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Fractional order differential systems involving right Caputo and left Riemann–Liouville fractional derivatives with nonlocal coupled conditions

Bashir Ahmad[1*], Sotiris K. Ntouyas[1,2] and Ahmed Alsaedi[1]

*Correspondence:
bashirahmad_qau@yahoo.com
[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we introduce and study a new kind of coupled fractional differential system involving right Caputo and left Riemann–Liouville fractional derivatives, supplemented with nonlocal three-point coupled boundary conditions. Existence and uniqueness results for the given problem are derived with the aid of modern techniques of functional analysis. An example illustrating the existence of a unique solution is presented.

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1 Introduction

In recent years, fractional order calculus has been one of the most rapidly developing areas of mathematical analysis. In fact, a natural phenomenon may not only depend on the current time but also on its previous time history. Fractional calculus facilitates modeling of such phenomena via nonlocal fractional differential and and integral operators. Fractional order differential equations naturally appear in the mathematical modeling of systems with memory. One can find numerous applications of fractional calculus in diverse fields such as mathematics, physics, chemistry, optimal control theory, finance, biology, engineering, and so on [1–6].

Fractional differential equations including both left and right fractional derivatives are also attracting much attention as they appear as the Euler–Lagrange equations in the study of variational principles, for details, see [7] and the references cited therein. Some recent results on the topic, obtained by means of different methods such as fixed point theorems, upper and lower solutions method, variational methods, etc., can be found in the papers [8–12]. In [9], the existence of extremal solutions to a nonlinear system with the right Riemann–Liouville fractional derivative was addressed.

In [10], the authors studied the existence of solutions for a nonlinear fractional oscillator equation with both Riemann–Liouville and Caputo fractional derivatives subject to natural boundary conditions:

$$\omega^2 u - {}^cD_{1-}^p D_{0+}^q u = f\big(t, u(t)\big) = 0, \quad \omega \in \mathbb{R} - \{0\}, 0 \le t \le 1,$$
$$u(0) = 0, \qquad D_{0+}^q u(1) = 0,$$

where ${}^cD_{1-}^p$ and D_{0+}^q respectively denote the right Caputo fractional derivative of order $p \in (0,1)$ and the left Riemann–Liouville fractional derivative of order $q \in (0,1)$.

In [12], the authors used the Krasnoselskii fixed point theorem to prove the existence of solutions to the following problem involving both left Riemann–Liouville and right Caputo fractional derivatives:

$$ {}^cD_{1-}^\alpha \big(D_{0+}^\beta u(t)\big) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = u(1) = 0,$$

where ${}^cD_{1-}^\alpha$ and D_{0+}^β respectively denote the right Caputo fractional derivative of order $\alpha \in (0,1)$ and the left Riemann–Liouville fractional derivative of order $\beta \in (1,2)$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$.

In [13], a nonlocal boundary value problem involving both Caputo and Riemann–Liouville fractional derivatives was studied:

$$\begin{cases} {}^cD_{1-}^\alpha D_{0+}^\beta y(t) = f(t, y(t)), \quad t \in J := [0,1], \\ y(0) = y'(0) = 0, \qquad y(1) = \delta y(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

where ${}^cD_{1-}^\alpha$ and D_{0+}^β denote the right Caputo fractional derivative of order $\alpha \in (1,2]$ and the left Riemann–Liouville fractional derivative of order $\beta \in (0,1]$ respectively, and $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function.

In this paper, we investigate a new coupled system of nonlinear fractional differential equations involving both right Caputo and left Riemann–Liouville fractional derivatives, equipped with nonlocal coupled boundary conditions given by

$$\begin{cases} {}^cD_{1-}^\alpha D_{0+}^\beta x(t) = f(t, x(t), y(t)), \quad t \in J := [0,1], \\ {}^cD_{1-}^p D_{0+}^q y(t) = g(t, x(t), y(t)), \quad t \in J := [0,1], \\ x(0) = x'(0) = 0, \qquad x(1) = \gamma y(\eta), \quad 0 < \eta < 1, \\ y(0) = y'(0) = 0, \qquad y(1) = \delta x(\theta), \quad 0 < \theta < 1, \end{cases} \tag{1.2}$$

