big[(N-1) g(|x'|) + |x'| g'(|x|)\big] \,dx' - \frac{R}{p} \int_{B_R^0} |x|^{-ap} |\nabla u|^p \,dx \\ &= \frac{-R}{r+1} \int_{\partial B_R^+} f(|x|) |u|^{r+1} \,dS + \frac{R}{p} \int_{S_R^+} |x|^{-ap} |\nabla u|^p \,dS \\ &\quad - \frac{R}{q+1} \int_{\partial B_R^0} g(|x|) |u|^{q+1} \,d\sigma. \end{aligned} \tag{3.2}$$

Proof Multiplying (1.1) by $x \cdot \nabla u$ and integrating, we get

$$-\sum_{j=1}^N \int_{B_R^+} \frac{\partial}{\partial x_j} \left(|x|^{-ap} |\nabla u|^{p-2} \frac{\partial u}{\partial x_j}\right) x \cdot \nabla u \,dx = \sum_{i=1}^N \int_{B_R^+} f(|x|) |u|^{r-1} u x_i \frac{\partial u}{\partial x_i} \,dx. \tag{3.3}$$

For the right-hand side of (3.3), we have

$$\begin{aligned} &\sum_{i=1}^N \int_{B_R^+} f(|x|) |u|^{r-1} u x_i \frac{\partial u}{\partial x_i} \,dx \\ &= \frac{1}{r+1} \sum_{i=1}^N \int_{B_R^+} \frac{\partial}{\partial x_i} \big(x_i f(|x|) |u|^{r+1}\big) \,dx \\ &\quad - \frac{N}{r+1} \int_{B_R^+} f(|x|) |u|^{r+1} \,dx - \sum_{i=1}^N \int_{B_R^+} \left(\int_0^u x_i f'(|x|) \frac{x_i}{|x|} |s|^{r-1} s \,ds\right) dx \\ &= \frac{1}{r+1} \sum_{i=1}^N \int_{\partial B_R^+} x_i \cdot \upsilon_i f(|x|) |u|^{r+1} \,dS - \frac{N}{r+1} \int_{B_R^+} f(|x|) |u|^{r+1} \,dx \\ &\quad - \frac{1}{r+1} \int_{B_R^+} f'(|x|) |x| |u|^{r+1} \,dx \\ &= \frac{R}{r+1} \int_{\partial B_R^+} f(|x|) |u|^{r+1} \,dS \\ &\quad - \frac{1}{r+1} \int_{B_R^+} \big(Nf(|x|) + |x| f'(|x|) |u|^{r+1}\big) \,dx. \end{aligned} \tag{3.4}$$

For the left part of (3.3), we have

$$
\begin{aligned}
&-\sum_{j=1}^{N}\int_{B_R^+}\frac{\partial}{\partial x_j}\left(|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\right)x\cdot\nabla u\,dx\\
&\quad=-\sum_{j=1}^{N}\int_{B_R^+}\frac{\partial}{\partial x_j}\left(|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}x\cdot\nabla u\right)dx\\
&\qquad+\int_{B_R^+}|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\sum_{i=1}^{N}\left(\delta_{ij}\frac{\partial u}{\partial x_i}+x_i\frac{\partial^2 u}{\partial x_i\partial x_j}\right)dx\\
&\quad=-\sum_{j=1}^{N}\int_{\partial B_R^+}\left(|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\upsilon_j x\cdot\nabla u\right)dS+\int_{B_R^+}|x|^{-ap}|\nabla u|^{p}\,dx\\
&\qquad+\sum_{j=1}^{N}\int_{B_R^+}|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\sum_{i=1}^{N}x_i\frac{\partial^2 u}{\partial x_i\partial x_j}\,dx.
\end{aligned}
\tag{3.5}
$$

