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Convergence rates of nonlinear Stokes problems in homogenization

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Abstract

In this paper, we study the convergence rates of solutions in homogenization of nonlinear Stokes Dirichlet problems. The main difficulty of this work is twofold. On the one hand, the nonlinear Stokes problems do not fit the standard framework of second-order elliptic equations in divergence form. On the other hand, nonlinear problems may cause new difficulties in the estimation of the quantity as well as first-order approximate term. As a consequence, we establish the sharp rates of convergence in H^1 and L^2. This work may be regarded as an extension of the approach for the linear Stokes problems to the nonlinear case.

MSC: Primary 35J15; secondary 35J25

Keywords: Homogenization; Convergence rates; Stokes problems; Smoothing operators

1 Introduction

The main purpose of this paper is to establish the sharp rates of convergence in H^1 and L^2 for nonlinear Stokes problems with the Dirichlet boundary condition. More precisely, let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $n \geq 3$. Let $u_\varepsilon \in H^1(\Omega;\mathbb{R}^n)$ be a weak solution to the following problem, which arose in fluid dynamics with porous media:

$$\begin{cases} L_\varepsilon u_\varepsilon + \nabla p_\varepsilon = F & \text{in } \Omega, \\ \operatorname{div} u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with the compatibility condition

$$\int_\Omega f\,dx = \int_{\partial\Omega} g\cdot n\,d\sigma, \tag{1.2}$$

where n is the outward unit normal to $\partial\Omega$.

Throughout this paper, the summation convention is used. The nonlinear operator L_ε is defined by

$$L_\varepsilon = -\operatorname{div} A(x/\varepsilon, \nabla). \tag{1.3}$$

We will assume that the function A satisfies the periodicity condition

$$A(y+Y,\xi)=A(y,\xi) \quad \text{for } Y=[0,1)^n \simeq \mathbb{R}^n/\mathbb{Z}^n, \tag{1.4}$$

coerciveness and growth conditions

$$\langle A(y,\xi)-A(y,\xi'),\xi-\xi'\rangle \geq \lambda_1|\xi-\xi'|^2, \tag{1.5}$$

$$|A(y,\xi)-A(y,\xi')| \leq \frac{1}{\lambda_1}|\xi-\xi'|, \tag{1.6}$$

for all $y\in\mathbb{R}^n$ and $\xi,\xi'\in\mathbb{R}^n$, where $\lambda_1>0$. We impose the smoothness condition

$$|A(y,\xi)-A(y',\xi)| \leq \lambda_2|y-y'|^{\alpha}, \quad F\in H^{-1}(\Omega), f\in L^2(\Omega), g\in H^{1/2}(\partial\Omega), \tag{1.7}$$

where $\lambda_2>0$ and $0<\alpha\leq 1$. Without loss of generality, we also assume that

$$\frac{1}{|\Omega|}\int_\Omega p_\varepsilon\,dx=0. \tag{1.8}$$

Associated with (1.1) is the homogenized problem

$$\begin{cases} L_0u_0+\nabla p_0=F & \text{in } \Omega,\\ \operatorname{div} u_0=f & \text{in } \Omega,\\ u_0=g & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

The homogenized operator is defined by

$$L_0=-\operatorname{div} Q(\nabla),$$

where the function Q is given, for each $\xi\in\mathbb{R}^n$, by

$$Q(\xi)=\int_Y A\big(y,\xi+\nabla_y N(y,\xi)\big)\,dy. \tag{1.10}$$

The periodic functions $(N,\chi)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ are the so-called correctors, satisfying the following cell problem:

$$\begin{cases} \operatorname{div} A(y,\xi+\nabla_y N(y,\xi))-\nabla\chi=0 & \text{in } Y,\\ N(y+Y,\xi)=N(y,\xi),\\ \int_Y N(y,\xi)\,dy=0. \end{cases} \tag{1.11}$$

It is well known that, by the homogenization theory of Stokes problems, the solution $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega;\mathbb{R}^n)$, $p_\varepsilon \rightharpoonup p_0$ weakly in $L^2(\Omega)$, and $A(x/\varepsilon,\nabla u_\varepsilon)\rightharpoonup Q(\nabla u_0)$ weakly in $L^2(\Omega,\mathbb{R}^{n\times n})$, as $\varepsilon\to 0$. The existence and convergence results of the weak solution to problem (1.1) may be found in [4, 12].