where ${}^cD_{1-}^\alpha$, ${}^cD_{1-}^p$ denote the right Caputo fractional derivatives of order $\alpha, p \in (1,2]$ and D_{0+}^β, D_{0+}^q denote the left Riemann–Liouville fractional derivative of order $\beta, q \in (0,1]$ respectively, $f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions and $\gamma, \delta \in \mathbb{R}$ are appropriate constants. The existence and uniqueness of solutions for the given problem will be derived by applying the well-known methods of functional analysis such as the Banach fixed point theorem and the Leray–Schauder alternative.

The rest of the paper is organized as follows. In Sect. 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma, which plays a pivotal role in obtaining the main results presented in Sect. 3. We also discuss an example for illustration of the existence-uniqueness result.

2 Preliminaries

This section is devoted to the preliminary concepts of fractional calculus [4] that we need in the sequel.

Definition 2.1 We define the left and right Riemann–Liouville fractional integrals of order $\alpha > 0$ of a function $g : (0,\infty) \to \mathbb{R}$ respectively as

$$I_{0+}^{\alpha} g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\, ds, \tag{2.1}$$

$$I_{1-}^{\alpha} g(t) = \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} g(s)\, ds, \tag{2.2}$$

provided the right-hand sides are point-wise defined on $(0,\infty)$, where Γ is the gamma function.

Definition 2.2 The left Riemann–Liouville fractional derivative and the right Caputo fractional derivative of order $\alpha > 0$ of a continuous function $g : (0,\infty) \to \mathbb{R}$ such that $g \in C^n((0,\infty),\mathbb{R})$ are respectively given by

$$D_{0+}^{\alpha} g(t) = \frac{d^n}{dt^n}\big(I_{0+}^{n-\alpha} g\big)(t),$$
$${}^{c}D_{1-}^{\alpha} g(t) = (-1)^n I_{1-}^{n-\alpha} g^{(n)}(t),$$

where $n-1 < \alpha < n$.

The following lemma, dealing with a linear variant of problem (1.2), plays an important role in the forthcoming analysis.

Lemma 2.3 *Let $h, k \in C(J,\mathbb{R})$ and*

$$\Lambda := \frac{1}{\Gamma(\beta+2)\Gamma(q+2)}\big[1 - \gamma\delta\eta^{q+1}\theta^{\beta+1}\big] \neq 0.$$

Then the solution of the linear fractional differential system supplemented with nonlocal boundary conditions

$$\begin{cases} D_{1-}^{\alpha} D_{0+}^{\beta} x(t) = h(t), & t \in J := [0,1], \\ D_{1-}^{p} D_{0+}^{q} y(t) = k(t), & t \in J := [0,1], \\ x(0) = x'(0) = 0, & x(1) = \gamma y(\eta), \quad 0 < \eta < 1, \\ y(0) = y'(0) = 0, & y(1) = \delta x(\theta), \quad 0 < \theta < 1, \end{cases} \tag{2.3}$$

is equivalent to a system of integral equations given by

$$\begin{aligned} x(t) = {} & I_{0+}^{\beta} I_{1-}^{\alpha} h(t) + \frac{t^{\beta+1}}{\Lambda\Gamma(\beta+2)\Gamma(q+2)}\big\{\big[\gamma I_{0+}^{q} I_{1-}^{p} k(\eta) - I_{0+}^{\beta} I_{1-}^{\alpha} h(1)\big] \\ & + \gamma\eta^{q+1}\big[\delta I_{0+}^{\beta} I_{1-}^{\alpha} h(\theta) - I_{0+}^{q} I_{1-}^{p} k(1)\big]\big\}, \end{aligned} \tag{2.4}$$

$$y(t) = I_{0+}^{q} I_{1-}^{p} k(t) + \frac{t^{q+1}}{\Lambda \Gamma(\beta+2)\Gamma(q+2)} \big\{ \big[\delta I_{0+}^{\beta} I_{1-}^{\alpha} h(\theta) - I_{0+}^{q} I_{1-}^{p} k(1) \big] \\ + \delta \theta^{\beta+1} \big[\gamma I_{0+}^{q} I_{1-}^{p} k(\eta) - I_{0+}^{\beta} I_{1-}^{\alpha} h(1) \big] \big\}. \tag{2.5}$$

