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Existence of solutions for a nonhomogeneous Dirichlet problem involving $p(x)$-Laplacian operator and indefinite weight

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Abstract

We obtain multiplicity and uniqueness results in the weak sense for the following nonhomogeneous quasilinear equation involving the $p(x)$-Laplacian operator with Dirichlet boundary condition:

$$-\Delta_{p(x)}u + V(x)|u|^{q(x)-2}u = f(x,u) \quad \text{in } \Omega, u = 0 \text{ on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, V is a given function with an indefinite sign in a suitable variable exponent Lebesgue space, $f(x,t)$ is a Carathéodory function satisfying some growth conditions. Depending on the assumptions, the solutions set may consist of a bounded infinite sequence of solutions or a unique one. Our technique is based on a symmetric version of the mountain pass theorem.

MSC: 35B38; 35J20; 35J60; 35J66

Keywords: Generalized Lebesgue–Sobolev spaces; $p(x)$-Laplacian operator; Symmetric mountain pass lemma

1 Introduction

In this work, we study the existence of solutions for the following nonlinear Dirichlet problem involving the $p(x)$-Laplacian operator:

$$\begin{cases} -\Delta_{p(x)}u + V(x)|u|^{q(x)-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $p, q, s : \overline{\Omega} \to \mathbb{R}^+$ are continuous functions, $V \in L^{s(x)}(\Omega)$ has an indefinite sign, and $f(x,t)$ is a Carathéodory function. Let us recall that the $p(x)$-Laplacian operator $\Delta_{p(x)}$ is defined by

$$\Delta_{p(x)}u = \operatorname{div}\bigl(|\nabla u|^{p(x)-2}\nabla u\bigr)$$

and is an extension of the classical p-Laplacian operator obtained in the case where the function $p(x)$ is just a positive constant p. A growing interest in the study of the $p(x)$-Laplacian operator has arisen during the last two decades, in regard to its involvement in the modelings of a large number of phenomena. One can name for instance electrorheological fluids [30, 32, 36], elastic mechanics, flows in porous media and image processing [11], curl systems emanating from electromagnetism [4, 7]. For more details, we refer to Acerbi and Mingione [2] and Růžička [32] about electrorheological fluids, and to Antontsev and Shmarev [5] about nonlinear Darcy's law in porous media. Because of its nonhomogeneous feature, the $p(x)$-Laplacian operator is reasonably expected to be appropriate for modeling nonhomogeneous materials. Recently, several works devoted to existence problems for p-Laplacian operator have been naturally generalized to the case of $p(x)$-Laplacian operator. In relevance with the present work are the papers in [1, 8, 9, 16, 17, 20, 21, 25, 27, 29] and the references therein. To the best of our knowledge, the earliest paper that deals with the variable exponent case with $V \equiv 0$ in (1.1) is the article by Fan and Zhang (cf. [22]). Indeed, those authors studying the problem

$$\begin{cases} -\Delta_{p(x)} u = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

proved by the mountain pass theorem and by the fountain theorem that problem (1.2) admits respectively at least one nontrivial weak solution and infinitely many pairs of weak solutions. In an earlier work, Dinca et al. (cf. [17]), using variational and topological methods, proved the existence and multiplicity of weak solutions for problem (1.2) for the case $p(x) \equiv p$ (p is a constant). Their main tool was the mountain pass theorem of Ambrosetti and Rabinowitz (cf. [3]). The approach for dealing with the case where $V > 0$ is technically similar to that of $V = 0$. In [8, 9], the authors considered problem (1.1) with V bounded and $p(\cdot) = q(\cdot)$ and proved the existence of nonnegative solutions using a mountain pass theorem. Most of the results are derived from a $\beta(x)$-growth assumption on the nonlinearity $f(x,t)$ with additional assumptions preventing the range of $p(\cdot)$ to interfere with that of $\beta(\cdot)$. Motivated by the works in [8, 9, 17, 20, 22, 25, 27], we study here the nonlinear Dirichlet problem (1.1) when the function $V(\cdot)$ has an indefinite sign and belongs to the generalized Lebesgue space $L^{s(x)}(\Omega)$. We assume furthermore that $p(\cdot) \neq q(\cdot)$ and the nonlinearity $f(x,t)$ satisfies a $\beta(x)$-growth assumption. The interest in this work is twofold: the weight V is not bounded and may change sign, and besides the range of $p(\cdot)$ may interfere with that of $q(\cdot)$ or $\beta(\cdot)$ as well. To the best of our knowledge, our setting is more general than those of [8, 9, 22] and our method contrasts with other treatments of (1.1). The functional framework is the generalized Lebesgue and Sobolev spaces with variable exponents and our technique is based on a $\mathbb{Z}_2$ symmetric version of the mountain pass theorem.

