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Optimal decay result for Kirchhoff plate equations with nonlinear damping and very general type of relaxation functions

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Abstract

In this paper, we consider plate equations with viscoelastic damping localized on a part of the boundary and nonlinear damping in the domain. We establish general and optimal decay rate results for a wider class of relaxation functions. These results are obtained without imposing any restrictive growth assumption on the frictional damping term. Our results are more general than the earlier results.

Keywords: Optimal decay; Plate equations; Viscoelastic; Convexity

1 Introduction

In this paper, we consider the following Kirchhoff plate equations:

$$u_{tt} + \Delta^2 u + \eta(t)h(u_t) = 0, \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \Gamma_0 \times (0,\infty), \tag{2}$$

$$-u + \int_0^t k_1(t-s)\Phi_2 u(s)\,ds = 0, \quad \text{on } \Gamma_1 \times (0,\infty), \tag{3}$$

$$\frac{\partial u}{\partial \nu} + \int_0^t k_2(t-s)\Phi_1 u(s)\,ds = 0, \quad \text{on } \Gamma_1 \times (0,\infty), \tag{4}$$

$$u(x,0) = u_0(x,t), \qquad u_t(x,0) = u_1(x), \quad \text{in } \Omega. \tag{5}$$

In system (1)–(5), $u = u(x,t)$ is the transversal displacement of a thin vibrating plate subjected to boundary viscoelastic damping and an internal time-dependent fractional damping. The integral terms in (3) and (4) describe the memory effects. The causes of these memory effects are, for example, the interaction with another viscoelastic element. In the above system, $\eta \in C^1(0,\infty)$ is a positive nonincreasing function called the time-dependent coefficient of the frictional damping and u_0 and u_1 are the initial data. The functions $k_1, k_2 \in C^1(0,\infty)$ are positive and nonincreasing, called relaxation functions, and h is a function that satisfies some conditions. Denoting by Φ_1, Φ_2 the differential operators

$$\Phi_1 u = \Delta u + (1-\rho)D_1 u, \qquad \Phi_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\rho)\frac{\partial D_2 u}{\partial \delta},$$

where

$$D_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx}, \qquad D_2 u = \left(\nu_1^2 - \nu_2^2\right) u_{xy} + \nu_1\nu_2 (u_{yy} - u_{xx}),$$

and $\rho \in (0, \frac{1}{2})$ represents the Poisson coefficient. The vector $\nu = (\nu_1, \nu_2)$ denotes the unit outward normal and $\delta = (-\nu_2, \nu_1)$ denotes the external unit normal to the boundary of the domain. The stability of the Kirchhoff plate equations in which the boundary (internal) feedback is linear or nonlinear has been studied by several authors, such as Lagnese [1], Komornik [2], Lasiecka [3], Cavalcanti et al. [4], Ammari and Tucsnak [5], Komornik [6], Guzman and Tucsnak [7], Vasconcellos and Teixeira [8] and Pazoto et al. [9]. For the existence, multiplicity and asymptotic behavior of nonnegative solutions for a fractional Schrödinger–Poisson–Kirchhoff type system, we refer to Xiang and Wang [10]. There exist a large number of papers which discuss the plate equations when the memory effects are in the domain or at the boundary. Here, we refer to Lagnese [11] and Rivera et al. [12] for the internal viscoelastic damping. They proved that the energy decays exponentially (polynomially) if the relaxation function k decays exponentially (polynomially). Alabau-Boussouira et al. [13] obtained the same results but for an abstract problem. Regarding the internal damping, if the viscoelastic term does not exist and $\eta \equiv 1$, the problem (1) was studied and analyzed in the literature such as by Enrike [14] who established an exponential decay for the wave equation with linear damping term. This result was extended by Komornik [15] and Nakao [16] who used different methods and treated the problem when the damping term is nonlinear. For the boundary damping, Santos and Junior [17] showed that the energy decays exponentially if the resolvent kernels r decays exponentially and polynomially if r decays polynomially. In the presence of $\eta(t)$, Benaissa et al. [18] established energy decay results which depend on h and $\eta(t)$. In all the above work, the rates of decay in the relaxation function were either of exponential or of polynomial type. In 2008, Messaoudi in [19] and [20] gave general decay rates for an extended class of relaxation functions for which the exponential (polynomial) decay rates are just special cases. However, the optimal decay rates in the polynomial decay case were not obtained. Specifically, he considered a relaxation function k that satisfies

$$k'(t) \le -\xi(t) k^p(t), \quad t \ge 0, \tag{6}$$

where $p = 1$ and ξ is a positive nonincreasing differentiable function. Furthermore, he showed that the decay rates of the energy are the same rates of decay of the kernel k. However, the decay rate is not necessarily of exponential or polynomial decay type. After that, different papers appeared and used the condition (6) where $p = 1$; see, for instance, [21–30]. Lasiecka and Tataru [31] took one step forward and considered the following condition:

$$k'(t) \le -G\bigl(k(t)\bigr), \tag{7}$$

where G is a positive, strictly increasing and strictly convex function on $(0, R_0]$, and G satisfies $G(0) = G'(0) = 0$. Using the above condition and imposing additional constraints conditions on G, several authors in different approaches obtained general decay results in terms of G; see for example [32–36], and [37]. Later, the condition (6) was extended by Messaoudi and Al-Khulaifi [38] to the case $1 \le p < \frac{3}{2}$ only and they obtained general and optimal decay results. In [34], Lasiecka et al. established optimal decay rate for all $1 \le p < 2$, but with $\gamma(t) = 1$. Very recently, Mustafa [39] obtained optimal exponential and polynomial decay rates for all $1 \le p < 2$ and γ is a function of t. The work most closely related to our study is by Kang [40], Mustafa and Abusharkh [41] and Mustafa [42]. Kang [40] investigated the system (1)–(5) whereas $\eta(t) \equiv 1$ and

$$G_i\big(-r_i'(t)\big) = -r_i'(t), \quad \forall i = 1, 2, \tag{8}$$

and established general decay results. Mustafa and Abusharkh [41] considered the system (1)–(5). But with the condition

$$r_i''(t) \ge G\big(-r_i'(t)\big), \quad \forall i = 1, 2, \tag{9}$$

and $h(t) \equiv 0$. They established explicit and general decay rate results. Very recently, Mustafa [42] studied system (1)–(5). However, under the same condition (9) he obtained a general decay rate result. *Our contribution in this paper* is to investigate the system (1)–(5) under a very general assumption on the resolvent kernels r_i. This assumption is more general as it comprises the earlier results in [40, 41] and [42] in the presence of $\xi(t)$ and the very general assumption on the relaxation functions. Furthermore, we obtain our results without imposing any restrictive growth assumption on the damping and take into account the effect of the time-dependent coefficient $\eta(t)$. The rest of the paper is as follows: In Sect. 2, we give a literature review and in Sect. 3, we state our main results and provide some examples. In Sect. 4, some technical lemmas are presented and established. Finally, we prove and discuss our decay results.

2 Preliminaries

In this section, some important materials in the proofs of our results will be presented. In this paper, $L^2(\Omega)$ stands for the standard Lebesgue space and $H_0^1(\Omega)$ the Sobolev space. We use those spaces with their usual scalar products and norms. Moreover, we denote by W the following space: $W = \{w \in H^2(\Omega) : w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0\}$, and r_i is the resolvent kernel of $\frac{-k_i'}{k_i(0)}$, which satisfies

$$r_i(t) + \frac{1}{k_i(0)}\big(k_i' * r_i\big)(t) = -\frac{1}{k_i(0)}k_i'(t), \quad \forall i = 1, 2,$$

where $*$ represents the convolution product

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds.$$

From (3) and (4), we get the following Volterra equations:

$$\begin{aligned}
&\Phi_2 u + \frac{1}{k_1(0)}k_1' * \Phi_2 u = \frac{1}{k_1(0)}u_t, \\
&\Phi_1 u + \frac{1}{k_2(0)}k_2' * \Phi_1 u = -\frac{1}{k_2(0)}\frac{\partial u_t}{\partial \nu}.
\end{aligned}$$

Taking $\tau_i = \frac{1}{k_i(0)}$, for $i = 1,2$, and using the Volterra's inverse operator, we get

$$\Phi_2 u = \tau_1\{u_t + r_1 * u_t\}, \quad \text{on } \Gamma_1 \times (0,\infty),$$
$$\Phi_1 u = -\tau_2\left\{\frac{\partial u_t}{\partial \nu} + r_2 * \frac{\partial u_t}{\partial \nu}\right\}, \quad \text{on } \Gamma_1 \times (0,\infty),$$

In our paper, we assume that $u_0 \equiv 0$, so we have

$$\Phi_2 u = \tau_1\left\{u_t + r_1(0)u + r_1' * u\right\}, \quad \text{on } \Gamma_1 \times (0,\infty), \tag{10}$$
$$\Phi_1 u = -\tau_2\left\{\frac{\partial u_t}{\partial \nu} + r_2(0)\frac{\partial u}{\partial \nu} + r_2' * \frac{\partial u}{\partial \nu}\right\}, \quad \text{on } \Gamma_1 \times (0,\infty). \tag{11}$$

Throughout the paper, c is a generic positive constant and we use (10) and (11) instead of (3) and (4).

2.1 Assumptions

($A1$) We assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, the partitions Γ_0 and Γ_1 are closed and disjoint. We also assume that $\text{meas}(\Gamma_0) > 0$, and there exists a fixed point $x_0 \in \mathbb{R}^2$ such that $m \cdot \nu \le 0$ on Γ_0 and $m \cdot \nu > 0$ on Γ_1 where $m(x) := x - x_0$ and ν is the unit outward normal vector. This assumption leads to positive constants δ_0 and R such that

$$m \cdot \nu \ge \delta_0 > 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \left|m(x) \cdot \nu\right| \le R, \quad \forall x \in \Omega.$$

($A2$) We assume that $h : \mathbb{R} \to \mathbb{R}$ is a C^0 nondecreasing function and there exists a strictly increasing function $h_0 \in C^1(\mathbb{R}^+)$ with $h_0(0) = 0$ such that

$$\begin{aligned} &h_0\left(|s|\right) \le \left|h(s)\right| \le h_0^{-1}\left(|s|\right) \quad \text{for all } |s| \le \epsilon, \\ &c_1|s| \le \left|h(s)\right| \le c_2|s| \quad \text{for all } |s| \ge \epsilon, \end{aligned} \tag{12}$$

where c_1, c_2, ϵ are positive constants. In the case h_0 is nonlinear, we assume that the function H defined by $H(s) = \sqrt{s}h_0(\sqrt{s})$ is a strictly convex C^2 on $(0, r_0]$, where $r_0 > 0$.

