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Multiplicity results for biharmonic equations involving multiple Rellich-type potentials and critical exponents

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Abstract

In this paper, a biharmonic equation is investigated, which involves multiple Rellich-type potentials and a critical Sobolev exponent. By using variational methods and analytical techniques, the existence and multiplicity of nontrivial solutions to the equation are established.

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Keywords: Multiplicity; Biharmonic equations; Rellich-type potential; Critical Sobolev exponent

1 Introduction

In this paper, we study the following biharmonic equation:

$$\begin{cases} \Delta^2 u - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^4} u = |u|^{2^*-2}u + \lambda |u|^{q-2}u, & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{E_λ}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ is a smooth bounded domain such that the different points $a_i \in \Omega$, $i = 1, 2, \dots, k$, $k \geq 2$, $\frac{\partial}{\partial n}$ is the outward normal derivative, $0 \leq \mu_i < \bar{\mu} := (\frac{N(N-4)}{4})^2$, $\lambda > 0$, $1 \leq q < 2^*$, and $2^* := \frac{2N}{N-4}$ is the critical Sobolev exponent.

Equation (E_λ) is related to the following Rellich inequality [22]:

$$\int_\Omega \frac{u^2}{|x-a|^4}\,dx \leq \frac{1}{\bar{\mu}} \int_\Omega |\Delta u|^2\,dx, \quad \forall a \in \Omega, u \in H_0^2(\Omega), \tag{1.1}$$

where $H_0^2(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to $(\int_\Omega |\Delta \cdot|^2\,dx)^{1/2}$. Then the following best constant is well defined:

$$A_\mu(\Omega) := \inf_{H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\Delta u|^2 - \mu \frac{u^2}{|x-a|^4})\,dx}{(\int_\Omega |u|^{2^*}\,dx)^{\frac{2}{2^*}}}, \quad \forall a \in \Omega, \mu < \bar{\mu}.$$

Note that it is well known that $A_\mu(\Omega)$ is independent of Ω and that $A_\mu(\Omega)$ is not obtained except in the case with $\Omega = \mathbb{R}^N$. Moreover, the minimizers of $A_\mu(\Omega)$ have been investigated by some authors (e.g. [3, 10, 11, 19]). Thus, we will simply denote $A_\mu(\Omega) = A_\mu(\mathbb{R}^N) = A_\mu$.

In this paper, for $\sum_{i=1}^k \mu_i \in [0, \bar{\mu})$, we use $H_0^2(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega \left(|\Delta u|^2 - \sum_{i=1}^k \frac{\mu_i u^2}{|x-a_i|^4}\right) dx\right)^{\frac{1}{2}}.$$

By (1.1), this norm is equivalent to the usual norm $(\int_\Omega |\Delta u|^2\, dx)^{\frac{1}{2}}$.

It is easily to see that Eq. (E_λ) is variational and its solutions are critical points of the functional defined in $H_0^2(\Omega)$ by

$$J_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} - \frac{\lambda}{q}\int_\Omega |u|^q, \quad u \in H_0^2(\Omega).$$

Then $J_\lambda \in C^1(H_0^2(\Omega), \mathbb{R})$ and that

$$\langle J'_\lambda(u), v\rangle = \int_\Omega \left(\Delta u \Delta v - \sum_{i=1}^k \frac{\mu_i u v}{|x-a_i|^4}\right) - \int_\Omega |u|^{2^*-2}uv - \lambda\int_\Omega |u|^{q-2}uv, \quad \forall v \in H_0^2(\Omega).$$

In recent years problems related with the inequality (1.1) and the equations with biharmonic operator have been investigated in several works; we quote [1, 3, 6–10, 13, 18, 19]. On the other hand, the biharmonic problems involving a Rellich-type potential and a critical Sobolev exponent have seldom been studied; we only find some results in [10, 18, 19]. Thus it is necessary for us to investigate the related biharmonic problems deeply. Very recently, Hsu and Zhang [16] studied the existence and multiplicity of nontrivial solution for the following equation:

$$\begin{cases} \Delta^2 u - \frac{\mu}{|x|^4}u = \frac{|u|^{2^*(s)-2}}{|x|^s}u + \lambda\frac{|u|^{q-2}}{|x|^t}u, & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ is a smooth bounded domain such that $0 \in \Omega$, $0 \leq \mu < \bar{\mu}$, $0 \leq s$, $t < 4$, $1 \leq q < 2$, $\lambda > 0$.

In this paper, we study a biharmonic equation involving multiple Rellich-type potentials and a critical Sobolev exponent. It should be mentioned that the main technical difficulty to study equations like Eq. (E_λ) is the lack of knowledge of the explicit form minimizers to the best Rellich–Sobolev constant A_{μ_i}. However, as in [10] and [19], this difficulty can be overcome since the unique tool which is necessary to perform the needed asymptotic expansions is the asymptotic behavior at the origin and infinity of Rellich–Sobolev extremals and their first derivatives, which is established in Theorem 1.1 of [19]. We are only aware of the work in [18] which studied the existence and nonexistence of ground state solution to Eq. (E_λ) when $\Omega = \mathbb{R}^N$, $k \geq 2$ and $\lambda = 0$. Furthermore, Eq. (E_λ) have never been studied when Ω is a smooth bounded domain and $k \geq 2$, and our results are new.

For $0 \leq \mu_i < \bar{\mu}$ and $a_i \in \Omega$, $i = 1, 2, \dots, k$, we can define the constant:

$$A_{\mu_i} := \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\Delta u|^2 - \mu_i \frac{u^2}{|x-a_i|^4})\, dx}{(\int_\Omega |u|^{2^*}\, dx)^{\frac{2}{2^*}}}.$$

The authors in [10, 19] proved that A_{μ_i} is attained in $\mathbb{R}^N$ by the functions

$$\left\{y_\varepsilon^{\mu_i}(x-a_i) = \varepsilon^{\frac{4-N}{2}} U_{\mu_i}\left(\varepsilon^{-1}(x-a_i)\right), \varepsilon > 0\right\}, \tag{1.2}$$

where $U_{\mu_i}(x)$ is positive, radially symmetric, radially decreasing, and solves

$$\Delta^2 u - \mu_i \frac{u}{|x|^4} = |u|^{2^*-1}, \quad x \in \mathbb{R}^N \setminus \{0\}, u > 0,$$

which satisfies

$$\int_{\mathbb{R}^N} \left(\left|\Delta y_\varepsilon^{\mu_i}(x-a_i)\right|^2 - \mu_i \frac{|y_\varepsilon^{\mu_i}(x-a_i)|^2}{|x-a_i|^4} \right) dx = \int_{\mathbb{R}^N} \left|y_\varepsilon^{\mu_i}(x-a_i)\right|^{2^*} dx = A_{\mu_i}^{\frac{N}{4}}.$$

Moreover, by setting $\rho = |x|$,

$$U_\mu(\rho) = O_1\left(\rho^{-a(\mu)}\right), \quad \text{as } \rho \to 0,$$
$$U_\mu(\rho) = O_1\left(\rho^{-b(\mu)}\right), \qquad U_\mu'(\rho) = O_1\left(\rho^{-b(\mu)-1}\right), \quad \text{as } \rho \to +\infty,$$

where $a(\mu) := \frac{N-4}{2} f(\mu)$, $b(\mu) := \frac{N-4}{2}(2 - f(\mu))$ and $f : [0, \bar{\mu}] \to [0, 1]$ is defined as

$$f(\mu) := 1 - \frac{\sqrt{N^2 - 4N + 8 - 4\sqrt{(N-2)^2 + \mu}}}{N-4}, \quad \mu \in [0, \bar{\mu}].$$

From Lemma 2.1 in [18], it follows that for $\mu \in [0, \bar{\mu})$

$$0 \le a(\mu) \le \delta \le b(\mu) \le 2\delta, \quad \delta := \frac{N-4}{2}. \tag{1.3}$$

