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Boundedness in a quasilinear attraction–repulsion chemotaxis system with nonlinear sensitivity and logistic source

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Abstract

In this paper, we deal with the following quasilinear attraction–repulsion model:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\chi(v)\nabla v) + \nabla\cdot(F(u)\xi(w)\nabla w) + f(u), & x\in\Omega, t>0,\\ v_t = \Delta v + \beta u - \alpha v, & x\in\Omega, t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega, t>0,\\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x\in\Omega \end{cases}$$

with homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset R^n$ ($n\geq 2$). Let the chemotactic sensitivity $\chi(v)$ be a positive constant, and let the chemotactic sensitivity $\xi(w)$ be a nonlinear function. Under some assumptions, we prove that the system has a unique globally bounded classical solution.

Keywords: Attraction–repulsion; Boundedness; Nonlinear sensitivity; Logistic source

1 Introduction

In this paper, we consider a quasilinear attraction–repulsion chemotaxis system with nonlinear sensitivity and logistic source

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\chi(v)\nabla v) + \nabla\cdot(F(u)\xi(w)\nabla w) + f(u), & x\in\Omega, t>0,\\ v_t = \Delta v + \beta u - \alpha v, & x\in\Omega, t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega, t>0,\\ \frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = \frac{\partial w(x,t)}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^n$ ($n\geq 2$) is a bounded domain with smooth boundary, and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$, α, β, γ, and δ are positive parameters, and $\chi(v)$ and $\xi(w)$ represent chemosensitivity. We assume that the functions $\chi(v)$ and $\xi(w)$ satisfy the following hypotheses:

(H_1) the function $\chi(v) = \chi_0$, which is a positive constant;
(H_2) the function $\xi(w) = \frac{\xi_0}{w}$ for all $w > 0$, where ξ_0 is a positive constant.
Here χ_0 is the strength of the attraction, and ξ_0 is the strength of the repulsion, $u(x,t)$, $v(x,t)$, and $w(x,t)$ denote the cell density, the concentration of the chemoattractant, and the concentration of the chemorepellent. We assume that

$$D(u), S(u), F(u) \in C^2\big([0,\infty)\big) \tag{1.2}$$

and there exist constants $C_D > 0$ and $m \geq 1$ such that

$$D(u) \geq C_D(u+1)^{m-1}. \tag{1.3}$$

The function $f : [0,\infty) \to R$ is smooth and satisfies $f(0) \geq 0$ and

$$f(u) \leq a - bu^\eta \tag{1.4}$$

with $a \geq 0$, $b > 0$, and $\eta > 1$. The initial data comply with

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) & \text{with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) & \text{with } v_0 \geq 0 \text{ in } \Omega. \end{cases} \tag{1.5}$$

Chemotaxis describes the oriented movement of cells along the concentration gradient of a chemical signal produced by cells. The prototype of the chemotaxis model, known as the Keller–Segel model, was first proposed by Keller and Segel [3] in 1970:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi(v)\nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v + u - v, & x \in \Omega, t > 0. \end{cases} \tag{1.6}$$

When $\chi(v)$ is a positive constant, a global solution is studied by Osaki and Yagi [8] for $n = 1$; a global solution is investigated by Nagai et al. [7, 16] for $n \geq 2$; the blowup solutions are proved by Herrero ea al. [2, 12]. For the case where $\chi(v) \leq \frac{\chi_0}{(1+\alpha v)^k}$, $\alpha > 0$, and $k > 1$, the global classical solution is asserted by Winkler [17]. For the case $\chi(v) = \frac{\chi}{v}$ with a positive constant $\chi < \sqrt{\frac{2}{n}}$, a global classical solution is explored by Winkler [18].

Moreover, when $D(u) = 1$ and $f(u) = 0$, Tao and Wang [11] studied the following chemotaxis model:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u\chi(v)\nabla v) + \nabla \cdot (u\xi(w)\nabla w) + f(u), & x \in \Omega, t > 0, \\ 0 = \Delta v + u - v, & x \in \Omega, t > 0, \\ 0 = \Delta w + u - w, & x \in \Omega, t > 0. \end{cases} \tag{1.7}$$

The global boundedness of the solutions was obtained in high dimensions, and blowup solutions were identified in R^2.

In the case where $\chi(v)$ and $\xi(w)$ are positive parameters in (1.7), $D(u)$ satisfies (1.3), and $f(u)$ satisfies (1.4), a unique global bounded classical solution was deduced by Wang [15]. When $f(u) = 0$ in (1.7), $\chi(v)$ and $\xi(w)$ are positive functions, $D(u)$ satisfies (1.3), and

$f(u)$ satisfies (1.4), the global classical solutions are asserted by Wu and Wu [19], who obtained an important estimate of $\int_\Omega |\nabla v|^2\,dx$. Note that this method is not applicable for the general $f(u)$ in our paper. For more details about chemotaxis system, we refer the interested readers to [1, 5, 6, 9, 13, 14].

