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Multiple positive solutions to Kirchhoff equations with competing potential functions in $\mathbb{R}^3$

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Abstract

In this paper, we study the existence of multiple positive solutions to the following Kirchhoff equation with competing potential functions:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla v|^2)\Delta v + V(x)v = K(x)|v|^{p-1}v & \text{in } \mathbb{R}^3, \\ v > 0, \quad v \in H^1(\mathbb{R}^3), \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$ are constants, $3 < p < 5$. We relate the number of solutions with the topology of the global minima set of the function $V^{\frac{2}{p-1}-\frac{1}{2}}(x)/K^{\frac{2}{p-1}}(x)$. The Nehari manifold and Ljusternik–Schnirelmann category theory are applied in our study.

MSC: 35J60; 35J20; 35B38

Keywords: Kirchhoff equation; Multiplicity; Competing potential functions; Nonlocal problems

1 Introduction

In this paper, we study the existence of multiple positive solutions to the Kirchhoff equation with competing potential functions:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla v|^2)\Delta v + V(x)v = K(x)|v|^{p-1}v & \text{in } \mathbb{R}^3, \\ v > 0, \quad v \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$ are constants, $3 < p < 5$, $V(x)$ and $K(x)$ are positive continuous functions satisfying

(H) $\inf_{x\in\mathbb{R}^3} V(x) = \bar{V} > 0$, $K(x) > 0$ and $K(x)$ is bounded.

In recent years, the elliptic Kirchhoff type equations have been studied extensively by many authors, and they are related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_{\Omega}|\nabla u|^2\right)\Delta u = g(x,t) \tag{1.2}$$

proposed by Kirchhoff [14] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations.

Some early classical studies of Kirchhoff equations were those by Bernstein [4] and Pohozaev [21]. Equation (1.2) received much attention after Lions [18] had proposed an abstract framework to the problem. In 2006, Perera and Zhang [20, 31] obtained existence and multiplicity of solutions via variational methods. Recently several interesting results can be found in Azzollini [2], Li et al. [15], Li et al. [16], Liang et al. [17], Wu [29], Zhang [30], etc.

On the other hand, the well-known Schrödinger equation

$$-\varepsilon^2\Delta v + V(x)v = f(v) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

has been paid much attention to after the celebrated work of Floer and Weinstein [11]. Many famous mathematicians have obtained a lot of interesting results, we only refer to [5, 6, 9, 22, 26, 27] and the references therein.

Recently, many authors have studied the existence and concentration behavior of positive solutions for Kirchhoff type equations in $\mathbb{R}^3$. In [12], He and Zou studied (1.1) with subcritical nonlinearity. In [23], Sun and Zhang investigated the uniqueness of positive ground state solutions for Kirchhoff type equations with constant coefficients and then studied the existence and concentration behavior of Kirchhoff type problems in $\mathbb{R}^3$ with competing potentials. For more interesting results, we refer to [10, 13, 24, 25] etc.

In [8], Cingolani and Lazzo studied the existence of multiple positive solutions to the nonlinear Schrödinger equation with competing potential functions

$$-\varepsilon^2\Delta v + V(x)v = K(x)|v|^{p-2}v + Q(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

They related the number of solutions with the topology of the global minima set of a suitable ground energy function. If $Q(x) = 0$ in (1.4), the ground energy function is $V^{(2p+2N-Np)/(2p-4)}(x)/K^{2/(p-2)}(x)$.

Inspired by [8], we consider the existence of multiple positive solutions for the Kirchhoff equation (1.1) where a nonlocal term $\int_{\mathbb{R}^3}|\nabla v|^2$ appears in it. Because of the nonlocal term, the method of proof in [8] cannot work directly for our case, and several special difficulties would be faced. For example, we cannot get the Palais–Smale condition if we deal with it in a completely the same way as in [8], which forces us to develop new techniques to solve it. Moreover, the appearance of a competing potential function $K(x)$ and the nonlocal term in (1.1) will bring troubles to the uniform estimate in Sect. 4.

Let us denote by M the global minima set of the function $g(x) := \frac{V^{\frac{2}{p-1}-\frac{1}{2}}(x)}{K^{\frac{2}{p-1}}(x)}$, i.e.,

$$M = \Big\{\xi \in \mathbb{R}^3 : g(\xi) = \inf_{x\in\mathbb{R}^3} g(x)\Big\}. \tag{1.5}$$

We recall that, if Y is a closed subset of a topological space X, $\mathrm{cat}_X Y$ denotes the Ljusternik–Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y. For $\delta > 0$, we denote

$$M_\delta := \big\{x \in \mathbb{R}^3 : \mathrm{dist}(x, M) \le \delta\big\}.$$

Our main result is the following.

Theorem 1.1 *Suppose that (H) holds and*

$$\frac{\liminf_{|x|\to\infty} V^{\frac{2}{p-1}-\frac{1}{2}}(x)}{\limsup_{|x|\to\infty} K^{\frac{2}{p-1}}(x)} > \inf_{x\in\mathbb{R}^3} \frac{V^{\frac{2}{p-1}-\frac{1}{2}}(x)}{K^{\frac{2}{p-1}}(x)}, \tag{1.6}$$

then, for any $\delta > 0$, *there exists* $\varepsilon_\delta > 0$ *such that equation* (1.1) *has at least* $\operatorname{cat}_{M_\delta}(M)$ *solutions for* $\varepsilon \in (0, \varepsilon_\delta)$.

Remark 1.2 By assumption (1.6), the set M defined in (1.5) is not empty and is a bounded closed set in $\mathbb{R}^3$.

Remark 1.3 Consider the following Kirchhoff equation with constant coefficients:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla v|^2)\Delta v + mv = n|v|^{p-1}v & \text{in } \mathbb{R}^3, \\ v > 0, \quad v \in H^1(\mathbb{R}^3), \end{cases} \tag{1.7}$$

where $a, b > 0$ are constants, $3 < p < 5$, $m, n > 0$ are taken as variable parameters here. We define $c(m,n)$ by

$$c(m,n) := \inf_{v\in\mathcal{N}^{(m,n)}} I^{(m,n)}(v),$$

where $I^{(m,n)}$ is the energy functional and $\mathcal{N}^{(m,n)}$ is the Nehari manifold associated to (1.7), i.e.,

$$I^{(m,n)}(v) = \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla v|^2 + mv^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} n|v|^{p+1}$$

and

$$\mathcal{N}^{(m,n)} = \left\{ v \in H^1\left(\mathbb{R}^3\right)\backslash\{0\} : \int_{\mathbb{R}^3} \left(a|\nabla v|^2 + mv^2\right) + b\left(\int_{\mathbb{R}^3} |\nabla v|^2\right)^2 = \int_{\mathbb{R}^3} n|v|^{p+1} \right\}.$$

Then, by Lemma 3.6 in [23], we know that $c(m,n) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Moreover, let $m_1, m_2, n_1, n_2 > 0$, then

$$c(m_1,n_1) < c(m_2,n_2) \quad \text{if and only if} \quad m_1^{\frac{2}{p-1}-\frac{1}{2}}/n_1^{\frac{2}{p-1}} < m_2^{\frac{2}{p-1}-\frac{1}{2}}/n_2^{\frac{2}{p-1}}. \tag{1.8}$$

Now we define the ground energy function $G(\xi)$ which was first introduced in [27] by

$$G(\xi) := c\big(V(\xi), K(\xi)\big) \quad \text{for } \xi \in \mathbb{R}^3.$$

Then by (1.8) we know that $\xi \in \mathbb{R}^3$ satisfies $G(\xi) = \inf_{s\in\mathbb{R}^3} G(s)$ if and only if $\xi \in M$, where M is defined in (1.5). Now define $c_0 := \inf_{s\in\mathbb{R}^3} G(s)$.