The remainder of this paper is organized as follows. In Section 2, we introduce some technical results and formulate the required hypotheses on (1.1). Sections 3 and 4 are devoted to the statement of our main results along with some auxiliary results and to their proofs. In the appendix we present the appropriate version of the mountain pass theorem related to our problem.

2 Preliminaries and hypotheses

In order to study problem (1.1), some of the properties on variable exponent Lebesgue spaces and Sobolev spaces, $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ respectively, are required and listed below. We refer to [15, 23, 26] for exhaustive details on properties of those spaces.

Suppose that Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Let us denote

$$L_+^\infty(\Omega) = \Bigl\{p \in L^\infty(\Omega) : \operatorname*{ess\,inf}_{\Omega} p(x) \geq 1\Bigr\},$$

$$C_+(\overline{\Omega}) = \bigl\{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for every } x \in \overline{\Omega}\bigr\},$$

$$p^- = \min_{x\in\overline{\Omega}} p(x), \qquad p^+ = \max_{x\in\overline{\Omega}} p(x) \quad \text{for } p \in C_+(\overline{\Omega}),$$

$$M = \{u : \Omega \to \mathbb{R} : u \text{ is a measurable real-valued function}\}.$$

Definition 2.1 For any $p \in L_+^\infty(\Omega)$, the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \biggl\{u \in M : \int_\Omega |u|^{p(x)}\,dx < +\infty\biggr\}.$$

For any $u \in L^{p(x)}(\Omega)$, we define the so-called Luxemburg norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf\biggl\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1\biggr\}.$$

Throughout this paper, $L^{p(x)}(\Omega)$ will be endowed with this norm.

The modular, which is the mapping $\varrho_{p(\cdot)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx,$$

is at many aspects an important tool in studying generalized Lebesgue–Sobolev spaces.

Remark 2.2 Variable exponent Lebesgue spaces have many properties similar to those of classical Lebesgue spaces, namely they are separable Banach spaces and the Hölder inequality holds. The inclusions between Lebesgue spaces are also naturally generalized, that is, if $0 < \operatorname{mes}(\Omega) < \infty$ and p, q are variable exponents such that $p(x) < q(x)$ a.e. in Ω, then there exists a continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

We recall below some statements whose details can be found in [18, 20, 23].

Let us denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. There is a counterpart of the Hölder inequality for variable exponent Lebesgue spaces when $p \in L_+^\infty(\Omega)$ in the literature (cf. [18]). We give below a version relevant to the need of this work.

Proposition 2.3 (Hölder inequality)

$$\int_\Omega |uv|\,dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{p'(\cdot)}$$

for all $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$, *and* $p, q \in C_+(\overline{\Omega})$.

Proposition 2.4 *For $p \in C_+(\overline{\Omega})$, we have the following*:

$$\min\left(|u|_{p(\cdot)}^{p^-}, |u|_{p(\cdot)}^{p^+}\right) \leq \varrho_{p(\cdot)}(u) \leq \max\left(|u|_{p(\cdot)}^{p^-}, |u|_{p(\cdot)}^{p^+}\right).$$

It is worth noticing that this relation between the norm and the modular shows an equivalence between the topology defined by the norm and that defined by the modular.

Proposition 2.5 *Let p and q be measurable functions such that $p \in C_+(\overline{\Omega})$ and $pq \in L_+^{\infty}(\Omega)$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\begin{aligned} |u|_{p(\cdot)q(\cdot)} \leq 1 \quad &\Rightarrow \quad |u|_{p(\cdot)q(\cdot)}^{p^+} \leq \left||u|^{p(\cdot)}\right|_{q(\cdot)} \leq |u|_{p(\cdot)q(\cdot)}^{p^-}, \\ |u|_{p(\cdot)q(\cdot)} \geq 1 \quad &\Rightarrow \quad |u|_{p(\cdot)q(\cdot)}^{p^-} \leq \left||u|^{p(\cdot)}\right|_{q(\cdot)} \leq |u|_{p(\cdot)q(\cdot)}^{p^+}. \end{aligned}$$

In particular, if $p(x) = p$ is a constant, then

$$\left||u|^{p}\right|_{q(\cdot)} = |u|_{pq(\cdot)}^{p}.$$

Definition 2.6 The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)} + |u|_{p(\cdot)},$$

where $|\nabla u| = \sqrt{\sum_{i=1}^{N}(\frac{\partial u}{\partial x_i})^2}$.

Proposition 2.7 *$L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable Banach spaces when $p \in L_+^{\infty}(\Omega)$, reflexive and uniformly convex for $p \in L^{\infty}(\Omega)$ and* $\operatorname{ess\,inf}_{\Omega} p(x) > 1$.

Definition 2.8 For $p \in C_+(\overline{\Omega})$, let us define the so-called critical Sobolev exponent of p by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

for every $x \in \overline{\Omega}$.

