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Applications of maximum modulus method and Phragmén–Lindelöf method for second-order boundary value problems with respect to the Schrödinger operator

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Abstract

In this paper, we present a reliable combination of the maximum modulus method with respect to the Schrödinger operator (Meng in J. Syst. Sci. Complex. 16:446–452, 2003) and Phragmén–Lindelöf method (Shehu in Matematiche 64:57–66, 2015) to investigate the solution of a second-order boundary value problem with respect to the Schrödinger operator. We establish the uniqueness of the solution for this problem. The results reveal that this method is effective and simple.

Keywords: Maximum modulus; Schrödinger operator; Uniqueness; Boundary value problem

1 Introduction

Let $\gimel = (0, W) \times (0, w)$ be a bounded rectangular domain in $\mathbb{R}^2$, which represents a porous medium with Lipschitz boundary $\partial\gimel = \daleth_1 \cup \daleth_2$, where

$$\daleth_2 = \bigl(\{0\} \times [0, w]\bigr) \cup \bigl([0, W] \times \{w\}\bigr) \cup \bigl(\{W\} \times [0, w]\bigr)$$

is the part in contact with air or covered by fluid, and

$$\daleth_1 = [0, W] \times \{0\}$$

is the impervious part of $\partial\gimel$. Let $P = \gimel \times (0, M)$, where $M > 0$. Let ϕ be a nonnegative Lipschitz function defined in the closure of P, which we denote by $\overline{P}$.

Define

$$\Sigma_1 = \daleth_1 \times (0, M), \qquad \Sigma_2 = \daleth_2 \times (0, M), \qquad \Sigma_3 = \Sigma_2 \cap \{\phi > 0\},$$

and

$$\Sigma_4 = \Sigma_2 \cap \{\phi = 0\}.$$

Let χ be a function of the variable t satisfying

$$c_1 \leq \chi(t) \leq c_2, \quad \text{a.e. } t \in (0, W), \tag{1.1}$$

for two positive constants c_1 and c_2, and let ω_0 be a function of the variable t satisfying (see [3, 4])

$$0 \leq \omega_0(t) \leq 1, \quad \text{a.e. } t \in \daleth. \tag{1.2}$$

The governing equation in terms of the deflection function $f(t)$ is the elliptic boundary value problem with respect to the Schrödinger operator (see [2, 5, 9, 11, 16, 24]): Find $(f, \omega) \in L^2(0, T; H^1(\daleth)) \times L^\infty(P)$ such that

$$\begin{aligned} & f \geq 0, 0 \leq \omega \leq 1, \qquad f(1-\omega) = 0 \quad \text{a.e. in } P, \\ & f = \phi \quad \text{on } \Sigma_2, \\ & \int_P \bigl[\chi(t_1)(f_{t_2} + \omega)\varsigma_{t_2} - \omega\varsigma_s\bigr]\,dt\,ds \leq \int_\daleth \omega_0(t)\varsigma(t,0)\,dt, \\ & \forall \varsigma \in H^1(P), \quad \varsigma = 0 \quad \text{on } \Sigma_3, \\ & \varsigma \geq 0 \quad \text{on } \Sigma_4, \\ & \varsigma(t, M) = 0 \quad \text{for a.e. } t \in \daleth. \end{aligned} \tag{1.3}$$

For the existence of a solution of problem (1.3) in the homogeneous case, we refer to [10] and [19] in the incompressible and compressible cases, respectively. For the heterogeneous case, we refer to [6] in a more general framework under Assumptions (1.1) and (1.2) for both incompressible and compressible cases. For the incompressible case with nonlinear Darcy's law, we refer to [18] for Dirichlet, Neumann, and generalized boundary conditions. Extensions to the quasilinear and incompressible case were obtained in [10, 18] in both homogeneous and nonhomogeneous frameworks. Meng [20] was able to extend the above regularity result to a more general framework under weaker assumptions on the data. The uniqueness of a solution for a homogeneous dam with general geometry was established by the method of doubling variables in [8, 24], but it is not obvious whether it works in the heterogeneous situation. Extensions to the Schrödinger operator modeling incompressible fluid flow governed by the nonlinear Darcy law with Dirichlet or Neumann boundary conditions were obtained in [15, 20], respectively.

On the other hand, the following boundary value problem with respect to the Schrödinger operator corresponding to (1.3) is given by [7, 14, 18, 29, 33, 36]:

$$\begin{aligned} & f \geq 0, 0 \leq \omega \leq 1, \quad u(1-\omega) = 0 \quad \text{in } P, \\ & \chi(t_1)(f_{t_2} + \omega)_{t_2} - \omega_s = 0 \quad \text{in } P, \\ & u = \phi \quad \text{on } \Sigma_2, \\ & \omega(\cdot, 0) = \omega_0 \quad \text{in } \daleth, \\ & \chi(t_1)(f_{t_2} + \omega) \cdot \nu = 0 \quad \text{on } \Sigma_1, \\ & \chi(t_1)(f_{t_2} + \omega) \cdot \nu \leq 0 \quad \text{on } \Sigma_4. \end{aligned} \tag{1.4}$$

Regarding the existence of a solution of the problem with respect to the Schrödinger operator (1.4), we refer to [15] and [37], respectively, for the evolutionary dam problem with homogeneous coefficients and for a class of free boundary problem with nonlocal boundary condition in heterogeneous domain. The regularity of the solution of the problem with nonlocal boundary condition was discussed in [40] (also see, e.g., [13, 32]), where it was proved that $\omega \in C^0([0,T];L^p(\daleth))$ for all $p \in [1,+\infty)$ in the free boundary problems with respect to the Schrödinger operator of types

$$\operatorname{div}\big(\chi(t)\nabla f + \mathcal{H}(t)\omega\big) - \omega_s$$

and

$$\operatorname{div}\big(\chi(t)\nabla f + \mathcal{H}(t)\omega\big) - (f+\omega)_s,$$

and that $f \in C^0([0,M];L^p(\daleth))$ for all $p \in [1,2]$ in the second-order class.

The uniqueness of a solution of the Schrödinger equation in the homogeneous case for both incompressible and compressible fluids was obtained in [20] by using the method of doubling variables. In the case of a rectangular dam wet at the bottom and dry near to the top, the uniqueness was obtained in [17] and [35] by the fixed point theory (as one of the main tools in this subject; see [21, 25]) with respect to the Schrödinger operator, respectively, in homogeneous and heterogeneous porous media (see [28]). For the evolution free boundary problem in theory of the Schrödinger equation, we refer to [8, 15].

