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Multiple solutions for quasilinear elliptic problems with concave-convex nonlinearities in Orlicz–Sobolev spaces

Shujun Wu[1*]

*Correspondence:
wushujun@upc.edu.cn
[1]College of Science, China University of Petroleum, Qingdao, China

Abstract

Using variational arguments, we establish the existence of two nontrivial solutions for quasilinear elliptic problems in Orlicz–Sobolev spaces, where the nonlinear terms exhibit the combined effects of concave and convex without the Ambrosetti–Rabinowitz type condition.

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1 Introduction

In this article, we investigate a class of nonlinear problems in the Orlicz–Sobolev setting:

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)|\nabla u|) = \lambda g(u) + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive constant. $a(t)$ is such that

$$\varphi(t) := \begin{cases} a(|t|)t, & t \neq 0, \\ 0, & t = 0 \end{cases}$$

is an odd, increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. g is an odd, increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ with $(\varphi_0 - 1)$ sublinear (see condition (g_1)), $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0$ with $(\varphi^0 - 1)$ superlinear near infinity (see condition (f_3)).

When $a(|t|)t = |t|^2 t$ with $1 < p < \infty$, problem (1.1) reads as follows:

$$\begin{cases} -\Delta_p u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The key hypothesis imposed on f is the well-known Ambrosetti–Rabinowitz type condition (AR-condition for short) [1]: there exist $\tau > p$, $t_0 > 0$ such that

$$0 < \tau F(x,t) = \int_0^t f(x,s)\,ds \le tf(x,t), \quad \forall x \in \Omega, |t| \ge t_0. \tag{1.3}$$

It is noted that the AR-condition ensures that f is $(p-1)$ superlinear at infinity.

However, the AR-condition is restrictive for many nonlinearities. Consequently, there have been many efforts to remove (1.3). In the case of $p = 2$, Miyagaki and Souto [2] introduced the following monotone condition: there is $s_0 > 0$ such that

$$\frac{f(x,s)}{s} \text{ is increasing in } s \ge s_0 \text{ and decreasing in } s \le -s_0, \quad \forall x \in \Omega. \tag{1.4}$$

Li and Yang [3] developed (1.4) to the case of $p > 1$. Meanwhile, Li and Yang [3] proved that (1.4) implied the following weaker condition: there is $C_* > 0$ such that, for all $s \in [0,1]$,

$$\overline{F}(x,st) \le \overline{F}(x,t) + C_*, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \overline{F}(x,t) = tf(x,t) - pF(x,t), \tag{1.5}$$

which is due to Jeanjean [4] and is used in [5, 6] and so on.

Ambrosetti, Brezis, and Cerami [7] initiated the study of semilinear elliptic problems with concave and convex nonlinearities. They investigated (1.1) with nonlinearities of the type $\lambda_u^p + u^q$, $0 < q < 1 < p$ and obtained the existence of two positive solutions for small $\lambda > 0$ by using sub- and super-solutions. Wu [8] studied problem (1.1) in the case when nonlinear terms exhibit $u^p + \lambda f(x)u^q$ with $0 < q < 1 < p < 2^*$ and obtained two positive solutions by Nehari manifold. Later, Wu [9] considered semilinear problems (1.1) in $H^1(\mathbb{R}^N)$ and established existence results. Papageorgiou and Rocha [10] considered a p-Laplacian problem with nonlinearities of the form $m(x)|u|^{r-2}u + f(x,u)$ with $1 < r < p < \infty$ when f is $(p-1)$ superlinear near infinity but does not satisfy the AR-condition. They employed variational approach and the Ekeland variational principle [11] to show the existence of two nontrivial solutions.

Divergence operators $-\operatorname{div}(a(|\nabla u|)|\nabla u|)$ involved in problem (1.1) are more general than p-Laplacian operators, please see [12–22]. Such operators have been intensively studied due to numerous and relevant applications in many fields such as plasticity [23], eletrorheological fluids [24], image processing [25]. When the nonlinear terms satisfy the AR-condition, problems of type (1.1) have been considered in [23, 26].

In the case of $\lambda = 0$, Chung [27], Carvalho et al. [28] studied problem (1.1) when f is $(\varphi^0 - 1)$ superlinear near infinity without the AR-condition. By variational methods, Chung [27], Carvalho et al. [28] established existence results under different assumptions imposed on f.

In this paper, motivated by [12–14, 16–18], we investigate a class of quasilinear elliptic problems (1.1) with concave and convex nonlinearities which do not satisfy the AR-condition in Orlicz–Sobolev spaces. Using functional techniques and variational approach, combined with the Ekeland variational principle, we establish existence results of at least two nontrivial solutions for $\lambda > 0$ small enough. We emphasize that the extension from p-Laplacian operators to $-\operatorname{div}(a(|\nabla u|)\nabla u)$ is interesting and nontrivial, since the divergence operators $-\operatorname{div}(a(|\nabla u|)\nabla u)$ involved in (1.1) have a more complicated structure, for example, they are non-homogeneous. In the case of $\lambda = 0$, problem (1.1) is studied in [27, 28], but their hypotheses do not apply when the concave terms are present. Furthermore, multiplicity results are given in this paper, while [27, 28] are concerned with existence of a nontrivial weak solution under our assumptions. Summarily, our results complement and extend previous studies such as [10, 27, 28].

2 Preliminaries

$\Phi : \mathbb{R} \to [0,\infty)$ is called an $\mathcal{N}$-function [29–31] provided that Φ is even, continuous, and convex with $\Phi(t) > 0$ for $t > 0$, $\frac{\Phi(t)}{t} \to 0$ as $t \to 0$, and $\frac{\Phi(t)}{t} \to \infty$ as $t \to \infty$. Its complementary function $\tilde{\Phi}$ is defined as

$$\tilde{\Phi}(s) := \sup_{t>0}\bigl\{t|s| - \Phi(t)\bigr\}, \quad \forall s \in \mathbb{R},$$

then $\tilde{\Phi}$ is also an $\mathcal{N}$-function.

Young's inequality holds true:

$$st \le \Phi(t) + \tilde{\Phi}(s), \quad s,t \in \mathbb{R}.$$

If Φ_1, Φ_2 are two $\mathcal{N}$-functions, we say that Φ_1 increases more slowly than Φ_2 near infinity (in short, $\Phi_1 \prec \Phi_2(\infty)$) if there exist two positive constants K, t_0 such that $\Phi_1(t) \le \Phi_2(Kt)$, $\forall t \ge t_0$. We say that Φ_1 increases essentially more slowly than Φ_2 near infinity (in short, $\Phi_1 \prec\prec \Phi_2(\infty)$) provided $\lim_{t\to\infty} \frac{\Phi_1(kt)}{\Phi_2(t)} = 0$, $\forall k > 0$.