We also define the space $W_0^{1,p(x)}(\Omega)$ as the closure of the space $C_0^{\infty}(\Omega)$ (C^{∞}-functions with compact support in Ω) in the space $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$.

The dual space of $W_0^{1,p(x)}(\Omega)$ is denoted by $W^{-1,p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for every $x \in \overline{\Omega}$.

With respect to those spaces, we recall from [15, 23] the following.

Proposition 2.9

(i) *$W_0^{1,p(x)}(\Omega)$ is a separable Banach space when $p \in L_+^\infty(\Omega)$, reflexive and uniformly convex when $p \in L^\infty(\Omega)$ and $\operatorname{ess\,inf}_\Omega p(x) > 1$.*

Assume that $p, q \in C_+(\overline{\Omega})$. Then

(ii) *if $p(x) < N$ and $q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for every $x \in \overline{\Omega}$, then there is a compact and continuous embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega);$$

(iii) *if $p(x) > N$ for every $x \in \overline{\Omega}$, then*

$$W^{1,p(x)(\Omega) \hookrightarrow L^\infty(\Omega)};$$

(iv) *there is a constant $C > 0$ such that*

$$|u|_{p()} \leq C\|\nabla u\|_{p()}, \quad \textit{for all } u \in W_0^{1,p(x)}(\Omega).$$

Remark 2.10 Using estimate (iv) of Proposition 2.9, we derive that the norm $\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)} + |u|_{p(\cdot)}$ is equivalent to the norm $\|u\| = \|\nabla u\|_{p(\cdot)}$ in $W_0^{1,p(x)}(\Omega)$. Here and henceforth, we will consider the space $W_0^{1,p(x)}(\Omega)$ equipped with the norm $\|u\| = \|\nabla u\|_{p(\cdot)}$.

Moreover, one can prove (cf. [15]) that the norm $\|u\|$ is weakly sequentially lower semicontinuous.

Remark 2.11 If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for every $x \in \overline{\Omega}$, then the embedding of $W_0^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ is compact.

As in the case $p(x) \equiv p$ (a constant), we consider the $p(x)$-Laplacian operator

$$-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$$

defined by

$$\langle -\Delta_{p(x)} u, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx \quad \text{for all } u, v \in W_0^{1,p(x)}(\Omega),$$

where $\langle\, , \rangle$ denotes the duality pairing between $W_0^{1,p(x)}(\Omega)$ and $W^{-1,p'(x)}(\Omega)$.

We have the following properties (cf. [10, 22]).

Proposition 2.12

(i) *$-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a homeomorphism.*

(ii) *$-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a strictly monotone operator, that is,*

$$-\langle \Delta_{p(x)} u - \Delta_{p(x)} v, u - v\rangle > 0 \quad \textit{for all } u \neq v \in W_0^{1,p(x)}(\Omega).$$

(iii) *$-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a mapping of type (S_+), that is,*

$$\textit{if } u_n \rightharpoonup u \quad \textit{in } W_0^{1,p(x)}(\Omega) \quad \textit{and} \quad \limsup_{n\to\infty}\langle -\Delta_{p(x)} u_n, u_n - u\rangle \leq 0,$$

then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

Proposition 2.13 *The functional* $\Psi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ *defined by*

$$\Psi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx$$

is continuously Fréchet differentiable, sequentially weakly lower semicontinuous, and $\Psi'(u) = -\Delta_{p(x)} u$ *for all* $u \in W_0^{1,p(x)}(\Omega)$.

Note that, if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $u \in M$, then the function $N_f u : \Omega \to \mathbb{R}$ defined by $(N_f u)(x) = f(x, u(x))$ for $x \in \Omega$ is measurable in Ω. Thus, the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ generates an operator $N_f : M \to M$, which is called the Nemytskii operator. The properties of N_f are recalled through the propositions below (see [35] for details).

Proposition 2.14 *Suppose that* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function satisfying the following growth condition*:

$$\bigl|f(x,t)\bigr| \le c|t|^{\frac{\alpha(x)}{\beta(x)}} + h(x) \quad \textit{for every } x \in \Omega, t \in \mathbb{R},$$

where $\alpha, \beta \in C_+(\overline{\Omega})$, $c \ge 0$ *is constant, and* $h \in L^{\beta(x)}(\Omega)$. *Then* $N_f(L^{\alpha(x)}(\Omega)) \subseteq L^{\beta(x)}(\Omega)$. *Moreover,* N_f *is continuous from* $L^{\alpha(x)}(\Omega)$ *into* $L^{\beta(x)}(\Omega)$ *and maps a bounded set into a bounded set.*