Proof We first apply the right fractional integrals $I_{1-}^{\alpha}, I_{1-}^{p}$ to the fractional differential equations in (2.3) and then the left fractional integrals $I_{0+}^{\beta}, I_{0+}^{q}$ to the resulting equations, and using the properties of Caputo and Riemann–Liouville fractional derivatives, we get

$$\begin{aligned} x(t) &= I_{0+}^{\beta} \big(I_{1-}^{\alpha} h(t) + c_0 + c_1 t \big) + c_2 t^{\beta-1} \\ &= I_{0+}^{\beta} I_{1-}^{\alpha} h(t) + c_0 \frac{t^{\beta}}{\Gamma(\beta+1)} + c_1 \frac{t^{\beta+1}}{\Gamma(\beta+2)} + c_2 t^{\beta-1}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} y(t) &= I_{0+}^{q} \big(I_{1-}^{p} k(t) + d_0 + d_1 t \big) + d_2 t^{q-1} \\ &= I_{0+}^{q} I_{1-}^{p} k(t) + d_0 \frac{t^{q}}{\Gamma(q+1)} + d_1 \frac{t^{q+1}}{\Gamma(q+2)} + d_2 t^{q-1}. \end{aligned} \tag{2.7}$$

Using the conditions $x(0) = 0$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = 0$ in (2.6) and (2.7) yields $c_0 = 0$, $d_0 = 0$, $c_2 = 0$, $d_2 = 0$. In consequence, the system of equations (2.6) and (2.7) reduces to the form:

$$x(t) = I_{0+}^{\beta} I_{1-}^{\alpha} h(t) + c_1 \frac{t^{\beta+1}}{\Gamma(\beta+2)}, \tag{2.8}$$

$$y(t) = I_{0+}^{q} I_{1-}^{p} k(t) + d_1 \frac{t^{q+1}}{\Gamma(q+2)}. \tag{2.9}$$

Making use of the conditions $x(1) = \gamma y(\eta)$, $y(1) = \delta x(\theta)$ in (2.8) and (2.9) and solving the resulting equations for c_1 and d_1, we find that

$$c_1 = \frac{1}{\Lambda \Gamma(q+2)} \big\{ \big[\gamma I_{0+}^{q} I_{1-}^{p} k(\eta) - I_{0+}^{\beta} I_{1-}^{\alpha} h(1) \big] + \gamma \eta^{q+1} \big[\delta I_{0+}^{\beta} I_{1-}^{\alpha} h(\theta) - I_{0+}^{q} I_{1-}^{p} k(1) \big] \big\},$$

$$d_1 = \frac{1}{\Lambda \Gamma(\beta+2)} \big\{ \big[\delta I_{0+}^{\beta} I_{1-}^{\alpha} h(\theta) - I_{0+}^{q} I_{1-}^{p} k(1) \big] + \delta \theta^{\beta+1} \big[\gamma I_{0+}^{q} I_{1-}^{p} k(\eta) - I_{0+}^{\beta} I_{1-}^{\alpha} h(1) \big] \big\},$$

which, on substituting in (2.8) and (2.9), leads to the solution system (2.4)–(2.5). The converse follows by direct computation. The proof is completed. □