Motivated by [11, 15, 17–19], we consider a quasilinear attraction–repulsion chemotaxis system with nonlinear sensitivity and logistic source. Our main results are given as follows.

Theorem 1.1 *Assume that* (1.2)–(1.5), (H_1), *and* (H_2) *are valid. Moreover, suppose that*

$$0 \le S(u) \le C_S(u+1)^s, \qquad 0 \le F(u) = C_F(u+1)^\sigma, \tag{1.8}$$

and

$$0 \le s < \begin{cases} \frac{m+\eta}{2} - \frac{n-1}{n}, & \eta \in (1, \frac{n+2}{n}], \\ \frac{m}{2} + \frac{\eta(n+4)}{2(n+2)} - 1, & \eta \in (\frac{n+2}{n}, n+2), \\ \frac{m+\eta}{2}, & \eta \in [n+2, \infty). \end{cases}$$

(i) *If* $\sigma \in (1,\eta)$, *then* (1.1) *admits a bounded global classical solution.*
(ii) *If* $\sigma \in (\eta, m)$, *then* (1.1) *admits a bounded classical solution.*
(iii) *If* $m > \max\{1, \frac{n\sigma+2-2\sigma}{n+2}, \frac{n\sigma-2}{n}\}$, *then* (1.1) *admits a bounded global classical solution.*

The local existence and uniqueness of system (1.1) can be derived from Lemma 2.1 in [4], and hence we only state the result and omit its proof.

Lemma 1.1 ([4]) *Suppose that* (1.2)–(1.5) *are valid. Then there exist a maximal existence time* $T_{\max} \in (0,+\infty)$ *and a unique triplet* (u,v,w) *of functions that satisfy*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times [0,T_{\max})), \\ v \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times [0,T_{\max})) \cap L^\infty((0,T_{\max}); W^{1,l}(\Omega)), \\ w \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times [0,T_{\max})) \cap L^\infty((0,T_{\max}); W^{1,l}(\Omega)) \end{cases} \tag{1.9}$$

with $l > n$ *and*

$$u \ge 0, \qquad v \ge 0, \qquad w \ge 0 \quad \text{in } \Omega \times (0,T_{\max}).$$

In addition, if $T_{\max} < +\infty$, *then*

$$\lim_{t\to T_{\max}} \sup\bigl(\bigl\|u(\cdot,t)\bigr\|\bigr)_{L^\infty}(\Omega) + \bigl\|v(\cdot,t)\bigr\|_{W^{1,\infty}(\Omega)} + \bigl\|w(\cdot,t)\bigr\|_{W^{1,\infty}(\Omega)} = \infty. \tag{1.10}$$

Lemma 1.2 *Let* (u,v,w) *be the solution of system* (1.1). *Then there exist a constant* m^* *such that*

$$\int_\Omega u(x,t)\,dx \le m^* := \max\left\{\int_\Omega u_0, \frac{a+b}{b}|\Omega|\right\}, \quad t \in (0,T_{\max}). \tag{1.11}$$

Proof Integrating the first equation of system (1.1) over Ω, we have

$$\frac{d}{dt}\int_\Omega u\,dx = a|\Omega| - b\int_\Omega u^\eta\,dx. \tag{1.12}$$

Due to $\eta > 1$ and Young's inequality, we derive

$$u \le u^\eta + 1. \tag{1.13}$$

Combining with (1.12), we have

$$\frac{d}{dt}\int_\Omega u\,dx \le -b\int_\Omega u\,dx + (a+b)|\Omega|, \tag{1.14}$$

which yields (1.11). □

Lemma 1.3 (Gagliardo–Nirenberg inequality) *Let $r \in (0,\alpha)$ and $\psi \in W^{1,2}(\Omega) \cap L^r(\Omega)$. Then there exists a constant $C_{\mathrm{GN}} > 0$ such that*

$$\|\psi\|_{L^\alpha(\Omega)} \le C_{\mathrm{GN}}\bigl(\|\nabla\psi\|_{L^2(\Omega)}^{\lambda^*}\|\psi\|_{L^r(\Omega)}^{1-\lambda^*} + \|\psi\|_{L^r(\Omega)}\bigr) \tag{1.15}$$

with

$$\lambda^* = \frac{\frac{n}{r} - \frac{n}{\alpha}}{1 - \frac{n}{2} + \frac{n}{r}} \in (0,1).$$