As an application of the Phragmén–Lindelöf method related to a second-order boundary value problem with respect to the Schrödinger operator, in this paper, we consider the weak formulation of an evolution dam problem with heterogeneous coefficients (1.3) in $\daleth$. We establish the uniqueness of the solution for this problem with respect to the Schrödinger operator. This uniqueness result is new in the general framework of a heterogeneous and bounded rectangular domain.

2 Main result and its proof

In this section, we obtain our main results that a solution of problem with respect to the Schrödinger operator (1.3) is unique. We assume that

$$\chi \in C^1\big([0,W]\big). \tag{2.1}$$

Theorem 2.1 *Let* (f_1,ω_1) *and* (f_2,ω_2) *be two solutions of* (1.3). *Then*

$$\begin{aligned}\int_P \chi(t_1)\big\{\big(f_1(t,s)-f_2(t,s)\big)^+_{t_2} + \big(1-\omega_2(t,s)\big)\omega_{\{f_1>f_2\}} \\ + \big(\big(1-\omega_2(t,s)\big)+\big(1-f_{2t_2}(t,s)\big)\big)\omega_{\{f_1>0\}}\big\}\iota\varsigma_{t_2}\,dt\,ds \le 0,\end{aligned} \tag{2.2}$$

where

$$0 \le \varsigma \in \mathfrak{D}(\daleth), \qquad 0 \le \iota \in \mathfrak{D}(0,M).$$

Proof Note that

$$(f_1,\omega_1):(t,s,x,y)\mapsto \bigl(f_1(t,s),\omega_1(x,y)\bigr),$$
$$(f_2,\omega_2):(t,s,x,y)\mapsto \bigl(f_2(t,s),\omega_2(x,y)\bigr).$$

Define

$$\psi(t,s,x,y)=\varsigma\left(\frac{t_1+x_1}{3},\frac{t_2+x_2}{3}\right)\iota\left(\frac{y+s}{3}\right)\rho_{1,\sigma}\left(\frac{t_1-x_1}{3}\right)\rho_{2,\sigma}\left(\frac{t_2-x_2}{3}\right)\rho_{3,\sigma}\left(\frac{y-s}{3}\right)$$

for $(t,s,x,y)\in\overline{P\times P}$, where

$$\rho_{1,\sigma}(r)=\frac{1}{\sigma}\rho_1\left(\frac{r}{\sigma}\right),$$
$$\rho_{2,\sigma}(r)=\frac{1}{\sigma}\rho_2\left(\frac{r}{\sigma}\right),$$
$$\rho_{3,\sigma}(r)=\frac{1}{\sigma}\rho_3\left(\frac{r}{\sigma}\right)$$

with $\rho_1,\rho_2,\rho_3\in\mathfrak{D}(\mathbb{R})$, $\rho_1,\rho_2,\rho_3\geq 0$, $\operatorname{supp}(\rho_1),\operatorname{supp}(\rho_2),\operatorname{supp}(\rho_3)\subset(-1,1)$.

So

$$\psi(\cdot,\cdot,x,y)\in\mathfrak{D}(P), \tag{2.3}$$
$$\psi(t,s,\cdot,\cdot)\in\mathfrak{D}(P), \tag{2.4}$$

where $(x,y)\in P$ and $(t,s)\in P$.

Define

$$\vartheta(t,s,x,y)=\max\left(\frac{(f_1(t,s)-f_2(t,s))^+}{\epsilon},\psi(t,s,x,y)\right), \tag{2.5}$$

where ϵ is a positive real number (see [22]).

By applying the fixed point theory to (f_1,ω_1) with $k=f_2(x,y)$ and $\varsigma(t,s)=\vartheta(t,s,x,y)$ for almost every $(x,y)\in P$ we know that

$$\int_P\chi(t_1)(f_{1t_2}+\omega_1)\vartheta_{t_2}\,dt\,ds=0, \tag{2.6}$$

which, together with $f_1.(1-\omega_1)=0$ a.e. in P, gives that

$$\omega_1\chi(t_1)\left(\max\left(\frac{(f_1-f_2)^+}{\epsilon},\psi\right)\right)_{t_2}=\chi(t_1)\left(\max\left(\frac{(f_1-f_2)^+}{\epsilon},\psi\right)\right)_{t_2}$$

a.e. in P.

Meanwhile, (2.6) also yields that

$$\int_P\chi(t_1)(f_{1t_2}+1)\vartheta_{t_2}\,dt\,ds=0$$

and

$$\int_{P\times P} \chi(t_1)(f_{1t_2}+1)\vartheta_{t_2}\,d\varrho = 0, \tag{2.7}$$

where $d\varrho = dt\,ds\,dx\,dy$.

Now we apply the fixed point theory to (f_2,ω_2) with

$$k = f_1(t,s), \qquad \varsigma(x,y) = \vartheta(t,s,x,y)$$

to obtain that

$$\int_P \chi(x_1)(f_{2x_2}+\omega_2)\left(\vartheta - \min\left(\frac{f_1}{\epsilon},\psi\right)\right)_{x_2} dy\,ds = 0.$$

So

$$\int_{P\times P} \chi(x_1)(f_{2x_2}+\omega_2)\left(\vartheta - \min\left(\frac{f_1}{\epsilon},\psi\right)\right)_{x_2} dy\,ds\,dt\,ds = 0. \tag{2.8}$$

By subtracting (2.8) from (2.7) we have that

$$\begin{aligned}
&\int_{P\times P} \bigl[\chi(t_1)f_{1t_2}\vartheta_{t_2} - \chi(x_1)f_{2x_2}\vartheta_{x_2} \\
&\quad + \chi(t_1)\vartheta_{t_2} - \omega_2\chi(x_1)\vartheta_{x_2}\bigr]\,d\varrho \\
&\quad - \int_{P\times P} \chi(x_1)(f_{2x_2}+\omega_2)\min\left(\frac{f_1}{\epsilon},\psi\right)_{x_2} d\varrho = 0.
\end{aligned} \tag{2.9}$$

It follows from (2.3)–(2.5) that

$$\int_{P\times P} \chi(t_1)f_{1t_2}\vartheta_{x_2}\,d\varrho = 0, \tag{2.10}$$

$$\int_{P\times P} \chi(x_1)f_{2x_2}\vartheta_{t_2}\,d\varrho = 0, \tag{2.11}$$

$$\int_{P\times P} \chi(t_1)(\partial_{t_2}+\partial_{x_2})\vartheta\,d\varrho = 0, \tag{2.12}$$

$$\int_{P\times P} \omega_2\chi(x_1)\vartheta_{t_2}\,d\varrho = 0, \tag{2.13}$$

$$\int_{P\times P} \chi(x_1)(f_{2x_2}+\omega_2)\min\left(\frac{f_1}{\epsilon},\psi\right)_{t_2} d\varrho = 0, \tag{2.14}$$

$$\int_{P\times P} \chi(t_1)(\partial_{t_2}+\partial_{x_2})\min\left(\frac{f_1}{\epsilon},\psi\right) d\varrho = 0. \tag{2.15}$$