($A3$) We assume that $r_i : \mathbb{R}^+ \to \mathbb{R}^+$, for $i = 1,2$, is a C^2 function satisfies

$$\lim_{t\to\infty} r_i(t) = 0, \qquad r_i(0) > 0, \qquad r_i'(t) \le 0, \tag{13}$$

and there exists a positive, differentiable and nonincreasing function $\xi_i : \mathbb{R}^+ \to \mathbb{R}^+$. We also assume that there exists a positive function $G_i \in C^1(\mathbb{R}^+)$, G_i being a linear or strictly increasing and strictly convex C^2 function on $(0, R_i]$, $R_i > 0$, with $G_i(0) = G_i'(0) = 0$, such that

$$r_i''(t) \ge \xi_i(t)G_i\left(-r_i'(t)\right), \quad (i = 1,2)\ \forall t > 0. \tag{14}$$

Furthermore, we assume that the system ((1)–(5) has a unique solution

$$u \in L^\infty\left(\mathbb{R}^+; H^4(\Omega) \cap W\right) \cap W^{1,\infty}\left(\mathbb{R}^+; W\right) \cap W^{2,\infty}\left(\mathbb{R}^+; L^2(\Omega)\right).$$

This result can be obtained by using the Galerkin method as in Park and Kang [43] and Santos et al. [17].

Remark 2.1 It is worth noting that condition (12) was considered first in [31].

Remark 2.2 Using Assumption $(A2)$, one may notice that $\mathrm{sh}(s) > 0$, for all $s \neq 0$.

Remark 2.3 If G is a strictly increasing and strictly convex C^2 function on $(0, r_1]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is a strictly increasing and strictly convex C^2 function on $(0, \infty)$. For instance, if $G(r_1) = a$, $G'(r_1) = b$, $G''(r_1) = c$, we can define $\overline{G}$, for $t > r_1$, by

$$\overline{G}(t) = \frac{c}{2}t^2 + (b - cr_1)t + \left(a + \frac{c}{2}{r_1}^2 - br_1\right). \tag{15}$$

The same remark can be established for $\overline{H}$.

Now, we define the bilinear form $a(\cdot,\cdot)$ as follows:

$$a(u,v) = \int_\Omega \left\{u_{xx}v_{xx} + u_{yy}v_{yy} + \rho(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\rho)u_{xy}v_{xy}\right\} dx\,dy. \tag{16}$$

It is well known that $\sqrt{a(u,u)}$ is an equivalent norm on W, that is,

$$\beta_1 \|u\|^2_{H^2(\Omega)} \le a(u,u) \le \beta_2 \|u\|^2_{H^2(\Omega)}, \tag{17}$$

for some positive constants β_1 and β_2. From (17) and the Sobolev embedding theorem, we have, for some positive constants c_p and c_s,

$$\|u\|^2 \le c_p a(u,u), \quad \text{and} \quad \|\nabla u\|^2 \le c_s a(u,u), \quad \forall u \in H^2(\Omega). \tag{18}$$

The energy functional associated with (1)–(5) is

$$\begin{aligned} E(t) := {}& \frac{1}{2}\left[\int_\Omega |u_t|^2 + a(u,u) + \tau_1 \int_{\Gamma_1} \left(r_1(t)|u|^2 - \left(r_1' \circ u\right)\right) d\Gamma\right] \\ &+ \frac{1}{2}\left[\tau_2 \int_{\Gamma_1} \left(r_2(t)\left|\frac{\partial u}{\partial \nu}\right| - \left(r_2' \circ \frac{\partial u}{\partial \nu}\right)\right) d\Gamma\right], \end{aligned} \tag{19}$$

where $(f \circ g)(t) = \int_0^t f(t-s)|g(t) - g(s)|^2\, ds$.

Our main stability results are in the following two theorems.

3 The main results

Theorem 3.1 *Assume that* $(A1)$–$(A3)$ *are satisfied and* h_0 is linear. *Then the solution of* (1)–(5) *satisfies, for all* $t \ge t_1$,

$$E(t) \le c_1 e^{-c_2 \int_{t_1}^t \sigma(s)\, ds}, \quad \textit{if } G \textit{ is linear}, \tag{20}$$

$$E(t) \le m_2 G_4^{-1}\left(m_1 \int_{t_1}^t \sigma(s)\, ds\right), \quad \textit{if } G \textit{ is nonlinear}, \tag{21}$$

where c_1, c_2, m_1 and m_2 are strictly positive constants. $G_4(t) = \int_t^r \frac{1}{sG'(s)}\,ds$, $G = \min\{G_1, G_2\}$, *and* $\sigma(t) = \min\{\eta(t), \xi(t)\}$ *where* $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$. G_1, G_2 *and* $\xi_1(t)$, $\xi_2(t)$ *are defined in* $(A3)$.

Theorem 3.2 *Assume that* $(A1)$–$(A3)$ *are satisfied and* h_0 is nonlinear. *Then there exist strictly positive constants* $c_3, c_4, m_3, m_4, \varepsilon_1$ *and* ε_2 *such that the solution of* (1)–(5) *satisfies, for all* $t \geq t_1$,

$$E(t) \leq H_1^{-1}\left(c_3 \int_{t_1}^{t} \sigma(s)\,ds + c_4\right), \quad \text{if } G \text{ is linear,} \tag{22}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,ds$ *and* $H_2(t) = tH'(\varepsilon_1 t)$.

$$E(t) \leq m_4(t - t_1) W_2^{-1}\left(\frac{m_3}{(t - t_1)\int_{t_1}^{t} \sigma(s)\,ds}\right), \quad \text{if } G \text{ is nonlinear,} \tag{23}$$

where $W_2(t) = tW'(\varepsilon_2 t)$, $W = (\overline{G}^{-1} + \overline{H}^{-1})^{-1}$ *and* $\overline{G}, \overline{H}$ *are introduced in Remark* 2.3 .

Remark 3.1 ([44]) In (21), one can see that the decay rate of $E(t)$ is consistent with the decay rate of $(-r_i'(t))$ given by (14). So, the decay rate of $E(t)$ is optimal.

In fact, using the general assumption (14), and taking into account the fact that $G = \min\{G_1, G_2\}$ and $\sigma(t) = \min\{\eta(t), \xi(t)\}$, we have

$$-r_i'(t) \leq G_5^{-1}\left(\int_{-r_i'^{-1}(r)}^{t} \sigma(s)\,ds\right), \quad \forall t \geq -r_i'^{-1}(r),$$

where $G_5(t) = \int_t^r \frac{1}{G(s)}\,ds$. Using the properties of G, we get

$$G_4(t) = \int_t^r \frac{1}{sG'(s)}\,ds \leq \int_t^r \frac{1}{G(s)}\,ds = G_5(t).$$

Also, using the properties of G_4 and G_5, we have

$$G_4^{-1}(t) \leq G_5^{-1}(t).$$

This shows that (21) provides the best decay rates expected under the very general assumption (14).

Example 3.3

(1.A) h_0 and G are linear and $\eta(t) \equiv 1$.

Let $r_i'(t) = -a_i e^{-b_i(1+t)}$, where $b_i > 0$ and $a_i > 0$, $\forall i = 1, 2$, so that Assumption $(A3)$ is satisfied, then $r_i''(t) = \xi_i(t) G_i(-r_i'(t))$. We take $a = \min\{a_1, a_2\}$, $b = \min\{b_1, b_2\}$, $G = \min\{G_1, G_2\}$, $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ and $\sigma(t) = \min\{\eta(t), \xi(t)\}$. Hence, $G(t) = t$, $\xi(t) = b$ and we let $\sigma(t) := b_0 = \min\{1, b\}$. For the nonlinear case, assume that

$h_0(t) = ct$ and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$. Therefore, we can use (20) to deduce

$$E(t) \leq c_1 e^{-c_2 t}, \tag{24}$$

which is the exponential decay.

(1.B) h_0 and G are linear and $\eta(t) = \frac{b}{t+1}$.

Let $r_i'(t) = -a_i e^{-b_i(1+t)}$, where $b_i > 0$ and $a_i > 0$, $\forall i = 1,2$, so that Assumption $(A3)$ is satisfied, then $r_i''(t) = \xi_i(t)G_i(-r_i'(t))$. We take $a = \min\{a_1, a_2\}$, $b = \min\{b_1, b_2\}$, $G = \min\{G_1, G_2\}$, $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ and $\sigma(t) = \min\{\eta(t), \xi(t)\}$. Hence, $G(t) = t$, $\xi(t) = b$ and $\sigma(t) = \frac{b}{t+1}$. For the nonlinear case, assume that $h_0(t) = ct$ and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$. Therefore, we can use (20) to deduce

$$E(t) \leq \frac{c}{1 + \ln(t+1)}. \tag{25}$$

(2) h_0 is linear, G is nonlinear and $\eta(t) \equiv 1$.

Let $r_i'(t) = -a_i e^{-t^q}$, where $0 < q < 1$ and $a_i > 0$, $\forall i = 1,2$, so that Assumption $(A3)$ is satisfied, then $r_i''(t) = \xi_i(t)G_i(-r_i'(t))$. We take $a = \min\{a_1, a_2\}$, $G = \min\{G_1, G_2\}$, $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ and $\sigma(t) = \min\{\eta(t), \xi(t)\}$. Hence, $\xi(t) = 1$ and $G(t) = \frac{q^t}{(\ln(a/t))^{\frac{1}{q}-1}}$. In this case, $\sigma(t) \equiv 1$. For, the boundary feedback, let $h_0(t) = ct$, and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$. Since

$$G'(t) = \frac{(1-q) + q\ln(a/t)}{(\ln(a/t))^{1/q}}$$

and

$$G''(t) = \frac{(1-q)(\ln(a/t) + 1/q)}{(\ln(a/t))^{\frac{1}{q}+1}},$$

the function G satisfies the condition $(A3)$ on $(0, r]$ for any $r > 0$. We have

$$\begin{aligned}
G_4(t) &= \int_t^r \frac{1}{sG'(s)}\,ds = \int_t^r \frac{[\ln\frac{a}{s}]^{\frac{1}{q}}}{s[1 - q + q\ln\frac{a}{s}]}\,ds \\
&= \int_{\ln\frac{a}{r}}^{\ln\frac{a}{t}} \frac{u^{\frac{1}{q}}}{1 - q + qu}\,du \\
&= \frac{1}{q}\int_{\ln\frac{a}{r}}^{\ln\frac{a}{t}} u^{\frac{1}{q}-1}\left[\frac{u}{\frac{1-q}{q} + u}\right]du \\
&\leq \frac{1}{q}\int_{\ln\frac{a}{r}}^{\ln\frac{a}{t}} u^{\frac{1}{q}-1}\,du \leq \left(\ln\frac{a}{t}\right)^{\frac{1}{q}}.
\end{aligned}$$

Then (21) gives

$$E(t) \leq ke^{-kt^q}, \tag{26}$$

which is the optimal decay.