Furthermore, there exist positive constants $\mathcal{C}_1(\mu)$ and $\mathcal{C}_2(\mu)$ such that

$$0 < \mathcal{C}_1(\mu) \le U_\mu(x)\left(|x|^{\frac{a(\mu)}{\delta}} + |x|^{\frac{b(\mu)}{\delta}}\right)^\delta \le \mathcal{C}_2(\mu), \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

Without loss of generality, throughout this paper we assume that
$(\mathcal{H})$ $0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_k < \bar{\mu}$, $\sum_{i=1}^k \mu_i < \bar{\mu}$, and $2^* := \frac{2N}{N-4}$.
In this paper, we define the following constants and notations:

$$\|u\|^2 = \int_\Omega \left(|\Delta u|^2 - \sum_{i=1}^k \frac{\mu_i u^2}{|x-a_i|^4} \right) dx \text{ is the norm in } H_0^2(\Omega);$$

$$H^{-2}(\Omega): \text{ the dual space of } H_0^2(\Omega);$$

$$\langle \cdot, \cdot \rangle: \text{ the usual scalar product in } H_0^2(\Omega);$$

$$B_r(a) = \left\{x : |x-a| < r\right\}, \qquad \overline{B_r(a)} = \left\{x : |x-a| \le r\right\}, \quad a \in \mathbb{R}^{\mathrm{N}}, \quad r > 0;$$

$$\mu^* := \frac{1}{16}\left(N^2 - 16\right)\left(N^2 - 8N\right), \quad N \ge 9;$$

$$S = \inf_{u \in H_0^2(\Omega) \setminus \{0\}} \frac{\int_\Omega (|\Delta u|^2 - \sum_{i=1}^k \mu_i \frac{u^2}{|x-a_i|^4})\, dx}{(\int_\Omega |u|^{2^*} dx)^{\frac{2}{2^*}}}; \tag{1.4}$$

$$\Lambda_0 := \left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}} \left(\frac{2^*-2}{2^*-q}\right)|\Omega|^{-\frac{2^*-q}{2^*}} S^{\frac{(2-q)N}{8}+\frac{q}{2}}; \tag{1.5}$$

$$\lambda_1 := \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\Delta u|^2 - \sum_{i=1}^k \mu_i \frac{u^2}{|x-a_i|^4})\,dx}{\int_\Omega |u|^2\,dx}. \tag{1.6}$$

Since the embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact, by choosing a minimizing sequence, we easily infer that λ_1 can be obtained in $H_0^2(\Omega)$, and $\lambda_1 > 0$. $C, C_1, C_2, \dots$ denote various positive constants. For all $\varepsilon > 0$, $\tau > 0$, $O(\varepsilon^\tau)$ denotes the quantity satisfying $|O(\varepsilon^\tau)/\varepsilon^\tau| \le C$ and $o(\varepsilon^\tau)$ means $|o(\varepsilon^\tau)/\varepsilon^\tau| \to 0$ as $\varepsilon \to \varepsilon_0$, $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$ and $O_1(\varepsilon^\tau)$ ($\varepsilon \to \varepsilon_0$) means that there exist the constants $C_1, C_2 > 0$ such that $C_1\varepsilon^\tau \le O_1(\varepsilon^\tau) \le C_2\varepsilon^\tau$ as $\varepsilon \to \varepsilon_0$. $|\Omega|$ denotes the Lebesgue measure of Ω and omit dx in integrals for convenience.

Let $1 \le q < 2^*$, by the Hölder inequality and (1.4), for all $u \in H_0^2(\Omega)$, we obtain

$$\int_\Omega |u|^q \le \left(\int_\Omega 1\right)^{\frac{2^*-q}{2^*}} \left(\int_\Omega |u|^{2^*}\right)^{\frac{q}{2^*}} \le |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \|u\|^q. \tag{1.7}$$

We are now ready to state our main results.

Theorem 1.1 *Let $N \ge 5$, $1 \le q < 2$ and assume that $(\mathcal{H})$ holds, then we have the following results.*

(i) *If $\lambda \in (0, \Lambda_0)$, then Eq. ($E_\lambda$) has at least one nontrivial solution.*
(ii) *If $\lambda \in (0, \frac{q}{2}\Lambda_0)$, then Eq. ($E_\lambda$) has at least two nontrivial solutions.*

Theorem 1.2 *Let $N \ge 5$, $2 \le q < 2^*$ and assume that $(\mathcal{H})$ and one of the following conditions holds:*

(i) $\lambda > 0$, $\overline{q} < q < 2^*$, *where*

$$\overline{q} = \max\left\{2, \frac{N}{b(\mu_k)}, \frac{4(N-2-b(\mu_k))}{N-4}\right\}.$$

(ii) $N \ge 8$, $0 < \lambda < \lambda_1$, $q = 2$, $0 \le \mu_k \le \mu^*$.

Then Eq. (E_λ) has at least one nontrivial solution.

This paper is organized as follows. In Sect. 2, we give some properties of Nehari manifold. In Sects. 3 and 4, we prove Theorem 1.1. In Sect. 5, we prove Theorem 1.2.

2 Nehari manifold

In this section, we will give some properties of Nehari manifold. As the energy functional J_λ is not bounded below on $H_0^2(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$\mathcal{M}_\lambda = \left\{u \in H_0^2(\Omega) \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0\right\}.$$

Thus, $u \in \mathcal{M}_\lambda$ if and only if

$$\langle J'_\lambda(u), u\rangle = \|u\|^2 - \int_\Omega |u|^{2^*} - \lambda \int_\Omega |u|^q = 0. \tag{2.1}$$

Note that $\mathcal{M}_\lambda$ contains every nonzero solution of Eq. (E_λ). Moreover, we have the following results.

Lemma 2.1 *Let $N \geq 5$, $1 \leq q < 2$ and $\lambda \in (0, \Lambda_0)$ where Λ_0 is the same as in (1.5). Then J_λ is coercive and bounded below on $\mathcal{M}_\lambda$.*

Proof If $u \in \mathcal{M}_\lambda$, then by (1.4), (2.1), and the Hölder inequality

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2^*}\left(\lambda \int_\Omega |u|^q - \|u\|^2\right) - \frac{\lambda}{q}\int_\Omega |u|^q \\ &= \frac{2^*-2}{22^*}\|u\|^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega |u|^q \qquad (2.2)\\ &\geq \frac{2}{N}\|u\|^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\|u\|^q. \qquad (2.3) \end{aligned}$$

Thus, J_λ is coercive and bounded below on $\mathcal{M}_\lambda$. □

Define $\psi_\lambda : H_0^2(\Omega) \to \mathbb{R}$, by $\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle$, that is,

$$\psi_\lambda(u) = \|u\|^2 - \int_\Omega |u|^{2^*} - \lambda\int_\Omega |u|^q.$$

Then we see that $\psi_\lambda \in C^1(H_0^2(\Omega), \mathbb{R})$, $\mathcal{M}_\lambda = \psi_\lambda^{-1}(0) \setminus \{0\}$, and for all $u \in \mathcal{M}_\lambda$,

$$\begin{aligned} \langle \psi'_\lambda(u), u\rangle &= 2\|u\|^2 - 2^*\int_\Omega |u|^{2^*} - \lambda q\int_\Omega |u|^q \\ &= (2-q)\|u\|^2 - (2^*-q)\int_\Omega |u|^{2^*} \qquad (2.4)\\ &= (2-2^*)\|u\|^2 - \lambda(q-2^*)\int_\Omega |u|^q. \qquad (2.5) \end{aligned}$$

We split $\mathcal{M}_\lambda$ into three parts:

$$\begin{aligned} \mathcal{M}_\lambda^+ &= \{u \in \mathcal{M}_\lambda : \langle \psi'_\lambda(u), u\rangle > 0\}, \\ \mathcal{M}_\lambda^0 &= \{u \in \mathcal{M}_\lambda : \langle \psi'_\lambda(u), u\rangle = 0\}, \\ \mathcal{M}_\lambda^- &= \{u \in \mathcal{M}_\lambda : \langle \psi'_\lambda(u), u\rangle < 0\}. \end{aligned}$$

We now derive some basic properties of $\mathcal{M}_\lambda^+$, $\mathcal{M}_\lambda^0$ and $\mathcal{M}_\lambda^-$.