3 Main results

Let us introduce the space $\mathcal{X} = \{x(t) | x(t) \in C([0,1], \mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$ and note that $(\mathcal{X}, \|\cdot\|)$ is a Banach space. Then the product space $(\mathcal{X} \times \mathcal{X}, \|(x,y)\|)$ is also a Banach space equipped with the norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 2.3, we define an operator $T : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by

$$T(x,y)(t) = \begin{pmatrix} T_1(x,y)(t) \\ T_2(x,y)(t) \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned}
T_1(x,y)(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u,x(u),y(u)\big)\,du\,ds \\
& + \frac{t^{\beta+1}}{\Lambda\Gamma(\beta+2)\Gamma(q+2)} \\
& \times \biggl\{ \biggl[\gamma \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} g\big(u,x(u),y(u)\big)\,du\,ds \\
& - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u,x(u),y(u)\big)\,du\,ds \biggr] \\
& + \gamma\eta^{q+1} \biggl[\delta \int_0^\theta \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u,x(u),y(u)\big)\,du\,ds \\
& - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} g\big(u,x(u),y(u)\big)\,du\,ds \biggr] \biggr\}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
T_2(x,y)(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} g\big(u,x(u),y(u)\big)\,du\,ds \\
& + \frac{t^{q+1}}{\Lambda\Gamma(\beta+2)\Gamma(q+2)} \\
& \times \biggl\{ \biggl[\delta \int_0^\theta \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u,x(u),y(u)\big)\,du\,ds \\
& - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} g\big(u,x(u),y(u)\big)\,du\,ds \biggr] \\
& + \delta\theta^{\beta+1} \biggl[\gamma \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} g\big(u,x(u),y(u)\big)\,du\,ds \\
& - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u,x(u),y(u)\big)\,du\,ds \biggr] \biggr\}.
\end{aligned} \tag{3.3}$$

For computational convenience, we set

$$Q_1 = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \biggl[1 + \frac{1}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)} \big(1 + |\gamma||\delta|\eta^{q+1}\theta^\beta\big) \biggr], \tag{3.4}$$

$$Q_2 = \frac{|\gamma|\eta^q}{|\Lambda|\Gamma(p+1)\Gamma(q+1)\Gamma(\beta+2)\Gamma(q+2)}(1+\eta), \tag{3.5}$$

$$Q_3 = \frac{|\delta|\theta^\beta}{|\Lambda|\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\beta+2)\Gamma(q+2)}(1+\theta), \tag{3.6}$$

$$Q_4 = \frac{1}{\Gamma(p+1)\Gamma(q+1)} \biggl[1 + \frac{1}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)} \big(1 + |\gamma||\delta|\eta^q\theta^{\beta+1}\big) \biggr]. \tag{3.7}$$

Now we are ready to present our main results. In the first result, we prove the existence and uniqueness of solutions to system (1.2) via the Banach contraction mapping principle.

Theorem 3.1 *Assume that*:

(H_1) $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist positive constants* ℓ_1 *and* ℓ_2 *such that, for all* $t \in [0,1]$ *and* $x_i, y_i \in \mathbb{R}$, $i = 1, 2$,

$$\begin{aligned} \big|f(t,x_1,x_2) - f(t,y_1,y_2)\big| &\leq \ell_1\big(|x_1 - y_1| + |x_2 - y_2|\big), \\ \big|g(t,x_1,x_2) - g(t,y_1,y_2)\big| &\leq \ell_2\big(|x_1 - y_1| + |x_2 - y_2|\big). \end{aligned}$$

If

$$(Q_1 + Q_3)\ell_1 + (Q_2 + Q_4)\ell_2 < 1,$$

where Q_i, $i = 1,2,3,4$ *are given by* (3.4)–(3.7), *then system* (1.2) *has a unique solution* $[0,1]$.

Proof Let us define a positive number r as follows:

$$r > \frac{(Q_1 + Q_3)N_1 + (Q_2 + Q_4)N_2}{1 - (Q_1 + Q_3)\ell_1 - (Q_2 + Q_4)\ell_2},$$