Lemma 1.4 *Let Ω be a bounded domain in R^n with smooth boundary, and let $v_0 \in W^{1,\infty}(\Omega)$. Suppose that there exists a constant C_1 such that*

$$\|u\|_{L^k(\Omega)} \le C_1, \quad t \in (0,T).$$

For the problem

$$\begin{cases} v_t = \Delta v + \beta u - \alpha v, & x \in \Omega, t > 0,\\ \frac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

(i) *if $1 \le k < n$, then*

$$\bigl\|v(t)\bigr\|_{W^{1,j}(\Omega)} \le C \quad \text{for all } j \in \left(0, \frac{nk}{n-k}\right); \tag{1.16}$$

(ii) *if $k = n$, then (1.16) holds for all $j \in (0,\infty)$;*
(iii) *if $k > n$, then (1.16) holds for $j = \infty$.*

Lemma 1.5 ([20]) *For any $h \in [1, \frac{n}{n-1})$, there exists a constant $C_2 > 0$ such that*

$$\bigl\|\nabla v(\cdot,t)\bigr\|_{L^h} \le C_2, \quad t \in (0,T_{\max}). \tag{1.17}$$

Lemma 1.6 ([21]) *For any $h \in [1, \frac{n\eta}{(n+2-\eta)^+})$, there exists a constant $C_3 > 0$ such that*

$$\bigl\|\nabla v(\cdot,t)\bigr\|_{L^h} \le C_3, \quad t \in (0,T_{\max}). \tag{1.18}$$

2 A priori estimates

Lemma 2.1 *Suppose*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (u+1)^k\,dx+\frac{d}{dt}\int_\Omega |\nabla v|^{2\beta}\,dx+D_1\int_\Omega \bigl|\nabla (u+1)^{\frac{k+m-1}{2}}\bigr|^2\,dx\\&\quad+\frac{bk}{2^{\eta+1}}\int_\Omega (u+1)^{k+\eta-1}\,dx+\int_\Omega |\nabla v|^{2\beta}\,dx\\&\le D_2\int_\Omega (u+1)^{k+\sigma-1}\,dx+D_3,\end{aligned}\tag{2.1}$$

where

$$D_1=\frac{2C_Dk(k-1)}{(k+m-1)^2},\qquad D_2=\frac{C_F\xi_0k(k-1)}{k+\sigma-1},\qquad D_3 \textit{ is a constant.}$$

If $\sigma\in(1,\eta)$, *then there exist constants* $E_1>0$ *and* $E_2>0$ *such that*

$$\frac{d}{dt}\biggl(\int_\Omega (u+1)^k\,dx+\int_\Omega |\nabla v|^{2\beta}\,dx\biggr)+E_1\biggl(\int_\Omega (u+1)^k\,dx+\int_\Omega |\nabla v|^{2\beta}\,dx\biggr)\le E_2\tag{2.2}$$

for sufficiently large k.

Proof Since $\sigma\in(1,\eta)$, by Young's inequality we have

$$\int_\Omega (u+1)^{k+\sigma-1}\,dx\le C_4\int_\Omega (u+1)^{k+\eta-1}\,dx+C_5\tag{2.3}$$

and

$$\int_\Omega (u+1)^k\,dx\le C_6\int_\Omega (u+1)^{k+\eta-1}\,dx+C_7.\tag{2.4}$$

Combining (2.1), (2.3), and (2.4), we get that there are positive constants E_1 and E_2 such that

$$\frac{d}{dt}\biggl(\int_\Omega (u+1)^k\,dx+\int_\Omega |\nabla v|^{2\beta}\,dx\biggr)+E_1\biggl(\int_\Omega (u+1)^k\,dx+\int_\Omega |\nabla v|^{2\beta}\,dx\biggr)\le E_2.\qquad\square$$

Lemma 2.2 *Suppose*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (u+1)^k\,dx+\frac{d}{dt}\int_\Omega |\nabla v|^{2\beta}\,dx+D_1\int_\Omega \bigl|\nabla (u+1)^{\frac{k+m-1}{2}}\bigr|^2\,dx\\&\quad+\frac{bk}{2^{\eta+1}}\int_\Omega (u+1)^{k+\eta-1}\,dx+\int_\Omega |\nabla v|^{2\beta}\,dx\\&\le D_2\int_\Omega (u+1)^{k+\sigma-1}\,dx+D_3,\end{aligned}\tag{2.5}$$

where

$$D_1=\frac{2C_Dk(k-1)}{(k+m-1)^2},\qquad D_2=\frac{C_F\xi_0k(k-1)}{k+\sigma-1},\qquad D_3 \textit{ is a constant.}$$