Φ is said to satisfy Δ_2-condition near infinity (in short, $\Phi \in \Delta_2(\infty)$) provided that there exist positive constants K, t_0 such that

$$\Phi(2t) \le K\Phi(t) \quad \forall t \ge t_0.$$

$\Phi \in \nabla_2(\infty)$ provided that $\tilde{\Phi} \in \Delta_2(\infty)$.

For a measurable function $u : \Omega \to \mathbb{R}$, denoted as $u \in \tilde{L}$, we define Orlicz space $L_\Phi(\Omega)$ by

$$L_\Phi(\Omega) = \left\{u \in \tilde{L} : \int_\Omega \Phi\bigl(\lambda u(x)\bigr)\,dx < \infty \text{ for some } \lambda > 0\right\}$$

endowed with Luxemburg norm

$$\|u\|_{(\Phi)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{u(x)}{\lambda}\right)dx \le 1\right\}.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{(\Phi)})$ forms a Banach space.

In the sequel, we always assume that [30]

$$\int_0^1 \frac{\Phi^{-1}(t)}{t^{\frac{N+1}{N}}}\,dt < \infty, \qquad \int_1^\infty \frac{\Phi^{-1}(t)}{t^{\frac{N+1}{N}}}\,dt = \infty.$$

The Sobolev conjugate Φ_* of Φ is defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds, \quad t \ge 0.$$

Let $\Phi_*(-t) = \Phi_*(t)$ for all $t < 0$. Then Φ_* is an $\mathcal{N}$-function and $\Phi \prec\prec \Phi_*(\infty)$ (see [30, 32]).

An Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$ is defined by

$$W^{1,\Phi}(\Omega) = \{u \in L_\Phi(\Omega) : D^\alpha u \in L_\Phi(\Omega), |\alpha| \leq 1\}$$

endowed with

$$\|u\|_{W^{1,\Phi}} = \|u\|_{(\Phi)} + \|\nabla u\|_{(\Phi)}.$$

Then $(W^{1,\Phi}(\Omega), \|\cdot\|_{W^{1,\Phi}})$ forms a Banach space.

Let $W_0^{1,\Phi}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$. By Lemma 5.7 in [33], there exists a best positive constant λ_1 such that

$$\lambda_1 \int_\Omega \Phi(u(x))\,dx \leq \int_\Omega \Phi(|\nabla_u(x)|)\,dx, \quad \forall u \in W_0^{1,\Phi}(\Omega). \tag{2.1}$$

Therefore, $W_0^{1,\Phi}(\Omega)$ can be reformed by an equivalent norm $\|u\| := \|\nabla u\|_{(\Phi)}$. If $\Phi \in \Delta_2(\infty) \cap \nabla_2(\infty)$, then $L_\Phi(\Omega))$, $W^{1,\Phi}(\Omega)$, $W_0^{1,\Phi}(\Omega)$ are separable and reflexive Banach spaces (refer [30]).

In this paper, we always assume $\Phi(t) = \int_0^t \varphi(s)\,ds$, $\forall t \in \mathbb{R}$, and

$$1 < \varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < N < \infty. \tag{Φ_1}$$

We note that (Φ_1) yields $\Phi \in \Delta_2(\infty) \cap \nabla_2(\infty)$ (see [29]).

Lemma 2.1 ([23]) *For an $\mathcal{N}$-function Φ satisfying $1 \leq \varphi_0 \leq \varphi^0 < \infty$ for all $t > 0$ and for some φ_0, φ^0. Then*

(1) $\|u\|_{(\Phi)}^{\varphi_0} \leq \int_\Omega \Phi(u)\,dx \leq \|u\|_{(\Phi)}^{\varphi^0}$ $(\|u\|_{(\Phi)} > 1)$.

(2) $\|u\|_{(\Phi)}^{\varphi^0} \leq \int_\Omega \Phi(u)\,dx \leq \|u\|_{(\Phi)}^{\varphi_0}$ $(0 \leq \|u\|_{(\Phi)} \leq 1)$.

Lemma 2.2 ([30]) *Let Ω be an arbitrary domain. Then $W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega)$. Moreover, if Ω_0 is a bounded subdomain of Ω, then the imbedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L_B(\Omega_0)$ exists and is compact for any $\mathcal{N}$-function B with $B \prec\prec \Phi_*(\infty)$.*

Definition 2.1 ([34]) Let $(X, \|\cdot\|)$ be a real Banach space, $J \in C^1(X, \mathbb{R})$. We say J satisfies the C_c condition if any sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $\|J'(u_n)\|_*(1 + \|u_n\|) \to 0$ as $n \to \infty$ has a convergent subsequence. $\{u_n\}$ is called a Cerami sequence at the level $c \in \mathbb{R}$.

Lemma 2.3 ([35]) *Let $(X, \|\cdot\|)$ be a real Banach space, $J \in C^1(X, \mathbb{R})$ satisfies the C_c condition for any $c > 0$, $J(\theta) = 0$, and the following conditions hold:*

(1) *There exist two positive constants ρ, η such that $J(u) \geq \eta$ for any $u \in X$ with $\|u\| = \rho$.*

(2) *There exists a function $\phi \in X$ such that $\|\phi\| > \rho$ and $J(\phi) < 0$.*

Then the functional J has a critical value $c \geq \eta$, i.e., there exists $u \in X$ such that $J'(u) = \theta$ and $J(u) = c$.

We call $u \in W_0^{1,\Phi}(\Omega)$ a weak solution of problem (1.1) if, for all $v \in W_0^{1,\Phi}(\Omega)$,

$$\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v\,dx - \lambda \int_\Omega g(u)v\,dx - \int_\Omega f(x,u)v\,dx = 0.$$

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces. $X \hookrightarrow Y$ means $(X, \|\cdot\|_X)$ is continuously imbedded in $(Y, \|\cdot\|_Y)$. $X \hookrightarrow\hookrightarrow Y$ means $(X, \|\cdot\|_X)$ is compactly imbedded in $(Y, \|\cdot\|_Y)$.

3 Main results

For convenience, we give some conditions.