(3) h_0 is nonlinear, G is linear and $\eta(t) = \frac{b}{(t+e)\ln(t+e)}$.
Let $r_i'(t) = -a_i e^{-b_i(1+t)}$, where $b_i > 0$ and $a_i > 0$, $\forall i = 1,2$, so that Assumption ($A3$) is satisfied, then $r_i''(t) = \xi_i(t)G_i(-r_i'(t))$. We take $a = \min\{a_1,a_2\}$, $b = \min\{b_1,b_2\}$, $G = \min\{G_1,G_2\}$, $\xi(t) = \min\{\xi_1(t),\xi_2(t)\}$ and $\sigma(t) = \min\{\eta(t),\xi(t)\}$. Hence, $G(t) = t$ and $\xi(t) = b$. In this case, $\sigma(t) = \frac{b}{(t+e)\ln(t+e)}$. Also, assume that $h_0(t) = ct^q$, where $q > 1$ and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct^{\frac{q+1}{2}}$. Then

$$H_1{}^{-1}(t) = (ct+1)^{\frac{-2}{q-1}}.$$

Therefore, applying (22), we obtain

$$E(t) \le \frac{c}{[1+\ln(\ln(t+e))]^{\frac{2}{q-1}}}. \tag{27}$$

(4) h_0 is nonlinear, G is non-linear and $\eta(t) \equiv 1$.
Let $r_i'(t) = \frac{-a_i}{(1+t)^2}$, where $a_i > 0$, $\forall i = 1,2$, is chosen so that Assumption ($A3$) holds. We choose $a = \min\{a_1,a_2\}$, then $r_i''(t) = b_i G_i(-r_i'(t))$. We select $b = \min\{b_1,b_2\}$, $G = \min\{G_1,G_2\}$, $\xi(t) = \min\{\xi_1(t),\xi_2(t)\}$ and $\sigma(t) = \min\{\eta(t),\xi(t)\}$. In this example, $G(s) = s^{\frac{3}{2}}$, $\xi(t) = b$. For the boundary feedback, let $h_0(t) = ct^5$ and $H(t) = ct^3$. Then

$$W(s) = \left(G^{-1} + H^{-1}\right)^{-1} = \left(\frac{-1+\sqrt{1+4s}}{2}\right)^3$$

and

$$\begin{aligned} W_2(s) &= \frac{3s}{\sqrt{1+4s}}\left(\frac{-1+\sqrt{1+4s}}{2}\right)^2 \\ &= \frac{3s}{2\sqrt{1+4s}} + \frac{3s^2}{\sqrt{1+4s}} - \frac{3s}{2} \\ &\le \frac{3s}{2} + \frac{3s^2}{2\sqrt{s}} - \frac{3s}{2} = cs^{\frac{3}{2}}. \end{aligned}$$

Therefore, applying (23), we obtain

$$E(t) \le \frac{c}{(t-t_1)^{\frac{1}{3}}}.$$

For the proofs of our main results, we state and establish several lemmas in the following section.

4 Technical lemmas

In this section, we introduce some lemmas which are important in our proofs of our main results.

Lemma 4.1 ([1]) *Let u and v be functions in $H^4(\Omega)$ and $\rho \in \mathbb{R}$. Then we have*

$$\int_\Omega (\Delta^2 u)v\,dx = a(u,v) + \int_{\Gamma_1}\left\{(\Phi_2 u)v - (\Phi_1 u)\frac{\partial u}{\partial \nu}\right\}d\Gamma \tag{28}$$

and

$$\int_{\Omega}(m.\nabla v)\Delta^2 v\,dx = a(v,v) + \frac{1}{2}\int_{\Gamma} m.\nu\big[v_{xx}^2 + v_{yy}^2 + 2\rho v_{xx}v_{yy} + 2(1-\rho)v_{xy}^2\big]\,d\Gamma + \int_{\Gamma}\left[(\Phi_2 v)m.\nabla v - (\Phi_1 v)\frac{\partial}{\partial\nu}(m.\nabla v)\right]d\Gamma. \tag{29}$$

Lemma 4.2 *Under Assumptions* $(A1)$–$(A3)$ *and considering Remark* 2.2, *the energy functional* E *satisfies, along the solution of* (1)–(5), *the estimate*

$$\begin{aligned} E'(t) &= -\frac{\tau_1}{2}\int_{\Gamma_1}\big(2|u_t|^2 - r_1'(t)|u|^2 + r_1''\circ u\big)\,d\Gamma \\ &\quad - \frac{\tau_2}{2}\int_{\Gamma_1}\left(2\left|\frac{\partial u_t}{\partial\nu}\right|^2 - r_2'(t)\left|\frac{\partial u}{\partial\nu}\right|^2 + r_2''\circ\frac{\partial u}{\partial\nu}\right)d\Gamma - \eta(t)\int_{\Omega}u_t h(u_t)\,dx \\ &\le 0. \end{aligned} \tag{30}$$

Proof The proof can be established by multiplying Eq. (1) by u_t, integrating by parts over Ω, and using (28) and the boundary conditions (10) and (11). With the help of the ideas in [44], one can establish the following two helpful lemmas.

Lemma 4.3 *For* $i = 1,2$, $0 < \alpha_i < 1$, *and for*

$$C_{\alpha_i} := \int_0^\infty \frac{r_i'^2(s)}{r_i'' - \alpha_i r_i'(s)}\,ds \quad \textit{and} \quad \theta_i(t) := r_i''(t) - \alpha_i r_i'(t), \tag{31}$$

we have

$$\left(\int_0^t r_1'(t-s)\big|u(s)-u(t)\big|\,ds\right)^2 \le C_{\alpha_1}(\theta_1\circ u)(t), \tag{32}$$

$$\left(\int_0^t r_2'(t-s)\left|\frac{\partial u(s)}{\partial\nu} - \frac{\partial u(t)}{\partial\nu}\right|ds\right)^2 \le C_{\alpha_2}\left(\theta_2\circ\frac{\partial u}{\partial\nu}\right)(t). \tag{33}$$

Lemma 4.4 *There exist positive constants* d_1, d_2 *and* t_1 *such that*

$$r_i''(t) \ge -d_i r_i'(t), \quad (i=1,2)\ \forall t\in[0,t_1]. \tag{34}$$

Lemma 4.5 *Under Assumptions* $(A1)$–$(A3)$, *the functional*

$$\psi_1(t) := \int_{\Omega}(m.\nabla u)u_t\,dx \tag{35}$$

satisfies, along the solution of (1)–(5), *the estimate*

$$\begin{aligned} \psi_1'(t) &\le \frac{1}{2}\int_{\Gamma_1} m.\nu|u_t|^2\,d\Gamma - \frac{1}{2}\int_{\Omega}|u_t|^2\,dx - \left(1 - \frac{c_0}{2} - \frac{\varepsilon c}{2}\right)a(u,u) \\ &\quad + \frac{{\tau_1}^2}{2\varepsilon}\int_{\Gamma_1}\big[|u_t|^2 + r_1^2(t)|u|^2\big]\,d\Gamma + \frac{{\tau_1}^2 C_{\alpha_1}}{2\varepsilon}\int_{\Gamma_1}(\theta_1\circ u)\,d\Gamma \end{aligned}$$

$$+\frac{\tau_2^2}{2\varepsilon}\int_{\Gamma_1}\left[\left|\frac{\partial u_t}{\partial \nu}\right| + r_2^2(t)\left|\frac{\partial u}{\partial \nu}\right|\right]d\Gamma + \frac{{\tau_2}^2 C_{\alpha_2}}{2\varepsilon}\int_{\Gamma_1}\left(\theta_2\circ\frac{\partial u}{\partial \nu}\right)d\Gamma + \frac{c}{2}\int_\Omega h^2(u_t)\,dx$$
$$-\left[\frac{1}{2}-\frac{\varepsilon c}{2}\right]\int_{\Gamma_1} m.\nu\left[u_{xx}^2 + u_{yy}^2 + 2\rho u_{xx}u_{yy} + 2(1-\rho)u_{xy}^2\right]d\Gamma. \tag{36}$$

Proof By direct integrations, using (1), and using (29) with $v = u$, we obtain

$$\begin{aligned}\psi_1'(t) &= \int_\Omega (m\cdot\nabla u_t)u_t\,dx + \int_\Omega (m\cdot\nabla u)u_{tt}\,dx\\ &= \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2\,d\Gamma - \frac{1}{2}\int_\Omega |u_t|^2\,dx - a(u,u) - \eta(t)\int_\Omega h(u_t)(m\cdot\nabla u)\,dx\\ &\quad -\int_\Gamma\left[(\Phi_2 u)(m\cdot\nabla u) - (\Phi_1 u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)\right]d\Gamma\\ &\quad -\frac{1}{2}\int_\Gamma m.\nu\left[u_{xx}^2 + u_{yy}^2 + 2\rho u_{xx}u_{yy} + 2(1-\rho)u_{xy}^2\right]d\Gamma.\end{aligned} \tag{37}$$

Since $u_{xx}u_{yy} - (u_{xy})^2 = 0$ on Γ_0, we have

$$u_{xx}u_{yy} + 2(1-\rho)u_{xy}^2 = (\Delta u)^2 \quad \text{on } \Gamma_0. \tag{38}$$