If $\sigma \in (\eta, m)$, then there exist constants $E_3 > 0$ and $E_4 > 0$ such that

$$\frac{d}{dt}\left(\int_\Omega (u+1)^k\,dx + \int_\Omega |\nabla v|^{2\beta}\,dx\right) + E_3\left(\int_\Omega (u+1)^k\,dx + \int_\Omega |\nabla v|^{2\beta}\,dx\right) \le E_4. \quad (2.6)$$

Proof By Lemma 1.2 and the Gagliardo–Nirenberg inequality there exists a constant $C_8 > 0$ such that

$$\begin{aligned}
&\int_\Omega (u+1)^{k+m-1}\,dx \\
&\quad = \left\|(u+1)^{\frac{k+m-1}{2}}\right\|_{L^2}^2 \\
&\quad \le C_{\mathrm{GN}}\left(\left\|\nabla(u+1)^{\frac{k+m-1}{2}}\right\|_{L^2}^{2\lambda^*}\left\|(u+1)^{\frac{k+m-1}{2}}\right\|_{L^{\frac{2}{k+m-1}}}^{2(1-\lambda^*)} + \left\|(u+1)^{\frac{k+m-1}{2}}\right\|_{L^{\frac{2}{k+m-1}}}^2\right) \\
&\quad \le C_8\left(\left\|\nabla(u+1)^{\frac{k+m-1}{2}}\right\|_{L^2}^{2\lambda^*} + 1\right),
\end{aligned} \quad (2.7)$$

where

$$\lambda^* = \frac{\frac{k+m-1}{2} - \frac{1}{2}}{\frac{k+m-1}{2} + \frac{1}{n} - \frac{1}{2}} \in (0,1).$$

By Young's inequality we obtain

$$\int_\Omega (u+1)^{k+m-1}\,dx \le C_9\int_\Omega \left|\nabla(u+1)^{\frac{k+m-1}{2}}\right|^2 dx + C_{10}. \quad (2.8)$$

Since $\sigma \in (\eta, m)$, by Young's inequality there exist $C_{11} > 0$ and $C_{12} > 0$ such that

$$\int_\Omega (u+1)^{k+\sigma-1}\,dx \le C_{11}\int_\Omega (u+1)^{k+m-1}\,dx + C_{12}. \quad (2.9)$$

Hence, combining (2.5), (2.8), and (2.9), we obtain (2.6). □

Lemma 2.3 *Suppose*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^k\,dx + \frac{d}{dt}\int_\Omega |\nabla v|^{2\beta}\,dx + D_1\int_\Omega \left|\nabla(u+1)^{\frac{k+m-1}{2}}\right|^2 dx \\
&\quad + \frac{bk}{2^{\eta+1}}\int_\Omega (u+1)^{k+\eta-1}\,dx + \int_\Omega |\nabla v|^{2\beta}\,dx \\
&\quad \le D_2\int_\Omega (u+1)^{k+\sigma-1}\,dx + D_3,
\end{aligned} \quad (2.10)$$

where

$$D_1 = \frac{2C_D k(k-1)}{(k+m-1)^2}, \qquad D_2 = \frac{C_F\xi_0 k(k-1)}{k+\sigma-1}, \qquad D_3 \text{ is a constant.}$$

If $m > \max\{1, \frac{n\sigma+2-2\sigma}{n+2}, \frac{n\sigma-2}{n}\}$, then there exist constants $E_5 > 0$ and $E_6 > 0$ such that

$$\frac{d}{dt}\left(\int_\Omega (u+1)^k\,dx + \int_\Omega |\nabla v|^{2\beta}\,dx\right) + E_5\left(\int_\Omega (u+1)^k\,dx + \int_\Omega |\nabla v|^{2\beta}\,dx\right) \le E_6. \quad (2.11)$$

Proof By the Gagliardo–Nirenberg inequality there exists $C_{13} > 0$ such that

$$\begin{aligned}
\int (u+1)^{k+\sigma-1}\,dx &= \bigl\|(u+1)^{\frac{k+m-1}{2}}\bigr\|_{L^{\frac{2(k+\sigma-1)}{k+m-1}}}^{\frac{2(k+\sigma-1)}{k+m-1}} \\
&\le C_{\mathrm{GN}}\Bigl(\bigl\|\nabla(u+1)^{\frac{k+m-1}{2}}\bigr\|_{L^2}^{\lambda_1}\bigl\|(u+1)^{\frac{k+m-1}{2}}\bigr\|_{L^{\frac{2}{k+m-1}}}^{(1-\lambda_1)} \\
&\quad + \bigl\|(u+1)^{\frac{k+m-1}{2}}\bigr\|_{L^{\frac{2}{k+m-1}}}^{2}\Bigr)^{\frac{2(k+\sigma-1)}{k+m-1}} \\
&\le C_{13}\Bigl(\bigl\|\nabla(u+1)^{\frac{k+m-1}{2}}\bigr\|_{L^2}^{\lambda_1\cdot\frac{2(k+\sigma-1)}{k+m-1}} + 1\Bigr),
\end{aligned} \tag{2.12}$$