Let V_∞ and K_∞ be defined as

$$V_\infty := \liminf_{|x|\to\infty} V(x), \qquad K_\infty := \limsup_{|x|\to\infty} K(x),$$

and let $c_\infty := c(V_\infty, K_\infty)$. If $V_\infty = +\infty$, define $c_\infty := +\infty$. Then, by (1.8), we get that condition (1.6) is equivalent to

$$c_0 < c_\infty. \tag{1.9}$$

2 Preliminaries

First let $u(x) = v(\varepsilon x)$, then equation (1.1) becomes the following equivalent equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(\varepsilon x)u = K(\varepsilon x)|u|^{p-1}u \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

Let $E_\varepsilon := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 < +\infty\}$ be the Hilbert subspace of $H^1(\mathbb{R}^3)$ with the norm

$$\|u\|_{E_\varepsilon} := \left(\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(\varepsilon x)u^2\right)\right)^{1/2}.$$

Then a weak solution of problem (2.1) in E_ε is a critical point of the energy functional $I_\varepsilon : E_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(\varepsilon x)u^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} K(\varepsilon x)|u|^{p+1}.$$

Moreover, $I_\varepsilon \in C^1(E_\varepsilon, \mathbb{R})$. We define the Nehari manifold for (2.1) by

$$\mathcal{N}_\varepsilon := \left\{u \in E_\varepsilon\backslash\{0\} : \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(\varepsilon x)u^2\right) + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 = \int_{\mathbb{R}^3} K(\varepsilon x)|u|^{p+1}\right\}.$$

That is,

$$\mathcal{N}_\varepsilon = \left\{u \in E_\varepsilon\backslash\{0\} : \left\langle I'_\varepsilon(u), u\right\rangle = 0\right\}.$$

By [23], we have

Lemma 2.1 *For any $u \in E_\varepsilon\backslash\{0\}$, there exists unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}_\varepsilon$ and the maximum of $I_\varepsilon(tu)$ for $t \geq 0$ is achieved at $t = t(u)$.*

We denote by $S(u) := \langle I'_\varepsilon(u), u\rangle$, then we have the following.

Lemma 2.2 *For any $\varepsilon > 0$, there exist $\sigma_\varepsilon, \tau_\varepsilon > 0$ such that, for any $u \in \mathcal{N}_\varepsilon$,*

$$\|u\|_{E_\varepsilon} \geq \sigma_\varepsilon, \qquad \left\langle S'(u), u\right\rangle \leq -\tau_\varepsilon. \tag{2.2}$$

Proof Since the embedding $E_\varepsilon \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for $2 \le r \le 6$, then we have that, for any $u \in \mathcal{N}_\varepsilon$,

$$0 < \|u\|_{E_\varepsilon}^2 \le \int_{\mathbb{R}^3} K(\varepsilon x)|u|^{p+1} \le C_1 \|K\|_\infty \|u\|_{E_\varepsilon}^{p+1},$$

where C_1 is a positive constant, and which implies the first inequality in (2.2). Furthermore,

$$\begin{aligned}\langle S'(u), u\rangle &= 2\|u\|_{E_\varepsilon}^2 + 4b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - (p+1)\int_{\mathbb{R}^3} K(\varepsilon x)|u|^{p+1} \\ &= -2\|u\|_{E_\varepsilon}^2 - (p-3)\int_{\mathbb{R}^3} K(\varepsilon x)|u|^{p+1} \le -2\|u\|_{E_\varepsilon}^2 \le -2\sigma_\varepsilon =: -\tau_\varepsilon,\end{aligned}$$

which gives the second inequality in (2.2). □

By Lemma 2.2, we know that (see Chap. 6 of [1]) $\mathcal{N}_\varepsilon$ is a C^1 manifold of codimension one in E_ε and $\mathcal{N}_\varepsilon$ is a natural constraint for I_ε, i.e., if u is a critical point of $I_\varepsilon|_{\mathcal{N}_\varepsilon}$ (I_ε constrained on $\mathcal{N}_\varepsilon$), then u is a weak solution of (2.1) in E_ε.

Now define the ground energy c_ε of functional I_ε by $c_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u)$. By Lemma 3.4 of [23], we know that there exists $\bar{c} > 0$ such that $c_\varepsilon > \bar{c}$ for each $\varepsilon > 0$ and $\limsup_{\varepsilon\to 0^+} c_\varepsilon \le c_0$, where c_0 is defined in Remark 1.3. Moreover, we can prove the following.

Lemma 2.3

$$\liminf_{\varepsilon\to 0^+} c_\varepsilon \ge c_0.$$

Proof By [23], there exists a positive ground state solution u_ε of (2.1) which satisfies $I_\varepsilon(u_\varepsilon) = c_\varepsilon$ for sufficiently small $\varepsilon > 0$. Now, by contradiction, we assume that there exist $d_0 > 0$ and a subsequence $\{u_{\varepsilon_k}\}$ of $\{u_\varepsilon\}$ such that $c_{\varepsilon_k} = I_{\varepsilon_k}(u_{\varepsilon_k}) \to c_0 - d_0$, i.e.,

$$\int_{\mathbb{R}^3} \left[\frac{1}{4}\left(a|\nabla u_{\varepsilon_k}|^2 + V(\varepsilon_k x)u_{\varepsilon_k}^2\right) + \left(\frac{1}{4} - \frac{1}{p+1}\right)K(\varepsilon_k x)|u_{\varepsilon_k}|^{p+1}\right] \to c_0 - d_0. \tag{2.3}$$

From [23], we know there exists $\{y_{\varepsilon_k}\} \subset \mathbb{R}^3$ such that $\varepsilon_k y_{\varepsilon_k} \to x_0 \in M$ (defined in (1.5)), and if we let $w_k(x) := u_{\varepsilon_k}(x + y_{\varepsilon_k})$, then $w_k \to w_0$ in $H^1(\mathbb{R}^3)$, where w_0 is the unique positive ground state solution of

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla w_0|^2\right)\Delta w_0 + V(x_0)w_0 = K(x_0)w_0^p.$$

Then $c_0 = \int_{\mathbb{R}^3}[\frac{1}{4}(a|\nabla w_0|^2 + V(x_0)w_0^2) + (\frac{1}{4} - \frac{1}{p+1})K(x_0)|w_0|^{p+1}]$ and there exists $\rho_0 > 0$ such that

$$\int_{B_{\rho_0}(0)} \left[\frac{1}{4}\left(a|\nabla w_0|^2 + V(x_0)w_0^2\right) + \left(\frac{1}{4} - \frac{1}{p+1}\right)K(x_0)|w_0|^{p+1}\right] > c_0 - \frac{1}{3}d_0. \tag{2.4}$$

By (2.3) and $w_k(x) = u_{\varepsilon_k}(x + y_{\varepsilon_k})$, we can choose large and fixed $\rho_1 > \rho_0$ such that

$$\begin{aligned}\int_{B_{\rho_1(0)}} \biggl[\frac{1}{4}\bigl(a|\nabla w_k|^2 + V(\varepsilon_k x + \varepsilon_k y_{\varepsilon_k})w_k^2\bigr) \\ + \biggl(\frac{1}{4} - \frac{1}{p+1}\biggr)K(\varepsilon_k x + \varepsilon_k y_{\varepsilon_k})|w_k|^{p+1}\biggr] < c_0 - \frac{2}{3}d_0.\end{aligned} \tag{2.5}$$