Proposition 2.15 *Suppose that* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function satisfying the following growth condition*:

$$\bigl|f(x,t)\bigr| \le c|t|^{\alpha(x)-1} + h(x) \quad \textit{for every } x \in \Omega, t \in \mathbb{R},$$

where $c \ge 0$ *is constant,* $\alpha \in C_+(\overline{\Omega})$, $h \in L^{\alpha'(x)}(\Omega)$ *with* α' *the conjugate exponent of* α, *i.e.,* $\alpha'(x) = \frac{\alpha(x)}{\alpha(x)-1}$. *Let* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$F(x,t) = \int_0^t f(x,s)\, ds.$$

Then

(i) *F is a Carathéodory function and there exist a constant* $c_1 \ge 0$ *and* $\sigma \in L^1(\Omega)$ *such that*

$$\bigl|F(x,t)\bigr| \le c_1|t|^{\alpha(x)} + \sigma(x) \quad \textit{for all } x \in \Omega, t \in \mathbb{R}.$$

(ii) *The functional* $\Phi : L^{\alpha(x)}(\Omega) \to \mathbb{R}$ *defined by* $\Phi(u) = \int_\Omega F(x, u(x))\, dx$ *is continuously Fréchet differentiable and* $\Phi'(u) = N_f(u)$ *for all* $u \in L^{\alpha(x)}(\Omega)$.

Remark 2.16 Since the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ is compact for $\alpha \in C_+(\overline{\Omega})$ with $\alpha(x) < p^*(x)$ for every $x \in \overline{\Omega}$, we derive from Proposition 2.15 the following diagram:

$$W_0^{1,p(x)}(\Omega) \overset{I}{\hookrightarrow} L^{\alpha(x)}(\Omega) \overset{N_f}{\to} L^{\alpha'(x)}(\Omega) \overset{I^*}{\hookrightarrow} W^{-1,p'(x)}(\Omega),$$

which shows that $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is strongly continuous on $W_0^{1,p(x)}(\Omega)$. Using the same argument, the functional $\Phi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by $\Phi(u) = \int_\Omega F(x,u(x))\,dx$ is also strongly continuous on $W_0^{1,p(x)}(\Omega)$.

Throughout this work, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $V \in L^{s(x)}(\Omega)$. Let us denote

$$\Omega_- = \{x \in \Omega, V(x) < 0\}.$$

We require the following assumptions on problem (1.1):

(A1) p, q, s and $\beta \in C_+(\overline{\Omega})$ such that, for all $x \in \overline{\Omega}$,

$$q(x) < p(x) \le N, \qquad \beta(x) < p(x) \le N, \qquad N < s(x) \quad \text{for } V \in L^{s(x)}(\Omega). \tag{2.1}$$

(A1′) Moreover, we assume that

$$q^+_{\Omega_-} = \sup_{\Omega_-} q(x) < p^- \quad \text{and} \quad \beta^+ < p^-. \tag{2.2}$$

(A2)

$$\left|f(x,t)\right| \le c|t|^{\beta(x)-1} + h(x) \quad \text{for all } x \in \Omega, t \in \mathbb{R}, \tag{2.3}$$

where $c \ge 0$ is constant, $\beta \in C_+(\overline{\Omega})$ with $\beta(x) < p^*(x)$ for every $x \in \overline{\Omega}$, and $h \in L^\infty(\Omega)$.

(A3) There exist $\gamma \in L^\infty(\Omega)$, $\gamma > 0$, and θ satisfying $\theta < p^-$ such that

$$F(x,t) \ge \gamma(x)|t|^\theta \quad \text{for all } x \in \Omega \text{ and any } t \in [0,1[, \tag{2.4}$$

where $F(x,s) = \int_0^s f(x,t)\,dt$.

(A4) $f(x,-t) = -f(x,t)$ for $x \in \Omega$, $t \in \mathbb{R}$.

3 Main results

In this section, we give some auxiliary results prior to the establishment of our main results. Here and henceforth, we denote by X the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$ equipped with the norm $\|\cdot\|$, X^* its dual space, and we define the continuous function α by

$$\alpha(x) = \frac{s(x)q(x)}{s(x) - q(x)}. \tag{3.1}$$

From assumptions (A1) on the functions p, q, s and from (3.1), a straightforward computation gives $q(x) < p^*(x)$, $\beta(x) < p^*(x)$, $s'(x)q(x) < p^*(x)$, $\alpha(x) < p^*(x)$ for every $x \in \overline{\Omega}$. Hence, we have the following remark.