Lemma 2.2 *Assume that u_0 is a local minimizer for J_λ on $\mathcal{M}_\lambda$ and $u_0 \notin \mathcal{M}_\lambda^0$. Then $J'_\lambda(u_0) = 0$ in $H^{-2}(\Omega)$.*

Proof See [5, Theorem 2.3]. □

Moreover, we have the following result.

Lemma 2.3 *If $\lambda \in (0, \Lambda_0)$, then $\mathcal{M}_\lambda^0 = \emptyset$.*

Proof Arguing by contradiction, we assume that there exists a $\lambda \in (0, \Lambda_0)$ such that $\mathcal{M}_\lambda^0 \neq \emptyset$. Then, for $u \in \mathcal{M}_\lambda^0$ by (1.4) and (2.4), we have

$$\frac{2-q}{2^*-q}\|u\|^2 = \int_\Omega |u|^{2^*} \leq S^{-\frac{2^*}{2}}\|u\|^{2^*}$$

and so

$$\|u\| \geq \left(\frac{2-q}{2^*-q}\right)^{\frac{1}{2^*-2}} S^{\frac{2^*}{2(2^*-2)}}.$$

Similarly, using (1.7), (2.5), and the Hölder inequality, we have

$$\|u\|^2 = \lambda\frac{2^*-q}{2^*-2}\int_\Omega |u|^q \leq \lambda\frac{2^*-q}{2^*-2}|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\|u\|^q,$$

which implies

$$\|u\| \leq \left[\lambda\frac{2^*-q}{2^*-2}|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\right]^{\frac{1}{2-q}}.$$

Hence, we must have

$$\lambda \geq \left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}}\left(\frac{2^*-2}{2^*-q}\right)|\Omega|^{-\frac{2^*-q}{2^*}} S^{\frac{(2-q)N}{8}+\frac{q}{2}} = \Lambda_0,$$

which is a contradiction. This completes the proof. □

For each $u \in H_0^2(\Omega) \setminus \{0\}$, let

$$\tau_{\max} = \left(\frac{(2-q)\|u\|^2}{(2^*-q)\int_\Omega |u|^{2^*}}\right)^{\frac{1}{2^*-2}} > 0.$$

Similar to Lemma 2.7 in [14], we can get the following result.

Lemma 2.4 *If $\lambda \in (0, \Lambda_0)$, then, for each $u \in H_0^2(\Omega) \setminus \{0\}$, the set $\{\tau u : \tau > 0\}$ intersects $\mathcal{M}_\lambda$ exactly twice. More specifically, there exist a unique $\tau^- = \tau^-(u) > 0$ such that $\tau^- u \in \mathcal{M}_\lambda^-$ and a unique $\tau^+ = \tau^+(u) > 0$ such that $\tau^+ u \in \mathcal{M}_\lambda^+$. Moreover, $\tau^+ < \tau_{\max} < \tau^-$ and*

$$J_\lambda(\tau^+ u) = \inf_{0 \leq \tau \leq \tau_{\max}} J_\lambda(\tau u), \qquad J_\lambda(\tau^- u) = \sup_{\tau \geq \tau_{\max}} J_\lambda(\tau u).$$

Proof The proof is similar to that of [14, Lemma 2.7] and is omitted. □

3 Existence of ground state solutions in the case of $1 \leq q < 2$

First, we remark that it follows from Lemma 2.3 that

$$\mathcal{M}_\lambda = \mathcal{M}_\lambda^+ \cup \mathcal{M}_\lambda^-$$

for all $\lambda \in (0, \Lambda_0)$. Furthermore, by Lemma 2.4 it follows that $\mathcal{M}_\lambda^+$ and $\mathcal{M}_\lambda^-$ are non-empty and by Lemma 2.1 we may define

$$\alpha_\lambda = \inf_{u\in\mathcal{M}_\lambda} J_\lambda(u); \qquad \alpha_\lambda^+ = \inf_{u\in\mathcal{M}_\lambda^+} J_\lambda(u); \qquad \alpha_\lambda^- = \inf_{u\in\mathcal{M}_\lambda^-} J_\lambda(u).$$

Lemma 3.1 *The following facts hold.*

(i) *If $\lambda \in (0, \Lambda_0)$, then $\alpha_\lambda \le \alpha_\lambda^+ < 0$.*

(ii) *If $\lambda \in (0, \frac{q}{2}\Lambda_0)$, then $\alpha_\lambda^- > c_0$ for some $c_0 > 0$.*

In particular, for each $\lambda \in (0, \frac{q}{2}\Lambda_0)$, we have $\alpha_\lambda^+ = \alpha_\lambda$.

Proof (i) Let $u \in \mathcal{M}_\lambda^+$. By (2.4)

$$\frac{2-q}{2^*-q}\|u\|^2 > \int_\Omega |u|^{2^*}$$

and so

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 + \left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega |u|^{2^*} \\ &< \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\frac{2-q}{2^*-q}\right)\right]\|u\|^2 \\ &= -\frac{(2^*-2)(2-q)}{22^*q}\|u\|^2 < 0. \end{aligned}$$

Therefore, from the definition of α_λ and α_λ^+, we can deduce that $\alpha_\lambda \le \alpha_\lambda^+ < 0$.

(ii) Let $u \in \mathcal{M}_\lambda^-$. By (2.4)

$$\frac{2-q}{2^*-q}\|u\|^2 < \int_\Omega |u|^{2^*}.$$

Moreover, by (1.4) we have

$$\int_\Omega |u|^{2^*} \le S^{-\frac{2^*}{2}}\|u\|^{2^*}.$$

This implies

$$\|u\| > \left(\frac{2-q}{2^*-q}\right)^{\frac{1}{2^*-2}} S^{\frac{N}{8}} \quad \text{for all } u \in \mathcal{M}_\lambda^-. \tag{3.1}$$

By (2.3) and (3.1), we have

$$\begin{aligned} J_\lambda(u) &\ge \|u\|^q\left[\frac{2}{N}\|u\|^{2-q} - \lambda\left(\frac{2^*-q}{2^*q}\right)|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\right] \\ &> \left(\frac{2-q}{2^*-q}\right)^{\frac{q}{2^*-2}} S^{\frac{qN}{8}}\left[\frac{2}{N}\left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}} S^{\frac{(2-q)N}{8}} - \lambda\left(\frac{2^*-q}{2^*q}\right)|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\right] \\ &= \left(\frac{q}{2}\Lambda_0 - \lambda\right)\left(\frac{2-q}{2^*-q}\right)^{\frac{q}{2^*-2}}\left(\frac{2^*-q}{2^*q}\right)|\Omega|^{\frac{2^*-q}{2^*}} S^{\frac{(N-4)q}{8}}. \end{aligned}$$

Thus, if $\lambda \in (0, \frac{q}{2}\Lambda_0)$, then there exists $c_0 > 0$ such that

$$J_\lambda(u) > c_0 \quad \text{for all } u \in \mathcal{M}_\lambda^-.$$

Consequently, this completes the proof. □

Remark 3.2

(i) If $\lambda \in (0, \Lambda_0)$, then, by (1.7), (2.5), and the Hölder inequality, for each $u \in \mathcal{M}_\lambda^+$ we have

$$\begin{aligned}\|u\|^2 &< \lambda \frac{2^* - q}{2^* - 2} \int_\Omega |u|^q \\ &\leq \lambda \frac{2^* - q}{2^* - 2} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \|u\|^q\end{aligned}$$

and so

$$\|u\| < \left[\lambda \frac{2^* - q}{2^* - 2} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\right]^{\frac{1}{2-q}} \quad \text{for all } u \in \mathcal{M}_\lambda^+. \tag{3.2}$$

(ii) If $\lambda \in (0, \frac{q}{2}\Lambda_0)$, then, by Lemma 2.4 and Lemma 3.1(ii), for each $u \in \mathcal{M}_\lambda^-$ we have

$$J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu).$$

We define the Palais–Smale (indicated simply by the prefix "(PS)-") sequences, (PS)-values, and (PS)-conditions in $H_0^2(\Omega)$ for J_λ as follows.