Thus letting $k \to \infty$ in (2.5), by $\varepsilon_k y_{\varepsilon_k} \to x_0$ and $w_k \to w_0$ in H^1, we have

$$\frac{1}{4}\int_{B_{\rho_1}(0)} \bigl(a|\nabla w_0|^2 + V(x_0)w_0^2\bigr) + \biggl(\frac{1}{4} - \frac{1}{p+1}\biggr)\int_{B_{\rho_1}(0)} K(x_0)|w_0|^{p+1} \le c_0 - \frac{2}{3}d_0,$$

which contradicts (2.4). □

To obtain the multiplicity result of problem (2.1), we need the following two results:

Lemma 2.4 (see Theorem 5.20 of [28]) *If $I_\varepsilon|_{\mathcal{N}_\varepsilon}$ is bounded frow below and satisfies the $(PS)_c$ condition for any $c \in [\inf_{\mathcal{N}_\varepsilon} I_\varepsilon, d]$, then $I_\varepsilon|_{\mathcal{N}_\varepsilon}$ has a minimum and I_ε^d contains at least $\operatorname{cat}_{I_\varepsilon^d} I_\varepsilon^d$ critical points of $I_\varepsilon|_{\mathcal{N}_\varepsilon}$, where $I_\varepsilon^d := \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \le d\}$.*

Lemma 2.5 (see Lemma 4.3 of [3]) *Let Γ, Ω^+, Ω^- be closed sets with $\Omega^- \subset \Omega^+$. Let $\Phi : \Omega^- \to \Gamma$, $\beta : \Gamma \to \Omega^+$ be two continuous maps such that $\beta \circ \Phi$ is homotopically equivalent to the embedding $\mathrm{Id} : \Omega^- \to \Omega^+$. Then $\operatorname{cat}_\Gamma(\Gamma) \ge \operatorname{cat}_{\Omega^+}(\Omega^-)$.*

3 Palais–Smale condition

In this section, we prove that the functional I_ε satisfies the Palais–Smale condition on $\mathcal{N}_\varepsilon$. We say that $I_\varepsilon|_{\mathcal{N}_\varepsilon}$ satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $I_\varepsilon(u_n) \to c$ and $\|I'_\varepsilon(u_n)\|_* \to 0$ contains a convergent subsequence. Here $\|I'_\varepsilon(u_n)\|_*$ denotes the norm of the derivative of I_ε restricted to $\mathcal{N}_\varepsilon$ at the point $u_n \in \mathcal{N}_\varepsilon$.

Lemma 3.1 *For $\varepsilon > 0$ sufficiently small, the constrained functional $I_\varepsilon|_{\mathcal{N}_\varepsilon}$ satisfies the $(PS)_c$ condition for $c < c_\infty$, where c_∞ is defined in Remark 1.3.*

Proof Since the ground energy c_ε of functional I_ε satisfies $\limsup_{\varepsilon\to 0^+} c_\varepsilon \le c_0$ and $c_0 < c_\infty$ by (1.9), we know that the set $\{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) < c_\infty\}$ is not empty for $\varepsilon > 0$ sufficiently small.

Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be such that

$$I_\varepsilon(u_n) \to c \quad \text{and} \quad \bigl\|I'_\varepsilon(u_n)\bigr\|_* \to 0. \tag{3.1}$$

As $\{u_n\} \subset \mathcal{N}_\varepsilon$, we have

$$\begin{aligned} I_\varepsilon(u_n) &= \biggl(\frac{1}{2} - \frac{1}{p+1}\biggr)\int_{\mathbb{R}^3} a|\nabla u_n|^2 + V(\varepsilon x)u_n^2 + \biggl(\frac{1}{4} - \frac{1}{p+1}\biggr)b\biggl(\int_{\mathbb{R}^3} |\nabla u_n|^2\biggr)^2 \\ &\ge \biggl(\frac{1}{2} - \frac{1}{p+1}\biggr)\|u_n\|_{E_\varepsilon}^2. \end{aligned}$$

Then by $I_\varepsilon(u_n) \to c$ and $c < c_\infty$ we know that $\{u_n\}$ is bounded in E_ε. Thus there exists $u \in E_\varepsilon$ and if necessary a subsequence of $\{u_n\}$ such that $u_n \rightharpoonup u$ in E_ε, $u_n \to u$ in $L^\tau_{\mathrm{loc}}(\mathbb{R}^3)$

for $1 \le \tau < 6$, and $u_n \to u$ a.e. on $\mathbb{R}^3$. We have to prove that $u_n \to u$ strongly in E_ε and $u \in \mathcal{N}_\varepsilon$.

First we show that if $\|I'_\varepsilon(u_n)\|_* \to 0$ then $I'_\varepsilon(u_n) \to 0$, which implies that $\{u_n\}$ is a $(PS)_c$ sequence for the unconstrained functional I_ε. Indeed, by $\|I'_\varepsilon(u_n)\|_* \to 0$, there exists $\mu_n \in \mathbb{R}$ such that $I'_\varepsilon(u_n) - \mu_n S'(u_n) \to 0$, where $S(u) = \langle I'_\varepsilon(u), u\rangle$. Then we have

$$0 = S(u_n) = \big\langle I'_\varepsilon(u_n), u_n\big\rangle = \mu_n\big\langle S'(u_n), u_n\big\rangle + o(1).$$

From Lemma 2.2, there exists $\tau_\varepsilon > 0$ such that $\langle S'(u_n), u_n\rangle \le -\tau_\varepsilon$, then by the above equality we have that $\mu_n \to 0$ as $n \to \infty$. By the definition of $S(u)$ and the boundedness of $\{u_n\}$ in E_ε, we know that $\|S'(u_n)\|$ is bounded. Thus from $I'_\varepsilon(u_n) = \mu_n S'(u_n) + o(1)$ we can get $I'_\varepsilon(u_n) \to 0$, as $n \to \infty$.

Now we prove if $u_n \to u$ strongly in E_ε, then $u \in \mathcal{N}_\varepsilon$. Since $u_n \in \mathcal{N}_\varepsilon$, we have

$$\int_{\mathbb{R}^3}\big(a|\nabla u_n|^2 + V(\varepsilon x)u_n^2\big) + b\bigg(\int_{\mathbb{R}^3}|\nabla u_n|^2\bigg)^2 = \int_{\mathbb{R}^3}K(\varepsilon x)|u_n|^{p+1}.$$

If $u_n \to u$ in E_ε, then passing to a limit in the above equality, we have

$$\int_{\mathbb{R}^3}\big(a|\nabla u|^2 + V(\varepsilon x)u^2\big) + b\bigg(\int_{\mathbb{R}^3}|\nabla u|^2\bigg)^2 = \int_{\mathbb{R}^3}K(\varepsilon x)|u|^{p+1},$$

which implies that $u \in \mathcal{N}_\varepsilon$.