Now, as $u = \frac{\partial u}{\partial \nu} = 0$ on Γ_0, we have $D_1 u = D_2 u = 0$ on Γ_0 and

$$\frac{\partial}{\partial\nu}(m.\nabla u) = (m.\nu)\Delta u. \tag{39}$$

Combining (37), (38) and (39), (37) becomes

$$\begin{aligned}\psi_1'(t) &= \frac{1}{2}\int_{\Gamma_1} m.\nu|u_t|^2\,d\Gamma - \frac{1}{2}\int_\Omega |u_t|^2\,dx - a(u,u) - \eta(t)\int_\Omega (m\cdot\nabla u)h(u_t)\,dx\\ &\quad + \frac{1}{2}\int_{\Gamma_0} m.\nu(\Delta u)^2\,d\Gamma - \frac{1}{2}\int_{\Gamma_1} m.\nu\left[u_{xx}^2 + u_{yy}^2 + 2\rho u_{xx}u_{yy} + 2(1-\rho)u_{xy}^2\right]d\Gamma\\ &\quad - \int_{\Gamma_1}(\Phi_2 u)(m.\nabla u)\,d\Gamma + \int_{\Gamma_1}(\Phi_1 u)\frac{\partial}{\partial\nu}(m.\nabla u)\,d\Gamma.\end{aligned} \tag{40}$$

Now, Young's inequality leads to

$$\left|\int_{\Gamma_1}(\Phi_2 u)(m.\nabla u)\,d\Gamma\right| \le \frac{1}{2\varepsilon}\int_{\Gamma_1}|\Phi_2 u|^2\,d\Gamma + \frac{\varepsilon}{2}\int_{\Gamma_1}|m.\nabla u|^2\,d\Gamma, \tag{41}$$

$$\left|\int_{\Gamma_1}(\Phi_1 u)\frac{\partial}{\partial\nu}(m.\nabla u)\,d\Gamma\right| \le \frac{1}{2\varepsilon}\int_{\Gamma_1}|\Phi_1 u|^2\,d\Gamma + \frac{\varepsilon}{2}\int_{\Gamma_1}\left|\frac{\partial}{\partial\nu}(m.\nabla u)\right|^2 d\Gamma, \tag{42}$$

where ε is a positive constant. Using (17) and (18), the fact $|\, m(x)\, |\le R$, and the trace theory, we obtain

$$\begin{aligned}&\int_{\Gamma_1}|m.\nabla u|^2\,d\Gamma + \int_{\Gamma_1}\left|\frac{\partial}{\partial\nu}(m.\nabla u)\right|^2 d\Gamma\\ &\quad \le R^2 c_s a(u,u) + R\int_{\Gamma_1} m.\nu\left[u_{xx}^2 + u_{yy}^2 + 2\rho u_{xx}u_{yy} + 2(1-\rho)u_{xy}^2\right]d\Gamma.\end{aligned} \tag{43}$$

Furthermore, using (17) and (18) and the property of the function $\eta(t)$, we have

$$\left|\eta(t)\int_\Omega h(u_t)m.\nabla u\,dx\right| \le \frac{c}{2}\int_\Omega h^2(u_t)\,dx + \frac{R^2c_s}{2}a(u,u). \tag{44}$$

Combining (40)–(44), we have

$$\begin{aligned}\psi_1'(t) \le{}& \frac{1}{2}\int_{\Gamma_1} m.\nu|u_t|^2\,d\Gamma - \frac{1}{2}\int_\Omega |u_t|^2\,dx - \left(1-\frac{\lambda_0}{2}-\frac{\varepsilon\lambda_0}{2}\right)a(u,u)\\ &+ \frac{1}{2\varepsilon}\int_{\Gamma_1}|\Phi_1 u|^2\,d\Gamma + \frac{1}{2\varepsilon}\int_{\Gamma_1}|\Phi_2 u|^2\,d\Gamma + \frac{c}{2}\int_\Omega h^2(u_t)\,dx\\ &- \left[\frac{1}{2}-\frac{\varepsilon R}{2}\right]\int_{\Gamma_1} m.\nu\left[u_{xx}^2+u_{yy}^2+2\rho u_{xx}u_{yy}+2(1-\rho)u_{xy}^2\right]d\Gamma,\end{aligned} \tag{45}$$

where $\lambda_0 = R^2c_s$. By direct computation and using (4.3), we arrive at

$$\begin{aligned}(r_1' * u)(t) &= \int_0^t r_1'(t-s)u(s)\,ds = \int_0^t r_1'(t-s)\left[u(s)-u(t)+u(t)\right]ds\\ &= \int_0^t r_1'(t-s)\left[u(s)-u(t)\right]ds + \int_0^t r_1'(t-s)u(t)\,ds\\ &= -\int_0^t r_1'(t-s)\left[u(t)-u(s)\right]ds + \int_0^t r_1'(t-s)u(t)\,ds\\ &= -\int_0^t r_1'(t-s)\left[u(t)-u(s)\right]ds + r_1(t)u(t) - r_1(0)u(t)\\ &\le \left[C_{\alpha_1}(\theta_1\circ u)(t)\right]^{\frac{1}{2}} + r_1(t)u(t) - r_1(0)u(t),\end{aligned} \tag{46}$$

similarly, we can show that

$$\left(r_2' * \frac{\partial u}{\partial \nu}\right)(t) \le \left[C_{\alpha_2}\left(\theta_2\circ\frac{\partial u}{\partial\nu}\right)(t)\right]^{\frac{1}{2}} + r_2(t)\frac{\partial u(t)}{\partial\nu} - r_2(0)\frac{\partial u(t)}{\partial\nu}, \tag{47}$$

then from the boundary conditions (10), (11) and using (46) and (47), we have

$$\begin{aligned}&\Phi_2 u \le \tau_1\left\{u_t + r_1(t)u + \left[C_{\alpha_1}(\theta_1\circ u)(t)\right]^{\frac{1}{2}}\right\},\\ &\Phi_1 u \le -\tau_2\left\{\frac{\partial u_t}{\partial\nu} + r_2(t)\frac{\partial u}{\partial\nu} + \left[C_{\alpha_2}\left(\theta_2\circ\frac{\partial u_t}{\partial\nu}\right)(t)\right]^{\frac{1}{2}}\right\}.\end{aligned} \tag{48}$$

Substituting the inequalities (48) in (45) and using the fact $m.\nu \le 0$ on Γ_0, (36) is achieved. □

Lemma 4.6 *Under Assumptions* (*A*1)–(*A*3), *the functionals*

$$\begin{aligned}&\psi_2(t) = \int_{\Gamma_1}\int_0^t \mu_1(t-s)\left|u(s)\right|^2\,ds\,dx,\\ &\psi_3(t) = \int_{\Gamma_1}\int_0^t \mu_2(t-s)\left|\frac{\partial u(s)}{\partial\nu}\right|^2\,ds\,dx,\end{aligned} \tag{49}$$

satisfy, along the solution of (1)–(5), *the estimates*

$$\psi_2'(t) \le \frac{1}{2}\big(r_1' \circ u\big)(t) + 3r_1(0)\int_{\Gamma_1} \big|u(t)\big|^2\,dx,$$
$$\psi_3'(t) \le \frac{1}{2}\left(r_2' \circ \frac{\partial u}{\partial \nu}\right)(t) + 3r_2(0)\int_{\Gamma_1} \left|\frac{\partial u(t)}{\partial \nu}\right|^2\,dx, \tag{50}$$

where $\mu_i(t) = \int_t^{+\infty}(-r_i'(s))\,ds$, $i = 1, 2$.

Proof Taking the derivative of the first equation in (49) and using the fact $\mu_1'(t) = r_1'(t)$, we have

$$\begin{aligned}\psi_2'(t) &= r_1(0)\int_{\Gamma_1}\big|u(t)\big|^2\,dx + \int_{\Gamma_1}\int_0^t r_1'(t-s)\big|u(s)\big|^2\,dx \\ &= \int_{\Gamma_1}\int_0^t r_1'(t-s)\big|u(s)-u(t)\big|^2\,ds\,dx \\ &\quad + 2\int_{\Gamma_1} u(t)\int_0^t r_1'(t-s)\big(u(s)-u(t)\big)\,ds\,dx + r_1(t)\int_{\Gamma_1}\big|u(t)\big|^2\,dx.\end{aligned} \tag{51}$$

Using the fact $\lim_{t\to\infty} r_1(t) = 0$, and Young's inequality we have the following:

$$\begin{aligned}&2\int_{\Gamma_1} u(t)\int_0^t r_1'(t-s)\big(u(s)-u(t)\big)\,ds\,dx \\ &\quad\le 2\gamma\int_{\Gamma_1}\big|u(s)\big|^2\,dx + \frac{\int_0^t r_1'(s)}{2\gamma}\int_{\Gamma_1}\int_0^t r_1'(t-s)\big|u(s)-u(t)\big|^2\,ds\,dx \\ &\quad\le 2\gamma\int_{\Gamma_1}\big|u(s)\big|^2\,dx + \frac{\int_0^\infty r_1'(s)}{2\gamma}\int_{\Gamma_1}\int_0^t r_1'(t-s)\big|u(s)-u(t)\big|^2\,ds\,dx \\ &\quad\le 2\gamma\int_{\Gamma_1}\big|u(t)\big|^2\,dx - \frac{r_1(0)}{2\gamma}\int_{\Gamma_1}\int_0^t r_1'(t-s)\big|u(s)-u(t)\big|^2\,ds\,dx \\ &\quad\le 2r_1(0)\int_{\Gamma_1}\big|u(t)\big|^2\,dx - \frac{1}{2}\int_{\Gamma_1}\int_0^t r_1'(t-s)\big|u(s)-u(t)\big|^2\,ds\,dx.\end{aligned} \tag{52}$$

Combining (51) and (52) and using the fact that $\mu_1(t) \le \mu_1(0) = r_1(0)$, the first estimate in (50) is established. Similarly, we can establish the second estimate in (50). □

Lemma 4.7 *Under Assumptions* $(A1)$–$(A3)$, *the functional* $L(t) := NE(t) + N_1\psi_1(t) + n_0E(t)$, *where* $N, N_1, n_0 > 0$, *satisfies along the solution of* (1)–(5) *the following estimate*:

$$\begin{aligned}L'(t) &\le -mE(t) - \frac{1}{4}\int_{t_1}^t r_1'(t-s)\int_{\Gamma_1}\big|u(t)-u(s)\big|^2\,dx\,d\Gamma \\ &\quad - \frac{1}{4}\int_{t_1}^t r_2'(t-s)\int_{\Gamma_1}\left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(s)}{\partial \nu}\right|^2\,d\Gamma + c\int_\Omega h^2(u_t)\,dx, \quad \forall t \ge t_1.\end{aligned} \tag{53}$$