The convergence rate estimate is one of the fundamental issues in quantitative homogenization. There are many such classic works about convergence results of solutions in homogenization of second-order elliptic equations with the various settings. In 2011, Gérard and Masmoudi [6] got the L^2 convergence for the Neumann boundary layer problems. In 2012, Kenig, Lin, and Shen [13] established L^2 as well as $H^{\frac{1}{2}}$ convergence in Lipschitz domains for Dirichlet and Neumann problems. In 2013, Aleksanyan, Shahgholian, and Sjölin [1, 2] proved pointwise and L^p convergence estimates for fixed operators and oscillating Dirichlet boundary data. In 2014, Kenig, Lin, and Shen [14] also obtained $W^{k,p}$ convergence rates of Dirichlet or Neumann problems for the second-order equations with rapidly oscillating periodic coefficients by using the asymptotic estimates of the Green or Neumann functions. In 2015, the second author [24] proved the pointwise as well as $W^{1,p}$ convergence estimates for the fixed operators and oscillating Neumann boundary data by utilizing oscillation integral estimates in Fourier analysis. In 2016, Shen [17] proved the L^q convergence rates with Dirichlet or Neumann problems with no smoothness assumption on the coefficients. In 2018, Shen and Zhuge [19] got the L^2 convergence rate for the Neumann problems with first-order oscillating Neumann boundary data.

For the case of Stokes problems, some outstanding results about regularity and convergence of solutions in homogenization were established by Gu and Shen in a series of papers. The uniform interior estimates and boundary Hölder estimates for the Dirichlet problem have been established in [8]. Then, the authors in [9] obtained the sharp boundary regularity estimates in homogenization of Dirichlet problem. In 2015, Gu [7] also proved convergence rates in L^2 and H^1 of Dirichlet problems for linear Stokes systems. In 2017, Gu and Shen [10] got the asymptotic behaviors of the Green functions as well as the convergence rates in L^p and L^∞ for solutions. Recently, other authors have also been interested in the regularity estimates for the Stokes problems, see [3, 5, 11, 23] and their references for more results.

The main difficulty of this work is twofold. On the one hand, the nonlinear Stokes problems do not fit the standard framework of second-order elliptic equations in divergence form, which is caused by the pressure term. On the other hand, nonlinear problems may cause new difficulties in the estimation of the quantity as well as first-order approximate term.

The motivation for studying this paper is inspired by the technology used to deal with linear Stokes problems studied by Gu in [7]. The novelty of this paper lies in that it may be regarded as an extension of the approach for the linear Stokes problems to the nonlinear case. As the author knows, very few convergence rate results are known in the field of nonlinear Stokes problems.

The following are the main results of this paper.

Theorem 1 *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$. Let $u_\varepsilon \in H^1(\Omega;\mathbb{R}^n)$ and $u_0 \in H^2(\Omega;\mathbb{R}^n)$ be the weak solutions of the mixed boundary value problems* (1.1) *and* (1.9), *respectively. Then, under assumptions* (1.2)–(1.8), *there exists a constant C such that*

$$\left\| u_\varepsilon - u_0 - \varepsilon N\big(x/\varepsilon, T_\varepsilon(\nabla \widetilde{u}_0)\big) + \omega_\varepsilon \right\|_{H_0^1(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1.12}$$

where T_ε is the smoothing operator, $\widetilde{u}_0$ is an extension of u_0, and ω_ε is an approximate function.

Theorem 2 *Under the same assumptions as Theorem 1, there exists a constant C such that*

$$\left\| u_\varepsilon - u_0 - \varepsilon N\big(x/\varepsilon, T_\varepsilon(\nabla \widetilde{u}_0)\big) \right\|_{H^1(\Omega)} \le C\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)}.$$

Theorem 3 *Under the same conditions as Theorem 1, there exists a constant C such that*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}.$$

The rest of the paper is organized as follows. Section 2 contains some basic definitions and useful propositions which will play important roles in obtaining convergence rates. In Sect. 3, we show that the solution u_ε of nonlinear Stokes problems is convergent to the solution u_0 of the corresponding homogenized problems, this is based on using of a smoothing operator as well as homogenization tools.