(g_1) $G \prec \Phi(\infty)$, $\lim_{t\to 0} \frac{\Phi(t)}{G(t)} = 0$, where $G(t) := \int_0^t g(s)\,ds$, $\forall t \in \mathbb{R}$.

(f_1) $|f(x,t)| \leq C(1 + h(|t|))$, $\forall (x,t) \in \Omega \times \mathbb{R}$,

where C is a positive constant, $h : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, $H(t) := \int_0^t h(s)\,ds$ satisfies $H \prec\prec \Phi_*(\infty)$ and $h_0 := \inf_{t>0} \frac{th(t)}{H(t)} > \varphi^0$.

(f_2) $\limsup_{t\to 0} \frac{f(x,t)}{|\varphi(t)|} < \lambda_1$ uniformly for almost all $x \in \Omega$, where λ_1 is defined in (2.1).

(f_3) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{\varphi^0-2}t} = +\infty$ uniformly for almost all $x \in \Omega$.

(f_4) There exist $D_1 \geq 1$ and $\alpha(x) \in L^1(\Omega)$ such that, for all $s \in [0,1]$,

$$\overline{F}(x,st) \leq D_1 \overline{F}(x,t) + \alpha(x), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\overline{F}(x,t) := tf(x,t) - \varphi^0 F(x,t)$, $F(x,t) = \int_0^t f(x,s)\,ds$.

(Φ_2) There exists $\beta(x) \in L^1(\Omega)$ such that, for all $s \in [0,1]$,

$$\overline{\Phi}(st) \leq D_1 \overline{\Phi}(t) + \beta(x), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\overline{\Phi}(t) = \varphi^0 \Phi(t) - t\varphi(t)$.

The main result of this paper is given by the following theorem.

Theorem 3.1 *Given Φ satisfies (Φ_1) and (Φ_2), g satisfies (g_1), f satisfies (f_1)–(f_4). Then there exists $\lambda_* > 0$ such that, for each $\lambda \in (0, \lambda_*)$, problem (1.1) has two nontrivial weak solutions.*

Remark 3.1 From (g_1) and (f_4), it follows that $W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$, $W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L_G(\Omega)$, and $W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L_H(\Omega)$.

For any $\lambda > 0$, we define $\mathcal{J}_\lambda : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ by

$$\mathcal{J}_\lambda(u) = \int_\Omega \Phi\big(|\nabla u|\big)\,dx - \lambda \int_\Omega G(u)\,dx - \int_\Omega F(x,u)\,dx.$$

Analogous to that in [32], we can deduce that $\mathcal{J}_\lambda \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$, $\mathcal{J}'_\lambda : W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ and the derivative is given by, for all $u, v \in W_0^{1,\Phi}(\Omega)$,

$$\big\langle \mathcal{J}'_\lambda(u), v\big\rangle = \int_\Omega a\big(|\nabla u|\big)\nabla u \cdot \nabla v\,dx - \lambda \int_\Omega g(u)v\,dx - \int_\Omega f(x,u)v\,dx.$$

So, critical points of the functional $\mathcal{J}_\lambda$ are weak solutions of problem (1.1).

Lemma 3.2 *Given that (Φ_1), (g_1), (f_1), and (f_2) hold, then there exist positive constants λ_*, ρ, η such that, for each $\lambda \in (0, \lambda_*)$, $\mathcal{J}_\lambda(u) \geq \eta$ for any $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\| = \rho$.*

Proof By conditions (f_1), (g_1) and Remark 3.1, there exists a positive constant C_1 such that

$$\|u\|_{(G)} \leq C_1\|u\|, \qquad \|u\|_{(H)} \leq C_1\|u\|, \quad \forall u \in W_0^{1,\Phi}(\Omega). \tag{3.1}$$

Let $\rho \in (0, \min\{1, 1/C_1\})$ for each $u \in S_\rho := \{u \in W_0^{1,\Phi}(\Omega) : \|u\| = \rho\}$, (3.1) implies that $\|u\|_{(G)} < 1$, $\int_\Omega G(u(x))\,dx < 1$, and $\|u\|_{(H)} < 1$.

From condition (f_2), we deduce that there exist $\varepsilon_0 \in (0, \lambda_1)$, $\delta > 0$ such that

$$\left|F(x,t)\right| \le (\lambda_1 - \varepsilon_0)\Phi(t) \quad \forall x \in \Omega, |t| < \delta. \tag{3.2}$$

By (f_1), one has $|F(x,t)| \le C|t| + CH(t)$ for all $x \in \Omega$, $|t| \ge \delta$. Since $\frac{H(t)}{t}$ is increasing on $[\delta, +\infty)$, we conclude $\frac{H(t)}{|t|} \ge \frac{H(\delta)}{\delta}$ for $|t| \ge \delta$. Combined with (3.2), we get

$$\left|F(x,t)\right| \le (\lambda_1 - \varepsilon_0)\Phi(t) + C_2 H(t), \quad \forall x \in \Omega, t \in \mathbb{R}. \tag{3.3}$$

By Lemma 2.1 and (3.3), for all $u \in S_\rho$,

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \int_\Omega \Phi\bigl(|\nabla u|\bigr)\,dx - \lambda \int_\Omega G(u)\,dx - \int_\Omega F(x,u)\,dx \\
&\ge \int_\Omega \Phi\bigl(|\nabla u|\bigr)\,dx - \lambda \int_\Omega G(u)\,dx \\
&\quad - (\lambda_1 - \varepsilon_0)\int_\Omega \Phi(u)\,dx - C_2 \int_\Omega H(u)\,dx \\
&\ge \left(1 - \frac{(\lambda_1 - \varepsilon_0)}{\lambda_1}\right)\int_\Omega \Phi\bigl(|\nabla u|\bigr)\,dx - \lambda - C_2 \|u\|_{(H)}^{h_0} \\
&\ge \frac{\varepsilon_0}{\lambda_1}\|u\|^{\varphi^0} - \lambda - C_3\|u\|^{h_0}.
\end{aligned} \tag{3.4}$$