Proof Using $L'(t) = NE'(t) + N_1\psi_1'(t) + n_0E'(t)$, combining (30) and (36), using the properties of r_i and r_i' given in Assumption ($A3$) and using $| m \cdot \nu | \leq R$, we obtain

$$\begin{aligned}
L'(t) \leq{}& -\left(\tau_1 N - \frac{RN_1}{2} - \frac{N_1\tau_1^2}{2\varepsilon}\right)\int_{\Gamma_1}|u_t|^2\,d\Gamma - \left(\tau_2 N - \frac{N_1\tau_1^2}{2\varepsilon}\right)\int_{\Gamma_1}\left|\frac{\partial u_t}{\partial \nu}\right|^2 d\Gamma \\
&- N_1\left(1 - \frac{\lambda_0}{2} - \frac{\varepsilon\lambda_0}{2}\right)a(u,u) + \frac{N_1\tau_1^2}{2\varepsilon}\int_{\Gamma_1} r_1^2(t)|u|^2\,d\Gamma \\
&+ \frac{N_1\tau_2^2}{2\varepsilon}\int_{\Gamma_1} r_2^2(t)\left|\frac{\partial u_t}{\partial \nu}\right|^2 d\Gamma - \frac{N_1}{2}\int_\Omega |u_t|^2\,dx \\
&+ \frac{N_1\tau_1^2 C_{\alpha_1}}{2\epsilon}\int_{\Gamma_1}(\theta_1 \circ u)\,d\Gamma + \frac{N_1\tau_2^2 C_{\alpha_1}}{2\epsilon}\int_{\Gamma_1}\left(\theta_2 \circ \frac{\partial u_t}{\partial \nu}\right) d\Gamma \\
&- N_1\left(\frac{1}{2} - \frac{\varepsilon R}{2}\right)\int_{\Gamma_1} m.\nu\left[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2\right] d\Gamma \\
&- \frac{N_1\tau_1}{2}\int_{\Gamma_1}\left(r_1'' \circ u\right) d\Gamma - \frac{N_1\tau_2}{2}\int_{\Gamma_1}\left(r_2'' \circ \frac{\partial u}{\partial \nu}\right) d\Gamma \\
&+ n_0E'(t) + \frac{N_1 c}{2}\int_\Omega h^2(u_t)\,dx.
\end{aligned} \tag{54}$$

Then choosing $0 < \varepsilon \leq \min\{\frac{1}{R}, \frac{2-\lambda_0}{\lambda_0}\}$ so that $\frac{1}{2} - \frac{\varepsilon R}{2} > 0$ and $c_0 := 1 - \frac{\lambda_0}{2} - \frac{\varepsilon\lambda_0}{2} > 0$ and using $\lim_{t\to\infty} r_i(t) = 0$, for $i = 1, 2$, we obtain

$$\begin{aligned}
L'(t) \leq{}& -\left(\tau_1 N - \frac{RN_1}{2} - \frac{N_1\tau_1^2}{2\varepsilon}\right)\int_{\Gamma_1}|u_t|^2\,d\Gamma - \left(\tau_2 N - \frac{N_1\tau_1^2}{2\varepsilon}\right)\int_{\Gamma_1}\left|\frac{\partial u_t}{\partial \nu}\right|^2 d\Gamma \\
&- \frac{N_1}{2}\int_\Omega |u_t|^2\,dx - N_1c_0a(u,u) + \frac{N_1\tau_1^2 C_{\alpha_1}}{2\epsilon}\int_{\Gamma_1}(\theta_1 \circ u)\,d\Gamma \\
&+ \frac{N_1 c}{2}\int_\Omega h^2(u_t)\,dx + \frac{N_1\tau_2^2 C_{\alpha_2}}{2\epsilon}\int_{\Gamma_1}\left(\theta_2 \circ \frac{\partial u_t}{\partial \nu}\right) d\Gamma - \frac{N_1\tau_1}{2}\int_{\Gamma_1}\left(r_1'' \circ u\right) d\Gamma \\
&- \frac{N_1\tau_2}{2}\int_{\Gamma_1}\left(r_2'' \circ \frac{\partial u}{\partial \nu}\right) d\Gamma + n_0E'(t).
\end{aligned} \tag{55}$$

In this case, we choice N large enough so that

$$\begin{aligned}
&\tau_2 N - \frac{N_1\tau_1^2}{2\varepsilon} > 0, \\
&\tau_1 N - \frac{RN_1}{2} - \frac{N_1\tau_1^2}{2\varepsilon} > 0.
\end{aligned} \tag{56}$$

Then (55) reduces to

$$\begin{aligned}
L'(t) \leq{}& -\frac{N_1}{2}\int_\Omega |u_t|^2\,dx - N_1c_0a(u,u) + \frac{N_1\tau_1^2 C_{\alpha_1}}{2\epsilon}\int_{\Gamma_1}(\theta_1 \circ u)\,d\Gamma \\
&+ \frac{N_1 c}{2}\int_\Omega h^2(u_t)\,dx + \frac{N_1\tau_2^2 C_{\alpha_2}}{2\epsilon}\int_{\Gamma_1}\left(\theta_2 \circ \frac{\partial u_t}{\partial \nu}\right) d\Gamma \\
&- \frac{N_1\tau_1}{2}\int_{\Gamma_1}\left(r_1'' \circ u\right) d\Gamma - \frac{N_1\tau_2}{2}\int_{\Gamma_1}\left(r_2'' \circ \frac{\partial u}{\partial \nu}\right) d\Gamma + n_0E'(t).
\end{aligned} \tag{57}$$

Recall that $r_i'' = \alpha r_i' + \theta_i$, $i = 1,2$, and use (19), to obtain

$$\begin{aligned} L'(t) \le & -\frac{N_1}{2}\int_\Omega |u_t|^2\,dx - N_1 c_0 a(u,u) - \left(\frac{N_1\tau_1}{2} - \frac{N_1\tau_1^2 C_{\alpha_1}}{2\epsilon}\right)\int_{\Gamma_1}(\theta_1 \circ u)\,d\Gamma \\ & - \left(\frac{N_1\tau_2}{2} - \frac{N_1\tau_2^2 C_{\alpha_2}}{2\epsilon}\right)\int_{\Gamma_1}\left(\theta_2 \circ \frac{\partial u}{\partial \nu}\right)d\Gamma + \frac{N_1 c}{2}\int_\Omega h^2(u_t)\,dx, \\ & - \frac{\tau_1 N_1 \alpha}{2}\int_{\Gamma_1}\left(r_1' \circ u\right)d\Gamma - \frac{\tau_2 N_1\alpha}{2}\int_{\Gamma_1}\left(r_2' \circ \frac{\partial u}{\partial \nu}\right)d\Gamma + n_0 E'(t) \quad \forall t \ge t_1. \end{aligned} \tag{58}$$

Now, our purpose is to have, for $i = 1,2$,

$$\frac{N_i\tau_i}{2} - C_{\alpha_i}\left(\frac{N_i\tau_i^2}{2\epsilon}\right) > \frac{N_i\tau_i}{4}. \tag{59}$$

As in [44], we can deduce that $\alpha_i C_{\alpha_i} \to 0$ when $\alpha_i \to 0$. Then there exists $0 < \alpha_{0i} < 1$ such that if $\alpha_i < \alpha_{0i}$, then

$$C_{\alpha_i} < \frac{\epsilon}{4\alpha_i \tau_i^2 N_i}.$$

Now, we choose $0 < \alpha_i = \frac{1}{2N_i\tau_i} < 1$, to obtain

$$C_{\alpha_i}\left(\frac{N_i\tau_i^2}{2\epsilon}\right) < \frac{1}{8\alpha_i} = \frac{N_i\tau_i}{4}, \tag{60}$$

and hence, we have

$$N_1\left(\frac{\tau_i}{2} - \frac{\tau_i^2 C_{\alpha_i}}{2\epsilon}\right) > 0, \quad i = 1,2, \tag{61}$$

and then (58) becomes

$$\begin{aligned} L'(t) \le & -\frac{N_1}{2}\int_\Omega |u_t|^2\,dx - N_1 c_0 a(u,u) + \frac{N_1 c}{2}\int_\Omega h^2(u_t)\,dx, \\ & - \frac{1}{4}\int_{\Gamma_1}\left(r_1' \circ u\right)d\Gamma - \frac{1}{4}\int_{\Gamma_1}\left(r_2' \circ \frac{\partial u}{\partial \nu}\right)d\Gamma + n_0 E'(t). \end{aligned} \tag{62}$$

From (34) and (30), we notice that, for all $t \ge t_1$,

$$\begin{aligned} & -\int_0^{t_1} r_1'(s)\int_{\Gamma_1}\left|u(t) - u(t-s)\right|^2 d\Gamma\,ds \\ & \quad \le \frac{1}{d_1}\int_0^{t_1} r_1''(s)\int_{\Gamma_1}\left|u(t) - u(t-s)\right|^2 d\Gamma\,ds \le -cE'(t), \\ & -\int_0^{t_1} r_2'(s)\int_{\Gamma_1}\left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\,ds \\ & \quad \le \frac{1}{d_2}\int_0^{t_1} r_2''(s)\left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\,ds \le -cE'(t). \end{aligned} \tag{63}$$

Then, using (62) and (63), we have for all $t \geq t_1$

$$
\begin{aligned}
L'(t) \leq &-\frac{N_1}{2}\int_\Omega |u_t|^2\,dx - N_1 c_0 a(u,u) - \frac{1}{4}\int_{t_1}^t r_1'(s)\int_{\Gamma_1}\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds \\
&-\frac{1}{4}\int_{t_1}^t r_2'(s)\int_{\Gamma_1}\left|\frac{\partial u(t)}{\partial\nu}-\frac{\partial u(t-s)}{\partial\nu}\right|\,d\Gamma\,ds \\
&+\frac{N_1 c}{2}\int_\Omega h^2(u_t)\,dx + (n_0-c)E'(t).
\end{aligned}
\tag{64}
$$