Definition 3.3

(i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence in $H_0^2(\Omega)$ for J_λ if $J_\lambda(u_n) = c + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ strongly in $H^{-2}(\Omega)$ as $n \to \infty$.

(ii) $c \in \mathbb{R}$ is a (PS)-value in $H_0^2(\Omega)$ for J_λ if there exists a $(\mathrm{PS})_c$-sequence in $H_0^2(\Omega)$ for J_λ.

(iii) J_λ satisfies the $(\mathrm{PS})_c$-condition in $H_0^2(\Omega)$ if any $(\mathrm{PS})_c$-sequence $\{u_n\}$ in $H_0^2(\Omega)$ for J_λ contains a convergent subsequence.

Now, we use the Ekeland variational principle [12] to get the following results.

Proposition 3.4

(i) *If $\lambda \in (0, \Lambda_0)$, then there exists a $(\mathrm{PS})_{\alpha_\lambda}$-sequence $\{u_n\} \subset \mathcal{M}_\lambda$ in $H_0^2(\Omega)$ for J_λ.*

(ii) *If $\lambda \in (0, \frac{q}{2}\Lambda_0)$, then there exists a $(\mathrm{PS})_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{M}_\lambda^-$ in $H_0^2(\Omega)$ for J_λ.*

Proof The proof is similar to that of [14, Proposition 3.3] and is omitted. □

Now, we establish the existence of a local minimum for J_λ on $\mathcal{M}_\lambda$.

Theorem 3.5 *Let $N \geq 5$, $1 \leq q < 2$ and assume that the condition $(\mathcal{H})$ holds. If $\lambda \in (0, \Lambda_0)$, then J_λ has a minimizer u_λ in $\mathcal{M}_\lambda^+$ and we have the following results.*

(i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$.

(ii) *u_λ is a nontrivial solution of Eq. (E_λ).*

(iii) $\|u_\lambda\| \to 0$ *as* $\lambda \to 0^+$.

Proof By Proposition 3.4(i), there is a minimizing sequence $\{u_n\}$ for J_λ on $\mathcal{M}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o_n(1) \quad \text{and} \quad J'_\lambda(u_n) = o_n(1) \quad \text{in } H^{-2}(\Omega). \tag{3.3}$$

Since J_λ is coercive on $\mathcal{M}_\lambda$ (see Lemma 2.1), we see that $\{u_n\}$ is bounded in $H_0^2(\Omega)$. Thus, passing a subsequence if necessary, there exists $u_\lambda \in H_0^2(\Omega)$ such that as $n \to \infty$

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{weakly in } H_0^2(\Omega), \\ u_n \to u_\lambda & \text{strongly in } L^q(\Omega) \text{ for } 1 \le q < 2^*, \\ u_n \to u_\lambda & \text{almost everywhere in } \Omega. \end{cases} \tag{3.4}$$

It follows that

$$\lambda \int_\Omega |u_n|^q \to \lambda \int_\Omega |u_\lambda|^q \quad \text{as } n \to \infty, \forall 1 \le q < 2. \tag{3.5}$$

By (3.3), (3.4) and (3.5), it is easy to see that u_λ is a weak solution of Eq. (E_λ). From $\{u_n\} \subset \mathcal{M}_\lambda$, (2.2) and (3.5), we deduce that

$$\begin{aligned} J_\lambda(u_n) &= \frac{2^*-2}{22^*}\|u_n\|^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega |u_n|^q \\ &\ge -\lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega |u_n|^q \\ &\to -\lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega |u_\lambda|^q. \end{aligned}$$

This and $J_\lambda(u_n) \to \alpha_\lambda < 0$ (see Lemma 3.1(i)) yield $\int_\Omega |u_\lambda|^q > 0$, that is, $u_\lambda \not\equiv 0$. We use $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{M}_\lambda$. Thus by Lemma 2.2, we may assume that u_λ is a nontrivial nonnegative solution of Eq. (E_λ).

Now we prove that up to a subsequence, $u_n \to u_\lambda$ strongly in $H_0^2(\Omega)$ and $J_\lambda(u_\lambda) = \alpha_\lambda$. From the fact $u_n, u \in \mathcal{M}_\lambda$ and Fatou's lemma, we have

$$\begin{aligned} \alpha_\lambda \le J_\lambda(u_\lambda) &= \frac{2^*-2}{22^*}\|u_\lambda\|^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega |u_\lambda|^q \\ &\le \liminf_{n\to\infty}\left[\frac{2^*-2}{22^*}\|u_n\|^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega |u_n|^q\right] \\ &= \liminf_{n\to\infty} J_\lambda(u_n) \\ &= \alpha_\lambda, \end{aligned}$$

which implies that $J_\lambda(u_\lambda) = \alpha_\lambda$ and $\lim_{n\to\infty}\|u_n\|^2 = \|u_\lambda\|^2$. Standard argument shows that $u_n \to u_\lambda$ strongly in $H_0^2(\Omega)$.

Next, we claim $u_\lambda \in \mathcal{M}_\lambda^+$. Indeed, if $u_\lambda \in \mathcal{M}_\lambda^-$, by Lemma 2.4, there exist unique τ_λ^+ and τ_λ^- such that $\tau_\lambda^+ u_\lambda \in \mathcal{M}_\lambda^+$, $\tau_\lambda^- u_\lambda \in \mathcal{M}_\lambda^-$ and $\tau_\lambda^+ < \tau_\lambda^- = 1$. Since

$$\frac{d}{d\tau}J_\lambda\left(\tau_\lambda^+ u_\lambda\right) = 0 \quad \text{and} \quad \frac{d^2}{d\tau^2}J_\lambda\left(\tau_\lambda^+ u_\lambda\right) > 0,$$

there exists $\bar{\tau} \in (\tau_\lambda^+, \tau_\lambda^-)$ such that $J_\lambda(\tau_\lambda^+ u_\lambda) < J_\lambda(\bar{\tau} u_\lambda)$. By Lemma 2.4 we get

$$J_\lambda(\tau_\lambda^+ u_\lambda) < J_\lambda(\bar{\tau} u_\lambda) \le J_\lambda(\tau_\lambda^- u_\lambda) = J_\lambda(u_\lambda),$$

which contradicts $J_\lambda(u_\lambda) = \alpha_\lambda$. Consequently, $u_\lambda \in \mathcal{M}_\lambda^+$.

Finally, by $u_\lambda \in \mathcal{M}_\lambda^+$ and (3.2), we obtain

$$\|u_\lambda\| < \left[\lambda \frac{2^* - q}{2^* - 2} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\right]^{\frac{1}{2-q}} \quad \text{for all } u \in \mathcal{M}_\lambda^+.$$

This implies that $\|u_\lambda\| \to 0$ as $\lambda \to 0^+$, and completes the proof. □

4 Multiplicity of nontrivial solutions in the case of $1 \le q < 2$

In this section, we will establish the existence of the second nontrivial solution of Eq. (E_λ) by proving that J_λ attains a local minimum on $\mathcal{M}_\lambda^-$.