$N_1 = \sup_{t\in[0,1]} |f(t,0,0)| < \infty$, $N_2 = \sup_{t\in[0,1]} |g(t,0,0)| = N_2 < \infty$ and show that $TB_r \subset B_r$, where $B_r = \{(x,y) \in \mathcal{X} \times \mathcal{X} : \|(x,y)\| \leq r\}$ is a closed ball. By assumption (H_1), for $(x,y) \in B_r$, $t \in [0,1]$, we have

$$\begin{aligned} \big|f\big(t,x(t),y(t)\big)\big| &\leq \big|f\big(t,x(t),y(t)\big) - f(t,0,0)\big| + \big|f(t,0,0)\big| \\ &\leq \ell_1\big(\big|x(t)\big| + \big|y(t)\big|\big) + N_1 \\ &\leq \ell_1\big(\|x\| + \|y\|\big) + N_1 \leq \ell_1 r + N_1, \end{aligned}$$

and

$$\big|g\big(t,x(t),y(t)\big)\big| \leq \ell_2\big(\|x\| + \|y\|\big) + N_2 \leq \ell_2 r + N_2.$$

Also note that

$$\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,du\,ds \leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)},$$

where we have used the fact that $(1-s)^\alpha \leq 1$ for $1 < \alpha \leq 2$. Using the above arguments, we have

$$\begin{aligned} &\big|T_1(x,y)(t)\big| \\ &\quad\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}(\ell_1 r + N_1)\,du\,ds \\ &\qquad+ \frac{1}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)}\biggl\{\biggl[|\gamma| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)}(\ell_2 r + N_2)\,du\,ds \\ &\qquad+ \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}(\ell_1 r + N_1)\,du\,ds\biggr] \end{aligned}$$

$$
\begin{aligned}
&+|\gamma|\eta^{q+1}\biggl[|\delta|\int_0^\theta \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}(\ell_1 r+N_1)\,du\,ds \\
&+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)}(\ell_2 r+N_2)\,du\,ds\biggr]\biggr\} \\
\leq{}& \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}(\ell_1 r+N_1) \\
&+\frac{1}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)}\biggl\{\biggl[|\gamma|\frac{\eta^q}{\Gamma(q+1)\Gamma(p+1)}(\ell_2 r+N_2) \\
&+\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}(\ell_1 r+N_1)\biggr]+|\gamma|\eta^{q+1}\biggl[|\delta|\frac{\theta^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}(\ell_1 r+N_1) \\
&+\frac{1}{\Gamma(p+1)\Gamma)q+1)}(\ell_2 r+N_2)\biggr]\biggr\} \\
={}&(Q_1\ell_1+Q_2\ell_2)r+Q_1N_1+Q_2N_2,
\end{aligned}
$$

which implies that

$$
\|T_1(x,y)\|\leq(Q_1\ell_1+Q_2\ell_2)r+Q_1N_1+Q_2N_2.
$$

In the same way, we can obtain that

$$
\|T_2(x,y)\|\leq(Q_3\ell_1+Q_4\ell_2)r+Q_3N_1+Q_4N_2.
$$

Consequently, we get

$$
\|T(x,y)\|\leq\bigl[(Q_1+Q_3)\ell_1+(Q_2+Q_4)\ell_2\bigr]r+(Q_1+Q_3)N_1+(Q_2+Q_4)N_2\leq r.
$$

Since $(x,y)\in B_r$ is arbitrary, therefore it follows that $TB_r\subset B_r$.

Now, for $(x_2,y_2),(x_1,y_1)\in\mathcal{X}\times\mathcal{X}$ and for any $t\in[0,1]$, we get

$$
\begin{aligned}
&\bigl|T_1(x_2,y_2)(t)-T_1(x_1,y_1)(t)\bigr| \\
&\leq\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\ell_1\bigl(\|x_2-x_1\|+\|y_2-y_1\|\bigr)\,du\,ds \\
&\quad+\frac{1}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)} \\
&\quad\times\biggl\{\biggl[|\gamma|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)}\ell_2\bigl(\|x_2-x_1\|+\|y_2-y_1\|\bigr)\,du\,ds \\
&\quad+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\ell_1\bigl(\|x_2-x_1\|+\|y_2-y_1\|\bigr)\,du\,ds\biggr] \\
&\quad+|\gamma|\eta^{q+1}\biggl[|\delta|\int_0^\theta \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\ell_1\bigl(\|x_2-x_1\|+\|y_2-y_1\|\bigr)\,du\,ds \\
&\quad+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)}\ell_2\bigl(\|x_2-x_1\|+\|y_2-y_1\|\bigr)\,du\,ds\biggr]\biggr\} \\
&\leq(Q_1\ell_1+Q_2\ell_2)\bigl(\|x_2-x_1\|+\|y_2-y_1\|\bigr),
\end{aligned}
$$