2 Preliminaries

We begin by specifying some of our notations.

Let $B_r(x)$ denote an open ball with center x and radius r. $\Omega_\varepsilon = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \partial\Omega) \le \varepsilon\}$. Since Ω is Lipschitz, then there exists a bounded extension operator $E : H^2(\Omega) \to H^2(\mathbb{R}^n)$ such that $\widetilde{u}_0$ is an extension of u_0 satisfying $\|\widetilde{u}_0\|_{H^2(\mathbb{R}^n)} \le C\|u_0\|_{H^2(\Omega)}$. We also set $\varphi \in C_0^\infty(\Omega; \mathbb{R}^n)$ is a smooth function and $\|\varphi\|_{H^1(\mathbb{R}^n)} \le C\|\varphi\|_{H^1(\Omega)}$. We choose a cut-off function $\eta_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, which satisfies the conditions: $\mathrm{supp}(\eta_\varepsilon) \subset \Omega_\varepsilon$, $\eta_\varepsilon|_{\partial\Omega} \equiv 1$ and $|\nabla\eta_\varepsilon| \le C/\varepsilon$. In this paper, C always denotes a positive constant which may vary in different formulas.

Associated with operator L_ε in (1.1), the homogenized operator is

$$L_0 = -\operatorname{div} Q(\nabla) \quad \text{in } \Omega, \tag{2.1}$$

the function Q and corrector function $N(x/\varepsilon, \nabla u_0)$ are defined in (1.10), (1.11), respectively, and they satisfy the following properties.

Proposition 2.1 *The function Q defined in (1.10) satisfies the analogous properties as function A*:

$$\left|Q(\xi) - Q(\xi')\right| \le C\left|\xi - \xi'\right|$$

and

$$\left\langle Q(\xi) - Q(\xi'), \xi - \xi'\right\rangle \ge C\left|\xi - \xi'\right|^2$$

for all $\xi, \xi' \in \mathbb{R}^n$.

Proof The proof could be found in [16], which is similar to the linear operator case. Obviously, this shows that the homogenized operator L_0 still satisfies the same coerciveness and growth conditions. □

Proposition 2.2 *The function* $N(\cdot, \xi) \in H^1(Y)$ *is a weak solution to* (1.11). *Then we have*

$$\left\|N(y, \xi)\right\|_{L^\infty(Y)} \le C|\xi|, \qquad \int_Y \left|\nabla_\xi N(y, \xi)\right|^2 dy \le C$$

and

$$\int_Y \left|N(y, \xi)\right|^2 dy + \int_Y \left|\nabla_y N(y, \xi)\right|^2 dy \le C|\xi|^2$$

for all $y, \xi \in \mathbb{R}^n$.

Proof These estimates have been proved in [16] and [22]. Multiplying both sides of (1.11) by $N(y,\xi)$ and integrating by parts, one could get the desired results. □

The next proposition is the special relation between Q and A in homogenization. We also call them the flux correctors.

Proposition 2.3 *Let*

$$F(y,\xi) = Q(\xi) - A\big(y, \xi + \nabla N(y,\xi)\big),$$

where $y \in Y$ and $\xi \in \mathbb{R}^n$. Together with (1.10) and (1.11), it is easy to know that $F(\cdot,\xi)$ satisfies conditions $\int_Y F(y,\xi)\,dy = 0$ and $\operatorname{div}_y F(y,\xi) = \nabla \chi$. Then there exists $\Phi_{ij}(\cdot,\xi) \in H^1(\mathbb{R}^n)$ such that

$$\Phi_{ij}(y,\xi) = -\Phi_{ji}(y,\xi) \quad \textit{and} \quad F_j(y,\xi) = \frac{\partial \Phi_{ij}(y,\xi)}{\partial y_i} + \chi_j.$$

Moreover,

$$\int_Y \big|\Phi_{ij}(y,\xi)\big|^2\,dy + \int_Y \big|\nabla_\xi \Phi_{ij}(y,\xi)\big|^2\,dy \le C.$$