Combining (2.10)–(2.15) and (2.9), we get that

$$\begin{aligned}&\int_{P\times P}\big[\big(\chi(t_1)(\partial_{t_2}+\partial_{x_2})f_1-\chi(x_1)(\partial_{t_2}+\partial_{x_2})f_2\big)(\partial_{t_2}+\partial_{x_2})\vartheta\\&\quad+\big(\chi(t_1)-\omega_2\chi(x_1)\big)(\partial_{t_2}+\partial_{x_2})\vartheta\big]\,d\varrho\\&\quad+\int_{P\times P}\big(\chi(t_1)-\chi(x_1)(\partial_{t_2}+\partial_{x_2})f_2\big)(\partial_{t_2}+\partial_{x_2})\min\Big(\frac{f_1}{\epsilon},\psi\Big)\,d\varrho\\&\quad+\int_{P\times P}\big(\chi(t_1)-\omega_2\chi(x_1)\big)(\partial_{t_2}+\partial_{x_2})\min\Big(\frac{f_1}{\epsilon},\psi\Big)\,d\varrho=0.\end{aligned}\tag{2.16}$$

Put

$$\frac{t+x}{3}=z,\qquad\frac{t-x}{3}=\xi,\qquad\frac{y+s}{3}=\kappa,\qquad\frac{y-s}{3}=\tau.\tag{2.17}$$

Note that $(z,\kappa)\in P$ and

$$(\xi,\tau)\in\Big(-\frac{W}{3},\frac{W}{3}\Big)\times\Big(-\frac{w}{3},\frac{w}{3}\Big)\times\Big(-\frac{M}{3},\frac{M}{3}\Big):=\daleth_1\times\Big(-\frac{M}{3},\frac{M}{3}\Big):=P_1.$$

We have that

$$\begin{aligned}&\int_{P\times P_1}\big(\chi(z_1+\xi_1)f_{1z_2}(z+\xi,\kappa+\tau)\\&\quad-\chi(z_1-\xi_1)f_{2z_2}(z-\xi,\kappa-\tau)\big)\vartheta_{z_2}\,d\mu\\&\quad+\int_{P\times P_1}\big(\chi(z_1+\xi_1)-\omega_2(z-\xi,\kappa-\tau)\chi(z_1-\xi_1)\big)\vartheta_{z_2}\,d\mu\\&\quad+\int_{P\times P_1}\big(\chi(z_1+\xi_1)-\omega_2(z-\xi,\kappa-\tau)\chi(z_1-\xi_1)\big)\\&\quad\times\min\Big(\frac{f_1}{\epsilon},\psi\Big)_{z_2}\,d\mu\\&\quad+\int_{P\times P_1}\big(\chi(z_1+\xi_1)-f_{2z_2}(z-\xi,\kappa-\tau)\chi(z_1-\xi_1)\big)\\&\quad\times\min\Big(\frac{f_1}{\epsilon},\psi\Big)_{z_2}\,d\mu=0\end{aligned}\tag{2.18}$$

from (2.16)–(2.17), where $d\mu=dz\,d\kappa\,d\xi\,d\tau$.

Put

$$\begin{aligned}\mathcal{I}_{\epsilon,\sigma}&=\int_{P\times P_1}\big(\chi(z_1+\xi_1)f_{1z_2}(z+\xi,\kappa+\tau)\\&\qquad-\chi(z_1-\xi_1)f_{2z_2}(z-\xi,\kappa-\tau)\big)\vartheta_{z_2}\,dz\,d\kappa\,d\xi\,d\tau,\\\mathcal{J}_{\epsilon,\sigma}&=\int_{P\times P_1}\big(\chi(z_1+\xi_1)-\omega_2(z-\xi,\kappa-\tau)\chi(z_1-\xi_1)\big)\vartheta_{z_2}\,d\mu,\end{aligned}$$

$$\mathcal{K}^1_{\epsilon,\sigma} = \int_{P\times P_1} \bigl(\chi(z_1+\xi_1) - \omega_2(z-\xi,\kappa-\tau)\chi(z_1-\xi_1)\bigr) \times \min\left(\frac{f_1}{\epsilon},\psi\right)_{z_2} d\mu,$$

$$\mathcal{K}^2_{\epsilon,\sigma} = \int_{P\times P_1} \bigl(\chi(z_1+\xi_1) - f_{2z_2}(z-\xi,\kappa-\tau)\chi(z_1-\xi_1)\bigr) \times \min\left(\frac{f_1}{\epsilon},\psi\right)_{z_2} d\mu.$$

Lemma 2.2

$$\lim_{\sigma\to 0}\Bigl(\lim_{\epsilon\to 0}\mathcal{J}_{\epsilon,\sigma}\Bigr) = \int_P \iota\omega_{\{f_1>f_2\}}\chi(z_1)\bigl(1-\omega_2(z,\kappa)\bigr)\varsigma_{z_2}\,dz\,d\kappa, \tag{2.19}$$

$$\lim_{\sigma\to 0}\Bigl(\lim_{\epsilon\to 0}\mathcal{K}^1_{\epsilon,\sigma}\Bigr) = \int_P \iota\omega_{\{f_1>0\}}\chi(z_1)\bigl(1-\omega_2(z,\kappa)\bigr)\varsigma_{z_2}\,dz\,d\kappa, \tag{2.20}$$

$$\lim_{\sigma\to 0}\Bigl(\lim_{\epsilon\to 0}\mathcal{K}^2_{\epsilon,\sigma}\Bigr) = \int_P \iota\omega_{\{f_1>0\}}\chi(z_1)\bigl(1-f_{2z_2}(z,\kappa)\bigr)\varsigma_{z_2}\,dz\,d\kappa. \tag{2.21}$$

Proof Set

$$\begin{aligned}\mathcal{J}_{\epsilon,\sigma} &= \int_{P\times P_1} \bigl(\chi(z_1+\xi_1) - \chi(z_1-\xi_1)\bigr)\vartheta_{z_2}\,d\mu \\ &\quad + \int_{P\times P_1} \chi(z_1-\xi_1)\bigl(1-\omega_2(z-\xi,\kappa-\tau)\bigr)\vartheta_{z_2}\,d\mu \\ &:= \mathcal{J}^1_{\epsilon,\sigma} + \mathcal{J}^2_{\epsilon,\sigma}.\end{aligned} \tag{2.22}$$