Lemma 4.1 *If $\{u_n\} \subset H_0^2(\Omega)$ is a $(\mathrm{PS})_c$-sequence for J_λ, then $\{u_n\}$ is bounded in $H_0^2(\Omega)$.*

Proof The proof is similar to that of [15, Lemma 4.1] and is omitted. □

We recall that

$$A_{\mu_i} := \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\Delta u|^2 - \mu_i \frac{u^2}{|x-a_i|^4})\,dx}{(\int_\Omega |u|^{2^*}\,dx)^{\frac{2}{2^*}}}.$$

Lemma 4.2 *Let $N \ge 5$, $1 \le q < 2$ and assume that $(\mathcal{H})$ holds. If $\{u_n\} \subset H_0^2(\Omega)$ is a $(\mathrm{PS})_c$-sequence for J_λ with $c \in (0, \frac{2}{N} A_{\mu_k}^{\frac{N}{4}})$, then there exists a subsequence of $\{u_n\}$ converging weakly to a nonzero solution of Eq. (E_λ).*

Proof Let $\{u_n\} \subset H_0^2(\Omega)$ be a $(\mathrm{PS})_c$-sequence for J_λ with $c \in (0, \frac{2}{N} A_{\mu_k}^{\frac{N}{4}})$. We know from Lemma 4.1 that $\{u_n\}$ is bounded in $H_0^2(\Omega)$. Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_0 \in H_0^2(\Omega)$ such that $u_n \rightharpoonup u_0$ in $H_0^2(\Omega)$, $u_n \to u_0$ almost everywhere in Ω, and $u_n \to u_0$ in $L^q(\Omega)$ for any $1 \le q < 2^*$ as $n \to \infty$. It is easy to see that $J_\lambda'(u_0) = 0$ and

$$\lambda \int_\Omega |u_n|^q = \lambda \int_\Omega |u_0|^q + o_n(1). \tag{4.1}$$

Next we verify that $u_0 \not\equiv 0$. Arguing by contradiction, we assume $u_0 \equiv 0$. By the concentration compactness principle (see [20, 21]) there exists a subsequence, still denoted by $\{u_n\}$, an at most countable set $\mathcal{J}$, a set of different points $\{x_j\}_{j\in\mathcal{J}} \subset \Omega \setminus \{a_1, a_2, \dots, a_k\}$, nonnegative real numbers $\widetilde{\mu_{x_j}}$, $\widetilde{\nu_{x_j}}$, $j \in \mathcal{J}$ and $\widetilde{\mu_{a_i}}$, $\widetilde{\gamma_{a_i}}$, $\widetilde{\nu_{a_i}}$ $(1 \le i \le k)$ such that

$$\begin{cases} |\Delta u_n|^2 \rightharpoonup d\widetilde{\mu} \ge |\Delta u_0|^2 + \sum_{j\in\mathcal{J}} \widetilde{\mu_{x_j}} \delta_{x_j} + \sum_{i=1}^k \widetilde{\mu_{a_i}} \delta_{a_i}, \\ \mu_i \frac{u_n^2}{|x-a_i|^4} \rightharpoonup d\widetilde{\gamma_{a_i}} = \mu_i \frac{u_0^2}{|x-a_i|^4} + \widetilde{\gamma_{a_i}} \delta_{a_i}, \\ |u_n|^{2^*} \rightharpoonup d\widetilde{\nu} = |u_0|^{2^*} + \sum_{j\in\mathcal{J}} \widetilde{\nu_{x_j}} \delta_{x_j} + \sum_{i=1}^k \widetilde{\nu_{a_i}} \delta_{a_i}, \end{cases} \tag{4.2}$$

where δ_x is the Dirac mass at x. By the Rellich inequalities, we get

$$\widetilde{\mu_{a_i}} - \mu_i \widetilde{\gamma_{a_i}} \geq A_{\mu_i} \widetilde{\nu_{a_i}}^{\frac{2}{2^*}}, \quad 1 \leq i \leq k.$$

Claim 1. We claim that $\mathcal{J}$ is finite and for any $j \in \mathcal{J}$, either

$$\widetilde{\nu_{x_j}} = 0 \quad \text{or} \quad \widetilde{\nu_{x_j}} \geq A_0^{\frac{N}{4}}.$$

In fact, let $\varepsilon > 0$ be small enough such that $a_i \notin B_{2\varepsilon}(x_j)$ for all $1 \leq i \leq k$ and $B_{2\varepsilon}(x_i) \cap B_{2\varepsilon}(x_j) = \varnothing$ for $i \neq j$, $i, j \in \mathcal{J}$. Let ϕ_ε^j be a smooth cut-off function centered at x_j such that $0 \leq \phi_\varepsilon^j \leq 1$, $\phi_\varepsilon^j = 1$ for $|x - x_j| \leq \varepsilon$, $\phi_\varepsilon^j = 0$ for $|x - x_j| \geq 2\varepsilon$, $|\nabla \phi_\varepsilon^j| \leq \frac{2}{\varepsilon}$ and $|\Delta \phi_\varepsilon^j| \leq \frac{2}{\varepsilon^2}$. Consider the sequence $\{\phi_\varepsilon^j u_n\}$; it is obvious that this sequence is bounded in $H_0^2(\Omega)$. Then (4.1) implies

$$\lim_{n\to\infty} \langle J_\lambda'(u_n), \phi_\varepsilon^j u_n \rangle = 0.$$

Moreover, by (4.2) we deduce

$$\int_\Omega \sum_{i=1}^k \phi_\varepsilon^j \, d\widetilde{\gamma_{a_i}} + \int_\Omega \phi_\varepsilon^j \, d\widetilde{\nu} + \lambda \int_\Omega |u_0|^q \phi_\varepsilon^j \, dx = \lim_{n\to\infty} \int_\Omega \Delta u_n \Delta\bigl(u_n \phi_\varepsilon^j\bigr) \, dx. \tag{4.3}$$

Then

$$\begin{cases} \lim_{\varepsilon\to 0} \int_\Omega \sum_{i=1}^k \phi_\varepsilon^j \, d\widetilde{\gamma} = \lim_{\varepsilon\to 0} \int_\Omega \sum_{i=1}^k \mu_i \frac{u_0^2 \phi_\varepsilon^j}{|x-a_i|^4} = 0, \\ \lim_{\varepsilon\to 0} \int_\Omega \phi_\varepsilon^j \, d\widetilde{\nu} = \lim_{\varepsilon\to 0} (\int_\Omega |u_0|^{2^*} \phi_\varepsilon^j + \widetilde{\nu_{x_j}}) = \widetilde{\nu_{x_j}}, \\ \lim_{\varepsilon\to 0} \lambda \int_\Omega |u_0|^q \phi_\varepsilon^j \, dx = 0. \end{cases} \tag{4.4}$$

On the other hand, by (4.2) and the weak convergence we can obtain

$$\lim_{n\to\infty} \int_\Omega \Delta u_n \Delta\bigl(u_n \phi_\varepsilon^j\bigr) \, dx = \int_\Omega \phi_\varepsilon^j \, d\widetilde{\mu} + \lim_{n\to\infty} \int_\Omega \Delta u_n \bigl(2\nabla u_n \nabla \phi_\varepsilon^j + u_n \Delta \phi_\varepsilon^j\bigr) \, dx. \tag{4.5}$$

Now, by (4.2) it is easy to see that

$$\lim_{\varepsilon\to 0} \int_\Omega \phi_\varepsilon^j \, d\widetilde{\mu} \geq \widetilde{\mu_{x_j}}. \tag{4.6}$$

By the Hölder inequality, we get

$$\begin{aligned} 0 &\leq \overline{\lim_{n\to\infty}} \left| \int_\Omega \Delta u_n \bigl(\nabla u_n \nabla \phi_\varepsilon^j\bigr) \, dx \right| \\ &\leq \overline{\lim_{n\to\infty}} \left[\left(\int_\Omega |\Delta u_n|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_n|^2 \bigl|\nabla \phi_\varepsilon^j\bigr|^2 \, dx \right)^{\frac{1}{2}} \right] \\ &\leq C \left(\int_{B_{2\varepsilon}(x_j)} |\nabla u_0|^2 \bigl|\nabla \phi_\varepsilon^j\bigr|^2 \, dx \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq C\left(\int_{B_{2\varepsilon}(x_j)} \left|\nabla\phi_\varepsilon^j\right|^N dx\right)^{\frac{1}{N}} \left(\int_{B_{2\varepsilon}(x_j)} |\nabla u_0|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}} \\
&\leq C\left(\int_{B_{2\varepsilon}(x_j)} |\nabla u_0|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}} \to 0 \quad \text{as } \varepsilon \to 0
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
0 &\leq \overline{\lim_{n\to\infty}}\left|\int_\Omega \Delta u_n u_n \Delta\phi_\varepsilon^j \, dx\right| \\
&\leq \overline{\lim_{n\to\infty}}\left[\left(\int_\Omega |\Delta u_n|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\Delta\phi_\varepsilon^j\right|^2 |u_n|^2 dx\right)^{\frac{1}{2}}\right] \\
&\leq C\left(\int_{B_{2\varepsilon}(x_j)} \left|\Delta\phi_\varepsilon^j\right|^2 |u_0|^2 dx\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{B_{2\varepsilon}(x_j)} \left|\Delta\phi_\varepsilon^j\right|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \left(\int_{B_{2\varepsilon}(x_j)} |u_0|^{\frac{2N}{N-4}} dx\right)^{\frac{N-4}{2N}} \\
&\leq C\left(\int_{B_{2\varepsilon}(x_j)} |u_0|^{2^*} dx\right)^{\frac{1}{2^*}} \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{4.8}$$