Remark 3.1 From (ii) of Proposition 2.9, the embeddings $X \hookrightarrow L^{q(x)}(\Omega)$, $X \hookrightarrow L^{\alpha(x)}(\Omega)$, $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$, and $X \hookrightarrow L^{\beta(x)}(\Omega)$ are compact and continuous. Therefore, there exists a positive constant C such that

$$|u|_{q(\cdot)} \leq C\|u\|, \qquad |u|_{\alpha(\cdot)} \leq C\|u\|, \qquad |u|_{s'(\cdot)q(\cdot)} \leq C\|u\|, \qquad |u|_{\beta(\cdot)} \leq C\|u\| \tag{3.2}$$

for all $u \in X$. Without any loss of generality, we can suppose that $C > 1$.

We are interested in the investigation of weak solution of (1.1), say a function $u \in X$ satisfying

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx + \int_\Omega V(x)|u|^{q(x)-2}uv \, dx = \int_\Omega f(x,u)v \, dx, \quad \forall v \in X. \tag{3.3}$$

Let us consider the Euler–Lagrange functional or the energy functional $H : X \to \mathbb{R}$ associated with problem (1.1) defined by

$$H(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \, dx + \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)} \, dx - \int_\Omega F(x,u) \, dx.$$

Let us denote $J(u) = \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)} \, dx$.

Then the energy functional H can be written as

$$H(u) = \Psi(u) + J(u) - \Phi(u),$$

where we recall that

$$\Psi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \, dx \quad \text{and} \quad \Phi(u) = \int_\Omega F(x,u) \, dx.$$

The functional H is obviously well defined and satisfies the following.

Proposition 3.2 *The functional H is continuously Fréchet differentiable and is weakly lower semicontinuous. Moreover, $u \in X$ is a critical point of H if and only if u is a weak solution of* (1.1).

Proof By Proposition 2.13 and Proposition 2.15, we have that the functional $H \in C^1(X,\mathbb{R})$ and its derivative function is given by

$$\langle dH(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx + \int_\Omega V(x)|u|^{q(x)-2}uv \, dx - \int_\Omega f(x,u)v \, dx$$

for all $u, v \in X$. Now, let u be a critical point of H, then we have $dH(u) = 0_{X^*}$, which implies that

$$\langle dH(u), v\rangle = 0 \quad \text{for all } v \in X.$$

Consequently,

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx + \int_\Omega V(x)|u|^{q(x)-2}uv \, dx = \int_\Omega f(x,u)v \, dx, \quad \forall v \in X.$$

It follows that u is a weak solution of (1.1). On the other hand, if u is a weak solution of (1.1), by definition, we have

$$\int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int_{\Omega} V(x)|u|^{q(x)-2}uv\,dx = \int_{\Omega} f(x,u)v\,dx, \quad \forall v \in X,$$

which implies that

$$\langle dH(u), v\rangle = 0 \quad \text{for all } v \in X.$$

So, $dH(u) = 0_{X^*}$ and hence u is a critical point of H.

Let us show that H is sequentially weakly lower semicontinuous.

Consider a weakly convergent sequence (u_n) to u in X, we have $\Psi(u) \leq \liminf_{n\to+\infty} \Psi(u_n)$. By using the Hölder inequality, Proposition 2.5, and inequality (3.2), for any $n \in \mathbb{N}$, we deduce that

$$\begin{aligned}\int_{\Omega} \frac{1}{q(x)}\bigl|V(x)\bigr||u_n|^{q(x)}\,dx &\leq \frac{2}{q^-}|V|_{s(x)} \max\bigl\{|u_n|^{q^-}_{s'(x)q(x)}, |u_n|^{q^+}_{s'(x)q(x)}\bigr\} \\ &\leq \frac{2}{q^-}|V|_{s(x)} \max\bigl\{C^{q^-}\|u_n\|^{q^-}, C^{q^+}\|u_n\|^{q^+}\bigr\}.\end{aligned} \tag{3.4}$$

Recalling assumption (A2) and integrating the growth condition with respect to t, we obtain

$$\left|\int_{\Omega} F(x,u)\,dx\right| \leq a\varrho_{\beta(\cdot)}(u) + |h|_{\infty}|u|, \tag{3.5}$$

where $a \geq 0$ is a constant depending on the real numbers c, β^- provided by (A2). Recalling the compact embeddings $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{\beta(x)}(\Omega)$, we deduce that $J(u_n) \to J(u)$ and $\Phi(u) \leq \liminf_{n\to+\infty} \Phi(u_n)$; and hence, H is sequentially weakly lower semicontinuous. The proof is complete. □

Next, we have the following.