In order to prove $u_n \to u$ in E_ε, it suffices to show that, for any $\delta > 0$, there exists $R > 0$ such that

$$\int_{|x|\ge R}\big(a|\nabla u_n|^2 + V(\varepsilon x)u_n^2\big) < \delta \quad \text{for each } n \in \mathbb{N}^+. \tag{3.2}$$

Indeed, by (3.2), we first show that $u_n \to u$ in $L^{p+1}(\mathbb{R}^3)$. For any $\delta > 0$, by (3.2), there exists $R > 0$ such that

$$\bigg(\int_{|x|\ge R}|u_n|^{p+1}\bigg)^{\frac{1}{p+1}} \le C\bigg(\int_{|x|\ge R}a|\nabla u_n|^2 + V(\varepsilon x)u_n^2\bigg)^{\frac{1}{2}} \le C\delta^{\frac{1}{2}}, \tag{3.3}$$

where $C > 0$ is a constant which is not dependent on R and n. Since $u_n \to u$ in $L^{p+1}_{\mathrm{loc}}(\mathbb{R}^3)$, we have that for the fixed δ and R in (3.3), there exists $N \in \mathbb{N}^+$ such that, for $n > N$,

$$\bigg(\int_{|x|\le R}|u_n - u|^{p+1}\bigg)^{\frac{1}{p+1}} \le \delta. \tag{3.4}$$

Combining (3.3) and (3.4), we can know that $u_n \to u$ in $L^{p+1}(\mathbb{R}^3)$. Next we show that by (3.2), we can prove $u_n \to u$ in E_ε. Note that

$$\begin{aligned}&\big\langle I'_\varepsilon(u_n) - I'_\varepsilon(u), u_n - u\big\rangle \\ &\quad = \bigg(a + b\int_{\mathbb{R}^3}|\nabla u_n|^2\bigg)\int_{\mathbb{R}^3}\nabla u_n \cdot \nabla(u_n - u)\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3} V(\varepsilon x)(u_n-u)^2-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)\\
&-\int_{\mathbb{R}^3} K(\varepsilon x)\big(|u_n|^{p-1}u_n-|u|^{p-1}u\big)(u_n-u)\\
=&\left(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2\right)\cdot\int_{\mathbb{R}^3}\big|\nabla(u_n-u)\big|^2+\int_{\mathbb{R}^3} V(\varepsilon x)(u_n-u)^2\\
&+b\left(\int_{\mathbb{R}^3}|\nabla u_n|^2-\int_{\mathbb{R}^3}|\nabla u|^2\right)\cdot\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)\\
&-\int_{\mathbb{R}^3} K(\varepsilon x)\big(|u_n|^{p-1}u_n-|u|^{p-1}u\big)(u_n-u)\\
\geq&\ \|u_n-u\|_\varepsilon^2-b\left(\int_{\mathbb{R}^3}|\nabla u|^2-\int_{\mathbb{R}^3}|\nabla u_n|^2\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)\\
&-\int_{\mathbb{R}^3} K(\varepsilon x)\big(|u_n|^{p-1}u_n-|u|^{p-1}u\big)(u_n-u),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|u_n-u\|_\varepsilon^2\leq{}&\big\langle I_\varepsilon'(u_n)-I_\varepsilon'(u),u_n-u\big\rangle+b\left(\int_{\mathbb{R}^3}|\nabla u|^2-\int_{\mathbb{R}^3}|\nabla u_n|^2\right)\\
&\cdot\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)+\int_{\mathbb{R}^3} K(\varepsilon x)\big(|u_n|^{p-1}u_n-|u|^{p-1}u\big)(u_n-u).
\end{aligned}
$$

Since $u_n\rightharpoonup u$ and $I_\varepsilon'(u_n)\to 0$, we have $\langle I_\varepsilon'(u_n)-I_\varepsilon'(u),u_n-u\rangle\to 0$ as $n\to\infty$. By the boundedness of $\{u_n\}$ in E_ε, we have

$$
b\left(\int_{\mathbb{R}^3}|\nabla u|^2-\int_{\mathbb{R}^3}|\nabla u_n|^2\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)\to 0,
$$

as $n\to\infty$. Furthermore,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} K(\varepsilon x)\big(|u_n|^{p-1}u_n-|u|^{p-1}u\big)(u_n-u)\right|\\
&\quad\leq\|K\|_\infty\left(\int_{\mathbb{R}^3}\big||u_n|^{p-1}u_n-|u|^{p-1}u\big|^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\left(\int_{\mathbb{R}^3}|u_n-u|^{p+1}\right)^{\frac{1}{p+1}}.
\end{aligned}
$$

Since $\{u_n\}$ is bounded in $L^{p+1}(\mathbb{R}^3)$ and $u_n\to u$ in $L^{p+1}(\mathbb{R}^3)$, we have

$$
\int_{\mathbb{R}^3} K(\varepsilon x)\big(|u_n|^{p-1}u_n-|u|^{p-1}u\big)(u_n-u)\to 0,
$$

as $n\to\infty$. Thus we have $\|u_n-u\|_\varepsilon\to 0$ as $n\to\infty$, i.e., $u_n\to u$ in E_ε.

Now we are in a position to prove (3.2) to complete the proof of Lemma 3.1. By contradiction assume that for some subsequence $\{u_{n_k}\}$ (we denote $\{u_k\}$ for the simplicity of notations) and some $\delta_0>0$

$$
\int_{|x|\geq k} a|\nabla u_k|^2+V(\varepsilon x)u_k^2\geq\delta_0 \tag{3.5}
$$

for any k. By the choice of c and Remark 1.3, there exists $\eta>0$ such that $c<c(V_\infty-\eta,K_\infty+\eta)=:c_\eta$ and $c_\eta<c_\infty$. Let $R(\eta)>0$ be an integer and such that $V(\varepsilon x)\geq V_\infty-\eta$ and $K(\varepsilon x)\leq$

$K_\infty + \eta$ for $|x| \geq R(\eta)$. For any $r > 0$, we define $A_r := \{x \in \mathbb{R}^3 : r \leq |x| \leq r+1\}$. Then as in [8], we can know that there exists $r > R(\eta)$ and if necessary a subsequence of $\{u_k\}$ such that

$$\int_{A_r} a|\nabla u_k|^2 + V(\varepsilon x)u_k^2 < \eta. \tag{3.6}$$

Now we fix $r = r(\eta) > R(\eta)$ so that (3.6) holds. Let $\rho \in C^\infty(\mathbb{R}^3)$ be such that $\rho(x) = 0$ for $|x| \leq r$, $\rho(x) = 1$ for $|x| \geq r+1$, $0 \leq \rho \leq 1$, and $|\nabla\rho(x)| \leq 2$ for any $x \in \mathbb{R}^3$. Define $w_k := \rho u_k$. As $u_k \in \mathcal{N}_\varepsilon$, we have

$$I_\varepsilon(u_k) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a|\nabla u_k|^2 + V(\varepsilon x)u_k^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)b\left(\int_{\mathbb{R}^3}|\nabla u_k|^2\right)^2.$$

Define

$$\bar{I}_\varepsilon(w_k) := \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a|\nabla w_k|^2 + V(\varepsilon x)w_k^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)b\left(\int_{\mathbb{R}^3}|\nabla w_k|^2\right)^2,$$

then by the definition of w_k and (3.6), we have

$$\bar{I}_\varepsilon(w_k) \leq I_\varepsilon(u_k) + O(\eta), \tag{3.7}$$

where $|O(\eta)| < C\eta$ and $C > 0$ is a constant.