Thus, from (4.3)–(4.8) it follows that

$$\widetilde{\mu_{x_j}} \leq \widetilde{\nu_{x_j}}.$$

By the Sobolev inequality, $S_0 \widetilde{\nu_{x_j}}^{\frac{2}{2^*}} \leq \widetilde{\mu_{x_j}}$, hence we deduce that

$$\widetilde{\nu_{x_j}} = 0 \quad \text{or} \quad \widetilde{\nu_{x_j}} \geq A_0^{\frac{N}{4}},$$

which implies that $\mathcal{J}$ is finite. Claim 1 is proved.

Claim 2. We claim that

$$\text{for each} \quad i = 1, 2, \dots, k \quad \text{either} \quad \widetilde{\nu_{a_i}} = 0 \quad \text{or} \quad \widetilde{\nu_{a_i}} \geq A_{\mu_i}^{\frac{N}{4}}.$$

In order to prove claim 2, for each $i = 1, 2, \dots, k$, we consider the possibility of concentration at points a_i $(1 \leq i \leq k)$. For $\varepsilon > 0$ be small enough such that $x_j \notin B_\varepsilon(a_i)$ for all $j \in \mathcal{J}$ and $B_\varepsilon(a_i) \cap B_\varepsilon(a_j) = \varnothing$ for $i \neq j$ and $1 \leq i, j \leq k$. Let φ_ε^i be a smooth cut-off function centered at a_i such that $0 \leq \varphi_\varepsilon^i \leq 1$, $\varphi_\varepsilon^i = 1$ for $|x - a_i| \leq \varepsilon$, $\varphi_\varepsilon^i = 0$ for $|x - a_i| \geq 2\varepsilon$, $|\nabla\varphi_\varepsilon^i| \leq \frac{2}{\varepsilon}$ and $|\Delta\varphi_\varepsilon^i| \leq \frac{2}{\varepsilon^2}$. Then, by (4.2) and similar arguments to the proof of claim 1, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \Delta u_n \Delta\left(u_n\varphi_\varepsilon^j\right) = \lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon^i \, d\widetilde{\mu} \geq \lim_{\varepsilon\to 0}\left(\int_\Omega |\Delta u_0|^2 \varphi_\varepsilon^i + \widetilde{\mu_{a_i}}\right) = \widetilde{\mu_{a_i}}, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \mu_i \frac{u_n^2}{|x - a_i|^4}\varphi_\varepsilon^i = \lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon^i \, d\widetilde{\gamma} = \lim_{\varepsilon\to 0}\left(\int_\Omega \mu_i \frac{u_0^2}{|x - a_i|^4}\varphi_\varepsilon^i + \widetilde{\gamma_{a_i}}\right) = \widetilde{\gamma_{a_i}}, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^{2^*}\varphi_\varepsilon^i = \lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon^i \, d\widetilde{\nu} = \lim_{\varepsilon\to 0}\left(\int_\Omega |u_0|^{2^*}\varphi_\varepsilon^i + \widetilde{\nu_{a_i}}\right) = \widetilde{\nu_{a_i}}, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \mu_j \frac{u_n^2}{|x - a_j|^4}\varphi_\varepsilon^i = 0 \quad \text{for } j \neq i.
\end{aligned}$$

Thus we have

$$0=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'_\lambda(u_n),u_n\varphi^i_\varepsilon\rangle\geq\widetilde{\mu_{a_i}}-\mu_i\widetilde{\gamma_{a_i}}-\widetilde{\nu_{a_i}}.$$

From (4.5) and (4.6) we derive that $A_{\mu_i}\widetilde{\nu_{a_i}}^{\frac{2}{2^*}}\leq\widetilde{\nu_{a_i}}$ for all $1\leq i\leq k$, and then

$$\text{either}\quad \widetilde{\nu_{a_i}}=0\quad\text{or}\quad \widetilde{\nu_{a_i}}\geq A_{\mu_i}^{\frac{N}{4}}.$$

Claim 2 is thereby proved.

From the above arguments and (4.1), we conclude that

$$\begin{aligned}
c&=\lim_{n\to\infty}\left(J_\lambda(u_n)-\frac{1}{2}\langle J'_\lambda(u_n),u_n\rangle\right)\\
&=\frac{2}{N}\lim_{n\to\infty}\int_\Omega|u_n|^{2^*}+\left(\frac{1}{2}-\frac{1}{q}\right)\lambda\int_\Omega|u_0|^q\\
&=\frac{2}{N}\left(\int_\Omega|u_0|^{2^*}+\sum_{j\in\mathcal{J}}\widetilde{\nu_{x_j}}+\sum_{i=1}^k\widetilde{\nu_{a_i}}\right)+\left(\frac{1}{2}-\frac{1}{q}\right)\lambda\int_\Omega|u_0|^q\\
&=\frac{2}{N}\left(\sum_{j\in\mathcal{J}}\widetilde{\nu_{x_j}}+\sum_{i=1}^k\widetilde{\nu_{a_i}}\right).
\end{aligned}$$

If $\widetilde{\nu_{a_i}}=\widetilde{\nu_{x_j}}=0$ for all $i\in\{1,2,\dots,k\}$ and $j\in\mathcal{J}$, then $c=0$ which contradicts the assumption that $c>0$. On the other hand, if there exists an $i\in\{1,2,\dots,k\}$ such that $\widetilde{\nu_{a_i}}\neq 0$ or there exists a $j\in\mathcal{J}$ with $\widetilde{\nu_{x_j}}\neq 0$, then we infer that

$$c\geq\frac{2}{N}\min\left\{A_0^{\frac{N}{4}},A_{\mu_1}^{\frac{N}{4}},A_{\mu_2}^{\frac{N}{4}},\dots,A_{\mu_k}^{\frac{N}{4}}\right\}=\frac{2}{N}A_{\mu_k}^{\frac{N}{4}},$$

which also contradicts the assumption that $c<\frac{2}{N}A_{\mu_k}^{\frac{N}{4}}$. Therefore u_0 is a nonzero solution of Eq. (E_λ). □

Take $\delta_0>0$ small enough such that $B_{2\delta_0}(a_k)\subset\Omega$. Choose the radial cut-off function $\eta(x)=\eta(|x|)\in C_0^\infty(B_{2\delta_0}(0))$ such that $0\leq\eta(x)\leq 1$ in $B_{2\delta_0}(0)$ and $\eta(x)=1$ in $B_{\delta_0}(0)$. Set $u_\varepsilon(x)=\eta(x-a_k)y_\varepsilon^{\mu_k}(x-a_k)$, where $y_\varepsilon^{\mu_k}(x)$ is the same function as in (1.2). The following asymptotic properties hold.