where

$$\lambda_1 = \frac{\frac{n(k+m-1)}{2} - \frac{n(k+m-1)}{2(k+\sigma-1)}}{1-\frac{n}{2}+\frac{n(k+m-1)}{2}} = \frac{n(k+m-1)(k+\sigma-1) - n(k+m-1)}{(k+\sigma-1)[2-n+n(k+m-1)]}.$$

The condition $m > \max\{1, \frac{n\sigma+2-2\sigma}{n+2}\}$ and sufficiently large k guarantee that

$$\begin{aligned}
&(k+f-1)\bigl[2-n+n(k+m-1)\bigr] \\
&\quad = n(k+\sigma-1)(k+m-1) + (k+\sigma-1)(2-n) \\
&\quad = n(k+\sigma-1)(k+m-1) + (k+m-1+\sigma-m)(2-n) \\
&\quad \ge n(k+\sigma-1)(k+m-1) - n(k+m-1) + (2-n)(\sigma-m) + 2(2m-1) \\
&\quad \ge n(k+\sigma-1)(k+m-1) - n(k+m-1).
\end{aligned}$$

Hence $\lambda_1 \in (0,1)$.

Since $m > \max\{1, \frac{n\sigma-2}{n}\}$, we obtain

$$\frac{k+\sigma-1}{k+m-1}\cdot\lambda_1 \in (0,1).$$

By Young's inequality we derive

$$\int_\Omega (u+1)^{k+m-1}\,dx \le C_{14}\int_\Omega \bigl|\nabla(u+1)^{\frac{k+m-1}{2}}\bigr|^2\,dx + C_{15}. \tag{2.13}$$

Therefore (2.10) and (2.13) yield (2.11). □

Lemma 2.4 *Let $n \ge 2$. Defining*

$$\lambda_2 = \frac{2(k+\eta-1)}{\eta+m-2s}, \qquad \lambda_3 = \frac{2(\beta-1)(k+\eta-1)}{k+\eta-3}, \tag{2.14}$$

and

$$\delta_i(k,\beta;h) = \frac{\frac{\beta}{h}-\frac{\beta}{\lambda_i}}{\frac{1}{n}-\frac{1}{2}+\frac{\beta}{h}}, \qquad w_i(k,\beta;h) = \frac{\delta_i\lambda_i}{\beta} = \frac{\frac{\lambda_i}{h}-1}{\frac{1}{n}-\frac{1}{2}+\frac{\beta}{h}}, \quad i=2,3, \tag{2.15}$$

we have

(a) *if* $\eta \in (1, \frac{n+2}{n}]$, $s < \frac{m+\eta}{2} - \frac{n-1}{n}$, *then for sufficiently large* k, *there exist* $\beta > 2$ *and* $h \in [1, \frac{n}{n-1})$ *such that*

$$\delta_i(k, \beta; h) \in (0,1) \quad \textit{and} \quad w_i(k, \beta; h) < 2, \quad i = 2, 3. \tag{2.16}$$

(b) *if* $\eta \in (\frac{n+2}{n}, n+2)$, $s < \frac{m}{2} + \frac{\eta(n+4)}{2(n+2)} - 1$, *then for sufficiently large* k, *there exist* $\beta > 2$ *and* $h \in (\frac{n}{n-1}, \frac{n\eta}{n+2-\eta})$ *such that* (2.16) *holds.*

Proof By computation we verify that (2.16) is equivalent to

$$\lambda_i > h, \qquad \beta > \frac{\lambda_i}{2} - \frac{\lambda_i}{n}, \qquad \beta > \frac{\lambda_i}{2} - \frac{h}{n}, \quad i = 2, 3.$$

Thus it is sufficient to ensure that

$$\lambda_i > h, \qquad \beta > \frac{\lambda_i}{2} - \frac{h}{n}, \quad i = 2, 3. \tag{2.17}$$

(a) For $h \in [1, \frac{n}{n-1}]$, by the continuity of h it suffices to prove the case $h = \frac{n}{n-1}$. To prove (2.17), we need to prove

$$\frac{k+\eta-1}{\eta+m-2s} - \frac{1}{n-1} < \beta < \frac{k+\eta-1}{2} + \frac{k+\eta-3}{2(n-1)}, \tag{2.18}$$

$$k > \frac{n(m+n-2s)}{2(n-1)} + 1 - \eta, \qquad \beta > \frac{n(k+\eta-3)}{2(n-1)(k+\eta-1)} + 1. \tag{2.19}$$