Now let $\theta_k > 0$ be such that $\theta_k w_k \in \mathcal{N}_\varepsilon$. If $\theta_k \leq 1$ (up to a subsequence) for $k = 1, 2, 3, \ldots$, then by (3.7) we have

$$\begin{aligned} &I_\varepsilon(\theta_k w_k) \\ &\quad= \left(\frac{1}{2} - \frac{1}{p+1}\right)\theta_k^2\int_{\mathbb{R}^3} a|\nabla w_k|^2 + V(\varepsilon x)w_k^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)b\theta_k^4\left(\int_{\mathbb{R}^3}|\nabla w_k|^2\right)^2 \\ &\quad\leq \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a|\nabla w_k|^2 + V(\varepsilon x)w_k^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)b\left(\int_{\mathbb{R}^3}|\nabla w_k|^2\right)^2 \\ &\quad= \bar{I}_\varepsilon(w_k) \leq I_\varepsilon(u_k) + O(\eta). \end{aligned} \tag{3.8}$$

Now we assume $\theta_k > 1$ for each k. Since $\langle I'_\varepsilon(\theta_k w_k), \theta_k w_k\rangle = 0$, we have $\langle I'_\varepsilon(w_k), w_k\rangle > 0$ by Lemma 2.1. Denote $\tilde{I}(w_k)$ by

$$\tilde{I}(w_k) = \int_{\mathbb{R}^3} a|\nabla w_k|^2 + V(\varepsilon x)w_k^2 + b\int_{\mathbb{R}^3}|\nabla u_k|^2\int_{\mathbb{R}^3}|\nabla w_k|^2 - \int_{\mathbb{R}^3} K(\varepsilon x)|w_k|^{p+1},$$

then we have

$$\left|\langle I'_\varepsilon(u_k), w_k\rangle - \tilde{I}(w_k)\right| \leq C_1\int_{A_r} a|\nabla u_k|^2 + V(\varepsilon x)u_k^2,$$

where $C_1 > 0$ is a constant which does not depend on r. Then, by (3.1) and (3.6), we have $\tilde{I}(w_k) = O(\eta) + o(1)$. Since $\langle I_\varepsilon'(w_k), w_k\rangle > 0$, we have

$$\begin{aligned}\langle I_\varepsilon'(w_k), w_k\rangle &= \tilde{I}(w_k) + b\left(\int_{\mathbb{R}^3} |\nabla w_k|^2 - \int_{\mathbb{R}^3} |\nabla u_k|^2\right)\int_{\mathbb{R}^3} |\nabla w_k|^2 \\ &\le \tilde{I}(w_k) + b\int_{A_r} |\nabla w_k|^2 \int_{\mathbb{R}^3} |\nabla w_k|^2 \\ &\le \tilde{I}(w_k) + O(\eta) = O(\eta) + o(1).\end{aligned} \tag{3.9}$$

By the definition of w_k and (3.5), we have

$$\int_{\mathbb{R}^3} a|\nabla w_k|^2 + V(\varepsilon x)w_k^2 \ge \delta_0 + O(\eta). \tag{3.10}$$

Then by $\theta_k w_k \in \mathcal{N}_\varepsilon$, (3.9) and (3.10), we have that $\{\theta_k\}$ is bounded and (see the similar result (6.13) in [8])

$$\theta_k = 1 + O(\eta) + o(1). \tag{3.11}$$

Thus by (3.7) and (3.11) we have

$$I_\varepsilon(\theta_k w_k) \le I_\varepsilon(u_k) + O(\eta). \tag{3.12}$$

From (3.8) and (3.12), up to a subsequence of $\{w_k\}$, we have

$$I_\varepsilon(\theta_k w_k) \le I_\varepsilon(u_k) + O(\eta). \tag{3.13}$$

Let $\tilde{w}_k := \theta_k w_k$, and let $\tilde{\theta}_k$ be such that $\tilde{\theta}_k \tilde{w}_k \in \mathcal{N}_\eta$, the Nehari manifold defined as in Remark 1.3, with $m = V_\infty - \eta$ and $n = K_\infty + \eta$ in (1.7). From

$$\begin{aligned}&\int_{\mathbb{R}^3} a|\nabla \tilde{w}_k|^2 + (V_\infty - \eta)\tilde{w}_k^2 + b\left(\int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2\right)^2 \\ &\quad\le \int_{\mathbb{R}^3} a|\nabla \tilde{w}_k|^2 + V(\varepsilon x)\tilde{w}_k^2 + b\left(\int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2\right)^2 \\ &\quad= \int_{\mathbb{R}^3} K(\varepsilon x)|\tilde{w}_k|^{p+1} \\ &\quad\le \int_{\mathbb{R}^3} (K_\infty + \eta)|\tilde{w}_k|^{p+1},\end{aligned}$$

we can know that $\tilde{\theta}_k \le 1$, the above equality holds because $\tilde{w}_k = \theta_k w_k \in \mathcal{N}_\varepsilon$. Now, by Lemma 2.1, the function

$$\begin{aligned}h(t) := {}&\frac{t^2}{2}\int_{\mathbb{R}^3} \left(a|\nabla \tilde{w}_k|^2 + V(\varepsilon x)\tilde{w}_k^2\right) + \frac{t^4}{4} b\left(\int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2\right)^2 \\ &- \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3} K(\varepsilon x)|\tilde{w}_k|^{p+1}\end{aligned}$$

is nondecreasing for $t \in (0,1)$. Thus, by (3.13) and (3.1),

$$\begin{aligned} c_\eta &\le \frac{\tilde\theta_k^2}{2}\int_{\mathbb{R}^3} a|\nabla \tilde w_k|^2 + (V_\infty - \eta)\tilde w_k^2 + \frac{b}{4}\tilde\theta_k^4\left(\int_{\mathbb{R}^3}|\nabla \tilde w_k|^2\right)^2 \\ &\quad - \frac{\tilde\theta_k^{p+1}}{p+1}\int_{\mathbb{R}^3}(K_\infty + \eta)|\tilde w_k|^{p+1} \le \frac{\tilde\theta_k^2}{2}\int_{\mathbb{R}^3} a|\nabla \tilde w_k|^2 + V(\varepsilon x)\tilde w_k^2 \\ &\quad + \frac{b}{4}\tilde\theta_k^4\left(\int_{\mathbb{R}^3}|\nabla \tilde w_k|^2\right)^2 - \frac{\tilde\theta_k^{p+1}}{p+1}\int_{\mathbb{R}^3} K(\varepsilon x)|\tilde w_k|^{p+1} \\ &\le \frac{1}{2}\int_{\mathbb{R}^3} a|\nabla \tilde w_k|^2 + V(\varepsilon x)\tilde w_k^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla \tilde w_k|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} K(\varepsilon x)|\tilde w_k|^{p+1} \\ &= I_\varepsilon(\tilde w_k) = I_\varepsilon(\theta_k w_k) \le I_\varepsilon(u_k) + O(\eta) \le c + O(\eta) + o(1). \end{aligned}$$

Letting $k \to \infty$, $\eta \to 0$ and by the continuity of c_η with respect to η (see Remark 1.3), we know that $c_\infty \le c$, a contradiction which concludes the proof. □

4 The maps Φ_ε and β_ε

In this section we construct two mappings Φ_ε and β_ε in order to apply Lemma 2.5 to prove Theorem 1.1.