Proposition 3.3 *Under assumptions* (A1), (A1′), (A2), *the energy functional* (H) *is coercive.*

Proof Suppose that $\|u\| > 1$, then

$$\begin{aligned}H(u) &= \int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\Omega} \frac{V(x)}{q(x)}|u|^{q(x)}\,dx - \int_{\Omega} F(x,u)\,dx \\ &\geq \frac{1}{p^+}\int_{\Omega} |\nabla u|^{p(x)}\,dx - \frac{1}{q^-}\int_{\Omega_-} V(x)|u|^{q(x)}\,dx - aC^{\beta^+}\|u\|^{\beta^+} - b\|u\| \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{1}{q^-}K^{q^+_{\Omega_-}}\|u\|^{q^+_{\Omega_-}} - aC^{\beta^+}\|u\|^{\beta^+} - b\|u\|,\end{aligned} \tag{3.6}$$

where K, C, and b are positive constants depending on the Sobolev embeddings V and F. Since $q^+_{\Omega_-} < p^-$ and $\beta^+ < p^-$, we get that H is coercive. □

We are now ready to state the first existence result of this work.

Theorem 3.4 *Under assumptions* (A1), (A1′), *and* (A2), *problem* (1.1) *has a weak solution*.

Proof Since H is differentiable, coercive, weakly lower semicontinuous, H has a critical minimum point u in X which is a weak solution of (1.1) and then the proof is complete. □

Remark 3.5 Notice that when V is positive, the first condition in (A1′) is not necessary and the proof of the coercivity of H is of course trivial.

Corollary 3.6 *Suppose that hypotheses of Theorem* 3.4 *are satisfied*. *If besides* $V \in L^{s(x)}(\Omega)$ *is a nonnegative function and* $f(x,\cdot): \mathbb{R} \to \mathbb{R}$ *defined by* $f(x,\cdot)(t) = f(x,t)$ *is a decreasing function for a*.*e*. $x \in \Omega$, *then problem* (1.1) *has a unique weak solution in* X.

Proof Let $u_1, u_2 \in E$ such that

$$\langle dH(u_1), v\rangle = \langle dH(u_2), v\rangle = 0 \quad \text{for all } v \in X,$$

then we have

$$\langle dH(u_1) - dH(u_2), u_1 - u_2\rangle = 0. \tag{3.7}$$

From the strict monotonicity of $-\Delta_{p(x)}$, we have

$$\begin{aligned} 0 &\le -\langle \Delta_{p(x)} u_1 - \Delta_{p(x)} u_2, u_1 - u_2\rangle \\ &= \int_\Omega \bigl(f\bigl(x,u_1(x)\bigr) - f\bigl(x,u_2(x)\bigr)\bigr)\bigl(u_1(x) - u_2(x)\bigr)\,dx \\ &\quad - \int_\Omega V(x)\bigl(\bigl|u_1(x)\bigr|^{q(x)-2}u_1(x) - \bigl|u_2(x)\bigr|^{q(x)-2}u_2(x)\bigr)\bigl(u_1(x) - u_2(x)\bigr)\,dx. \end{aligned} \tag{3.8}$$

Since the potential V is nonnegative and the function $f(x,\cdot)$ is decreasing for a.e. $x \in \Omega$, the right-hand side of (3.8) is nonpositive, and then we get

$$-\langle \Delta_{p(x)} u_1 - \Delta_{p(x)} u_2, u_1 - u_2\rangle = 0.$$

Recalling again the strict monotonicity of $-\Delta_{p(x)}$, we get $u_1 = u_2$. □

Remark 3.7 In addition to the assumptions in Corollary 3.6, if the Carathéodory function $f: \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (A4), then $u \equiv 0$ is a trivial solution, and since the solution is unique when it exists, then $u \equiv 0$ is the only one solution in X for Dirichlet problem (1.1).

In the next step, we will suppose that condition (A1′) is no longer satisfied and of course the coercivity of the functional H fails since V is a sign-changing function. In this case, we prove that H satisfies the Palais–Smale (PS) condition and show multiplicity results for our problem.

Definition 3.8 The C^1-functional H is said to satisfy the Palais–Smale condition, in short the (PS) condition, if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$, for which $(H(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $dH(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Proposition 3.9 *Under assumptions* (A1) *and* (A2), *the functional* H *satisfies the* (PS) *condition.*

The following lemma plays a key role in the proof of Proposition (3.9). The constant exponent version ($p(x) \equiv p$) of the lemma can be found in [14].

Lemma 3.10 *Let* ω *be positive on* Ω *and satisfy* $\omega \in L^{r(x)}(\Omega)$ *with* $r(x) > \frac{N}{q(x)}$ *if* $1 < q(x) \leq N$ *or* $r(x) = 1$ *if* $q(x) > N$, $x \in \overline{\Omega}$, *and let* $T > 0$ *be a constant. Then there are three positive constants* C_1, C_2, C_3 *depending only on* ω *and* T *such that*

$$\varrho_{p(\cdot)}(\nabla u) \leq C_1 H(u) + C_2 \min\bigl(|u|^{q^+}_{\omega,q(\cdot)}, |u|^{q^-}_{\omega,q(\cdot)}\bigr) + C_3 \min\bigl(|u|^{\beta^+}_{\beta(\cdot)}, |u|^{\beta^-}_{\beta(\cdot)}\bigr) \tag{3.9}$$

for every $V \in L^{s(x)}(\Omega)$ *such that* $|V|_{s(\cdot)} \leq T$, $r \in L^\infty_+(\Omega)$, $p, q, \beta \in C_+(\overline{\Omega})$, *and for* $u \in X$, *where* $|u|_{\omega,q(\cdot)} = \inf\{\lambda > 0 : \int_\Omega \omega |\frac{u}{\lambda}|^{q(x)}\,dx \leq 1\}$.