Since $s < \frac{m+\eta}{2} - \frac{n-1}{n}$, there exists

$$k > \max\left\{1, m+1-2s, 3-\eta, \frac{n(m+n-2s)}{2(n-1)} + 1 - \eta\right\}$$

such that

$$\frac{k+\eta-1}{\eta+m-2s} - \frac{1}{n-1} < \frac{k+\eta-1}{2} + \frac{k+\eta-3}{2(n-1)},$$

so (2.18) and (2.19) are satisfied. Hence (2.17) holds.

(b) We note that $\eta \in (\frac{n+2}{n}, n+2)$ ensures the interval $h \in (\frac{n}{n-1}, \frac{n\eta}{n+2-\eta})$. By the continuity of h, let $h = \frac{n\eta}{n+2-\eta}$. To prove (2.17), we need to show that

$$\frac{k+\eta-1}{\eta+m-2s} - \frac{\eta}{n+2-\eta} < \beta < \frac{n+2}{2(n+2-\eta)} \cdot k - \frac{n+2}{2(n+2-\eta)} + \frac{n\eta}{2n+2-\eta} \tag{2.20}$$

and

$$k > \frac{n\eta(\eta+m-2s)}{2(n+2-\eta)} - \eta + 1, \qquad \beta > \frac{\eta(n-2)+2(n+2)}{2(n+2-\eta)}. \tag{2.21}$$

Since $s < \frac{m}{2} + \frac{\eta(n+4)}{2(n+2)} - 1$, there exists

$$k > \max\left\{1, m+1-2s, 3-\eta, \frac{n\eta+2(n+2)}{2(n+2-\eta)} \cdot (\eta+m-2s) + 1 - \eta\right\}$$

such that

$$\frac{k+\eta-1}{\eta+m-2s}-\frac{\eta}{n+2-\eta}<\frac{n+2}{2(n+2-\eta)}\cdot k-\frac{n+2}{2(n+2-\eta)}+\frac{n\eta}{2n+2-\eta}. \tag{2.22}$$

Then (2.20) and (2.21) are satisfied, and hence (2.17) holds. □

Lemma 2.5 *For the second equation in* (1.1), $E>0$, *and* $\beta>2$ *we have*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega|\nabla v|^{2\beta}\,dx+\frac{\beta-1}{\beta}\int_\Omega\big|\nabla|\nabla v|^\beta\big|^2\,dx\\&\quad\le\big[4\beta(\beta-1)+\beta n\big]\int_\Omega u^2|\nabla v|^{2\beta-2}\,dx+E\end{aligned} \tag{2.23}$$

for all $t\in[0,T_{\max})$.

Proof The proof can be found in [18]. □

Lemma 2.6 *Under assumptions* (1.2)–(1.5), (H_1), *and* (H_2), *let* $n\ge2$ *satisfy*

$$0\le s<\begin{cases}\frac{m+\eta}{2}-\frac{n-1}{n} & \text{for } \eta\in(1,\frac{n+2}{n}],\\ \frac{m}{2}+\frac{\eta(n+4)}{2(n+2)}-1 & \text{for } \eta\in(\frac{n+2}{n},n+2),\\ \frac{m+\eta}{2} & \text{for } \eta\in[n+2,\infty),\end{cases}$$

and let $S(u)$ *and* $F(u)$ *satisfy* (1.8). *If* $\sigma\in(1,\eta)$, *there exist sufficiently large* k *and* $t\in[0,T_{\max})$ *such that*

$$\|u\|_{L^k(\Omega)}\le C. \tag{2.24}$$

Proof Multiplying by $(u+1)^{k-1}$ the both sides of the first equation in (1.1), we have

$$\begin{aligned}&\frac{1}{k}\frac{d}{dt}\int_\Omega(u+1)^k\,dx+C_D(k-1)\int_\Omega(u+1)^{k+m-3}|\nabla u|^2\,dx\\&\quad\le\chi_0(k-1)\int_\Omega S(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx\\&\qquad-C_F(k-1)\int_\Omega(u+1)^f\frac{\xi_0}{w}(u+1)^{k-2}\nabla u\cdot\nabla w\,dx\\&\qquad+a\int_\Omega(u+1)^{k-1}\,dx-b\int_\Omega(u+1)^{k-1}u^\eta\,dx\end{aligned} \tag{2.25}$$

for all $t\in(0,T_{\max})$. Since $(u+1)^\eta\le2^{\eta-1}(u^\eta+1)$ for $\eta>1$, this implies that