Now, we choose n_0 so that $n_0 - c > 0$, then (53) is established. Moreover, we can choose N even larger (if needed) so that

$$
L(t) \sim E(t). \tag{65}
$$

□

Lemma 4.8 [45] *Under Assumptions* $(A1)$–$(A3)$, *the solution satisfies the estimates*

$$
\int_{\Omega_1} h^2(u_t)\,dx \leq c\int_{\Omega_1} u_t h(u_t)\,dx, \quad \textit{if } h_0 \textit{ is linear}, \tag{66}
$$

$$
\int_{\Omega_1} h^2(u_t)\,dx \leq cH^{-1}\bigl(J(t)\bigr) - cE'(t), \quad \textit{if } h_0 \textit{ is nonlinear}, \tag{67}
$$

where

$$
J(t) := \int_{\Omega_1} u_t(t)h\bigl(u_t(t)\bigr)\,dx \leq -cE'(t) \tag{68}
$$

and

$$
\Omega_1 = \bigl\{x \in \Omega : \bigl|u_t(t)\bigr| \leq \varepsilon_1\bigr\}.
$$

Lemma 4.9 *Assume that* $(A1)$–$(A3)$ *hold and* h_0 is linear. *Then the energy functional satisfies the following estimate*:

$$
\int_0^{+\infty} E(s)\,ds < \infty. \tag{69}
$$

Proof Let $F(t) = L(t) + \psi_2(t) + \psi_3(t)$, using (50) and (64), and using the trace theory, we obtain for all $t \geq t_1$

$$
\begin{aligned}
F'(t) \leq &-\frac{N_1}{2}\int_\Omega |u_t|\,dx - N_1 c_0 a(u,u) + \frac{1}{4}\bigl(r_1' \circ u\bigr)(t) + \frac{1}{4}\left(r_2' \circ \frac{\partial u}{\partial\nu}\right)(t) \\
&+\frac{N_1 c}{2}\int_{\Gamma_1} h^2(u_t)\,d\Gamma + 3r_1(0)\int_\Omega \bigl|u(t)\bigr|^2\,dx + 3r_2(0)\int_\Omega\left|\frac{\partial u(t)}{\partial\nu}\right|^2\,dx.
\end{aligned}
\tag{70}
$$

Using (17) and (18), we arrive at

$$
\begin{aligned}
F'(t) \leq &-\frac{N_1}{2}\int_\Omega |u_t|\,dx - (N_1 c_0 - c_r)a(u,u) + \frac{1}{4}\bigl(r_1' \circ u\bigr)(t) + \frac{1}{4}\left(r_2' \circ \frac{\partial u}{\partial\nu}\right)(t) \\
&+ c\int_{\Gamma_1} h^2(u_t)\,d\Gamma,
\end{aligned}
\tag{71}
$$

where $c_r = (3c_p r_1(0) + 3c_s r_2(0))$ and c_p, c_s are given in (18). Here, we choose N_1 large enough so that $N_1 c_0 - c_r > 0$. After that, we can choose N even larger (if needed) so that (56) holds. Now, we have

$$\begin{aligned} F'(t) &\le -bE(t) + c\int_\Omega u_t h(u_t)\,dx \\ &\le -bE(t) - cE'(t), \end{aligned}$$

where b is a positive constant. Therefore,

$$b\int_{t_1}^t E(s)\,ds \le F_1(t_1) - F_1(t) \le F_1(t_1) < \infty, \tag{72}$$

where $F_1(t) = F(t) + cE(t) \sim E$. □

Now, we define

$$\begin{aligned} I_1(t) &:= \int_{t_1}^t r_1''(s)\int_{\Gamma_1} \left|u(t) - u(t-s)\right|^2 d\Gamma\,ds \le -cE'(t), \\ I_2(t) &:= \int_{t_1}^t r_2''(s)\int_{\Gamma_1} \left|\frac{\partial u(t)}{\partial\nu} - \frac{\partial u(t-s)}{\partial\nu}\right|^2 d\Gamma\,ds \le -cE'(t). \end{aligned} \tag{73}$$

Lemma 4.10 *Under Assumptions* $(A1)$–$(A3)$, *and if* h_0 is linear, *we have the following estimates*:

$$\int_{t_1}^t -r_1'(s)\int_\Omega \left|u(t) - u(t-s)\right|^2 dx\,ds \le \frac{1}{q}\overline{G_1}^{-1}\left(\frac{qI_1(t)}{\xi_1(t)}\right) \tag{74}$$

and

$$\int_{t_1}^t -r_2'(s)\int_\Omega \left|\frac{\partial u(t)}{\partial\nu} - \frac{\partial u(t-s)}{\partial\nu}\right|^2 d\Gamma\,ds \le \frac{1}{q}\overline{G_2}^{-1}\left(\frac{qI_2(t)}{\xi_2(t)}\right), \tag{75}$$

and if h_0 is nonlinear, *we have the following estimates*:

$$\int_{t_1}^t -r_1'(s)\int_\Omega \left|u(t) - u(t-s)\right|^2 dx\,ds \le \frac{(t-t_1)}{q}\overline{G_1}^{-1}\left(\frac{qI_1(t)}{(t-t_1)\xi_1(t)}\right), \tag{76}$$

$$\int_{t_1}^t -r_2'(s)\int_\Omega \left|\frac{\partial u(t)}{\partial\nu} - \frac{\partial u(t-s)}{\partial\nu}\right|^2 d\Gamma\,ds \le \frac{(t-t_1)}{q}\overline{G_2}^{-1}\left(\frac{qI_2(t)}{(t-t_1)\xi_2(t)}\right), \tag{77}$$

where $q \in (0,1)$, $\overline{G_1}$ *and* $\overline{G_2}$ *are the extensions of* G_1 *and* G_2, *respectively, such that* $\overline{G_1}$ *and* $\overline{G_2}$ *are strictly increasing and strictly convex* C^2 *functions on* $(0,\infty)$

Proof *Case I: if* h_0 *is linear:* we define the following quantities:

$$\begin{aligned} \lambda_1(t) &:= q\int_{t_1}^t \int_{\Gamma_1} \left|u(t) - u(t-s)\right|^2 d\Gamma\,ds, \\ \lambda_2(t) &:= q\int_{t_1}^t \int_{\Gamma_1} \left|\frac{\partial u(t)}{\partial\nu} - \frac{\partial u(t-s)}{\partial\nu}\right|^2 d\Gamma\,ds, \end{aligned} \tag{78}$$

where by (69), (19) and (16) we can choose q so small such that $\forall t \geq t_1$,

$$\lambda_i(t) < 1, \quad i = 1, 2. \tag{79}$$

Since G_i is strictly convex on $(0, R_i]$ and $G_i(0) = 0$, we have

$$G_i(\theta z) \leq \theta G_i(z), \quad 0 \leq \theta \leq 1 \text{ and } z \in (0, r], \tag{80}$$

where $r = \min\{R_1, R_2\}$. Without loss of generality, for all $t \geq t_1$, we assume that $I_i(t) > 0$, $i = 1, 2$, otherwise we get an exponential decay from (53). Using (14), (79), (80) and Jensen's inequality, we have

$$\begin{aligned}
I_1(t) &= \frac{1}{q\lambda_1(t)} \int_{t_1}^t \lambda_1(t) r_1''(s) \int_{\Gamma_1} q \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\
&\geq \frac{1}{q\lambda_1(t)} \int_{t_1}^t \lambda_1(t) \xi_1(s) G_1\left(-r_1'(s)\right) \int_{\Gamma_1} q \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\
&\geq \frac{1}{q\lambda_1(t)} \int_{t_1}^t \xi_1(s) G_1\left(-\lambda_1(t) r_1'(s)\right) \int_{\Gamma_1} q \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\
&\geq \frac{\xi_1(t)}{q\lambda_1(t)} \int_{t_1}^t G_1\left(-\lambda_1(t) r_1'(s)\right) \int_{\Gamma_1} q \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\
&\geq \frac{\xi_1(t)}{q\lambda_1(t)} \lambda_1(t) G_1\left(q \int_{t_1}^t -r_1'(s) \int_{\Gamma_1} \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds\right) \\
&= \frac{\xi_1(t)}{q} \overline{G_1}\left(q \int_{t_1}^t -r_1'(s) \int_{\Gamma_1} \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds\right).
\end{aligned}$$

This gives

$$\int_{t_1}^t -r_1'(s) \int_{\Gamma_1} \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \leq \frac{1}{q} \overline{G_1}^{-1}\left(\frac{qI_1(t)}{\xi_1(t)}\right).$$

Similarly, we can show that

$$\int_{t_1}^t -r_2'(s) \int_{\Gamma_1} \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds \leq \frac{1}{q} \overline{G_2}^{-1}\left(\frac{qI_2(t)}{\xi_2(t)}\right).$$

Case II: if h_0 is nonlinear: we introduce the following functionals:

$$\begin{aligned}
\lambda_3(t) &:= \frac{q}{t - t_1} \int_{t_1}^t \int_{\Gamma_1} \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds, \\
\lambda_4(t) &:= \frac{q}{t - t_1} \int_{t_1}^t \int_{\Gamma_1} \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds,
\end{aligned} \tag{81}$$

then using (16), (19) and (69), we can choose q so small enough so that $\forall t \geq t_1$,

$$\lambda_i(t) < 1, \quad i = 3, 4. \tag{82}$$

Using (14), (80), (82) and Jensen's inequality, we get

$$\begin{aligned} I_1(t) &= \frac{1}{q\lambda_3(t)} \int_{t_1}^{t} \lambda_3(t) r_1''(s) \int_{\Gamma} q\left|u(t)-u(t-s)\right|^2 \, d\Gamma \, ds \\ &\geq \frac{1}{q\lambda_3(t)} \int_{t_1}^{t} \lambda_3(t)\xi_1(s) G_1\left(-r_1'\right) \int_{\Gamma} q\left|u(t)-u(t-s)\right|^2 \, d\Gamma \, ds \\ &\geq \frac{1}{q\lambda_3(t)} \int_{t_1}^{t} \xi_1(s) G_1\left(-\lambda_3(t) r_1'(s)\right) \int_{\Gamma} q\left|u(t)-u(t-s)\right|^2 \, d\Gamma \, ds \\ &\geq \frac{\xi_1(t)}{q\lambda_3(t)} \int_{t_1}^{t} G_1\left(-\lambda_3(t) r_1'(s)\right) \int_{\Gamma} q\left|u(t)-u(t-s)\right|^2 \, d\Gamma \, ds \\ &\geq \frac{(t-t_1)\xi_1(t)}{q\lambda_3(t)} \lambda_3(t) G_1\left(\frac{q}{(t-t_1)} \int_{t_1}^{t} -r_1'(s) \int_{\Gamma} \left|u(t)-u(t-s)\right|^2 \, d\Gamma \, ds\right) \\ &= \frac{(t-t_1)\xi_1(t)}{q} \overline{G_1}\left(\frac{q}{(t-t_1)} \int_{t_1}^{t} -r_1'(s) \int_{\Gamma} \left|u(t)-u(t-s)\right|^2 \, d\Gamma \, ds\right). \end{aligned}$$

This gives

$$\int_{t_1}^{t} -r_1' \int_{\Gamma} \left|u(t)-u(t-s)\right|^2 \, d\Gamma \, ds \leq \frac{(t-t_1)}{q} \overline{G_1}^{-1}\left(\frac{qI_1(t)}{(t-t_1)\xi_1(t)}\right).$$

Similarly, we can have

$$\int_{t_1}^{t} -r_2'(s) \int_{\Gamma_1} \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 \, d\Gamma \, ds \leq \frac{(t-t_1)}{q} \overline{G_2}^{-1}\left(\frac{qI_2(t)}{(t-t_1)\xi_1(t)}\right). \qquad \square$$

5 Proofs of our main results

Here, we prove the main results of our work given in Theorem 3.1 and 3.2.