Lemma 4.3 *Assume that $N\geq 5$, $\mu_k\in[0,\bar{\mu})$, $\delta=\frac{N-4}{2}$ and $1\leq q<2^*$. Then, as $\varepsilon\to 0$, we have the following estimates*:

$$\int_\Omega\left(|\Delta u_\varepsilon|^2-\mu_k\frac{|u_\varepsilon|^2}{|x-a_k|^4}\right)=A_{\mu_k}^{\frac{N}{4}}+O\left(\varepsilon^{2(b(\mu_k)-\delta)}\right),\tag{4.9}$$

$$\int_\Omega|u_\varepsilon|^{2^*}=A_{\mu_k}^{\frac{N}{4}}+O\left(\varepsilon^{2^*(b(\mu_k)-\delta)}\right),\tag{4.10}$$

and

$$\int_\Omega |u_\varepsilon|^q = \begin{cases} O_1(\varepsilon^{N-q\delta}), & \text{if } \frac{N}{b(\mu_k)} < q < 2^*, \\ O_1(\varepsilon^{N-q\delta})|\ln\varepsilon|, & \text{if } q = \frac{N}{b(\mu_k)}, \\ O_1(\varepsilon^{q(b(\mu_k)-\delta)}), & \text{if } 1 \le q < \frac{N}{b(\mu_k)}. \end{cases} \tag{4.11}$$

Moreover, for all $N \ge 8$, *as* $\varepsilon \to 0$, *we have*

$$\int_\Omega |u_\varepsilon|^2 = \begin{cases} O_1(\varepsilon^4), & \text{if } 0 \le \mu_k < \mu^*, \\ O_1(\varepsilon^4|\ln\varepsilon|), & \text{if } \mu_k = \mu^*, \end{cases} \tag{4.12}$$

where $\mu^* := \frac{1}{16}(N^2-16)(N^2-8N)$.

Proof See Kang-Xu [19, Lemma 3.2]. □

Lemma 4.4 *Let* $N \ge 5$, $1 \le q < 2$ *and assume that* $(\mathcal{H})$ *holds. Then, for any* $\lambda > 0$, *there exists a* $v_\lambda \in H_0^2(\Omega)$ *such that*

$$\sup_{t\ge 0} J_\lambda(tv_\lambda) < \frac{2}{N} A_{\mu_k}^{\frac{N}{4}}. \tag{4.13}$$

In particular, $\alpha_\lambda^- < \frac{2}{N} A_{\mu_k}^{\frac{N}{4}}$ *for all* $\lambda \in (0, \Lambda_0)$.

Proof For $t \ge 0$, we consider the functions

$$\begin{aligned} g(t) &:= J_\lambda(tu_\varepsilon) \\ &= \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}\int_\Omega |u_\varepsilon|^{2^*} - \lambda\frac{t^q}{q}\int_\Omega |u_\varepsilon|^q \\ &\le \frac{t^2}{2}\int_\Omega \left(|\Delta u_\varepsilon|^2 - \mu_k \frac{u_\varepsilon^2}{|x-a_k|^4}\right) - \frac{t^{2^*}}{2^*}\int_\Omega |u_\varepsilon|^{2^*} - \lambda\frac{t^q}{q}\int_\Omega |u_\varepsilon|^q \end{aligned}$$

and

$$\begin{aligned} \bar{g}(t) &:= \frac{t^2}{2}\int_\Omega \left(|\Delta u_\varepsilon|^2 - \mu_k \frac{u_\varepsilon^2}{|x-a_k|^4}\right) - \frac{t^{2^*}}{2^*}\int_\Omega |u_\varepsilon|^{2^*} \\ &= \frac{t^2}{2}\|u_\varepsilon\|_{\mu_k}^2 - \frac{t^{2^*}}{2^*}\int_\Omega |u_\varepsilon|^{2^*}, \end{aligned}$$

where $\|u_\varepsilon\|_{\mu_k}^2 := \int_\Omega (|\Delta u_\varepsilon|^2 - \mu_k \frac{u_\varepsilon^2}{|x-a_k|^4})$.

Using the definitions of g and u_ε, we get

$$g(t) = J_\lambda(tu_\varepsilon) \le \frac{t^2}{2}\|u_\varepsilon\|_{\mu_k}^2, \quad \text{for all } t \ge 0 \text{ and } \lambda > 0.$$

Combining this with (4.9), let $\varepsilon \in (0,1)$, then there exists $t_0 \in (0,1)$ not depending on ε such that

$$\sup_{0\le t\le t_0} g(t) < \frac{2}{N} A_{\mu_k}^{\frac{N}{4}}, \quad \text{for all } \lambda > 0 \text{ and } \varepsilon \in (0,1). \tag{4.14}$$

On the other hand, by the fact that

$$\max_{t\geq 0}\left(\frac{t^2}{2}B_1-\frac{t^{2^*}}{2^*}B_2\right)=\frac{2}{N}B_1^{\frac{N}{4}}B_2^{\frac{4-N}{4}},\quad B_1>0,B_2>0,$$

and by (4.9) and (4.10), we can get

$$\begin{aligned}\max_{t\geq 0}\bar{g}(t)&=\frac{2}{N}\|u_\varepsilon\|_{\mu_k}^{\frac{N}{4}}\left(\int_\Omega|u_\varepsilon|^{2^*}\right)^{\frac{4-N}{4}}\\&=\frac{2}{N}\left(A_{\mu_k}^{\frac{N}{4}}+O\left(\varepsilon^{2(b(\mu_k)-\delta)}\right)\right)^{\frac{N}{4}}\left(A_{\mu_k}^{\frac{N}{4}}+O\left(\varepsilon^{2^*(b(\mu_k)-\delta)}\right)\right)^{\frac{4-N}{4}}\\&=\frac{2}{N}A_{\mu_k}^{\frac{N}{4}}+O\left(\varepsilon^{2(b(\mu_k)-\delta)}\right).\end{aligned}\tag{4.15}$$

Hence, for all $\lambda>0$, $1\leq q<2$, by (4.15) we have

$$\begin{aligned}\sup_{t\geq t_0}g(t)&\leq\sup_{t\geq t_0}\left(\bar{g}(t)-\lambda\frac{t^q}{q}\int_\Omega|u_\varepsilon|^q\right)\\&\leq\frac{2}{N}A_{\mu_k}^{\frac{N}{4}}+O\left(\varepsilon^{2(b(\mu_k)-\delta)}\right)-\lambda\frac{t_0^q}{q}\int_\Omega|u_\varepsilon|^q.\end{aligned}\tag{4.16}$$

Now, we need to distinguish two cases.

Case (i): $1\leq q<\frac{N}{b(\mu)}$ and $q<2$. By (1.3) and (4.11) we have as $\varepsilon\to 0$

$$\int_\Omega|u_\varepsilon|^q=O_1\left(\varepsilon^{q(b(\mu)-\delta)}\right)>O\left(\varepsilon^{2(b(\mu)-\delta)}\right).$$

Combining this with (4.14) and (4.16), for any $\lambda>0$, we can choose ε_λ small enough such that

$$\sup_{t\geq 0}J_\lambda(tu_{\varepsilon_\lambda})<\frac{2}{N}A_{\mu_k}^{\frac{N}{4}}.$$

Case (ii): $\frac{N}{b(\mu)}\leq q<2$. By (4.11) we have

$$\int_\Omega|u_\varepsilon|^q=\begin{cases}O_1(\varepsilon^{N-q\delta}), & \text{if } q>\frac{N}{b(\mu)},\\ O_1(\varepsilon^{N-q\delta}|\ln\varepsilon|), & \text{if } q=\frac{N}{b(\mu)}.\end{cases}$$

Moreover, it follows from $b(\mu)>\delta$ and $q\geq\frac{N}{b(\mu)}$ that

$$2\left(b(\mu)-\delta\right)>q\left(b(\mu)-\delta\right)\geq N-q\delta.$$

Combining this with (4.14) and (4.16), for any $\lambda>0$, we can choose ε_λ small enough such that

$$\sup_{t\geq 0}J_\lambda(tu_{\varepsilon_\lambda})<\frac{2}{N}A_{\mu_k}^{\frac{N}{4}}.$$

From cases (i) and (ii), (4.13) holds by taking $v_\lambda = u_{\varepsilon_\lambda}$.

From Lemma 2.4, the definition of α_λ^- and (4.13), for any $\lambda \in (0, \Lambda_0)$, we see that there exists $t_\lambda^- > 0$ such that $t_\lambda^- v_\lambda \in \mathcal{M}_\lambda^-$ and

$$\alpha_\lambda^- \le J_\lambda\left(t_\lambda^- v_\lambda\right) \le \sup_{t\ge 0} J_\lambda(tv_\lambda) < \frac{2}{N} A_{\mu_k}^{\frac{N}{4}}.$$

The proof is thus completed. □

Now, we establish the existence of a local minimum of J_λ on $\mathcal{M}_\lambda^-$.