Combining (2.3)–(2.5) and the fact that $\chi(z_1+\xi_1) - \chi(z_1-\xi_1)$ does not depend on z_2, we have that

$$J^1_{\epsilon,\sigma} = 0.$$

So

$$\lim_{\sigma\to 0}\Bigl(\lim_{\epsilon\to 0}\mathcal{J}^1_{\epsilon,\sigma}\Bigr) = 0. \tag{2.23}$$

Put

$$\lim_{\sigma\to 0}\Bigl(\lim_{\epsilon\to 0}\mathcal{J}^2_{\epsilon,\sigma}\Bigr) = \int_P \iota\chi(z_1)\bigl(1-\omega_2(z,\kappa)\bigr)\varsigma_{z_2}\,dz\,d\kappa. \tag{2.24}$$

Set

$$\mathcal{A}_\epsilon = \bigl\{(f_1-f_2)^+ > \epsilon\psi\bigr\}$$

and

$$\mathcal{B}_\epsilon = \{0 < f_1 - f_2 \le \epsilon\psi\}.$$

We know that

$$\begin{aligned}\mathcal{J}^{2}_{\epsilon,\sigma} &= \int_{\mathcal{B}_\epsilon} \chi(z_1-\xi_1)\big(1-\omega_2(z-\xi,\kappa-\tau)\big)\left(\frac{f_1-f_2}{\epsilon}\right)_{z_2} d\mu \\ &\quad + \int_{\mathcal{A}_\epsilon} \chi(z_1-\xi_1)\big(1-\omega_2(z-\xi,\kappa-\tau)\big)\psi_{z_2}\, d\mu \\ &:= \mathcal{J}^{2,1}_{\epsilon,\sigma} + \mathcal{J}^{2,2}_{\epsilon,\sigma}. \end{aligned} \tag{2.25}$$

By applying (2.17) and that $(1-\omega_2)f_2 = 0$ and $f_{1x_2} = 0$ a.e. in P (see [1, 41]), we have that

$$\mathcal{J}^{2,1}_{\epsilon,\sigma} = \int_{\mathcal{B}_\epsilon} \chi(x_1)\big(1-\omega_2(x,y)\big)\frac{f_{1t_2}}{\epsilon}\, d\varrho.$$

By applying (2.3) and that the function $(x,y) \mapsto \chi(x_1)(1-\omega_2(x,y))$ does not depend on t_2 we have that

$$\begin{aligned}\mathcal{J}^{2,1}_{\epsilon,\sigma} &= \int_{P\times P} \chi(x_1)\big(1-\omega_2(x,y)\big)\left(\min\left(\frac{f_1}{\epsilon},\psi\right)\right)_{t_2} d\varrho \\ &\quad - \int_{\mathcal{A}_\epsilon} \chi(x_1)\big(1-\omega_2(x,y)\big)\psi_{t_2}\, dt\, ds\, dy\, ds \\ &= \int_{P\times P} \chi(x_1)\big(1-\omega_2(x,y)\big)\left(\min\left(\frac{f_1}{\epsilon},\psi\right)\right)_{t_2} d\varrho \\ &\quad - \int_{P\times P} \chi(x_1)\big(1-\omega_2(x,y)\big)\psi_{t_2}\, d\varrho \\ &\quad + \int_{\mathcal{B}_\epsilon} \chi(x_1)\big(1-\omega_2(x,y)\big)\psi_{t_2}\, d\varrho \\ &= \int_{\mathcal{B}_\epsilon} \chi(x_1)\big(1-\omega_2(x,y)\big)\psi_{t_2}\, d\varrho. \end{aligned}$$

Taking into account the fact that $\lim_{\epsilon\to 0} |\mathcal{B}_\epsilon| = 0$ (see [39]), we know that

$$\lim_{\epsilon\to 0} \mathcal{J}^{2,1}_{\epsilon,\sigma} = 0,$$

which yields that

$$\lim_{\sigma\to 0}\Big(\lim_{\epsilon\to 0} \mathcal{J}^{2,1}_{\epsilon,\sigma}\Big) = 0. \tag{2.26}$$

To estimate $\mathcal{J}^{2,2}_{\epsilon,\sigma}$, we obtain that

$$\lim_{\sigma\to 0}\Big(\lim_{\epsilon\to 0} \mathcal{J}^{2,2}_{\epsilon,\sigma}\Big) = \int_P \chi(z_1)\omega_{\{f_1>f_2\}}\big(1-\omega_2(z,\kappa)\big)\varsigma_{z_2}\iota\, dz\, d\kappa \tag{2.27}$$

by passing to the limit as $\epsilon \to 0$ and then as $\sigma \to 0$.

Hence, combining (2.26)–(2.27), we obtain (2.24) by letting $\epsilon \to 0$ and $\sigma \to 0$ in (2.25). Now we pass successively to the limit in (2.22) as $\epsilon \to 0$ and then as $\sigma \to 0$, and using (2.23)–(2.24), we obtain (2.19). Finally, arguing as in the proof (2.19), we obtain (2.20) and (2.21). □

Lemma 2.3

$$\lim_{\sigma\to 0}\Bigl(\lim_{\epsilon\to 0}\mathcal{I}_{\epsilon,\sigma}\Bigr) \geq \int_P \iota\chi(z_1)\bigl(f_1(z,\kappa)-f_2(z,\kappa)\bigr)^+_{z_2}\varsigma_{z_2}\,dz\,d\kappa. \tag{2.28}$$

Proof Set

$$\begin{aligned}\mathcal{I}_{\epsilon,\sigma} &= \int_{\mathcal{A}_\epsilon}\bigl(\chi(z_1+\xi_1)f_{1z_2}(z+\xi,\kappa+\tau)\\ &\quad -\chi(z_1-\xi_1)f_{2z_2}(z-\xi,\kappa-\tau)\bigr)\psi_{z_2}\,d\mu\\ &\quad +\int_{\mathcal{B}_\epsilon}\bigl(\chi(z_1+\xi_1)f_{1z_2}(z+\xi,\kappa+\tau)\\ &\quad -\chi(z_1-\xi_1)f_{2z_2}(z-\xi,\kappa-\tau)\bigr)\left(\frac{f_1-f_2}{\epsilon}\right)_{z_2}d\mu\\ &:= \mathcal{I}^1_{\epsilon,\sigma}+\mathcal{I}^2_{\epsilon,\sigma}.\end{aligned} \tag{2.29}$$