Remark 3.11 Clearly, in Lemma (3.10), $\int_\Omega \omega |u|^{q(x)}\,dx < \infty$ for any $u \in X$. The case $r(x) = 1$ with $q(x) > N$ is obvious since $p(x) > N$ and then $X \hookrightarrow L^\infty(\Omega)$.

For the case $r(x) > \frac{N}{q(x)}$ and $1 < q(x) \leq N$, Propositions 2.3 and 2.5 yield

$$\int_\Omega \omega |u|^{q(x)}\,dx \leq 2|\omega|_{r(\cdot)} \bigl||u|^{q(\cdot)}\bigr|_{r'(\cdot)} \leq \max\bigl(|\omega|_{r(\cdot)} |u|^{q^+}_{q(\cdot)r'(\cdot)}, |\omega|_{r(\cdot)} |u|^{q^-}_{q(\cdot)r'(\cdot)}\bigr).$$

And since $q(x)r'(x) < p^*(x)$ for any $x \in \overline{\Omega}$, $\int_\Omega \omega |u|^{q(x)}\,dx < \infty$ for any $u \in X$. Accordingly, $|u|_{\omega,q(\cdot)} = \inf\{\lambda > 0 : \int_\Omega \omega |\frac{u}{\lambda}|^{q(x)}\,dx \leq 1\}$ stands for the Luxembourg norm on the Orlicz–Musielak space L_M where $M(x,t) = \omega(x) t^{q(x)}$. Endowed with this norm, L_M is separable and reflexive (see [19, 24]).

Proof of Lemma 3.10 Recalling (3.4) and (3.5), we have

$$\left| \int_\Omega \frac{V}{q(x)} |u|^{q(x)}\,dx \right| \leq \frac{2}{q^-} |V|_{s(\cdot)} \max\bigl(|u|^{q^+}_{q(\cdot)s'(\cdot)}, |u|^{q^-}_{q(\cdot)s'(\cdot)}\bigr), \tag{3.10}$$

$$\left| \int_\Omega F(x,u)\,dx \right| \leq a \varrho_{\beta(\cdot)}(u) + |h|_\infty |u| \leq M \max\bigl(|u|^{\beta^+}_{\beta(\cdot)}, |u|^{\beta^-}_{\beta(\cdot)}\bigr), \tag{3.11}$$

where $M \geq 0$ is a constant depending on the real numbers $|h|_\infty$, c, β^- provided by (A2), and on Ω.

Let us claim that, for any $\epsilon > 0$, there are M_ϵ and M'_ϵ such that

$$\max\bigl(|u|^{q^+}_{q(\cdot)s'(\cdot)}, |u|^{q^-}_{q(\cdot)s'(\cdot)}\bigr) \leq \epsilon \varrho_{p(\cdot)}(\nabla u) + M_\epsilon \min\bigl(|u|^{q^+}_{\omega,q(\cdot)}, |u|^{q^-}_{\omega,q(\cdot)}\bigr), \tag{3.12}$$

$$\max\bigl(|u|^{\beta^+}_{\beta(\cdot)}, |u|^{\beta^-}_{\beta(\cdot)}\bigr) \leq \epsilon \varrho_{p(\cdot)}(\nabla u) + M'_\epsilon \min\bigl(|u|^{\beta^+}_{\beta(\cdot)}, |u|^{\beta^-}_{\beta(\cdot)}\bigr). \tag{3.13}$$

Let us deal first with (3.12) by assuming to the contrary that there exist $\epsilon_0 > 0$ and a sequence u_n in X such that $|u_n|_{q(\cdot)s'(\cdot)} = 1$ and

$$\epsilon_0 \varrho_{p(\cdot)}(\nabla u_n) + n \min\bigl(|u_n|^{q^+}_{\omega,q(\cdot)}, |u_n|^{q^-}_{\omega,q(\cdot)}\bigr) < 1.$$

Then (u_n) is a bounded sequence in X and up to a subsequence, (u_n) converges weakly to some $u_0 \in X$ and strongly in $L^{q(\cdot)s'(\cdot)}(\Omega)$. Consequently, $|u_0|_{q(\cdot)s'(\cdot)} = 1$, and then $\min(|u_0|^{q^+}_{\omega,q(\cdot)}, |u_0|^{q^-}_{\omega,q(\cdot)}) < 0$. Contradiction.