Proof of Theorem 3.1, case 1, G is linear We multiply (53) by the nonincreasing function $\sigma(t)$. We use (14), (30) and (66), and invoke (14) to have

$$\begin{aligned} \sigma(t)L'(t) &\leq -m\sigma(t)E(t) - c\sigma(t) \int_{t_1}^{t} r_1'(s) \int_{\Gamma_1} \left|u(t)-u(t-s)\right|^2 \, d\Gamma \, ds \\ &\quad - c\sigma(t) \int_{t_1}^{t} r_2'(s) \int_{\Gamma_1} \left|\frac{\partial u(t)}{\partial \nu} - \frac{u(t-s)}{\partial \nu}\right|^2 \, d\Gamma \, ds + c\sigma(t) \int_{\Omega} h^2(u_t) \, dx \quad \forall t \geq t_1 \\ &\leq -m\sigma(t)E(t) + c \int_{\Gamma_1} \left[\left(r_1'' \circ u\right)(t) + \left(r_2'' \circ \frac{\partial u(t)}{\partial \nu}\right)\right] d\Gamma + c\sigma(t) \int_{\Omega} h^2(u_t) \, dx \\ &\leq -m\sigma(t)E(t) - 2cE'(t). \end{aligned}$$

This gives

$$(\sigma L + 2cE)' \leq -m\sigma(t)E(t), \quad \forall t \geq t_1. \tag{83}$$

Using the fact $\sigma'(t) \leq 0$, we have $\sigma L + 2cE \sim E$, and we can obtain

$$E(t) \leq c_1 e^{-c_2 \int_{t_1}^{t} \sigma(s) \, ds}. \tag{84}$$

Proof of Theorem 3.1, *case* 2, *G is nonlinear* Using (53), (66), (75) and (74), we get

$$\begin{aligned} L'(t) &\le -mE(t) - c\int_{t_1}^{t} r_1'(s)\int_{\Gamma_1}\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds \\ &\quad - c\int_{t_1}^{t} r_2'(s)\int_{\Gamma_1}\left|\frac{\partial u(t)}{\partial\nu}-\frac{u(t-s)}{\partial\nu}\right|^2 d\Gamma\,ds + c\int_{\Omega} h^2(u_t)\,dx \quad \forall t\ge t_1 \\ &\le -mE(t) + \frac{1}{q}\overline{G}^{-1}\left(\frac{qI_1(t)}{\sigma(t)}\right) + \frac{1}{q}\overline{G}^{-1}\left(\frac{qI_2(t)}{\sigma(t)}\right) - cE'(t) \\ &\le -mE(t) + \frac{1}{q}\overline{G}^{-1}\left(\frac{qI(t)}{\sigma(t)}\right) - cE'(t), \end{aligned} \tag{85}$$

where $I(t) = \max\{I_1(t), I_2(t)\}$ $\forall t \ge t_1$. Let $\mathcal{F}_1(t) = L(t) + cE(t) \sim E$, then (85) becomes

$$\mathcal{F}_1'(t) \le -mE(t) + c(\overline{G})^{-1}\left(\frac{qI(t)}{\sigma(t)}\right), \tag{86}$$

we notice that the functional $\mathcal{F}_2$, defined by

$$\mathcal{F}_2(t) := \overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}_1(t)$$

satisfies

$$\alpha_1\mathcal{F}_2(t) \le E(t) \le \alpha_2\mathcal{F}_2(t) \tag{87}$$

where $\alpha_1, \alpha_2 > 0$, and

$$\begin{aligned} \mathcal{F}_2'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}\overline{G}''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}_1(t) + \overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right){\mathcal{F}_1}'(t) \\ &\le -mE(t)\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{G}^{-1}\left(\frac{qI(t)}{\sigma(t)}\right). \end{aligned} \tag{88}$$

As in the sense of Young (see [46]), let $\overline{G}^*$ be the convex conjugate of $\overline{G}$, then

$$\overline{G}^*(a) = a\bigl(\overline{G}'\bigr)^{-1}(a) - \overline{G}\bigl[\bigl(\overline{G}'\bigr)^{-1}(a)\bigr], \quad \text{if } a \in \bigl(0, \overline{G}'(r)\bigr] \tag{89}$$

and $\overline{G}^*$ satisfies the generalized Young inequality

$$AB \le \overline{G}^*(A) + \overline{G}(B), \quad \text{if } A \in \bigl(0, \overline{G}'(r)\bigr], B \in (0, r]. \tag{90}$$

So, with $A = \overline{G}'(\varepsilon_0\frac{E'(t)}{E(0)})$ and $B = \overline{G}^{-1}(\frac{qI(t)}{\sigma(t)})$ and using (19) and (88)–(90), we arrive at

$$\begin{aligned} \mathcal{F}_2'(t) &\le -mE(t)\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\overline{G}^*\left(\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\left(\frac{qI(t)}{\sigma(t)}\right) \\ &\le -mE(t)\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\varepsilon_0\frac{E(t)}{E(0)}\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\left(\frac{qI(t)}{\sigma(t)}\right). \end{aligned} \tag{91}$$

So, multiplying (91) by $\sigma(t)$ and using the fact that $\varepsilon_0 \frac{E(t)}{E(0)} < r$, $\overline{G}'(\varepsilon_0 \frac{E(t)}{E(0)}) = G'(\varepsilon_0 \frac{E(t)}{E(0)})$, gives

$$\begin{aligned}\sigma(t)\mathcal{F}_2'(t) &\le -m\sigma(t)E(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\sigma(t)\varepsilon_0\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cqI(t) \\ &\le -m\sigma(t)E(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\sigma(t)\varepsilon_0\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - cE'(t).\end{aligned}$$

Now, for all $t \ge t_1$ and with a good choice of ε_0, we obtain

$$\mathcal{F}_3'(t) \le -m_0\sigma(t)\left(\frac{E(t)}{E(0)}\right)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -m_0\sigma(t)G_3\left(\frac{E(t)}{E(0)}\right), \tag{92}$$

where $\mathcal{F}_3 = \sigma\mathcal{F}_2 + cE \sim E$ satisfies, for any $\beta_3, \beta_4 > 0$,

$$\beta_3\mathcal{F}_3(t) \le E(t) \le \beta_4\mathcal{F}_3(t), \tag{93}$$

and $G_3(t) = tG'(\varepsilon_0 t)$. Since $G_3'(t) = G'(\varepsilon_0 t) + \varepsilon_0 tG''(\varepsilon_0 t)$. Since G is strictly convex over $(0, r]$, we find that $G_3'(t), G_3(t) > 0$ on $(0, 1]$. Then, with

$$R(t) = \frac{\beta_3\mathcal{F}_3(t)}{E(0)},$$

using (93) and (92), we obtain

$$R(t) \sim E(t) \tag{94}$$

and then

$$R'(t) \le -m_1\sigma(t)G_3\big(R(t)\big), \quad \forall t \ge t_1,$$

where $m_1 > 0$. We, after integration over (t_1, t), get

$$\int_{t_1}^t \frac{-R'(s)}{G_3(R(s))}\,ds \ge m_1\int_{t_1}^t \sigma(s)\,ds.$$

Hence, by an appropriate change of variable, we get

$$\int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_1)} \frac{1}{\tau G'(\tau)}\,d\tau \ge m_1\int_{t_1}^t \sigma(s)\,ds.$$

Thus, we have

$$R(t) \le \frac{1}{\varepsilon_0}G_4^{-1}\left(m_1\int_{t_1}^t \sigma(s)\,ds\right), \tag{95}$$

where $G_4(t) = \int_t^r \frac{1}{sG'(s)}\,ds$. Here, we used the strictly decreasing property of G_4 over $(0, r]$. Therefore (21) is established by virtue of (94) and hence we finished the proof of Theorem 3.1. □

Proof of Theorem 3.2, *case* 1, *G is linear* Multiplying (53) by $\sigma(t)$, using (67), gives, as $\sigma(t)$ is nonincreasing, the following:

$$\begin{aligned}\sigma(t)L'(t) &\leq -m\sigma(t)E(t) + c\int_{\Gamma_1}\left[\left(r_1''\circ u\right)(t) + \left(r_2''\circ\frac{\partial u(t)}{\partial\nu}\right)\right]d\Gamma + c\sigma(t)\int_{\Omega}h^2(u_t)\,dx\\ &\leq -m\sigma(t)E(t) - 2cE'(t) + c\sigma(t)\int_{\Omega}h^2(u_t)\,dx\\ &\leq -m\sigma(t)E(t) - 2cE'(t) + c\sigma(t)\left(H^{-1}\left(J(t)\right) - cE'(t)\right)\\ &\leq -m\sigma(t)E(t) - 3cE'(t) + c\sigma(t)H^{-1}\left(J(t)\right),\\ (\sigma L + 3cE)' &\leq -m\sigma(t)E(t) + c\sigma(t)H^{-1}\left(J(t)\right), \quad \forall t\geq t_1.\end{aligned} \tag{96}$$