Denote $m(\rho) = \frac{\varepsilon_0}{\lambda_1} - C_3 \rho^{h_0 - \varphi^0}$, by $h_0 > \varphi^0$, we have $m(\rho) \to \frac{\varepsilon_0}{\lambda_1} > 0$ as $\rho \to 0^+$. Therefore, we can choose $\rho > 0$ small enough such that $m(\rho) > \frac{\varepsilon_0}{2\lambda_1}$. Set $\lambda_* := \frac{\varepsilon_0 \rho^{\varphi^0}}{4\lambda_1} > 0$, $\eta =: \frac{\varepsilon_0 \rho^{\varphi^0}}{4\lambda_1} > 0$. For all $\lambda \in (0, \lambda_*)$ and $u \in S_\rho$, applying (3.4), we obtain

$$\mathcal{J}_\lambda(u) \ge \frac{\varepsilon_0 \rho^{\varphi^0}}{4\lambda_1} = \eta > 0. \qquad \square$$

Lemma 3.3 *Given that (Φ_1), (g_1), and (f_3) hold. Then, for any $\lambda > 0$, $\rho > 0$, there exists a function $u_\lambda \in W_0^{1,\Phi}(\Omega)$ such that $\|u_\lambda\| > \rho$ and $\mathcal{J}_\lambda(u_\lambda) < 0$.*

Proof Take a compact set $S \subset \Omega$ with positive measure, we can define $u_0 \in C_c^\infty(\Omega)$ such that $u_0(x) = 1$ for $x \in S$, $0 \le u_0(x) \le 1$ for $x \in \Omega$ (please see [30]). Then $u_0 \in W_0^{1,\Phi}(\Omega)$.

By condition (f_3), we deduce that for $M_0 := \frac{2\|u_0\|^{\varphi^0}}{\mu S} > 0$ there exists $C_1 > 0$ such that

$$F(x,t) \ge M_0|t|^{\varphi^0} - C_1, \quad \forall x \in \Omega, t \in \mathbb{R}.$$

Let $t > 1$ large enough such that $\|tu_0\| > 1$, by Lemma 2.1,

$$\begin{aligned}
\mathcal{J}_\lambda(tu_0) &= \int_\Omega \Phi\bigl(|\nabla tu_0|\bigr)\,dx - \lambda \int_\Omega G(tu_0)\,dx - \int_\Omega F(x,tu_0)\,dx \\
&\le t^{\varphi^0}\|u_0\|^{\varphi^0} - M_0 t^{\varphi^0}\int_\Omega |u_0|^{\varphi^0}\,dx + C_1 \mu \Omega
\end{aligned}$$

$$
\begin{aligned}
&\leq t^{\varphi^0}\left(\|u_0\|^{\varphi^0} - \frac{2\|u_0\|^{\varphi^0}}{\mu S}\int_S |u_0|^{\varphi^0}\,dx\right) + C_1\mu\Omega \\
&= -t^{\varphi^0}\|u_0\|^{\varphi^0} + C_1\mu\Omega.
\end{aligned}
$$

Due to $\|u_0\| > 0$, we see $\mathcal{J}_\lambda(tu_0) \to -\infty$ as $t \to +\infty$.

Taking t large enough such that $t > \max\{1, \frac{\rho+1}{\|u_0\|}\}$, set $u_\lambda = tu_0$, which completes the proof. □

Lemma 3.4 *Given that* (Φ_1), (g_1), *and* (f_2) *hold. Then, for any* $\lambda > 0$, $\rho > 0$, *there exists a function* $\tilde{u}_\lambda \in W_0^{1,\Phi}(\Omega)$ *such that* $\|\tilde{u}_\lambda\| < \rho$ *and* $\mathcal{J}_\lambda(\tilde{u}_\lambda) < 0$.

Proof Take a compact set $\tilde{S} \subset \Omega$ with positive measure, we can define $\tilde{u}_0 \in C_c^\infty(\Omega)$ such that $\tilde{u}_0(x) = 1$ for $x \in \tilde{S}$, $0 \leq \tilde{u}_0(x) \leq 1$ for $x \in \Omega$ (please see [30]). Then $\tilde{u}_0 \in W_0^{1,\Phi}(\Omega)$.

We take $t \in (0,\delta)$ (where δ is defined in (3.2)) such that $\|t\tilde{u}_0\| < 1$ and $\|t\tilde{u}_0\|_{(G)} < 1$. By (3.2), we have $|F(x,t\tilde{u}_0(x))| \leq (\lambda_1 - \varepsilon_0)\Phi(t\tilde{u}_0(x))$ for all $x \in \Omega$. From Lemma 2.1 and (2.1), it follows

$$
\begin{aligned}
\mathcal{J}_\lambda(t\tilde{u}_0) &= \int_\Omega \Phi\big(|\nabla t\tilde{u}_0|\big)\,dx - \lambda\int_\Omega G(t\tilde{u}_0)\,dx - \int_\Omega F(x,t\tilde{u}_0)\,dx \\
&\leq \int_\Omega \Phi\big(|\nabla t\tilde{u}_0|\big)\,dx - \lambda\int_\Omega G(t\tilde{u}_0)\,dx - (\lambda_1-\varepsilon_0)\int_\Omega \Phi(t\tilde{u}_0)\,dx \\
&\leq \left(1 + \frac{(\lambda_1-\varepsilon_0)}{\lambda_1}\right)\int_\Omega \Phi\big(|\nabla t\tilde{u}_0|\big)\,dx - \lambda\int_\Omega G(t\tilde{u}_0)\,dx \\
&\leq \left(2 - \frac{\varepsilon_0}{\lambda_1}\right)\int_\Omega \Phi(C_1 t)\,dx - \lambda\int_{\tilde{S}} G(t)\,dx \\
&\leq C_2\Phi(t) - \lambda G(t)\mu\tilde{S} \\
&= G(t)\left[C_2\frac{\Phi(t)}{G(t)} - \lambda\mu\tilde{S}\right].
\end{aligned}
$$

Due to (g_1), we can find $t > 0$ small enough such that for $\|\tilde{u}_\lambda\| = \|t\tilde{u}_0\| < \rho$ and $\mathcal{J}_\lambda(t\tilde{u}_0) < 0$. □

Lemma 3.5 *Given that* (Φ_1), (Φ_2), (g_1), *and* (f_1)–(f_4) *hold. Then, for each* $\lambda > 0$, *the functional* $\mathcal{J}_\lambda$ *satisfies* C_c *condition for any* $c > 0$.

Proof Given $\lambda > 0$, $c > 0$. Let $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ be a Cerami sequence at the level c of $\mathcal{J}_\lambda$, i.e.,

$$
\mathcal{J}_\lambda(u_n) \to c \quad \text{and} \quad \big\|\mathcal{J}_\lambda'(u_n)\big\|_*\big(1 + \|u_n\|\big) \to 0, \quad n \to \infty. \tag{3.5}
$$

First, we shall show that $\{u_n\}$ is bounded.