Proof The linear operator case is well known (see, for example, [13], Lemma 3.1). This proposition is quite similar to the linear case. Let $f_j \in H^2(Y)$ be the solution to the cell problem $\triangle f_j + \chi_j = F_j$ in Y. Then, we could define $\Phi_{ij}(y,\xi) = \frac{\partial}{\partial y_i}[f_j(y,\xi)] - \frac{\partial}{\partial y_j}[f_i(y,\xi)]$. From every estimate and (1.11), we may get the desired properties. We refer the reader to [7] for more details. □

Recently, the smoothing operators were introduced by Suslina in [20, 21], which was used to establish the convergence estimate in L^2 for a broad class of elliptic or parabolic operators. This work seems to extend the usage of smoothing operators to the case of nonlinear Stokes problems.

Fix $\psi \in C_0^\infty(B_1(0))$ such that $\psi \ge 0$ and $\int_{\mathbb{R}^n} \psi\,dx = 1$. Define operator T_ε on L^2 as

$$T_\varepsilon(u)(x) = u * \psi_\varepsilon = \int_{\mathbb{R}^n} u(x-y)\psi_\varepsilon(y)\,dy,$$

where $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$.

Proposition 2.4 *If $u_0 \in H^2(\mathbb{R}^n)$, then*

$$\big\|\nabla u_0 - T_\varepsilon(\nabla u_0)\big\|_{L^2(\mathbb{R}^n)} \le C\varepsilon \big\|\nabla^2 u_0\big\|_{L^2(\mathbb{R}^n)}$$

and

$$\big\|T_\varepsilon\big(\nabla^2 u_0\big)\big\|_{L^2(\mathbb{R}^n)} \le C\big\|\nabla^2 u_0\big\|_{L^2(\mathbb{R}^n)}.$$

Proof By Parseval's theorem and Hölder's inequality, we could get the desired result. The proof could be found in [17]. □

Proposition 2.5 *If $u_0 \in H^2(\mathbb{R}^n)$, then*

$$\left\| T_\varepsilon(\nabla u_0)\right\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{1/2}\|u_0\|_{H^2(\mathbb{R}^n)}.$$

Proof This estimate could be proved by Fubini's theorem. See [15] or [18] for the detailed proof. □

3 Proofs of theorems

The goal of this section is to establish H^1 and L^2 convergence rates of solutions.

Proof of Theorem 1 Let $\omega_\varepsilon \in H^1(\Omega)$ be a weak solution of

$$\begin{cases} L_\varepsilon \omega_\varepsilon + \nabla(p_\varepsilon - p_0 - \chi) = 0 & \text{in } \Omega, \\ \operatorname{div} \omega_\varepsilon = \varepsilon \operatorname{div} N(x/\varepsilon, T_\varepsilon(\nabla \widetilde{u}_0)) & \text{in } \Omega, \\ \omega_\varepsilon = \varepsilon N(x/\varepsilon, T_\varepsilon(\nabla \widetilde{u}_0)) & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We will use ω_ε to approximate the difference of pressure term.

Introduce the first-order approximation of u_ε:

$$v_\varepsilon = u_0 + \varepsilon N\big(x/\varepsilon, T_\varepsilon(\nabla \widetilde{u}_0)\big) - \omega_\varepsilon.$$

Note that, for any $\varphi \in C_0^\infty(\Omega;\mathbb{R}^n)$,

$$\int_\Omega A(x/\varepsilon, \nabla u_\varepsilon)\cdot \nabla\varphi\, dx - \int_\Omega p_\varepsilon \operatorname{div}\varphi\, dx = \int_\Omega Q(\nabla u_0)\cdot\nabla\varphi\, dx - \int_\Omega p_0 \operatorname{div}\varphi\, dx.$$