which leads to the estimate

$$\left\| T_1(x_2,y_2) - T_1(x_1,y_1) \right\| \leq (Q_1\ell_1 + Q_2\ell_2)\big(\|x_2 - x_1\| + \|y_2 - y_1\|\big). \tag{3.8}$$

Similarly, we can find that

$$\left\| T_2(x_2,y_2)(t) - T_2(x_1,y_1) \right\| \leq (Q_3\ell_1 + Q_4\ell_2)\big(\|x_2 - x_1\| + \|y_2 - y_1\|\big). \tag{3.9}$$

Then it follows from (3.8) and (3.9) that

$$\left\| T(x_2,y_2) - T(x_1,y_1) \right\| \leq \big[(Q_1 + Q_3)\ell_1 + (Q_2 + Q_4)\ell_2\big]\big(\|x_2 - x_1\| + \|y_2 - y_1\|\big).$$

Since $(Q_1 + Q_3)\ell_1 + (Q_2 + Q_4)\ell_2 < 1$, therefore, T is a contraction. So, by the Banach fixed point theorem, the operator T has a unique fixed point, which corresponds to a unique solution of problem (1.2). This completes the proof. □

The second result is based on the Leray–Schauder alternative [14].

Lemma 3.2 (Leray–Schauder alternative [14, p. 4]) *Let $F : E \to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \big\{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\big\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.3 *Assume that:*

(H_3) *$f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist real constants $k_i, \gamma_i \geq 0$, $i = 0,1,2$, and $k_0 > 0$, $\gamma_0 > 0$ such that*

$$\begin{aligned} \left|f(t,x_1,x_2)\right| &\leq k_0 + k_1|x_1| + k_2|x_2|, \\ \left|g(t,x_1,x_2)\right| &\leq \gamma_0 + \gamma_1|x_1| + \gamma_2|x_2|, \quad \forall x_i \in \mathbb{R}, i = 1,2. \end{aligned}$$

Then system (1.2) has at least one solution on $[0,1]$ if

$$(Q_1 + Q_3)k_1 + (Q_2 + Q_4)\gamma_1 < 1 \quad \text{and} \quad (Q_1 + Q_3)k_2 + (Q_2 + Q_4)\gamma_2 < 1, \tag{3.10}$$

where Q_i, $i = 1,2,3,4$ are given by (3.4)–(3.7),

Proof We first show that the operator $T : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ defined by (3.1) is completely continuous. By the continuity of functions f and g, the operator T is continuous.

Let $\Omega \subset \mathcal{X} \times \mathcal{X}$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$\left|f\big(t,x(t),y(t)\big)\right| \leq L_1, \qquad \left|h\big(t,x(t),y(t)\big)\right| \leq L_2, \quad \forall (x,y) \in \Omega.$$

Then, for any $(x,y) \in \Omega$, we have

$$\begin{aligned}
\big|T_1(x,y)(t)\big| \le{}& L_1 \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,du\,ds \\
&+ \frac{1}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)} \biggl\{ \biggl[L_2|\gamma| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)}\,du\,ds \\
&+ L_1 \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,du\,ds \biggr] \\
&+ |\gamma|\eta^{q+1} \biggl[L_1|\delta| \int_0^\theta \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,du\,ds \\
&+ L_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)}\,du\,ds \biggr] \biggr\} \\
\le{}& Q_1L_1 + Q_2L_2,
\end{aligned}$$