Let $\delta > 0$ be fixed and $\eta \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \eta \le 1$, $\eta = 1$ on $B_1(0)$, $\eta = 0$ on $\mathbb{R}^3 \setminus B_2(0)$, $|\nabla \eta| \le C$ for some $C > 0$. For any $y \in M$ (defined in (1.5)), we define

$$\Psi_{\varepsilon,y}(x) = \eta\left(\frac{\varepsilon x - y}{\sqrt{\varepsilon}}\right) w^y\left(\frac{\varepsilon x - y}{\varepsilon}\right),$$

where w^y is the unique positive ground state solution (see [23]) of

$$\begin{cases} -(a + b\int_{\mathbb{R}^3}|\nabla v|^2)\Delta v + V(y)v = K(y)|v|^{p-1}v & \text{in } \mathbb{R}^3, \\ v > 0, \quad v \in H^1(\mathbb{R}^3). \end{cases} \tag{4.1}$$

Let w be such that $w^y = \lambda w(\mu x)$, where $\mu^2 = V(y)$ and $\lambda = (V(y)/K(y))^{\frac{1}{p-1}}$, then w satisfies

$$\begin{cases} -(a + b\frac{\lambda^2}{\mu}\int_{\mathbb{R}^3}|\nabla v|^2)\Delta v + v = |v|^{p-1}v & \text{in } \mathbb{R}^3, \\ v > 0, \quad v \in H^1(\mathbb{R}^3). \end{cases} \tag{4.2}$$

Since $\frac{\lambda^2}{\mu} = \frac{V^{\frac{2}{p-1}-\frac{1}{2}}(y)}{K^{\frac{2}{p-1}}(y)}$, then by the definition of M we know that, for any $y \in M$,

$$\frac{\lambda^2}{\mu} = \frac{V^{\frac{2}{p-1}-\frac{1}{2}}(y)}{K^{\frac{2}{p-1}}(y)} \equiv L,$$

where L is a positive constant. Thus we have that, for $y \in M$,

$$w^y = \lambda w(\mu x), \tag{4.3}$$

where w is the unique positive ground state solution of (4.2) with $\frac{\lambda^2}{\mu} = L$.

Now let $t_{\varepsilon,y} > 0$ be such that $I_\varepsilon(t_{\varepsilon,y}\Psi_{\varepsilon,y}) = \max_{t\geq 0} I_\varepsilon(t\Psi_{\varepsilon,y})$ and $\frac{dI_\varepsilon(t\Psi_{\varepsilon,y})}{dt}|_{t=t_{\varepsilon,y}} = 0$. We define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ by $\Phi_\varepsilon(y) := t_{\varepsilon,y}\Psi_{\varepsilon,y}$.

Lemma 4.1 *Uniformly for $y \in M$, we have*

$$\lim_{\varepsilon\to 0^+} I_\varepsilon\big(\Phi_\varepsilon(y)\big) = c_0, \tag{4.4}$$

where c_0 is defined in Remark 1.3.

Proof We first show that $t_{\varepsilon,y} \to 1$ as $\varepsilon \to 0^+$. Since $t_{\varepsilon,y}\Psi_{\varepsilon,y} = \Phi_\varepsilon(y) \in \mathcal{N}_\varepsilon$, we have

$$\int_{\mathbb{R}^3}\big(a|\nabla\Psi_{\varepsilon,y}|^2 + V(\varepsilon x)\Psi_{\varepsilon,y}^2\big) + t_{\varepsilon,y}^2 b\biggl(\int_{\mathbb{R}^3}|\nabla\Psi_{\varepsilon,y}|^2\biggr)^2 = t_{\varepsilon,y}^{p-1}\int_{\mathbb{R}^3} K(\varepsilon x)|\Psi_{\varepsilon,y}|^{p+1}.$$

By the definition of $\Psi_{\varepsilon,y}$ and (4.3), after a change of variable, we get

$$\begin{aligned}&\int_{B_{\frac{\mu}{\sqrt{\varepsilon}}}(0)}\biggl(La|\nabla w|^2 + \frac{L}{\mu^2}V\biggl(\frac{\varepsilon x}{\mu}+y\biggr)w^2\biggr) + o(1) + t_{\varepsilon,y}^2 b\biggl(L\int_{B_{\frac{\mu}{\sqrt{\varepsilon}}}(0)}|\nabla w|^2 + o(1)\biggr)^2\\ &\quad = t_{\varepsilon,y}^{p-1}\biggl(\frac{L}{K(y)}\int_{B_{\frac{\mu}{\sqrt{\varepsilon}}}(0)} K\biggl(\frac{\varepsilon x}{\mu}+y\biggr)w^{p+1} + o(1)\biggr).\end{aligned} \tag{4.5}$$

By the definition of μ and (H), we know that as $\varepsilon \to 0^+$, $\frac{\mu}{\sqrt{\varepsilon}} \to +\infty$ uniformly for $y \in M$. Moreover, for $|x| \leq \frac{\mu}{\sqrt{\varepsilon}}$, $|\frac{\varepsilon x}{\mu} + y|$ is bounded and $\frac{\varepsilon x}{\mu} + y \to y$ as $\varepsilon \to 0^+$ uniformly for $y \in M$. Then we have as $\varepsilon \to 0^+$ and uniformly for $y \in M$,

$$\begin{aligned}&\int_{B_{\frac{\mu}{\sqrt{\varepsilon}}}(0)} La|\nabla w|^2 \to La\int_{\mathbb{R}^3}|\nabla w|^2,\\ &\int_{B_{\frac{\mu}{\sqrt{\varepsilon}}}(0)}\frac{L}{\mu^2}V\biggl(\frac{\varepsilon x}{\mu}+y\biggr)w^2 \to \int_{\mathbb{R}^3}\frac{L}{\mu^2}V(y)w^2 = L\int_{\mathbb{R}^3}w^2,\\ &\frac{L}{K(y)}\int_{B_{\frac{\mu}{\sqrt{\varepsilon}}}(0)} K\biggl(\frac{\varepsilon x}{\mu}+y\biggr)w^{p+1} \to \frac{L}{K(y)}\int_{\mathbb{R}^3}K(y)w^{p+1} = L\int_{\mathbb{R}^3}w^{p+1}.\end{aligned}$$

Now assume that there exist t_0, T_0 such that $0 < t_0 \leq t_{\varepsilon,y} \leq T_0$, and let $t_{\varepsilon,y} \to T > 0$ as $\varepsilon \to 0^+$, then by the above estimates we have

$$\int_{\mathbb{R}^3}\big(a|\nabla w|^2 + w^2\big) + bT^2L\biggl(\int_{\mathbb{R}^3}|\nabla w|^2\biggr)^2 = T^{p-1}\int_{\mathbb{R}^3}w^{p+1}.$$

Since w is the ground state solution of (4.2), we have

$$\int_{\mathbb{R}^3}\big(a|\nabla w|^2 + w^2\big) + bL\biggl(\int_{\mathbb{R}^3}|\nabla w|^2\biggr)^2 = \int_{\mathbb{R}^3}w^{p+1},$$

these imply that

$$\big(T^{p-1}-1\big)\int_{\mathbb{R}^3}\big(a|\nabla w|^2 + w^2\big) + bL\big(T^{p-1}-T^2\big)\biggl(\int_{\mathbb{R}^3}|\nabla w|^2\biggr)^2 = 0.$$

If $T < 1$, then the left part of the above equality is less than 0, and if $T > 1$, it will be larger than 0, which yields that $T = 1$.

Now we prove that $t_{\varepsilon,y} \nrightarrow 0$. Otherwise, from (4.5), we have

$$\int_{\mathbb{R}^3} \left(a|\nabla w|^2 + w^2\right) = 0,$$

which is a contradiction. Also from (4.5), we have that $t_{\varepsilon,y} \nrightarrow +\infty$ as $p-1 > 2$. By the above arguments, we can see that $t_{\varepsilon,y} \to 1$ uniformly for $y \in M$ as $\varepsilon \to 0^+$.