Therefore, (96) becomes

$$\mathcal{L}'(t) \leq -m\sigma(t)E(t) + c\sigma(t)H^{-1}\left(J(t)\right), \quad \forall t\geq t_1, \tag{97}$$

where $\mathcal{L} := \sigma L + 3cE \sim E$. Now, for $\varepsilon_1 < r_0$ and $c_0 > 0$, using (97) and the fact that $E' \leq 0$, $H' > 0$, $H'' > 0$ on $(0, r_0]$, we notice that the functional $\mathcal{L}_1$, defined by

$$\mathcal{L}_1(t) := H'\left(\varepsilon_1\frac{E(t)}{E(0)}\right)\mathcal{L}(t) + c_0E(t)$$

satisfies, for some $\alpha_3, \alpha_4 > 0$.

$$\alpha_3\mathcal{L}_1(t) \leq E(t) \leq \alpha_4\mathcal{L}_1(t) \tag{98}$$

and

$$\begin{aligned}\mathcal{L}_1'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}H''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}(t) + H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}'(t) + c_0E'(t)\\ &\leq -mE(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\sigma(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)H^{-1}\left(J(t)\right) + c_0E'(t).\end{aligned} \tag{99}$$

Now, let H^* be the convex conjugate of H (see [46]), then, as in (89) and (90), with $A = H'(\varepsilon_1\frac{E(t)}{E(0)})$ and $B = H^{-1}(J(t))$, (99) gives

$$\begin{aligned}\mathcal{L}_1'(t) &\leq -mE(t)H'\left(\varepsilon_1\frac{E(t)}{E(0)}\right) + c\sigma(t)H^*\left(H'\left(\varepsilon_1\frac{E(t)}{E(0)}\right)\right) + c\sigma(t)J(t) + c_0E'(t)\\ &\leq -mE(t)H'\left(\varepsilon_1\frac{E(t)}{E(0)}\right) + c\varepsilon_1\sigma(t)\frac{E(t)}{E(0)}H'\left(\varepsilon_1\frac{E(t)}{E(0)}\right) - cE'(t) + c_0E'(t).\end{aligned}$$

Choosing suitable ε_1 and c_0, we find, for all $t \geq t_1$,

$$\mathcal{L}_1'(t) \leq -c\sigma(t)\frac{E'(t)}{E(0)}H'\left(\varepsilon_1\frac{E(t)}{E(0)}\right) = -c\sigma(t)H_2\left(\varepsilon_1\frac{E(t)}{E(0)}\right), \tag{100}$$

where $H_2(t) = tH'(\varepsilon_1 t)$. We have $H_2'(t) = H'(\varepsilon_1 t) + \varepsilon_1 tH''(\varepsilon_1 t)$. Since H is strictly convex over $(0, r_0]$, we find that $H_2'(t), H_2(t) > 0$ on $(0, 1]$. Then, with

$$R_1(t) = \frac{\alpha_3 \mathcal{L}_1(t)}{E(0)},$$

using (98) and (100), we have

$$R_1(t) \sim E(t), \tag{101}$$
$$R_1'(t) \leq -c_3\sigma(t)H_2\bigl(R_1(t)\bigr), \quad \forall t \geq t_1,$$

where $c_3 > 0$. Thus, we integrate over (t_1, t) to get

$$R_1(t) \leq H_1^{-1}\left(c_3 \int_{t_1}^{t} \sigma(s)\,ds + c_4\right), \quad \forall t \geq t_1, \tag{102}$$

where $c_4 > 0$, and $H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,ds$. □

Proof of Theorem 3.2, *case* 2, *G is nonlinear* Using (53), (67) and (77), we obtain

$$L'(t) \leq -mE(t) + c(t-t_1)(\overline{G})^{-1}\left(\frac{qI(t)}{(t-t_1)\sigma(t)}\right) + cH^{-1}\bigl(J(t)\bigr) - cE'(t). \tag{103}$$

Since $\lim_{t\to+\infty} \frac{1}{t-t_1} = 0$, there exists $t_2 > t_1$ such that $\frac{1}{t-t_1} < 1$ whenever $t > t_2$. By setting $\theta = \frac{1}{t-t_1} < 1$ and using (80), we obtain

$$\overline{H}^{-1}\bigl(J(t)\bigr) \leq (t-t_1)\overline{H}^{-1}\left(\frac{J(t)}{(t-t_1)}\right), \quad \forall t \geq t_2,$$

and then (103) becomes

$$L'(t) \leq -mE(t) + c(t-t_1)(\overline{G})^{-1}\left(\frac{qI(t)}{(t-t_1)\sigma(t)}\right) + c(t-t_1)\overline{H}^{-1}\left(\frac{J(t)}{(t-t_1)}\right)$$
$$- cE'(t), \quad \forall t \geq t_2. \tag{104}$$

Let $L_1(t) = L(t) + cE(t) \sim E$, then (104) takes the form

$$L_1'(t) \leq -mE(t) + c(t-t_1)(\overline{G})^{-1}\left(\frac{qI(t)}{(t-t_1)\sigma(t)}\right) + c(t-t_1)\overline{H}^{-1}\left(\frac{J(t)}{(t-t_1)}\right), \tag{105}$$

Let $r_3 = \min\{r, r_0\}$, $\chi(t) = \max\{\frac{qI(t)}{(t-t_1)\sigma(t)}, \frac{J(t)}{(t-t_1)}\}$ and $W = ((\overline{G})^{-1} + \overline{H}^{-1})^{-1}$.

So, (105) reduces to

$$L_1'(t) \leq -mE(t) + c(t-t_1)W^{-1}\bigl(\chi(t)\bigr), \quad \forall t \geq t_2. \tag{106}$$

Now, for $\varepsilon_2 < r_3$ and using (103) and the fact that $E' \leq 0$, $W' > 0$, $W'' > 0$ on $(0, r_3]$, we find that the functional L_2, defined by

$$L_2(t) := W'\left(\frac{\varepsilon_2}{t-t_1} \cdot \frac{E(t)}{E(0)}\right)L_1(t), \quad \forall t \geq t_2,$$

satisfies, for some $\alpha_5, \alpha_6 > 0$.

$$\alpha_5 L_2(t) \leq E(t) \leq \alpha_6 L_2(t) \tag{107}$$

and, for all $t \geq t_2$,

$$\begin{aligned} L_2'(t) &= \left(\frac{-\varepsilon_2}{(t-t_1)^2}\frac{E(t)}{E(0)} + \frac{\varepsilon_2}{(t-t_1)}\frac{E'(t)}{E(0)}\right) W''\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) L_1(t) \\ &\quad + W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) L_1'(t) \\ &\leq -mE(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c(t-t_1)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) W^{-1}\big(\chi(t)\big). \end{aligned} \tag{108}$$

Let W^* be the convex conjugate of W (see [46]), then as in (89) and (90),

$$W^*(a) = a\big(W'\big)^{-1}(a) - W\big[\big(W'\big)^{-1}(a)\big], \quad \text{if } a \in \big(0, W'(r_3)\big] \tag{109}$$

and W^* satisfies the Young inequality,

$$AB \leq W^*(A) + W(B), \quad \text{if } A \in \big(0, W'(r_3)\big], B \in (0, r_3]. \tag{110}$$

Therefore, taking $A = W'(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)})$ and $B = W^{-1}(\chi(t))$, (108) gives

$$\begin{aligned} L_2'(t) &\leq -mE(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c(t-t_1)W^*\left(W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\right) \\ &\quad + c(t-t_1)\chi(t) \\ &\leq -mE(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c(t-t_1)\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) \\ &\quad + c(t-t_1)\chi(t). \end{aligned} \tag{111}$$

Using (68) and (73), we observe that

$$(t-t_1)\sigma(t)\chi(t) \leq -cE'(t).$$

So, multiplying (111) by $\sigma(t)$, using the fact that $\varepsilon_2\frac{E(t)}{E(0)} < r_3$, gives

$$\begin{aligned} \sigma(t)L_2'(t) &\leq -m\sigma(t)E(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c\varepsilon_2\sigma(t)\cdot\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) \\ &\quad - cE'(t), \quad \forall t \geq t_2. \end{aligned}$$

Using the property of $\sigma(t)$, we obtain, for all $t \geq t_2$,

$$\begin{aligned} \big(\sigma(t)L_2 + cE\big)'(t) &\leq -m\sigma(t)E(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) \\ &\quad + c\varepsilon_2\sigma(t)\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right). \end{aligned}$$

Therefore, by setting $L_3(t) := \sigma(t)L_2(t) + cE(t) \sim E(t)$, we get

$$L_3'(t) \leq -m\sigma(t)E(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c\varepsilon_2\sigma(t)\cdot\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right).$$

This gives, for a suitable choice of ε_2,

$$L_3'(t) \leq -m_0\sigma(t)\left(\frac{E(t)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right), \quad \forall t \geq t_2,$$

or

$$m_0\left(\frac{E(t)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\sigma(t) \leq -L_3'(t), \quad \forall t \geq t_2. \tag{112}$$

An integration of (112) yields

$$\int_{t_2}^{t} m_0\left(\frac{E(s)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{s-t_1}\cdot\frac{E(s)}{E(0)}\right)\sigma(s)\,ds \leq -\int_{t_2}^{t} L_3'(s)\,ds \leq L_3(t_2). \tag{113}$$

Using the facts that $W', W'' > 0$ and the nonincreasing property of E, we deduce that the map $t \mapsto E(t)W'(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)})$ is nonincreasing and, consequently, we have

$$\begin{aligned} &m_0\left(\frac{E(t)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\int_{t_2}^{t}\sigma(s)\,ds \\ &\quad\leq \int_{t_2}^{t} m_0\left(\frac{E(s)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{s-t_1}\cdot\frac{E(s)}{E(0)}\right)\sigma(s)\,ds \leq L_3(t_2). \end{aligned} \tag{114}$$

Multiplying each side of (114) by $\frac{1}{t-t_1}$, we have

$$m_0\left(\frac{1}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\int_{t_2}^{t}\sigma(s)\,ds \leq \frac{m_3}{t-t_1}. \tag{115}$$

Next, we set $W_2(t) = tW'(\varepsilon_2 t)$ which is strictly increasing, then we obtain

$$m_0 W_2\left(\frac{1}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\int_{t_2}^{t}\sigma(s)\,ds \leq \frac{m_3}{t-t_1}. \tag{116}$$

Finally, for two positive constants m_3 and m_4, we obtain

$$E(t) \leq m_4(t-t_1)W_2^{-1}\left(\frac{m_3}{(t-t_1)\int_{t_2}^{t}\sigma(s)\,ds}\right). \tag{117}$$

This finishes the proof of Theorem 3.2. □

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