Note that $\mathcal{I}^2_{\epsilon,\sigma}$ can be rewritten as

$$\begin{aligned}\mathcal{I}^2_{\epsilon,\sigma} &= \frac{1}{\epsilon}\biggl\{\int_{\mathcal{B}_\epsilon}\bigl[\chi(z_1+\xi_1)f_{1z_2}(z+\xi,\kappa+\tau)f_{1z_2}(z+\xi,\kappa+\tau)\\ &\quad +\chi(z_1-\xi_1)f_{2z_2}(z-\xi,\kappa-\tau)f_{2z_2}(z-\xi,\kappa-\tau)\bigr]\,dz\,d\kappa\,d\xi\,d\tau\\ &\quad -\int_{\mathcal{B}_\epsilon}\chi(z_1-\xi_1)f_{2z_2}(z-\xi,\kappa-\tau)f_{1z_2}(z+\xi,\kappa+\tau)\,d\mu\\ &\quad -\int_{\mathcal{B}_\epsilon}(\chi(z_1+\xi_1)f_{1z_2}(z+\xi,\kappa+\tau)f_{2z_2}(z-\xi,\kappa-\tau)\,d\mu\biggr\}\\ &:= \mathcal{I}^{2,1}_{\epsilon,\sigma}-\mathcal{I}^{2,2}_{\epsilon,\sigma}-\mathcal{I}^{2,3}_{\epsilon,\sigma}.\end{aligned}$$

It follows from (1.1) that

$$\mathcal{I}^2_{\epsilon,\sigma} \geq -\mathcal{I}^{2,2}_{\epsilon,\sigma}-\mathcal{I}^{2,3}_{\epsilon,\sigma}. \tag{2.30}$$

Let us estimate $\mathcal{I}^{2,2}_{\epsilon,\sigma}$. It follows from (2.3) and (2.5) that

$$\begin{aligned}\mathcal{I}^{2,2}_{\epsilon,\sigma} &= \int_{\mathcal{B}_\epsilon}\chi(x_1)f_{2x_2}(x,y)\left(\frac{f_1(t,s)-f_2(x,y)}{\epsilon}\right)_{t_2}dt\,ds\,dy\,ds\\ &= \int_{P\times P}\chi(x_1)f_{2x_2}(x,y)\vartheta_{t_2}\,d\varrho\\ &\quad -\int_{\mathcal{A}_\epsilon}\chi(x_1)f_{2x_2}(x,y)\psi_{t_2}\,d\varrho\\ &= \int_{P\times P}\chi(x_1)f_{2x_2}(x,y)\vartheta_{t_2}\,d\varrho\\ &\quad -\int_{P\times P}\chi(x_1)f_{2x_2}(x,y)\psi_{t_2}\,d\varrho\end{aligned}$$

$$+\int_{\mathcal{B}_\epsilon}\chi(x_1)f_{2x_2}(x,y)\psi_{t_2}\,d\varrho$$
$$=\int_{\mathcal{B}_\epsilon}\chi(x_1)f_{2x_2}(x,y)\psi_{t_2}\,d\varrho.$$

Taking into account that $\lim_{\epsilon\to 0}|\mathcal{B}_\epsilon| = 0$ (see [39]), we obtain that

$$\lim_{\epsilon\to 0}\bigl(\mathcal{I}^{2,2}_{\epsilon,\sigma}\bigr) = 0. \tag{2.31}$$

Similarly,

$$\lim_{\epsilon\to 0}\bigl(\mathcal{I}^{2,3}_{\epsilon,\sigma}\bigr) = 0. \tag{2.32}$$

Combining (2.31)–(2.32), we obtain that

$$\lim_{\epsilon\to 0}\bigl(\mathcal{I}^{2}_{\epsilon,\sigma}\bigr) \geq 0 \tag{2.33}$$

by passing to the limit as $\epsilon \to 0$ in (2.30).

Let us estimate $\mathcal{I}^1_{\epsilon,\sigma}$. We obtain that

$$\lim_{\epsilon\to 0}\bigl(\mathcal{I}^{1}_{\epsilon,\sigma}\bigr) = \int_{P\times P_1}\omega_{\{f_1>f_2\}}\bigl(\chi(z_1+\xi_1)f_{1z_2}(z+\xi,\kappa+\tau)$$
$$-\chi(z_1-\xi_1)f_{2z_2}(z-\xi,\kappa-\tau)\bigr)\psi_{z_2}\,d\mu,$$

which yields that

$$\lim_{\epsilon\to 0}\bigl(\mathcal{I}^{1}_{\epsilon,\sigma}\bigr) = \int_{P\times P_1}\omega_{\{f_1>f_2\}}\chi(z_1+\xi_1)\bigl(f_1(z+\xi,\kappa+\tau)$$
$$-f_2(z-\xi,\kappa-\tau)\bigr)_{z_2}\psi_{z_2}\,d\mu$$
$$+\int_{P\times P_1}\omega_{\{f_1>f_2\}}\bigl(\chi(z_1+\xi_1)$$
$$-\chi(z_1-\xi_1)\bigr)f_{2z_2}(z-\xi,\kappa-\tau)\psi_{z_2}\,d\mu$$
$$=\mathcal{I}^{1,1}_\sigma+\mathcal{I}^{1,2}_\sigma. \tag{2.34}$$

By applying (2.1) and taking into account the fact that $\operatorname{supp}(\rho_{1,\sigma})\subset(-\sigma,\sigma)$ (see [39]), we obtain that

$$\bigl|\mathcal{I}^{1,2}_\sigma\bigr| \leq C\int_{P\times P_1}|\xi_1||f_{2z_2}||\varsigma_{z_2}|\iota\rho_{1,\sigma}(\xi_1)\rho_{2,\sigma}(\xi_1)\rho_{3,\sigma}(\tau)\,d\mu$$
$$\leq \sigma C\int_{P\times P_1}|f_{2z_2}||\varsigma_{z_2}|\iota\rho_{1,\sigma}(\xi_1)\rho_{2,\sigma}(\xi_1)\rho_{3,\sigma}(\tau)\,d\mu$$
$$:= \sigma C\mathcal{W}_\sigma$$

for some positive constant C.

So

$$\lim_{\sigma\to 0}\mathcal{I}_\sigma^{1,2}=0. \tag{2.35}$$

For $\mathcal{I}_\sigma^{1,1}$, we have that

$$\lim_{\sigma\to 0}\mathcal{I}_\sigma^{1,1}=\int_P \iota\chi(z_1)\big(f_1(z,\kappa)-f_2(z,\kappa)\big)^+_{z_2}\varsigma_{z_2}\,dz\,d\kappa \tag{2.36}$$

by passing to the limit as $\sigma\to 0$.