Otherwise, there is a subsequence, still denoted by $\{u_n\}$, such that $\lim_{n\to\infty}\|u_n\| = \infty$ and $\|u_n\| > 1$ $(\forall n \in \mathbb{N})$.

We denote $w_n(x) := \frac{u_n(x)}{\|u_n\|}$, $x \in \Omega$, $n = 1,2,\ldots$. Then $\{w_n\} \subset W_0^{1,\Phi}(\Omega)$ and $\|w_n\| = 1$ for every $n \in \mathbb{N}$. Applying the Eberlein–Smulian theorem, we may assume that there exists

$w \in W_0^{1,\Phi}(\Omega)$ such that w_n converges weakly to w. From Remark 3.1, it follows that

$$\|w_n - w\|_{L^1(\Omega)} \to 0, \qquad \|w_n - w\|_{(G)} \to 0, \qquad \|w_n - w\|_{(H)} \to 0, \quad n \to \infty, \tag{3.6}$$

$$w_n(x) \to w(x) \quad \text{a.e. } x \in \Omega, n \to \infty. \tag{3.7}$$

Claim: $w(x) = 0$ a.e. $x \in \Omega$.

We suppose $\mu\Omega_0 := \mu\{x \in \Omega : w(x) \neq 0\} > 0$. Given $x \in \Omega_0$, (3.7) implies that $|u_n(x)| = |w_n(x)| \cdot \|u_n\| \to \infty$ as $n \to \infty$. Furthermore, by (f_3) we obtain that, for given $x \in \Omega_0$,

$$\frac{F(x,u_n(x))}{\|u_n\|^{\varphi^0}} = \frac{F(x,u_n(x))}{|u_n(x)|^{\varphi^0}} |w_n(x)|^{\varphi^0} \to \infty, \quad n \to \infty. \tag{3.8}$$

From (f_3) and the continuity of F on $\overline{\Omega} \times \mathbb{R}$, there exists a constant C_1 such that

$$F(x,t) \geq C_1, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

which implies that

$$\frac{F(x,u_n(x)) - C_1}{\|u_n\|^{\varphi^0}} = \frac{F(x,u_n(x)) - C_1}{|u_n(x)|^{\varphi^0}} |w_n(x)|^{\varphi^0} \geq 0, \quad \forall x \in \Omega, \forall t \in \mathbb{R}. \tag{3.9}$$

From (3.5), it follows that

$$c + o(1) = \mathcal{J}_\lambda(u_n) = \int_\Omega \Phi(|\nabla u_n|)\,dx - \lambda \int_\Omega G(u_n)\,dx - \int_\Omega F(x,u_n)\,dx.$$

Dividing the above equality by $\|u_n\|^{\varphi^0}$, by Lemma 2.1 and $\|u_n\| > 1$,

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^{\varphi^0}}\,dx \\
&\quad= \frac{\liminf_{n\to\infty} \int_\Omega F(x,u_n(x))\,dx}{\|u_n\|^{\varphi^0}} \\
&\quad= \liminf_{n\to\infty} \left(\frac{\int_\Omega \Phi(|\nabla u_n|)\,dx}{\|u_n\|^{\varphi^0}} - \frac{\lambda \int_\Omega G(u_n)\,dx}{\|u_n\|^{\varphi^0}} - \frac{c+o(1)}{\|u_n\|^{\varphi^0}} \right) \\
&\quad\leq \liminf_{n\to\infty} \left(\frac{\int_\Omega \Phi(|\nabla u_n|)\,dx}{\|u_n\|^{\varphi^0}} - \frac{c+o(1)}{\|u_n\|^{\varphi^0}} \right) \leq 1.
\end{aligned} \tag{3.10}$$

By Fatou's lemma and (3.7)–(3.10),

$$\begin{aligned}
\infty &= \int_{\Omega_0} \lim_{n\to\infty} \frac{F(x,u_n(x)) - C_1}{\|u_n\|^{\varphi^0}}\,dx \\
&\leq \liminf_{n\to\infty} \int_{\Omega_0} \frac{F(x,u_n(x)) - C_1}{\|u_n\|^{\varphi^0}}\,dx \\
&\leq \liminf_{n\to\infty} \int_{\Omega} \frac{F(x,u_n(x)) - C_1}{\|u_n\|^{\varphi^0}}\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{n\to\infty} \frac{\int_\Omega F(x,u_n(x))\,dx}{\|u_n\|^{\varphi^0}} - \limsup_{n\to\infty} \frac{\int_\Omega C_1\,dx}{\|u_n\|^{\varphi^0}} \\
&= \liminf_{n\to\infty} \frac{\int_\Omega F(x,u_n(x))\,dx}{\|u_n\|^{\varphi^0}} \le 1.
\end{aligned}
$$

Consequently, we get a contradiction, which implies that $w(x) = 0$ a.e. $x \in \Omega$.

Since $\mathcal{J}_\lambda(tu_n)$ is continuous on $[0,1]$ for each $n \in \mathbb{N}$, there exists $t_n \in [0,1]$ such that $\mathcal{J}_\lambda(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}_\lambda(tu_n)$. Due to $\|\mathcal{J}'_\lambda(u_n)\|_*(1 + \|u_n\|) \to 0$, we deduce

$$
\langle \mathcal{J}'_\lambda(t_n u_n), t_n u_n \rangle \to 0, \quad n \to \infty. \tag{3.11}
$$

Take $\{s_k\}_{k=1}^\infty \subset (1,\infty)$ with $s_k \to +\infty$ as $k \to \infty$. Then, for each $n, k \in \mathbb{N}$, one has $\|s_k w_n\| = s_k > 1$. From (3.6) and the claim, combining conditions (g_1) and (f_1), we deduce

$$
\begin{aligned}
\int_\Omega F\bigl(x, s_k w_n(x)\bigr)\,dx &\le C \int_\Omega \bigl[|s_k w_n| + H(s_k w_n)\bigr]\,dx \\
&\le C\bigl(\|s_k w_n\|_{L^1(\Omega)} + \|s_k w_n\|_{(H)}\bigr) \to 0, \quad n \to \infty,
\end{aligned} \tag{3.12}
$$

and

$$
\int_\Omega G\bigl(s_k w_n(x)\bigr)\,dx \le \|s_k w_n\|_{(G)} \to 0, \quad n \to \infty. \tag{3.13}
$$

Due to $\lim_{n\to\infty}\|u_n\| = \infty$, given $k \in \mathbb{N}$, there exists $n_k \ge k$. For all $n \ge n_k \ge k$, one has $\|u_n\| > s_k$, i.e., $0 < \frac{s_k}{\|u_n\|} < 1$.