which implies that

$$\big\|T_1(x,y)\big\| \le Q_1L_1 + Q_2L_2.$$

In a similar manner, one can derive that

$$\big\|T_2(x,y)\big\| \le Q_3L_1 + Q_4L_2.$$

From the foregoing inequalities, we have $\|T(x,y)\| \le (Q_1+Q_3)L_1 + (Q_2+Q_4)L_2$, which implies that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\big|T_1\big(x(t_2),y(t_2)\big) - T_1\big(x(t_1),y(t_1)\big)\big| \\
&\quad\le L_1 \biggl| \int_0^{t_1} \frac{[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]}{\Gamma(\beta)\Gamma(\alpha+1)}\,ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}\,ds \biggr| \\
&\qquad+ \frac{|t_2^{\beta+1} - t_1^{\beta+1}|}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)} \biggl\{ \biggl[L_2|\gamma| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)}\,du\,ds \\
&\qquad+ L_1 \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,du\,ds \biggr] \\
&\qquad+ |\gamma|\eta^{q+1} \biggl[L_1|\delta| \int_0^\theta \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,du\,ds \\
&\qquad+ L_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)}\,du\,ds \biggr] \biggr\} \\
&\quad\le L_1 \frac{2(t_2-t_1)^\beta + t_2^\beta - t_1^\beta}{\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{|t_2^{\beta+1} - t_1^{\beta+1}|}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)} \biggl\{ \frac{L_2|\gamma|\eta^q(\eta+1)}{\Gamma(p+1)\Gamma(q+1)} \\
&\qquad+ \frac{L_1(1+|\gamma||\delta|\eta^{q+1}\theta^\beta)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \biggr\}.
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}&\bigl|T_2\bigl(x(t_2),y(t_2)\bigr)-T_2\bigl(x(t_1),y(t_1)\bigr)\bigr|\\&\quad\le L_2\frac{2(t_2-t_1)^q+t_2^q-t_1^q}{\Gamma(q+1)\Gamma(p+1)}+\frac{|t_2^{\beta+1}-t_1^{\beta+1}|}{|\Lambda|\Gamma(\beta+2)\Gamma(q+2)}\biggl\{\frac{L_1|\delta|\theta^\beta(\theta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\&\qquad+\frac{L_2(1+|\gamma||\delta|\eta^q\theta^{\beta+1})}{\Gamma(p+1)\Gamma(q+1)}\biggr\}.\end{aligned}$$

Therefore, the operator $T(x,y)$ is equicontinuous. In consequence, from the foregoing arguments, it follows that the operator $T(x,y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E}=\{(x,y)\in\mathcal{X}\times\mathcal{X}|(x,y)=\lambda T(x,y),0\le\lambda\le1\}$ is bounded. Let $(x,y)\in\mathcal{E}$ with $(x,y)=\lambda T(x,y)$. For any $t\in[0,1]$, we have

$$x(t)=\lambda T_1(x,y)(t),\qquad y(t)=\lambda T_2(x,y)(t).$$

Then

$$\begin{aligned}\bigl|x(t)\bigr|&\le Q_1\bigl(k_0+k_1|x|+k_2|y|\bigr)+Q_2\bigl(\gamma_0+\gamma_1|x|+\gamma_2|y|\bigr)\\&=Q_1k_0+Q_2\gamma_0+(Q_1k_1+Q_2\gamma_1)|x|+(Q_1k_2+Q_2\gamma_2)|y|\end{aligned}$$

and

$$\begin{aligned}\bigl|y(t)\bigr|&\le Q_3\bigl(k_0+k_1|x|+k_2|y|\bigr)+Q_4\bigl(\gamma_0+\gamma_1|x|+\gamma_2|y|\bigr)\\&=Q_3k_0+Q_4\gamma_0+(Q_3k_1+Q_4\gamma_1)|x|+(Q_3k_2+Q_4\gamma_2)|y|.\end{aligned}$$