Combining (2.35)–(2.36), we obtain that

$$\lim_{\sigma\to 0}\Big(\lim_{\epsilon\to 0}\mathcal{I}^1_{\epsilon,\sigma}\Big)=\int_P \iota\chi(z_1)\big(f_1(z,\kappa)-f_2(z,\kappa)\big)^+_{z_2}\varsigma_{z_2}\,dz\,d\kappa \tag{2.37}$$

by letting $\sigma\to 0$ in (2.34).

Finally, we pass successively to the limit in (2.29) as $\epsilon\to 0$ and then as $\sigma\to 0$, and using (2.33) and (2.37), we obtain (2.28). □

Now, using Lemmas 2.2 and 3.1 and letting successively $\epsilon\to 0$ and $\sigma\to 0$ in (2.18), we obtain (2.2). This completes the proof of Theorem 2.1. □

3 An application

In this section, we establish the existence of nontrivial positive solutions of (1.3) with $\varsigma=\varsigma^*$ by Theorem 2.1. To this end, we first construct a pair of proper super- and subsolutions of (1.3) with $\varsigma=\varsigma^*$ (see [23, 27]). We define the continuous functions (see [26, 30, 34, 38])

$$\mathcal{J}_+(\zeta)=\mathcal{J}_0,$$

$$\mathcal{J}_-(\zeta)=\begin{cases}\mathcal{J}_0-pe^{\lambda_3\zeta}, & \zeta<\zeta_1,\\ \epsilon e^{-\lambda_4\zeta}, & \zeta\ge\zeta_1,\end{cases}$$

$$\mathcal{K}_+(\zeta)=\begin{cases}-\varpi\zeta e^{\sigma^*\zeta}, & \zeta<\zeta_2,\\ (\frac{\varphi}{\gamma}-1)\mathcal{J}_0, & \zeta\ge\zeta_2,\end{cases}$$

$$\mathcal{K}_-(\zeta)=\begin{cases}-\varpi\zeta e^{\sigma^*\zeta}-L(-\zeta)^{1/2}e^{\sigma^*\zeta}, & \zeta<\zeta_3,\\ 0, & \zeta\ge\zeta_3,\end{cases}$$

where

$$\tau_1=\min\left\{\frac{\sigma^*}{2},\frac{\alpha^*}{2\mathfrak{d}_1}\right\},\qquad \tau_2=\sqrt{\varphi/\mathfrak{d}_1},$$

and $\zeta_1,\zeta_2,\zeta_3\in\mathbb{R}$, $p>\mathcal{J}_0$, $\epsilon>0$, $\varpi>0$, $L>0$ will be clarified later.

Lemma 3.1 *Assume that $\varphi>\gamma$. Then (1.3) with $\varsigma=\varsigma^*$ admits a solution satisfying*

$$\mathcal{J}_-(\zeta)\le\mathcal{S}(\zeta)\le\mathcal{J}_+(\zeta),\qquad \mathcal{K}_-(\zeta)\le I(\zeta)\le\mathcal{K}_+(\zeta),\quad \zeta\in\mathbb{R}.$$

Proof Define

$$m := \int_0^{\infty} \int_{\mathbb{R}} \mathcal{I}(t,s) e^{\lambda^*(t-c^*s)}\,dt\,ds,$$
$$n := \int_0^{\infty} \int_{\mathbb{R}} \mathcal{I}(t,s)\bigl(t-\alpha^* s\bigr) e^{\lambda^*(t-c^*s)}\,dt\,ds.$$

It is easy to see that m, n are bounded and

$$\zeta m + n \le (\zeta + K)m - \alpha^* \int_0^{\infty} \int_{\mathbb{R}} \mathcal{I}(t,s) s e^{\lambda^*(t-c^*s)}\,dt\,ds,$$

and so $\zeta m + n < 0$ for all $\zeta < -K$. Moreover, Lemma 2.3 indicates that

$$\eth_2 \lambda^{*2} - \alpha^* \lambda^* + \varphi m - \gamma = 0, \qquad 2\eth_2 \lambda^* - c^* + \varphi n = 0.$$

Note that if $h_1 \ge 0$, $h_2 \ge 0$ with $h_1 + h_2 > 0$, then

$$\frac{h_1 h_2}{h_1 + h_2} \le \min\{u, v\}.$$

If $\zeta < \zeta_2$, then $I_+(\zeta) = -\varpi \zeta e^{\sigma^* \zeta}$, and it suffices to prove that

$$\alpha^* \mathcal{K}_+'(\zeta) \ge \eth_2 \mathcal{K}_+''(\zeta) + \varphi(\mathcal{I} * \mathcal{K}_+)(\zeta) - \gamma \mathcal{K}_+(\zeta). \tag{3.1}$$

Note that

$$\begin{aligned}
(\mathcal{I} * \mathcal{K}_+)(\zeta) &= \int_0^{\infty} \int_{\mathbb{R}} \mathcal{I}(t,s) \mathcal{K}_+\bigl(\zeta - t - c^* s\bigr)\,dt\,ds \\
&\le \int_0^{\infty} \int_{\zeta-\zeta^*-\alpha^* s}^{+\infty} \mathcal{I}(t,s) I_+\bigl(\zeta - t - \alpha^* s\bigr)\,dt\,ds \\
&= -\varpi \int_0^{\infty} \int_{\mathbb{R}} \mathcal{I}(t,s)\bigl(\zeta - t - c^* s\bigr) e^{\lambda^*(\zeta - t - c^* s)}\,dt\,ds \\
&= -\varpi \int_0^{\infty} \int_{\mathbb{R}} \mathcal{I}(t,s)\bigl(\zeta + t - c^* s\bigr) e^{\lambda^*(\zeta + t - c^* s)}\,dt\,ds \\
&= -\varpi \zeta e^{\lambda^* \zeta} \int_0^{\infty} \int_{\mathbb{R}} \mathcal{I}(t,s) e^{\lambda^*(t-c^*s)}\,dt\,ds \\
&\quad - \varpi e^{\lambda^* \zeta} \int_0^{\infty} \int_{\mathbb{R}} \mathcal{I}(t,s)\bigl(t - c^* s\bigr) e^{\lambda^*(t-c^*s)}\,dt\,ds \\
&= -\varpi \zeta e^{\lambda^* \zeta} m - \varpi e^{\lambda^* \zeta} n.
\end{aligned}$$

It is obvious that (3.1) holds in the case where the following inequality holds.

$$\alpha^* \mathcal{K}_+'(\zeta) \ge \eth_2 \mathcal{K}_+''(\zeta) - \varphi \varpi \zeta e^{\sigma^* \zeta} m - \varphi \varpi e^{\sigma^* \zeta} n - \gamma \mathcal{K}_+(\zeta).$$

It follows that

$$\mathcal{K}_+'(\zeta) = -\varpi e^{\sigma^* \zeta}\bigl(1 + \sigma^* \zeta\bigr),$$
$$\mathcal{K}_+''(\zeta) = -\varpi e^{\sigma^* \zeta}\bigl(2\sigma^* + \bigl(\sigma^*\bigr)^2 \zeta\bigr)$$

for any $\zeta < \zeta_2$ by Lemma 2.2 and the definition of $I_+(\zeta)$.