Note that

$$\begin{aligned}
I_\varepsilon\big(\Phi_\varepsilon(y)\big) &= I_\varepsilon(t_{\varepsilon,y}\Psi_{\varepsilon,y}) \\
&= \frac{t_{\varepsilon,y}^2}{2}\left(L\int_{\mathbb{R}^3}\left(a|\nabla w|^2 + w^2\right) + o(1)\right) + \frac{t_{\varepsilon,y}^4}{4} b\left(\left(\int_{\mathbb{R}^3} L|\nabla w|^2\right)^2 + o(1)\right) \\
&\quad - \frac{t_{\varepsilon,y}^{p+1}}{p+1}\left(L\int_{\mathbb{R}^3} w^{p+1} + o(1)\right) = \frac{L}{2}\int_{\mathbb{R}^3}\left(a|\nabla w|^2 + w^2\right) + \frac{L^2}{4} b\left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^2 \\
&\quad - \frac{L}{p+1}\int_{\mathbb{R}^3} w^{p+1} + o(1) = LI_L(w) + o(1),
\end{aligned} \tag{4.6}$$

where I_L is the energy functional of equation (4.2) with $\frac{\lambda^2}{\mu} = L$ in it. Let I^y be the energy functional of (4.1), then we have

$$\begin{aligned}
c_0 &= I^y\big(w^y\big) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} a\left|\nabla w^y\right|^2 + V(y)\left(w^y\right)^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}\left|\nabla w^y\right|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} K(y)\left(w^y\right)^{p+1} \\
&= \frac{1}{2}\frac{\lambda^2}{\mu}\int_{\mathbb{R}^3} a|\nabla w|^2 + \frac{1}{2}\frac{\lambda^2}{\mu^3}\int_{\mathbb{R}^3} V(y)w^2 + \frac{b}{4}\frac{\lambda^4}{\mu^2}\left(\int_{\mathbb{R}^3}|\nabla w|^2\right)^2 \\
&\quad - \frac{1}{p+1}\frac{\lambda^{p+1}}{\mu^3}\int_{\mathbb{R}^3} K(y)w^{p+1} \\
&= L\left(\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla w|^2 + w^2\right) + \frac{L}{4} b\left(\int_{\mathbb{R}^3}|\nabla w|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} w^{p+1}\right) = LI_L(w),
\end{aligned}$$

thus from (4.6), we prove that $\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = c_0$. □

Remark 4.2 If there is no competing potential function $K(x)$ in (1.1), i.e., $K(x) \equiv 1$, then in equation (4.1), $K(y) \equiv 1$. In this case, for different $y \in M$, $V(y)$ is the same, then the positive ground state solution w^y of (4.1) is the same function for every $y \in M$. But in our case, because of the competing function $K(y)$ in (4.1), the ground state solution w^y may change for different $y \in M$, this causes troubles in the proof of (4.4), and we develop the technique of rescaling to solve the problem.

Let $\rho > 0$ be such that $M_\delta \subset B_\rho(0) := \{x \in \mathbb{R}^3 : |x| \leq \rho\}$. Define $\gamma : \mathbb{R}^3 \to \mathbb{R}^3$ by $\gamma(x) = x$ for $|x| \leq \rho$ and $\gamma(x) = \rho x/|x|$ for $|x| \geq \rho$. Consider the mapping $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^3$ given by $\beta_\varepsilon(u) := \frac{\int_{\mathbb{R}^3}\gamma(\varepsilon x)u^2}{\int_{\mathbb{R}^3}u^2}$, then as the proof in [8] and by (4.3) we have that

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon\big(\Phi_\varepsilon(y)\big) = y \quad \text{uniformly for } y \in M. \tag{4.7}$$

Now define $h(\varepsilon) := \sup_{y\in M} |I_\varepsilon(\Phi_\varepsilon(y)) - c_0|$, then Lemma 4.1 yields that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Let

$$\tilde{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \le c_0 + h(\varepsilon)\}, \tag{4.8}$$

then by the definition of $h(\varepsilon)$ we know that, for any $y \in M$ and $\varepsilon > 0$, $\Phi_\varepsilon(y) \in \tilde{\mathcal{N}}_\varepsilon$ and $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$.

Lemma 4.3 *Let $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$. Then there exists $\{y_n\} \subset \mathbb{R}^3$ such that the sequence $\{u_n(x + y_n)\}$ has a convergent subsequence in $H^1(\mathbb{R}^3)$ and $\varepsilon_n y_n \to y \in M$.*

Proof As in [23], for $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$, we define a measure μ_n on $\mathbb{R}^3$ by

$$\mu_n(\Omega) = \int_\Omega \left[\frac{1}{4}\big(a|\nabla u_n|^2 + V(\varepsilon_n x)u_n^2\big) + \left(\frac{1}{4} - \frac{1}{p+1}\right)K(\varepsilon_n x)|u_n|^{p+1}\right].$$

Since $0 \le \mu_n(\mathbb{R}^3) = I_{\varepsilon_n}(u_n) \le c_0 + h(\varepsilon_n)$, then along a subsequence if necessary, as $\varepsilon_n \to 0^+$,

$$\mu_n(\mathbb{R}^3) \to \tilde{c} \le c_0. \tag{4.9}$$

Moreover, let $\bar{V} = \inf_{x\in\mathbb{R}^3} V(x)$ and $\bar{K} = \sup_{x\in\mathbb{R}^3} K(x)$, then by Lemma 3.3 of [23], $\tilde{c} \ge \bar{c} > 0$ where $\bar{c} = c(\bar{V}, \bar{K})$ is defined in Remark 1.3.

By the concentration-compactness lemma of P.L. Lions in [19] and as the proof in Lemma 4.1 in [23], we know that there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that, for any $h > 0$, there is $\rho > 0$ such that

$$\int_{B_\rho(y_n)} d\mu_n \ge \tilde{c} - h. \tag{4.10}$$

Now we prove that $\{\varepsilon_n y_n\}$ is bounded. Otherwise, assume that $|\varepsilon_n y_n| \to \infty$ as $n \to \infty$. Since $\mu_n(\mathbb{R}^3)$ is bounded, we know that $w_n := u_n(x + y_n)$ is bounded in $H^1(\mathbb{R}^3)$. Therefore there exists $w_0 \in H^1(\mathbb{R}^3)$ such that up to a subsequence, $w_n \rightharpoonup w_0$ in $H^1(\mathbb{R}^3)$, $w_n \to w_0$ in $L^\tau_{\mathrm{loc}}(\mathbb{R}^3)$ for $1 \le \tau < 6$, and almost everywhere in $\mathbb{R}^3$. Furthermore, by (4.10), we can prove that $w_n \to w_0$ in $L^\tau(\mathbb{R}^3)$ for $1 \le \tau < 6$ and $w_0 \neq 0$ in $H^1(\mathbb{R}^3)$. Let $\theta_\infty > 0$ be such that $\theta_\infty w_0 \in \mathcal{N}_\infty$, the Nehari manifold associated to (1.7) with $m = V_\infty$ and $n = K_\infty$ in it. Then as the proof in the Appendix of [8], we have $\theta_\infty \le 1$; and furthermore, $\{\varepsilon_n y_n\}$ is bounded.