$$u^\eta\ge\frac{1}{2^{\eta-1}}(u+1)^\eta-1.$$

Then (2.25) can be rewritten as

$$\frac{1}{k}\frac{d}{dt}\int_\Omega(u+1)^k\,dx+C_D(k-1)\int_\Omega(u+1)^{k+m-3}|\nabla u|^2\,dx+\frac{b}{2^{\eta-1}}\int_\Omega(u+1)^{k+\eta-1}\,dx$$

$$
\begin{aligned}
&\le \chi_0(k-1)\int_\Omega S(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx \\
&\quad - C_F(k-1)\int_\Omega (u+1)^f\frac{\xi_0}{w}(u+1)^{k-2}\nabla u\cdot\nabla w\,dx + (a+b)\int_\Omega (u+1)^{k-1}\,dx \\
&= I_1 + I_2 + I_3,
\end{aligned}
\tag{2.26}
$$

where

$$
\begin{aligned}
I_1 &= \chi_0(k-1)\int_\Omega S(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx \\
&\le \chi_0 C_S(k-1)\int_\Omega (u+1)^{k+s-2}|\nabla u||\nabla v|\,dx \\
&\le \frac{C_D(k-1)}{2}\int_\Omega (u+1)^{k+m-3}|\nabla u|^2\,dx \\
&\quad + \frac{\chi_0^2 C_S^2(k-1)}{2C_D}\int_\Omega (u+1)^{k+2s-m-1}|\nabla v|^2\,dx.
\end{aligned}
\tag{2.27}
$$

Similarly, we have

$$
I_2 = -C_F(k-1)\int_\Omega (u+1)^{k+f-2}\frac{\xi_0}{w}\nabla u\nabla w\,dx = \frac{C_F(k-1)\xi_0}{k+f-1}\int_\Omega (u+1)^{k+f-1}\nabla\cdot\left(\frac{1}{w}\nabla w\,dx\right),
$$

and then

$$
\begin{aligned}
I_2 &= \frac{C_F(k-1)\xi_0}{k+f-1}\int_\Omega (u+1)^{k+f-1}\left(-\frac{1}{w^2}|\nabla w|^2 + \frac{1}{w}\Delta w\right)dx \\
&\le \frac{C_F(k-1)\xi_0}{k+f-1}\int_\Omega (u+1)^{k+f-1}\frac{1}{w}(\delta w - \gamma u)\,dx \\
&\le \frac{C_F(k-1)\xi_0}{k+f-1}\delta\int (u+1)^{k+f-1}\,dx.
\end{aligned}
\tag{2.28}
$$

For all $t \in (0, T_{\max})$ with $C_{16} > 0$, we obtain

$$
\begin{aligned}
I_3(a+b) &= \int_\Omega (u+1)^{k-1}\,dx \\
&\le \frac{b}{2^\eta}\int_\Omega (u+1)^{k+\eta-1}\,dx + C_{16}.
\end{aligned}
\tag{2.29}
$$

Combining (2.23), (2.27), (2.28), (2.29), and Young's inequality, we deduce

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^k\,dx + \frac{d}{dt}\int_\Omega |\nabla v|^{2\beta}\,dx + \frac{2C_D k(k-1)}{(k+m-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{k+m-1}{2}}\right|^2 dx \\
&\quad + \frac{bk}{2^\eta}\int_\Omega (u+1)^{k+\eta-1}\,dx + \frac{\beta-1}{\beta}\int_\Omega \left|\nabla|\nabla v|^\beta\right|^2 dx \\
&\le \frac{\chi_0^2 C_s^2 k(k-1)}{2C_D}\int_\Omega (u+1)^{k+2s-m-1}|\nabla v|^2\,dx + \frac{C_F\xi_0 k(k-1)}{k+f-1}\delta\int_\Omega (u+1)^{k+f-1}\,dx + C_{16} \\
&\quad + \left[4\beta(\beta-1)+\beta n\right]\int_\Omega u^2|\nabla v|^{2\beta-2}\,dx + E
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{bk}{2^{\eta+1}} \int_\Omega (u+1)^{k+\eta-1}\,dx + C_{17}\int_\Omega |\nabla v|^{\lambda_2}\,dx + C_{18}\int_\Omega |\nabla v|^{\lambda_3}\,dx \\
&\quad + \frac{C_F\xi_0 k(k-1)}{k+f-1}\delta\int_\delta (u+1)^{k+f-1}\,dx + C_{19}
\end{aligned} \tag{2.30}$$

with $C_{17}, C_{18}, C_{19} > 0$ and λ_2, λ_3 as shown in Lemma 2.4 for all $t \in (0, T_{\max})$. By Lemma 1.5, Lemma 1.6, and the Gagliardo–Nirenberg inequality we have