So

$$\begin{aligned}&\alpha^*\mathcal{K}_+'(\zeta) - \mathfrak{d}_2\mathcal{K}_+''(\zeta) + \varphi\varpi\zeta e^{\sigma^*\zeta}m + \varphi\varpi e^{\sigma^*\zeta}n + \gamma\mathcal{K}_+(\zeta)\\&\quad = \Lambda_\lambda(\sigma^*,\alpha^*)\varpi e^{\sigma^*\zeta} + \Lambda(\sigma^*,\alpha^*)\varpi\zeta e^{\sigma^*\zeta} = 0,\end{aligned}$$

which yields (3.1).

Put $\zeta > \zeta_2$. Then

$$\frac{\varphi\mathcal{J}_+(\zeta)(\mathcal{I}*\mathcal{K}_+)(\zeta)}{\mathcal{J}_+(\zeta) + (\mathcal{I}*\mathcal{K}_+)(\zeta)} \le \frac{\varphi\mathcal{J}_0(\frac{\varphi}{\gamma}-1)\mathcal{J}_0}{\mathcal{J}_0 + (\frac{\varphi}{\gamma}-1)\mathcal{J}_0} = \gamma\left(\frac{\varphi}{\gamma}-1\right)\mathcal{J}_0.$$

Denote

$$p = \mathcal{J}_0 e^{\tau_1(K-\zeta^*)} + \sup_{\zeta<0}\frac{-\varphi\varpi e^{(\sigma^*-\tau_1)\zeta}(\zeta m + n)}{\alpha^*\tau_1 - \mathfrak{d}_1\tau_1^2}$$

and

$$\mathcal{J}_0 - pe^{\lambda_3\zeta} = \epsilon e^{-\lambda_4\zeta}.$$

If $\zeta < \zeta_1$, then

$$\mathcal{J}_-(\zeta) = \mathcal{J}_0 - pe^{\tau_1\zeta} > 0$$

and

$$\alpha^*\mathcal{J}_-'(\zeta) \le \mathfrak{d}_1\mathcal{J}_-''(\zeta) - \varphi(\mathcal{I}*I_+)(\zeta),$$

where $\zeta < \zeta_1$.

It follows that

$$\begin{aligned}(\mathcal{I}*\mathcal{K}_+)(\zeta) &= \int_0^\infty\int_{\mathbb{R}}\mathcal{I}(t,s)I_+(\zeta - t - \alpha^*s)\,dt\,ds\\&\le \int_0^\infty\int_{\zeta-\zeta^*-\alpha^*s}^{+\infty}\mathcal{I}(t,s)I_+(\zeta - t - \alpha^*s)\,dt\,ds\\&= -\varpi\zeta e^{\sigma^*\zeta}m - \varpi e^{\sigma^*\zeta}n\end{aligned}$$

since $\zeta < \zeta_1 \le \zeta^* - \mathcal{K}$.

To verify that

$$\alpha^*\mathcal{J}_-'(\zeta) \le \mathfrak{d}_1\mathcal{J}_-''(\zeta) + \varphi\varpi\zeta e^{\sigma^*\zeta}m + \varphi\varpi e^{\sigma^*\zeta}n,$$

by simple calculations, we have

$$-\alpha^*\tau_1 pe^{\lambda_3\zeta} \le -\mathfrak{d}_1\lambda_3^2 pe^{\lambda_3\zeta} + \varphi\varpi\zeta e^{\sigma^*\zeta}m + \varphi\varpi e^{\sigma^*\zeta}n, \quad \zeta < \zeta_1,$$

which is true by the definition of p.

It suffices to confirm that

$$\alpha^* \mathcal{J}_-'(\zeta) \leq \mathfrak{d}_1 \mathcal{J}_-''(\zeta) - \varphi \mathcal{J}_-(\zeta),$$

which is equivalent to

$$-\alpha^* \lambda_4 \epsilon e^{-\lambda_4 \zeta} \leq \mathfrak{d}_1 \lambda_4^2 \epsilon e^{-\lambda_4 \zeta} - \varphi \epsilon e^{-\lambda_4 \zeta},$$

and this is also evident by the definition of λ_4.

Let $L \geq M_1 \geq \varpi \sqrt{-\zeta_2}$ be such that

$$\mathcal{J}_-(\zeta) \geq \mathcal{J}_0/2,$$

$$\zeta < \zeta_3 := -\left(\frac{L}{\varpi}\right)^2 \leq \zeta_2$$

for $\varpi > 0$ and $\zeta_2 < 0$.

It is obvious that the definition of ζ_3 implies that

$$\mathcal{K}_-(\zeta) \leq \mathcal{K}_+(\zeta)$$

for all $\zeta \in \mathbb{R}$.

If $\zeta < \zeta_3$, then

$$\mathcal{K}_-(\zeta) = -\varpi \zeta e^{\lambda^* \zeta} - L(-\zeta)^{1/2} e^{\lambda^* \zeta} \leq -\varpi \zeta e^{\lambda^* \zeta} = \mathcal{K}_+(\zeta),$$

which yields that

$$\begin{aligned}
(\mathcal{I} * \mathcal{K}_-)(\zeta) &= \int_0^\infty \int_{\mathbb{R}} \mathcal{I}(t,s) \mathcal{K}_-\bigl(\zeta - t - c^* s\bigr)\, dt\, ds \\
&\leq \int_0^\infty \int_{\mathbb{R}} \mathcal{I}(t,s) \mathcal{K}_+\bigl(\zeta - t - c^* s\bigr)\, dt\, ds \\
&= -\varpi \zeta e^{\lambda^* \zeta} m - \varpi e^{\lambda^* \zeta} n.
\end{aligned}$$