Assume that $\{\varepsilon_n y_n\}$ converges to some y (up to a subsequence), we now prove that $y \in M$ and that $w_n \to w$ strongly in $H^1(\mathbb{R}^3)$. Since $u_n \in \mathcal{N}_{\varepsilon_n}$ and $w_n = u_n(x + y_n)$, we have

$$\int_{\mathbb{R}^3} a|\nabla w_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n)w_n^2 + b\left(\int_{\mathbb{R}^3} |\nabla w_n|^2\right)^2 = \int_{\mathbb{R}^3} K(\varepsilon_n x + \varepsilon_n y_n)|w_n|^{p+1}.$$

Taking the lower limit of both sides of the above equality and by $\varepsilon_n y_n \to y$, we have

$$\int_{\mathbb{R}^3} a|\nabla w|^2 + V(y)w^2 + b\left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^2 \le \int_{\mathbb{R}^3} K(y)|w|^{p+1}.$$

Now let $\theta_y > 0$ be such that $\theta_y w \in \mathcal{N}_y$, the Nehari manifold associated to (4.1), we have that $\theta_y \le 1$ by Lemma 2.1. Let I^y be the energy functional associated to (4.1). Then

$$\begin{aligned} c_0 &\le c\big(V(y),K(y)\big) \le I^y(\theta_y w) \\ &= \frac{1}{4}\theta_y^2 \int_{\mathbb{R}^3} a|\nabla w|^2 + V(y)w^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\theta_y^{p+1} \int_{\mathbb{R}^3} K(y)|w|^{p+1} \\ &\le \frac{1}{4} \int_{\mathbb{R}^3} a|\nabla w|^2 + V(y)w^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) \int_{\mathbb{R}^3} K(y)|w|^{p+1} \\ &\le \liminf_{n\to\infty} \left[\frac{1}{4} \int_{\mathbb{R}^3} a|\nabla w_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n)w_n^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) \right. \\ &\quad \left. \cdot \int_{\mathbb{R}^3} K(\varepsilon_n x + \varepsilon_n y_n)|w_n|^{p+1}\right] = \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = \tilde{c} \le c_0, \end{aligned} \tag{4.11}$$

which implies that $\theta_y = 1$ and $c(V(y), K(y)) = c_0$. Thus we have $y \in M$. Moreover, $I_y(w) = c_0$, hence w is a ground state solution of (4.1). The strong convergence $w_n \to w$ in $L^\tau(\mathbb{R}^3)$ for $1 \le \tau < 6$ and (4.11) give

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} a|\nabla w_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n)w_n^2 = \int_{\mathbb{R}^3} a|\nabla w|^2 + V(y)w^2. \tag{4.12}$$

From (4.12) we can prove that $w_n \to w$ in $H^1(\mathbb{R}^3)$. □

Lemma 4.4 *For any* $\delta > 0$, *we have*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\tilde{\mathcal{N}}_\varepsilon} \operatorname{dist}\big(\beta_\varepsilon(u), M_\delta\big) = 0.$$

Proof The proof is similar to the proof of Lemma 5.1 in [8] or Lemma 4.7 in [13], we omit it here. □

5 Proof of Theorem 1.1

For $\delta > 0$, by Lemma 4.1, Lemma 4.4, and (4.7), there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \tilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined. Moreover, by (4.7), the mapping $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the inclusion $\mathrm{Id} : M \to M_\delta$. Now set $\tilde{\mathcal{N}}_\varepsilon^+ := \tilde{\mathcal{N}}_\varepsilon \cap \{u \in \mathcal{N}_\varepsilon : u \ge 0 \text{ in } \mathbb{R}^3\}$, then similar to [8] (or [7]), by Lemma 2.5 we have that $\operatorname{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon^+) \ge \operatorname{cat}_{M_\delta}(M)$; and furthermore, $\operatorname{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon) \ge 2\operatorname{cat}_{M_\delta}(M)$. Lemma 2.4 shows that I_ε has at least $2\operatorname{cat}_{M_\delta}(M)$ critical points on $\tilde{\mathcal{N}}_\varepsilon$. Now, in order to prove Theorem 1.1, we only need to show that the critical point $u \in \tilde{\mathcal{N}}_\varepsilon$ cannot change sign for sufficiently small $\varepsilon > 0$. Indeed, if $u = u^+ + u^-$ with $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. First, because $u \in \tilde{\mathcal{N}}_\varepsilon$, we have

$$I_\varepsilon(u) \le c_0 + h(\varepsilon), \tag{5.1}$$

where $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Since $u \in \mathcal{N}_\varepsilon$, we have

$$\begin{aligned}
I_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_{E_\varepsilon}^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\
&\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u^+\|_{E_\varepsilon}^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) b\left(\int_{\mathbb{R}^3} |\nabla u^+|^2\right)^2 \\
&\quad + \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u^-\|_{E_\varepsilon}^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) b\left(\int_{\mathbb{R}^3} |\nabla u^-|^2\right)^2 \\
&=: \tilde{I}_\varepsilon(u^+) + \tilde{I}_\varepsilon(u^-),
\end{aligned} \tag{5.2}$$

where $\tilde{I}_\varepsilon(u)$ is defined by $\tilde{I}_\varepsilon(u) := (\frac{1}{2} - \frac{1}{p+1})\|u\|_{E_\varepsilon}^2 + (\frac{1}{4} - \frac{1}{p+1}) b(\int_{\mathbb{R}^3} |\nabla u|^2)^2$.

Since $u^+ \not\equiv 0$, there exists $t^+ > 0$ such that $t^+u^+ \in \mathcal{N}_\varepsilon$. Multiplying equation (2.1) by u^+ and integrating over $\mathbb{R}^3$, we have

$$\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\int_{\mathbb{R}^3} |\nabla u^+|^2 + \int_{\mathbb{R}^3} V(\varepsilon x) u^{+2} - \int_{\mathbb{R}^3} K(\varepsilon x)|u^+|^{p+1} = 0,$$

which implies that

$$\begin{aligned}
\langle I_\varepsilon'(u^+), u^+\rangle &= \left(a + b\int_{\mathbb{R}^3} |\nabla u^+|^2\right)\int_{\mathbb{R}^3} |\nabla u^+|^2 + \int_{\mathbb{R}^3} V(\varepsilon x) u^{+2} \\
&\quad - \int_{\mathbb{R}^3} K(\varepsilon x)|u^+|^{p+1} < 0.
\end{aligned} \tag{5.3}$$

Since $t^+u^+ \in \mathcal{N}_\varepsilon$, we have $\langle I_\varepsilon{}'(t^+u^+), t^+u^+\rangle = 0$. Then from (5.3) we get that $0 < t^+ < 1$. Now

$$\begin{aligned}
c_\varepsilon &= \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u) \leq I_\varepsilon(t^+u^+) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right) t^{+2}\|u^+\|_{E_\varepsilon}^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) b t^{+4}\left(\int_{\mathbb{R}^3} |\nabla u^+|^2\right)^2 \\
&< \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u^+\|_{E_\varepsilon}^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) b\left(\int_{\mathbb{R}^3} |\nabla u^+|^2\right)^2 = \tilde{I}_\varepsilon(u^+).
\end{aligned} \tag{5.4}$$

Similar to (5.4), we can also prove that $\tilde{I}_\varepsilon(u^-) > c_\varepsilon$. Now by (5.2), we have that $I_\varepsilon(u) \geq \tilde{I}_\varepsilon(u^+) + \tilde{I}_\varepsilon(u^-) > 2c_\varepsilon$, which contradicts (5.1) by Lemma 2.3. Thus we can assume that there exist at least $\mathrm{cat}_{M_\delta}(M)$ critical points that are positive on $\mathbb{R}^3$ and by the maximum principle they are strictly positive. Now the proof of Theorem 1.1 is complete.

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