A simple calculation then gives that

$$\begin{aligned} &\int_\Omega \big[A(x/\varepsilon, \nabla u_\varepsilon) - A(x/\varepsilon, \nabla v_\varepsilon)\big]\cdot\nabla\varphi\, dx + \int_\Omega (p_0 - p_\varepsilon)\operatorname{div}\varphi\, dx \\ &\quad = \int_\Omega \big[Q(\nabla u_0) - Q\big(T_\varepsilon(\nabla\widetilde{u}_0)\big)\big]\cdot\nabla\varphi\, dx \\ &\qquad + \int_\Omega \big[Q\big(T_\varepsilon(\nabla\widetilde{u}_0)\big) - A\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0) + \nabla N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\big)\big]\cdot\nabla\varphi\, dx \\ &\qquad + \int_\Omega \big[A\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0) + \nabla N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\big) - A(x/\varepsilon, \nabla v_\varepsilon)\big]\cdot\nabla\varphi\, dx \\ &\quad \doteq I_1 + I_2 + I_3. \end{aligned} \tag{3.2}$$

To estimate I_1, we note that by Proposition 2.1 and Proposition 2.4,

$$\begin{aligned} |I_1| &\le C\int_\Omega \big|Q(\nabla u_0) - Q\big(T_\varepsilon(\nabla\widetilde{u}_0)\big)\big|\cdot|\nabla\varphi|\, dx \\ &\le C\big\|\nabla\widetilde{u}_0 - T_\varepsilon(\nabla\widetilde{u}_0)\big\|_{L^2(\mathbb{R}^n)}\|\nabla\varphi\|_{L^2(\Omega)} \\ &\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}. \end{aligned} \tag{3.3}$$

Next, we shall estimate I_2. Let

$$F\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big) = Q\big(T_\varepsilon(\nabla\widetilde{u}_0)\big) - A\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0) + \nabla N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\big).$$

Note that $F(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0))$ is a periodic function with respect to the first variable, and it satisfies the conditions of Proposition 2.3. Then there exists $\Phi_{ij}(\cdot,\xi) \in H^1(\mathbb{R}^n)$ satisfying

$$Q_j\big(T_\varepsilon(\nabla\widetilde{u}_0)\big) - A_j\big(y, T_\varepsilon(\nabla\widetilde{u}_0) + \nabla N\big(y, T_\varepsilon(\nabla\widetilde{u}_0)\big)\big) = \frac{\partial \Phi_{ij}(y, T_\varepsilon(\nabla\widetilde{u}_0))}{\partial y_i} + \chi_j.$$

Thus, it gives that

$$\begin{aligned} I_2 &= \int_\Omega F\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big) \cdot \nabla\varphi \, dx \\ &= \int_\Omega \left[\frac{\partial \Phi_{ij}(y, T_\varepsilon(\nabla\widetilde{u}_0))}{\partial y_i} + \chi_j\right] \frac{\partial\varphi}{\partial x_j} \, dx \\ &\doteq I_{21} + I_{22}. \end{aligned}$$

For the first term,

$$\begin{aligned} I_{21} &= \int_\Omega \frac{\partial}{\partial x_i}\big(\varepsilon\Phi_{ij}\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\big) \cdot \frac{\partial\varphi}{\partial x_j} \, dx - \int_\Omega \varepsilon \frac{\partial\Phi_{ij}(x/\varepsilon,\xi)}{\partial\xi_h} \frac{\partial\xi_h}{\partial x_i} \frac{\partial\varphi}{\partial x_j} \, dx \\ &= -\int_\Omega \varepsilon \frac{\partial\Phi_{ij}(x/\varepsilon,\xi)}{\partial\xi_h} \frac{\partial\xi_h}{\partial x_i} \frac{\partial\varphi}{\partial x_j} \, dx, \end{aligned}$$

where the first term vanishes in the last equality, which depends on the antisymmetry of Φ_{ij}.