So

$$\begin{aligned}
&\frac{\varphi \mathcal{J}_-(\zeta)(\mathcal{I} * \mathcal{K}_-)(\zeta)}{\mathcal{J}_-(\zeta) + (\mathcal{I} * \mathcal{K}_-)(\zeta)} - \varphi(\mathcal{I} * \mathcal{K}_-)(\zeta) \\
&\quad \geq \frac{\varphi \frac{\mathcal{J}_0}{2}(\mathcal{I} * \mathcal{K}_-)(\zeta)}{\frac{\mathcal{J}_0}{2} + (\mathcal{I} * \mathcal{K}_-)(\zeta)} - \varphi(\mathcal{I} * \mathcal{K}_-)(\zeta) \\
&\quad \geq -\frac{2\varphi}{\mathcal{J}_0}\bigl[(\mathcal{I} * \mathcal{K}_-)(\zeta)\bigr]^2 \\
&\quad \geq -\frac{2\varphi \varpi^2}{\mathcal{J}_0} e^{2\lambda^* \zeta} (\zeta m + n)^2,
\end{aligned}$$

which gives that

$$c^* \mathcal{K}_-'(\zeta) \leq \mathfrak{d}_2 \mathcal{K}_-''(\zeta) + \varphi(\mathcal{I} * \mathcal{K}_-)(\zeta) - \gamma \mathcal{K}_-(\zeta) - \frac{2\varphi \varpi^2}{\mathcal{J}_0} e^{2\lambda^* \zeta} (\zeta m + n)^2.$$

By a simple calculation we have that

$$\mathcal{K}_-'(\zeta) = \mathcal{K}_+'(\zeta) + Le^{\sigma^*\zeta}\left[\frac{1}{2}(-\zeta)^{-1/2} - \sigma^*(-\zeta)^{1/2}\right],$$
$$\mathcal{K}_-''(\zeta) = \mathcal{K}_+''(\zeta) + Le^{\sigma^*\zeta}\left[\frac{1}{4}(-\zeta)^{-3/2} + \sigma^*(-\zeta)^{-1/2} - (\sigma^*)^2(-\zeta)^{1/2}\right].$$

Since $-\zeta + y + \alpha^* s \geq 0$ for any $\zeta < \zeta_3$, $|y| < K$, and $s \geq 0$, there exists $\theta \in (0,1)$ such that

$$\begin{aligned}
&\left[-\zeta + (t + \alpha^* s)\right]^{1/2} \\
&\quad = (-\zeta)^{1/2} + \frac{1}{2}(-\zeta)^{-1/2}(t + \alpha^* s) - \frac{1}{8}\left[-\zeta + \theta(t + \alpha^* s)\right]^{-3/2}(t + \alpha^* s)^2 \\
&\quad \leq (-\zeta)^{1/2} + \frac{1}{2}(-\zeta)^{-1/2}(t + \alpha^* s)
\end{aligned}$$

by applying the Taylor theorem (see [12, 31]), which yields that

$$\begin{aligned}
(\mathcal{I} * \mathcal{K}_-)(\zeta) &= \int_0^\infty \int_{\mathbb{R}} \mathcal{I}(t,s) I_-(\zeta - t - \alpha^* s)\,dt\,ds \\
&= \int_0^\infty \int_{-K}^{K} \mathcal{I}(t,s) I_-(\zeta - t - \alpha^* s)\,dt\,ds \\
&\geq \int_0^\infty \int_{-K}^{K} \mathcal{I}(t,s)\big[-\varpi(\zeta - t - \alpha^* s)e^{\sigma^*(\zeta - t - \alpha^* s)} \\
&\quad - L\big(-(\zeta - t - \alpha^* s)\big)^{1/2} e^{\sigma^*(\zeta - t - \alpha^* s)}\big]\,dt\,ds \\
&\geq -\varpi \int_0^\infty \int_{-K}^{K} \mathcal{I}(t,s)(\zeta - t - \alpha^* s)e^{\sigma^*(\zeta - t - \alpha^* s)}\,dt\,ds \\
&\quad - L\int_0^\infty \int_{-K}^{K} \mathcal{I}(t,s)\left[(-\zeta)^{1/2} + \frac{1}{2}(-\zeta)^{-1/2}(t + \alpha^* s)\right]e^{\sigma^*(\zeta - t - \alpha^* s)}\,dt\,ds \\
&= -\varpi\zeta e^{\sigma^*\zeta} m - \varpi e^{\sigma^*\zeta} n - L(-\zeta)^{1/2} e^{\sigma^*\zeta} m + \frac{1}{2}L(-\zeta)^{-1/2} e^{\sigma^*\zeta} n.
\end{aligned}$$

So

$$\begin{aligned}
&\alpha^*\mathcal{K}_+'(\zeta) + L\alpha^* e^{\sigma^*\zeta}\left[\frac{1}{2}(-\zeta)^{-1/2} - \sigma^*(-\zeta)^{1/2}\right] \\
&\quad \leq \mathfrak{d}_2\mathcal{K}_+''(\zeta) + \mathfrak{d}_2 Le^{\sigma^*\zeta}\left[\frac{1}{4}(-\zeta)^{-3/2} + \sigma^*(-\zeta)^{-1/2} - (\sigma^*)^2(-\zeta)^{1/2}\right] \\
&\qquad - \varphi\varpi\zeta e^{\sigma^*\zeta} m - \varphi\varpi e^{\sigma^*\zeta} n - \varphi L(-\zeta)^{1/2} e^{\sigma^*\zeta} m + \frac{\varphi}{2} L(-\zeta)^{-1/2} e^{\sigma^*\zeta} n \\
&\qquad - \gamma\mathcal{K}_+(\zeta) + \gamma L(-\zeta)^{1/2} e^{\sigma^*\zeta} - \frac{2\varphi\varpi^2}{\mathcal{J}_0} e^{2\sigma^*\zeta}(\zeta m + n)^2,
\end{aligned}$$

which is true provided that

$$\mathfrak{d}_2 Le^{\sigma^*\zeta}\frac{1}{4}(-\zeta)^{-3/2} - \frac{2\varphi\varpi^2}{\mathcal{J}_0} e^{2\sigma^*\zeta}(\zeta m + n)^2 \geq 0.$$

Put

$$M_2 := \sup_{\zeta<0} \frac{8\varphi\varpi^2(\zeta m+n)^2(-\zeta)^{3/2}e^{\sigma^*\zeta}}{\mathfrak{d}_2\mathcal{J}_0} + 1$$

for any $\zeta < \zeta_3$, where $L := M_1 + M_2$. When $\zeta > \zeta_3$, it is straightforward to show the required result. The proof is complete. □

4 Conclusions

In this paper, we further studied the Phragmén–Lindelöf method related to a second-order order boundary value problem with respect to the Schrödinger operator. We also presented some mathematical consequences of the method including a stability result. The main technical tools used to develop the mathematical analysis are local and global bifurcation, monotonicity techniques, the augmented Phragmén–Lindelöf method, blow-up arguments, and some techniques used in the previous works. As an application, we proved the uniqueness of a solution for the definite problem of a parabolic variational inequality.

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