From $\|s_k w_n\| > 1$, Lemma 2.1 and (3.12), (3.13), for large $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathcal{J}_\lambda(t_n u_n) &= \max_{t\in[0,1]} \mathcal{J}_\lambda(tu_n) \ge \mathcal{J}_\lambda\left(\frac{s_k}{\|u_n\|} u_n\right) = \mathcal{J}_\lambda(s_k w_n) \\
&= \int_\Omega \Phi\bigl(|\nabla s_k w_n|\bigr)\,dx - \lambda \int_\Omega G(s_k w_n)\,dx - \int_\Omega F(x, s_k w_n)\,dx \\
&\ge \|s_k w_n\|^{\varphi_0} - \lambda \int_\Omega G(s_k w_n)\,dx - \int_\Omega F(x, s_k w_n)\,dx \\
&\ge \frac{1}{2}\|s_k w_n\|^{\varphi_0} = \frac{1}{2} s_k^{\varphi_0}.
\end{aligned}
$$

Let $s_k = \|u_k\|^\gamma > 1$, where $\gamma \in (\frac{\varphi^0}{\varphi_0}, +\infty)$ is a constant. For all $n \ge n_k \ge k$, one has

$$
\mathcal{J}_\lambda(t_n u_n) \ge \frac{1}{2}\|u_k\|^{\gamma\varphi_0}. \tag{3.14}
$$

Applying (3.11), (f_3), (f_4), and (Φ_2), for large $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathcal{J}_\lambda(t_n u_n) &= \mathcal{J}_\lambda(t_n u_n) - \frac{1}{\varphi^0}\langle \mathcal{J}'_\lambda(t_n u_n), t_n u_n \rangle + o(1) \\
&= \int_\Omega \Phi\bigl(|\nabla t_n u_n|\bigr)\,dx - \lambda \int_\Omega G(t_n u_n)\,dx - \int_\Omega F(x, t_n u_n)\,dx \\
&\quad - \frac{1}{\varphi^0} \int_\Omega \varphi\bigl(|\nabla t_n u_n|\bigr)|\nabla t_n u_n|\,dx + \frac{\lambda}{\varphi^0} \int_\Omega t_n u_n g(t_n u_n)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\varphi^0}\int_\Omega t_n u_n f(x,t_nu_n)\,dx+o(1)\\
&=\frac{1}{\varphi^0}\int_\Omega \overline{\Phi}\big(t_n|\nabla u_n|\big)\,dx+\frac{1}{\varphi^0}\int_\Omega \overline{F}(x,t_nu_n)\,dx\\
&\quad+\frac{\lambda}{\varphi^0}\int_\Omega\big[t_nu_ng(t_nu_n)-\varphi^0G(t_nu_n)\big]\,dx+o(1)\\
&\le\frac{1}{\varphi^0}\int_\Omega\big[D_1\overline{\Phi}\big(|\nabla u_n|\big)+\beta(x)\big]\,dx+\frac{1}{\varphi^0}\int_\Omega\big[D_1\overline{F}(x,u_n)+\alpha(x)\big]\,dx+o(1)\\
&=\frac{D_1}{\varphi^0}\int_\Omega\big[\overline{\Phi}\big(|\nabla u_n|\big)+\overline{F}(x,u_n)\big]\,dx+C_2+o(1)\\
&=D_1\mathcal{J}_\lambda(u_n)-\frac{D_1}{\varphi^0}\big\langle\mathcal{J}_\lambda'(u_n),u_n\big\rangle\\
&\quad+D_1\lambda\int_\Omega\bigg[G(u_n)-\frac{1}{\varphi^0}u_ng(u_n)\bigg]\,dx+C_2+o(1)\\
&\le D_1c+D_1\lambda\bigg(1-\frac{1}{\varphi^0}\bigg)\int_\Omega G(2u_n)\,dx+C_2+o(1)\\
&\le C_3+C_3\int_\Omega\Phi(2u_n)\,dx\le C_3+C_3\|u_n\|^{\varphi^0}.
\end{aligned}
$$

Combined with (3.14), we have $\frac{1}{2}\|u_k\|^{\gamma\varphi_0}-C_3\|u_n\|^{\varphi^0}\le C_3$. Letting $k\to\infty$, then $n\ge n_k\ge k\to\infty$. From $\gamma\varphi_0>\varphi^0$, we get $\infty\le C_3$. This contradiction shows that $\{\|u_n\|\}$ is bounded, that is, $\sup_{n\in\mathbb{N}}\|u_n\|:=K_0<\infty$.

Taking into account the reflexivity of $W_0^{1,\Phi}(\Omega)$ and the Eberlein–Smulian theorem, we may assume that u_n converges weakly to $u\in W_0^{1,\Phi}(\Omega)$. By using Remark 3.1, we obtain

$$
\|u_n-u\|_{L^1(\Omega)}\to 0,\qquad \|u_n-u\|_{(G)}\to 0,\qquad \|u_n-u\|_{(H)}\to 0,\quad n\to\infty. \tag{3.15}
$$

Using (f_1) and Hölder's inequality, we have

$$
\begin{aligned}
&\bigg|\lambda\int_\Omega g(u_n)(u_n-u)\,dx+\int_\Omega f(x,u_n)(u_n-u)\,dx\bigg|\\
&\quad\le\lambda\int_\Omega\big|g(u_n)(u_n-u)\big|\,dx+\int_\Omega\big[C|u_n-u|+C\big|h(x,u_n)(u_n-u)\big|\big]\,dx\\
&\quad\le 2\lambda\big\|g(u_n)\big\|_{(\tilde{G})}\|u_n-u\|_{(G)}+C\|u_n-u\|_{L^1(\Omega)}\\
&\qquad+2C\big\|h(u_n)\big\|_{(\tilde{H})}\|u_n-u\|_{(H)}.
\end{aligned} \tag{3.16}
$$

Now, we will show that both $\|g(u_n)\|_{(\tilde{G})}$ and $\|h(u_n)\|_{(\tilde{H})}$ are bounded.