Hence we have

$$\|x\|\le Q_1k_0+Q_2\gamma_0+(Q_1k_1+Q_2\gamma_1)\|x\|+(Q_1k_2+Q_2\gamma_2)\|y\|$$

and

$$\|y\|\le Q_3k_0+Q_4\gamma_0+(Q_3k_1+Q_4\gamma_1)\|x\|+(Q_3k_2+Q_4\gamma_2)\|y\|,$$

which imply that

$$\begin{aligned}\|x\|+\|y\|\le{}&(Q_1+Q_3)k_0+(Q_2+Q_4)\gamma_0+\bigl[(Q_1+Q_3)k_1+(Q_2+Q_4)\gamma_1\bigr]\|x\|\\&+\bigl[(Q_1+Q_3)k_2+(Q_2+Q_4)\gamma_2)\bigr]\|y\|.\end{aligned}$$

Consequently,

$$\bigl\|(x,y)\bigr\|\le\frac{(Q_1+Q_3)k_0+(Q_2+Q_4)\gamma_0}{M_0},$$

where $M_0=\min\{1-[(Q_1+Q_3)k_1+(Q_2+Q_4)\gamma_1],1-[(Q_1+Q_3)k_2+(Q_2+Q_4)\gamma_2)]\}$, which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.2, the operator T has at least one fixed point. Hence problem (1.2) has at least one solution on $[0,1]$. The proof is complete. □

Example 3.4 Consider the following system:

$$\begin{cases} D_{1-}^{3/2} D_{0+}^{1/2} x(t) = \frac{1}{8(t+2)^2} \frac{|x|}{1+|x|} + 1 + \frac{1}{36} \sin^2 y, & t \in J := [0,1], \\ D_{1-}^{5/2} D_{0+}^{1/2} y(t) = \frac{1}{32\pi} \sin(2\pi u) + \frac{|y|}{16(1+|y|)} + \frac{1}{2}, & t \in J := [0,1], \\ x(0) = x'(0) = 0, \qquad x(1) = (1/2)y(3/4), \\ y(0) = y'(0) = 0, \qquad y(1) = 4x(2/3). \end{cases} \tag{3.11}$$

Here $\alpha = 3/2$, $\beta = 1/2$, $p = 5/2$, $q = 1/2$, $\gamma = 1/2$, $\delta = 4$, $\eta = 3/4$, $\theta = 2/3$, $f(t,x,y) = \frac{1}{8(t+2)^2} \frac{|x|}{1+|x|} + 1 + \frac{1}{32} \sin^2 y$, and $g(t,x,y) = \frac{1}{32\pi} \sin(2\pi x) + \frac{|y|}{16(1+|y|)} + \frac{1}{2}$. With the given data, we find that $Q_1 \approx 6.125906$, $Q_2 \approx 0.439438$, $Q_3 \approx 13.668568$, $Q_4 \approx 2.874670$. Note that $|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \frac{1}{32}|x_1 - x_2| + \frac{1}{32}|y_1 - y_2|$, $|g(t,x_1,x_2) - g(t,y_1,y_2)| \leq \frac{1}{16}|x_1 - x_2| + \frac{1}{16}|y_1 - y_2|$, and $(Q_1 + Q_3)\ell_1 + (Q_2 + Q_4)\ell_2 \approx 0.825708 < 1$. Thus all the conditions of Theorem 3.1 are satisfied and, consequently, its conclusion applies to problem (3.11).

4 Conclusions

We have investigated the existence criteria for a coupled system of nonlinear fractional differential equations involving right Caputo and left Riemann–Liouville fractional derivatives, equipped with nonlocal three-point coupled boundary conditions. We apply the Banach contraction mapping principle and the Leray–Schauder alternative to obtain the desired results. We emphasize that the work accomplished in this paper is new and enhances the scope of the literature on the topic. Moreover, we obtain new existence results for the given coupled fractional differential system with the boundary conditions of the form: $x(0) = x'(0) = 0$, $x(1) = 0$, $y(0) = y'(0) = 0$, $y(1) = 0$ by taking $\gamma = 0 = \delta$ in the results of this paper.

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Authors' contributions

Each of the authors, BA, SKN, and AA, contributed equally to each part of this work. All authors read and approved the final manuscript.

Author details

[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia. [2]Department of Mathematics, University of Ioannina, Ioannina, Greece.

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