For the first and third terms of the right-hand side of (3.5), we get

$$
\begin{aligned}
&\sum_{j=1}^{N}\int_{\partial B_R^+}\left(|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\upsilon_j x\cdot\nabla u\right)dS\\
&\quad=\int_{S_R^+}|x|^{-ap}|\nabla u|^{p}\,dS\\
&\qquad+\sum_{i=1}^{N-1}\int_{B_R^0}g(|x'|)|u|^{q-1}ux_i\frac{\partial u}{\partial x_i}\,dx'\\
&\quad=\int_{S_R^+}|x|^{-ap}|\nabla u|^{p}\,dS+\frac{1}{q+1}\sum_{i=1}^{N-1}\int_{B_R^0}\left[\frac{\partial}{\partial x_i}\left(x_i g(|x'|)|u|^{q+1}\right)-|u|^{q+1}\frac{\partial}{\partial x_i}\left(x_i g(|x'|)\right)\right]dx'\\
&\quad=\int_{S_R^+}|x|^{-ap}|\nabla u|^{p}\,dS+\frac{1}{q+1}\sum_{i=1}^{N-1}\int_{\partial B_R^0}x_i g(|x|)|u|^{q+1}\upsilon_i\,d\sigma\\
&\qquad-\frac{1}{q+1}\sum_{i=1}^{N-1}\int_{B_R^0}|u|^{q+1}\frac{\partial}{\partial x_i}\left(x_i g(|x'|)\right)dx'\\
&\quad=\int_{S_R^+}|x|^{-ap}|\nabla u|^{p}\,dS+\frac{1}{q+1}\int_{\partial B_R^0}\left(x\cdot\upsilon g(|x|)|u|^{q+1}\right)d\sigma\\
&\qquad-\frac{1}{q+1}\sum_{i=1}^{N-1}\int_{B_R^0}|u|^{q+1}\left[g(|x'|)+x_i g'(|x'|)\frac{x_i}{|x|}\right]dx'\\
&\quad=\int_{S_R^+}|x|^{-ap}|\nabla u|^{p}\,dS++\frac{R}{q+1}\int_{\partial B_R^0}g(|x|)|u|^{q+1}\,d\sigma\\
&\qquad-\frac{1}{q+1}\int_{B_R^0}\left[(N-1)g(|x'|)+|x'|g'(|x'|)\right]dx'
\end{aligned}
\tag{3.6}
$$

and

$$
\sum_{j=1}^{N}\int_{B_R^+}|x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\sum_{i=1}^{N}x_i\frac{\partial^2 u}{\partial x_i\partial x_j}\,dx
$$

$$
\begin{aligned}
&= \frac{1}{p}\sum_{i=1}^{N}\int_{B_R^+}|x|^{-ap}x_i\frac{\partial}{\partial x_i}\big(|\nabla u|^p\big)\,dx \\
&= \frac{-N}{p}\int_{B_R^+}|x|^{-ap}|\nabla u|^p\,dx + \frac{1}{p}\sum_{i=1}^{N}\int_{\partial B_R^+}|x|^{-ap}|\nabla u|^p x_i\vartheta_i\,dS \\
&\quad + a\sum_{i=1}^{N}\int_{B_R^+}x_i|x|^{-ap-1}\frac{x_i}{|x|}|\nabla u|^p\,dx \\
&= \frac{-N}{p}\int_{B_R^+}|x|^{-ap}|\nabla u|^p\,dx + \frac{R}{p}\int_{\partial B_R^+}|x|^{-ap}|\nabla u|^p\,dS + a\int_{B_R^+}|x|^{-ap}|\nabla u|^p\,dx.
\end{aligned} \tag{3.7}
$$

Thus, (3.2) follows from (3.3)–(3.7). □

Now, we give the proof of Theorem 1.

Multiplying (1.1) by u and integrating, one gets that

$$
\int_{\mathbb{R}_+^N}|x|^{-ap}|\nabla u|^p\,dx = \int_{\mathbb{R}_+^N}f\big(|x|\big)|u|^{r+1}\,dx + \int_{\partial\mathbb{R}_+^N}g\big(|x'|\big)|u|^{q+1}\,dx'. \tag{3.8}
$$

We need to prove

$$
\liminf_{R\to\infty} R\left(\int_{\partial B_R^+}f\big(|x|\big)|u|^{r+1}\,dS + \int_{\partial B_R^+}|x|^{-ap}|\nabla u|^p\,dS + \int_{\partial B_R^0}g\big(|x|\big)|u|^{q+1}\,d\sigma\right) = 0. \tag{3.9}
$$