Theorem 4.5 *Assume that $N \ge 5$ and the condition $(\mathcal{H})$ holds. If $\lambda \in (0, \frac{q}{2}\Lambda_0)$, then J_λ has a minimizer U_λ in $\mathcal{M}_\lambda^-$ and such that*:

(i) $J_\lambda(U_\lambda) = \alpha_\lambda^-$.

(ii) *U_λ is a nontrivial solution of Eq.* (E_λ).

Proof If $\lambda \in (0, \frac{q}{2}\Lambda_0)$, then, by Lemma 3.1(ii), Proposition 3.4(ii) and Lemma 4.4, there exists a $(\mathrm{PS})_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{M}_\lambda^-$ in $H_0^2(\Omega)$ for J_λ with $\alpha_\lambda^- \in (0, \frac{2}{N} A_{\mu_k}^{\frac{N}{4}})$. Since J_λ is coercive on $\mathcal{M}_\lambda$ (see Lemma 4.1), we see that $\{u_n\}$ is bounded in $H_0^2(\Omega)$. From Lemma 4.2, there exist a subsequence still denoted by $\{u_n\}$ and a nonzero solution $U_\lambda \in H_0^2(\Omega)$ of Eq. (E_λ) such that $u_n \rightharpoonup U_\lambda$ weakly in $H_0^2(\Omega)$.

Now, we first prove that $U_\lambda \in \mathcal{M}_\lambda^-$. Arguing by contradiction, we assume $U_\lambda \in \mathcal{M}_\lambda^+$. Then, by Lemma 2.4, there exists a unique t_λ^- such that $t_\lambda^- U_\lambda \in \mathcal{M}_\lambda^-$. It follows that

$$\alpha_\lambda^- \le J_\lambda\left(t_\lambda^- U_\lambda\right) < \lim_{n\to\infty} J_\lambda\left(t_\lambda^- u_n\right) \le \lim_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^-.$$

This is a contradiction. Consequently, $U_\lambda \in \mathcal{M}_\lambda^-$.

Next, by the same argument as that in Theorem 3.5, we get $u_n \to U_\lambda$ strongly in $H_0^2(\Omega)$ and $J_\lambda(U_\lambda) = \alpha_\lambda^- > 0$ for all $\lambda \in (0, \frac{q}{2}\Lambda_0)$. Since $J_\lambda(U_\lambda) = J_\lambda(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{M}_\lambda^-$, by Lemma 2.2 we may assume that U_λ is a nontrivial nonnegative solution of Eq. (E_λ). The proof of this theorem is then completed. □

Proof of Theorem 1.1 The part (i) of Theorem 1.1 immediately follows from Theorem 3.5. When $0 < \lambda < \frac{q}{2}\Lambda_0 < \Lambda_0$, by Theorems 3.5, and 4.5, we see that Eq. (E_λ) has at least two nontrivial solutions u_λ and U_λ such that $u_\lambda \in \mathcal{M}_\lambda^+$ and $U_\lambda \in \mathcal{M}_\lambda^-$. Since $\mathcal{M}_\lambda^+ \cap \mathcal{M}_\lambda^- = \emptyset$, this implies that u_λ and U_λ are distinct. This completes the proof of Theorem 1.1. □

5 Existence of solutions in the case of $2 \le q < 2^*$

In order to prove Theorem 1.2, we first establish several lemmas.

Lemma 5.1 *Let $N \ge 5$ and assume that $(\mathcal{H})$ holds and one of the following conditions hold*:

(i) $\lambda > 0$, $2 < q < 2^*$.

(ii) $0 < \lambda < \lambda_1$, $q = 2$.

Then the functional J_λ satisfies the (PS) *condition for all $c < c^* := \frac{2}{N} A_{\mu_k}^{\frac{N}{4}}$.*

Proof The argument is standard and is omitted (e.g. [17]) □

Lemma 5.2 *Let $N \geq 5$ and assume that $(\mathcal{H})$ holds and one of the following conditions holds*:

(i) $\lambda > 0$, $\overline{q} < q < 2^*$, *where*

$$\overline{q} = \max\left\{2, \frac{N}{b(\mu_k)}, \frac{4(N-2-b(\mu_k))}{N-4}\right\}.$$

(ii) $N \geq 8$, $0 < \lambda < \lambda_1$, $q = 2$, $0 \leq \mu_k \leq \mu^*$.

Then as $\varepsilon \to 0^+$ we have

$$\sup_{t\geq 0} J_\lambda(tu_\varepsilon) < c^* = \frac{2}{N} A_{\mu_k}^{\frac{N}{4}}, \tag{5.1}$$

where λ_1 is the same as in (1.6) *and u_ε is the same function as in Lemma* 4.3.

Proof For $t \geq 0$, we define the functions $g(t) := J_\lambda(tu_\varepsilon)$ and

$$\bar{g}(t) := \frac{t^2}{2} \int_\Omega \left(|\Delta u_\varepsilon|^2 - \mu_k \frac{u_\varepsilon^2}{|x-a_k|^4}\right) - \frac{t^{2^*}}{2^*} \int_\Omega |u_\varepsilon|^{2^*}.$$

(i) Since $\lambda > 0$, $2 < q < 2^*$, a direct calculation shows that $\sup_{t\geq 0} g(t)$ can be obtained at finite $t_\varepsilon > 0$ such that

$$0 = g'(t_\varepsilon) = t_\varepsilon\left(\|u_\varepsilon\|^2 - t_\varepsilon^{2^*-2} \int_\Omega |u_\varepsilon|^{2^*} - \lambda t_\varepsilon^{q-2} \int_\Omega |u_\varepsilon|^q\right).$$

Furthermore, $t_\varepsilon \in [C_1, C_2]$, where C_1 and C_2 are positive constants independent of ε.

From the definitions of g, $\bar{g}$ and (4.15), it follows that

$$g(t_\varepsilon) \leq \bar{g}(t_\varepsilon) - \frac{\lambda}{q} t_\varepsilon^q \int_\Omega |u_\varepsilon|^q \leq c^* + O\big(\varepsilon^{2(b(\mu_k)-\delta)}\big) - C \int_\Omega |u_\varepsilon|^q. \tag{5.2}$$

If $\overline{q} < q < 2^*$, by (4.11) we have

$$\int_\Omega |u_\varepsilon|^q = O_1\big(\varepsilon^{N-q\delta}\big). \tag{5.3}$$

Since $2(b(\mu_k) - \delta)) > N - q\delta$, from (5.2) and (5.3) it follows that

$$\sup_{t\geq 0} J_\lambda(tu_\varepsilon) = g(t_\varepsilon) < c^*.$$

(ii) Suppose that $N \geq 8$, $0 < \lambda < \lambda_1$, $q = 2$, $0 \leq \mu_k \leq \mu^*$. A direct calculation shows that

$$0 \leq \mu_k \leq \mu^* \iff b(\mu_k) - \delta > 2, \qquad \mu_k = \mu^* \iff b(\mu_k) - \delta = 2.$$

Using a similar argument to (i), we can deduce that $\sup_{t\geq 0} g(t) < c^*$ is attained at finite $t_\varepsilon > 0$. Moreover,

$$g(t_\varepsilon) \leq \bar{g}(t_\varepsilon) - \lambda \frac{t_\varepsilon^2}{2} \int_\Omega |u_\varepsilon|^2 \leq c^* + O\big(\varepsilon^{2(b(\mu_k)-\delta)}\big) - C \int_\Omega |u_\varepsilon|^2. \tag{5.4}$$

Then by (4.12) and (5.4) it follows that (5.1) holds as $\varepsilon \to 0^+$. The proof is thus completed. □

Proof of Theorem 1.2 According to Lemmas 5.1 and 5.2 and applying the mountain-pass theorem [2, 4], Theorem 1.2 can be concluded to. □

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