Applying Lemma 2.1,

$$
\begin{aligned}
\int_\Omega\tilde{G}\big(g(u_n)\big)\,dx&\le\int_\Omega u_ng(u_n)\,dx\le\int_\Omega G(2u_n)\,dx\le C_4+\int_\Omega\Phi(2u_n)\,dx\\
&\le C_4+C_4\|u_n\|^{\varphi^0}<\infty.
\end{aligned}
$$

The definition of $\|\cdot\|_{(\tilde{G})}$ yields that $\|g(u_n)\|_{(\tilde{G})}\le C_4+C_4K_0^{\varphi^0}$, $n=1,2,\ldots$. On the other hand, due to $\lim_{t\to\infty}\frac{H(2t)}{\Phi_*(t)}=0$, there exists $t_0>0$ such that $H(2t)\le\Phi_*(t)$ for all $t\ge t_0$. By

Lemma 2.4 in [23], we have $d_0 := \sup_{t>0} \frac{t\Phi_*'(t)}{\Phi_*(t)} \leq \frac{N\varphi^0}{N-\varphi^0} < \infty$. Since $W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega)$,

$$\begin{aligned}\int_\Omega \tilde{H}\big(h(u_n)\big)\,dx &\leq \int_\Omega H(2u_n)\,dx \leq H_4(2t_0)\mu\Omega + \int_\Omega \Phi_*(u_n)\,dx\\ &\leq C_5 + C_5\|u_n\|_{(\Phi_*)}^{d_0} \leq C_6 + C_6K_0^{d_0} < \infty, \quad n = 1,2,\ldots.\end{aligned}$$

Hence, $\|h(u_n)\|_{(\tilde{H})} \leq C_6 + C_6K_0^{d_0} < \infty$, $n = 1,2,\ldots$.

Combining (3.15) and (3.16), we have

$$\int_\Omega g(u_n)(u_n - u)\,dx + \int_\Omega f(x,u_n)(u_n - u)\,dx \to 0, \quad n \to \infty. \tag{3.17}$$

From (3.5), it follows that $\int_\Omega a(|\nabla u_n|)\nabla u_n \cdot \nabla(u_n - u)\,dx \to 0$ as $n \to \infty$. Since u_n converges weakly to u, Theorem 4 in [36] implies that $\lim_{n\to\infty} \|u_n - u\| = 0$. Therefore, $\mathcal{J}_\lambda$ satisfies C_c condition. □

Next, we give the proof of our main result Theorem 3.1.

Proof $\lambda_* > 0$, $\eta > 0$, $\rho > 0$ are constants defined in Lemma 3.2. For all $\lambda \in (0,\lambda_*)$, Lemma 3.2, Lemma 3.3, and Lemma 3.5 show that the functional $\mathcal{J}_\lambda$ satisfies all the assumptions of Lemma 2.3. Then $\mathcal{J}_\lambda$ has a critical value $c \geq \eta > 0$. This shows that problem (1.1) has a nontrivial weak solution u with $\mathcal{J}_\lambda(u) = c$.

In the following, we prove there exists a second weak solution $\tilde{u} \neq u$.

Let $B_\rho := \{u \in W_0^{1,\Phi}(\Omega) : \|u\| \leq \rho\}$, $U_\rho := \{u \in W_0^{1,\Phi}(\Omega) : \|u\| < \rho\}$. Applying Lemma 3.4, we deduce that

$$-\infty < \tilde{c} := \inf_{B_\rho} \mathcal{J}_\lambda(u) < 0.$$

For each $\sigma \in (0, \inf_{S_\rho} \mathcal{J}_\lambda(u) - \inf_{U_\rho} \mathcal{J}_\lambda(u))$, by the Ekeland variational principle [11], there exists $u_\sigma \in B_\rho$ such that

$$\mathcal{J}_\lambda(u_\sigma) \leq \inf_{B_\rho} \mathcal{J}_\lambda(u) + \sigma$$

and

$$\mathcal{J}_\lambda(u_\sigma) < \mathcal{J}_\lambda(u) + \sigma\|u_\sigma - u\|, \quad \forall u \neq u_\sigma. \tag{3.18}$$

Therefore,

$$\mathcal{J}_\lambda(u_\sigma) \leq \inf_{B_\rho} \mathcal{J}_\lambda(u) + \sigma < \inf_{U_\rho} \mathcal{J}_\lambda(u) + \inf_{S_\rho} \mathcal{J}_\lambda(u) - \inf_{U_\rho} \mathcal{J}_\lambda(u) = \inf_{S_\rho} \mathcal{J}_\lambda(u),$$

which implies $u_\sigma \in U_\rho$.

$\forall v \in B_1$, take $h \in (0, \rho - \|u_\sigma\|)$, then $u_\sigma + hv \in B_\rho$. By (3.18), we have

$$\mathcal{J}_\lambda(u_\sigma) - \mathcal{J}_\lambda(u_\sigma + hv) \leq \sigma h\|v\|.$$

Dividing the above inequality by h and letting $h \to 0^+$, one has

$$\langle \mathcal{J}_\lambda'(u_\sigma), v \rangle \geq -\sigma \|v\|.$$

Replacing v with $-v$ in the above inequality, we deduce $\langle \mathcal{J}_\lambda'(u_\sigma), v\rangle \leq \sigma\|v\|$. Therefore, $\|\mathcal{J}_\lambda'(u_\sigma)\| \leq \sigma$.

Summarily, there exist $\{\tilde{u}_n\}_{n=1}^{\infty} \subset U_\rho$ such that $\mathcal{J}_\lambda(\tilde{u}_n) \to \tilde{c}$ and $\|\mathcal{J}_\lambda'(\tilde{u}_n)\| \leq \frac{1}{n} \to 0$ as $n \to \infty$. From the Eberlein–Smulian theorem, we may assume $\tilde{u}_n$ converges to $\tilde{u} \in B_\rho$. (3.17) and Theorem 4 in [36] imply that $\lim_{n\to\infty} \|\tilde{u}_n - \tilde{u}\| = 0$. Since $\mathcal{J}_\lambda \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ and $\|\mathcal{J}_\lambda'(\tilde{u}_n)\| \to 0$, one has $\mathcal{J}_\lambda'(\tilde{u}) = \lim_{n\to\infty} \mathcal{J}_\lambda'(\tilde{u}_n) = \theta$ and $\mathcal{J}_\lambda'(\tilde{u}) = \tilde{c}$, so $\tilde{u} \neq \theta$ and $\tilde{u} \neq u$, which completes the proof. □

By Lemma 3.2, Lemma 3.3, and Lemma 3.5, we can get the following corollary.