Assume on the contrary that (3.9) is wrong. Then there exists a constant $\delta > 0$ such that

$$
\liminf_{R\to\infty} R\left(\int_{\partial B_R^+}f\big(|x|\big)|u|^{r+1}\,dS + \int_{\partial B_R^+}|x|^{-ap}|\nabla u|^p\,dS + \int_{\partial B_R^0}g\big(|x|\big)|u|^{q+1}\,d\sigma\right) = \delta. \tag{3.10}
$$

Then, there exists $R_0 \in \mathbb{R}^+$ such that

$$
R\left(\int_{\partial B_R^+}f\big(|x|\big)|u|^{r+1}\,dS + \int_{\partial B_R^+}|x|^{-ap}|\nabla u|^p\,dS + \int_{\partial B_R^0}g\big(|x|\big)|u|^{q+1}\,d\sigma\right) > \delta/2 \tag{3.11}
$$

for all $R > R_0$.

Writing $R_n = R_0 + n$, $n = 1,2,\dots$, there exists $\zeta_n \in (R_{n-1}, R_n)$ such that for $n = 1,2,\dots$, there holds

$$
\begin{aligned}
&\int_{R_{n-1}}^{R_n}\left(\int_{\partial B_R^+}f\big(|x|\big)|u|^{r+1}\,dS + \int_{\partial B_R^+}|x|^{-ap}|\nabla u|^p\,dS + \int_{\partial B_R^0}g\big(|x|\big)|u|^{q+1}\,d\sigma\right)dR \\
&\quad = \zeta_n\left(\int_{\partial B_R^+}f\big(|x|\big)|u|^{r+1}\,dS + \int_{\partial B_R^+}|x|^{-ap}|\nabla u|^p\,dS + \int_{\partial B_R^0}g\big(|x|\big)|u|^{q+1}\,d\sigma\right) \\
&\quad > \delta/2.
\end{aligned} \tag{3.12}
$$

Furthermore, we get

$$
\int_0^{\infty}\left(\int_{\partial B_R^+}f\big(|x|\big)|u|^{r+1}\,dS + \int_{\partial B_R^+}|x|^{-ap}|\nabla u|^p\,dS + \int_{\partial B_R^0}g\big(|x|\big)|u|^{q+1}\,d\sigma\right)dR
$$

$$\geq \sum_{n=2}^{\infty} \int_{R_{n-1}}^{R_n} \left(\int_{\partial B_R^+} f(|x|)|u|^{r+1}\,dS + \int_{\partial B_R^+} |x|^{-ap}|\nabla u|^p\,dS + \int_{\partial B_R^0} g(|x|)|u|^{q+1}\,d\sigma \right) dR$$
$$= \infty, \tag{3.13}$$

which contracts the result of Lemma 2.2, and we get (3.9).

Thus, letting $R \to \infty$ in (3.2), it follows that

$$\left(\frac{N}{p} - 1 - a\right) \int_{\mathbb{R}_+^N} |x|^{-ap}|\nabla u|^p\,dx - \frac{1}{r+1} \int_{\mathbb{R}_+^N} |u|^{r+1}\big[Nf(|x|) + |x|f'(|x|)\big]\,dx$$
$$= \frac{1}{q+1} \int_{\partial\mathbb{R}_+^N} |u|^{q+1}\big[(N-1)g(|x'|) + |x'|g'(|x'|)\big]\,dx'. \tag{3.14}$$

For any $\eta \in \mathbb{R}$, we get from (3.8) and (3.14) that

$$\left(\frac{N}{p} - 1 - a - \eta\right) \int_{\mathbb{R}_+^N} |x|^{-ap}|\nabla u|^p\,dx - \left(\frac{N}{r+1} - \eta\right) \int_{\mathbb{R}_+^N} |u|^{r+1} f(|x|)\,dx$$
$$- \int_{\mathbb{R}_+^N} |u|^{r+1}|x|f'(|x|)\,dx$$
$$= \left(\frac{N-1}{q+1} - \eta\right) \int_{\partial\mathbb{R}_+^N} |u|^{q+1} g(|x'|)\,dx' + \frac{1}{q+1} \int_{\partial\mathbb{R}_+^N} |x'|g'(|x'|)\,dx'. \tag{3.15}$$

Particularly, when $\eta = \frac{N}{p} - 1 - a$, it follows from (A_3) that $u \equiv 0$ in $\mathbb{R}_+^N$. Thus, we complete the proof.

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Each author contributed equally to each part of this study, all authors read and approved the final manuscript.

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