$$\begin{aligned}
&\int_\Omega |\nabla v|^{\lambda_i}\,dx \\
&\quad = \big\| |\nabla v|^\beta \big\|_{L^{\frac{\lambda_i}{\beta}}}^{\frac{\lambda_i}{\beta}} \\
&\quad \leq C_{20}\Big(\big\| \nabla |\nabla v|^\beta \big\|_{L^2}^{\delta_i} \big\| |\nabla v|^\beta \big\|_{L^{\frac{h}{\beta}}}^{1-\delta_i} + \big\| |\nabla v|^\beta \big\|_{L^{\frac{h}{\beta}}}\Big)^{\frac{\lambda_i}{\beta}} \\
&\quad \leq C_{21} \big\| \nabla |\nabla v|^\beta \big\|_{L_2}^{\frac{\delta_i\lambda_i}{\beta}} + C_{22}
\end{aligned}$$

with λ_i, δ_i as in Lemma 2.4, where $i = 2, 3$. Since $w_i = \frac{\delta_i\lambda_i}{\beta} < 2$, by Young's inequality we have

$$\int_\Omega |\nabla v|^{\lambda_i}\,dx \leq C_{23}\int_\Omega \big|\nabla |\nabla v|^\beta\big|^2 + C_{24}. \tag{2.31}$$

From (2.30) and (2.31) we have that there exist constants $D_1, D_2, D_3 > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^k\,dx + \frac{d}{dt}\int_\Omega |\nabla v|^{2\beta}\,dx + D_1\int_\Omega \big|\nabla (u+1)^{\frac{k+m-1}{2}}\big|^2\,dx \\
&\quad + \frac{bk}{2^{\eta+1}}\int_\Omega (u+1)^{k+\eta-1}\,dx + \int_\Omega |\nabla v|^{2\beta}\,dx \\
&\leq D_2\int_\Omega (u+1)^{k+\sigma-1}\,dx + D_3,
\end{aligned} \tag{2.32}$$

where $D_1 = \frac{2C_D k(k-1)}{(k+m-1)^2}$ and $D_2 = \frac{C_F\xi_0 k(k-1)}{k+\sigma-1}$. By Lemma 2.1 we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega (u+1)^k\,dx + \int_\Omega |\nabla v|^{2\beta}\,dx\right) + C_{25}\left(\int_\Omega (u+1)^k\,dx + \int_\Omega |\nabla v|^{2\beta}\,dx\right) \\
&\quad \leq C_{26}
\end{aligned} \tag{2.33}$$

for all $t \in (0, T_{\max})$. By an ODE comparison argument we obtain (2.24).

For $\eta \in [n+2, \infty)$, from the Lemma 1.6 we have

$$\int_\Omega |\nabla v|^{\lambda_i}\,dx \leq C. \tag{2.34}$$

In addition, $s < \frac{m+\eta}{2}$ is equivalent to $k + 2s - m - 1 < k + \eta - 1$, so by (2.30), (2.34), and Lemma 2.1, using an ODE comparison argument, we derive (2.24). □

Remark 2.1 If $\sigma \in (\eta, m)$ in Theorem 1.1, then by (2.32), Lemma 2.2, and Lemma 2.4 we obtain (2.24).

Remark 2.2 If $m > \max\{1, \frac{n\sigma+2-2\sigma}{n+2}, \frac{n\sigma-2}{n}\}$ in Theorem 1.1, then by (2.32), Lemma 2.3, and Lemma 2.4 we obtain (2.24).

Proof of Theorem 1.1 For $k > \frac{n}{2}$, by Lemmas 1.4 and 2.6 there exists a positive constant C_{27} such that

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_{27}.$$

Using the elliptic regularity theory, we have

$$\|w(\cdot,t)\|_{w^{2,k}(\Omega)} \leq C_{28}. \tag{2.35}$$

Then, for a sufficiently large k, by the Sobolev embedding theorem there exists a positive constant C_{29} such that

$$\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{29}. \tag{2.36}$$

By using Lemma A.1 in [10] we conclude that u is uniformly bounded in $\Omega \times (0, T_{\max})$. Thus there exists a positive constant C_{30} such that

$$\|u(\cdot,t)\|_{L^\infty} \leq C_{30}, \quad t \in (0, T_{\max}), \tag{2.37}$$

that is, (u, v, w) is a global bounded classical solution to (1.1). □

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Authors' contributions
This work was carried out in collaboration between both authors. ZY designed the study and guided the research. LY performed the analysis and wrote the first draft of the manuscript. ZY and LY managed the analysis of the study. Both authors read and approved the final manuscript.

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