Corollary 3.6 ([27]) *Given that Φ satisfies (Φ_1) and (Φ_2), f satisfies (f_1)–(f_4). Then*

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has a nontrivial weak solution.

4 Conclusions

Using variational arguments, we establish the existence of two nontrivial solutions for quasilinear elliptic problems in Orlicz–Sobolev spaces, where the nonlinear terms exhibit the combined effects of concave and convex without the Ambrosetti–Rabinowitz type condition.

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References

1. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical points theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
2. Miyagaki, O.H., Souto, M.A.S.: Super-linear problems without Ambrosetti and Rabinowitz growth condition. J. Differ. Equ. **245**, 3628–3638 (2008)

3. Li, G., Yang, C.: The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti–Rabinowitz condition. Nonlinear Anal. **72**, 4602–4613 (2010)
4. Jeanjean, L.: On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbb{R}^N$. Proc. R. Soc. A **129**, 787–809 (1999)
5. Lam, N., Lu, G.: Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti–Rabinowitz condition. J. Geom. Anal. **24**, 118–143 (2014)
6. Mugnai, D., Papageorgiou, N.S.: Wang's multiplicity result for superliner (p,q)-equations without the Ambrosetti–Rabinowitz condition. Trans. Am. Math. Soc. **366**, 4919–4937 (2014)
7. Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)
8. Wu, T.F.: On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function. J. Math. Anal. Appl. **318**, 253–270 (2006)
9. Wu, T.F.: Multiple positive solutions for a class of concave-convex elliptic problems in $\mathbb{R}^N$ involving sign-changing weight. J. Funct. Anal. **258**, 99–131 (2010)
10. Papageorgiou, N.S., Rocha, E.M.: Pairs of positive solutions for p-Laplacian equations with sublinear and superlinear nonlinearities which do not satisfy the AR-condition. Nonlinear Anal. **70**, 3854–3863 (2009)
11. Ekeland, I.: On the variational principle. J. Math. Anal. Appl. **47**, 324–353 (1974)
12. Carvalho, M.L.M., da Silva, E.D., Goulart, C.: Quasilinear elliptic problems with concave-convex nonlinearities. Commun. Contemp. Math. **19**(6), 1650050 (2017). https://doi.org/10.1142/S0219199716500504
13. Chung, N.T.: Three solutions for a class of nonlocal problems in Orlicz–Sobolev spaces. J. Korean Math. Soc. **50**(6), 1257–1269 (2013)
14. Chung, N.T.: Multiple solutions for a class of $p(x)$-Laplacian problems involving concave-convex nonlinearities. Electron. J. Qual. Theory Differ. Equ. **2013**(26), 1 (2013)
15. Chung, N.T.: Existence of solutions for a class of Kirchhoff type problems in Orlicz–Sobolev spaces. Ann. Pol. Math. **113**, 283–294 (2015)
16. Bonanno, G., Molica Bisci, G., Rădulescu, V.: Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz–Sobolev spaces. C. R. Math. Acad. Sci. Paris **349**(5–6), 263–268 (2011)
17. Bonanno, G., Molica Bisci, G., Rădulescu, V.: Existence of three solutions for a non-homogeneous Neumann problem through Orlicz–Sobolev spaces. Nonlinear Anal. **74**(14), 4785–4795 (2011)
18. Bonanno, G., Molica Bisci, G., Rădulescu, V.: Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz–Sobolev spaces. Nonlinear Anal. **75**(12), 4441–4456 (2012)
19. Chlebicka, I.: A pocket guide to nonlinear differential equations in Musielak–Orlicz spaces. Nonlinear Anal. **175**, 1–27 (2018)
20. Jang, Y., Kim, Y.: An end point Orlicz type estimate for nonlinear elliptic equations. Nonlinear Anal. **177**(part B), 572–585 (2018)
21. Ho, K., Sim, I.: A-priori bounds and existence for solutions of weighted elliptic equations with a convection term. Adv. Nonlinear Anal. **6**(4), 427–445 (2017)
22. Mohammed, A., Porru, G.: Harnack inequality for non-divergence structure semi-linear elliptic equations. Adv. Nonlinear Anal. **7**(3), 259–269 (2018)
23. Fukagai, N., Ito, M., Narukawa, K.: Positive solutions of quasilinear elliptic equations with critical Orlicz–Sobolev nonlinearity on Ω. Funkc. Ekvacioj **49**, 235–267 (2006)
24. Mihuilescu, M., Radulescu, V.: A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. R. Soc. A **462**, 2625–2641 (2006)
25. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**(4), 1383–1406 (2006)
26. Clent, P., Garcia-Huidobro, M., Manaevich, R., Schmitt, K.: Mountain pass type solutions for quasilinear elliptic equations. Calc. Var. **11**, 33–62 (2000)
27. Chung, N.T., Toan, H.Q.: On a nonlinear and non-homogeneous problem without (A-R) type condition in Orlicz–Sobolev spaces. Appl. Math. Comput. **219**, 7820–7829 (2013)
28. Carvalho, M.L.M., Goncalves, J.V.A., da Silva, E.D.: On quasilinear elliptic problems without the Ambrosetti–Rabinowitz condition. J. Math. Anal. Appl. **426**, 466–483 (2015)
29. Krasnoselski, M., Rutickii, Y.: Convex Functions and Orlicz Space. Noordhoff, Groningen (1961)
30. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces, 2nd edn. Academic Press, New York (2003)
31. Chen, S.: Geometry of Orlicz Spaces. Polish Sci., Warszawa (1996)
32. Garcia-Huidobro, M., Le, V.K., Manasevich, R., Schmitt, K.: On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz–Sobolev space setting. Nonlinear Differ. Equ. Appl. **6**, 207–225 (1999)
33. Gossez, J.P.: Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. Trans. Am. Math. Soc. **190**, 163–205 (1974)
34. Cerami, G.: On the existence of eigenvalues for a nonlinear boundary value problem. Ann. Mat. Pura Appl. **124**, 161–179 (1980)
35. Costa, D.G., Magalhaes, C.A.: Existence results for perturbation of the p-Laplacian. Nonlinear Anal. **24**, 409–418 (1995)
36. Le, V.K.: A global bifurcation result for quasilinear elliptic equations in Orlicz–Sobolev spaces. Topol. Methods Nonlinear Anal. **15**, 301–327 (2000)