As a result, using Proposition 2.3 and Proposition 2.4, we get that

$$\begin{aligned} |I_{21}| &\le C\varepsilon \big\| T_\varepsilon\big(\nabla^2\widetilde{u}_0\big)\big\|_{L^2(\mathbb{R}^n)} \|\nabla\varphi\|_{L^2(\Omega)} \\ &\le C\varepsilon \|u_0\|_{H^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)}. \end{aligned} \tag{3.4}$$

For I_3, it follows from the growth condition of (1.4) as well as Proposition 2.4 that

$$\begin{aligned} |I_3| &\le C\big(\big\|\nabla\widetilde{u}_0 - T_\varepsilon(\nabla\widetilde{u}_0)\big\|_{L^2(\mathbb{R}^n)} + \varepsilon\big\|\nabla_\xi N(y,\xi)\big\|_{L^2(\Omega)} \big\| T_\varepsilon\big(\nabla^2\widetilde{u}_0\big)\big\|_{L^2(\mathbb{R}^n)}\big) \|\nabla\varphi\|_{L^2(\Omega)} \\ &\le C\varepsilon \|u_0\|_{H^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)}, \end{aligned} \tag{3.5}$$

where we have used Proposition 2.2.

Then we rearrange equation (3.1), together with (3.3)–(3.5), to show that

$$\begin{aligned} &\left| \int_\Omega \big[A(x/\varepsilon, \nabla u_\varepsilon) - A(x/\varepsilon, \nabla v_\varepsilon)\big] \cdot \nabla\varphi \, dx + \int_\Omega (p_0 - p_\varepsilon - \chi) \operatorname{div}\varphi \, dx \right| \\ &\quad \le C\varepsilon \|u_0\|_{H^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)}. \end{aligned}$$

Then let $\varphi = u_\varepsilon - v_\varepsilon = u_\varepsilon - u_0 - \varepsilon N(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)) + \omega_\varepsilon$. By the coercive condition and the equation satisfied by ω_ε, we can get the desired result, which completes the proof. □

Proof of Theorem 2 According to Theorem 1, we obtain the estimate

$$\left\| u_\varepsilon - u_0 - \varepsilon N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big) + \omega_\varepsilon \right\|_{H_0^1(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}.$$

Hence, it suffices to show that

$$\|\omega_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}.$$

In fact, by equation (3.1) and energy estimate, we obtain

$$\begin{aligned}
&\|\omega_\varepsilon\|_{H^1(\Omega)} \\
&\quad\le C\varepsilon\left\|\operatorname{div} N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\right\|_{L^2(\Omega)} + C\varepsilon\left\|N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\right\|_{H^{1/2}(\partial\Omega)} \\
&\quad\le C\varepsilon\left\|\eta_\varepsilon N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\right\|_{H^1(\Omega)} \\
&\quad\le C\varepsilon\left\|\nabla_\xi N(y,\xi) T_\varepsilon\big(\nabla^2\widetilde{u}_0\big)\right\|_{L^2(\Omega_\varepsilon)} + C\left\|\nabla_y N(y, T_\varepsilon(\nabla\widetilde{u}_0)\right\|_{L^2(\Omega_\varepsilon)} \\
&\qquad + C\varepsilon\left\|\eta_\varepsilon N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\right\|_{L^2(\Omega)} + C\left\|N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\right\|_{L^2(\Omega_\varepsilon)} \\
&\quad\le C\varepsilon\left\|T_\varepsilon\big(\nabla^2\widetilde{u}_0\big)\right\|_{L^2(\Omega)} + C\left\|T_\varepsilon(\nabla\widetilde{u}_0)\right\|_{L^2(\Omega_\varepsilon)} + C\varepsilon\left\|T_\varepsilon(\nabla\widetilde{u}_0)\right\|_{L^2(\Omega)} \\
&\quad\le C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)},
\end{aligned} \tag{3.6}$$

where we have used the estimates in Proposition 2.2, Proposition 2.4, and Proposition 2.5.

This completes the proof of Theorem 2. □

Proof of Theorem 3 It follows from Theorem 2 and Proposition 2.2, together with Minkowski's inequality, that

$$\begin{aligned}
\|u_\varepsilon - u_0\|_{L^2(\Omega)} &\le C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)} + \left\|\varepsilon N\big(x/\varepsilon, T_\varepsilon(\nabla\widetilde{u}_0)\big)\right\|_{L^2(\Omega)} \\
&\le C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)} + C\varepsilon\left\|T_\varepsilon(\nabla\widetilde{u}_0)\right\|_{L^2(\mathbb{R}^n)} \\
&\le C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)},
\end{aligned} \tag{3.7}$$

which completes the proof of Theorem 3. □

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