A similar approach enables us to get (3.13).

Let $T > 0$ be such that $\max(\frac{2}{q^-}|V_s(\cdot)|, M) < T$, and let ϵ satisfy $0 < \epsilon < T^{-1}$. By combining (3.10), (3.11), (3.12), we get

$$(1-\epsilon T)\varrho_{p(\cdot)}(\nabla u) \le H(u) + TM_\epsilon \min\bigl(|u|^{q^+}_{\omega,q(\cdot)}, |u|^{q^-}_{\omega,q(\cdot)}\bigr) + TM'_\epsilon \min\bigl(|u|^{\beta^+}_{\beta(\cdot)}, |u|^{\beta^-}_{\beta(\cdot)}\bigr), \quad (3.14)$$

and the proof of the lemma is complete. □

Proof of Proposition 3.9 Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a (PS) sequence for the functional H, i.e., there exists a positive constant $k > 0$ such that

$$\bigl|H(u_n)\bigr| \le k \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad \bigl|\bigl\langle dH(u_n), v\bigr\rangle\bigr| \le \epsilon_n \|v\| \quad \text{for } v \in X, \qquad (3.15)$$

and $\epsilon_n \to 0$ as $n \to \infty$.

Let us show that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X. By contradiction, assume that $\|u_n\| \to \infty$ as $n \to \infty$. Up to a subsequence, we have $\|u_n\| > 1$ for any $n \in \mathbb{N}$, and by choosing $\omega = 1$ in Lemma 3.10 and replacing u by u_n in (3.9), we obtain

$$\varrho_{p(\cdot)}(\nabla u_n) \le C_1 H(u_n) + C_2 \min\bigl(|u_n|^{q^-}_{q()}, |u_n|^{q^+}_{q()}\bigr) + C_3 \min\bigl(|u_n|^{\beta^-}_{\beta()}, |u_n|^{\beta^+}_{\beta()}\bigr),$$

and consequently

$$\|u_n\|^{p^-} \le C_1 k + C'_2 \|u_n\|^{q^-} + C'_3 \|u_n\|^{\beta^-}, \qquad (3.16)$$

where C'_2, C'_3 are constants depending on C_2, C_3 and the Sobolev embedding constants.

Dividing (3.16) by $\|u_n\|^{p^-}$, we have

$$1 \le \frac{C_1 k}{\|u_n\|^{p^-}} + C'_2 \|u_n\|^{q^- - p^-} + C'_3 \|u_n\|^{\beta^- - p^-},$$

and passing to the limit leads to a contradiction since $p^- > q^-$ and $p^- > \beta^-$. So (u_n) is bounded in X and up to a subsequence converges weakly in X and strongly in $L^{r(x)}$ with $1 < r(x) < p^*(x)$ to u_0.

Let us show now that $(u_n)_{n\in\mathbb{N}}$ converges strongly to u_0 in X. Indeed, since the functional H satisfies the Palais–Smale condition, we have

$$\begin{aligned}
&\langle -\Delta_{p(x)} u_n, u_n - u_0\rangle \\
&\quad = \bigl\langle dH(u_n), u_n - u_0\bigr\rangle - \int_\Omega V(x)|u_n|^{q(x)-2} u_n (u_n - u_0)\,dx + \int_\Omega f(x,u_n)(u_n - u_0)\,dx \\
&\quad \le \varepsilon_n \|u_n - u_0\| + M_0 |V|_{s(\cdot)} |u_n|^k_{q(\cdot)} |u_n - u_0|_{\alpha(\cdot)} + M_1 \bigl||u_n|^{\beta(\cdot)-1}\bigr|_{\frac{\beta(\cdot)}{\beta(\cdot)-1}} |u_n - u_0|_{\beta(\cdot)} \\
&\qquad + M_2 |h|_\infty |u_n - u_0|_{\beta(\cdot)},
\end{aligned}$$

where $\varepsilon_n \to 0$ and M_0, M_1, M_2, and $k \in \{q^- - 1, q^+ - 1\}$ are positive constants. Next, using the Sobolev compact embeddings, we obtain

$$\limsup_{n\to\infty} \langle -\Delta_{p(x)} u_n, u_n - u_0 \rangle \leq 0 \tag{3.17}$$

and then, by the (S_+) property of $-\Delta_{p(x)}$, we get that $u_n \to u_0$ strongly in X, and the proof is complete. □

We are now ready to state the following existence result for multiple solutions.

Theorem 3.12 *Under assumptions* (A1), (A2), (A3), *and* (A4), *problem* (1.1) *has a bounded sequence of weak solutions* $(u_n)_{n\in\mathbb{N}} \subseteq X$ *such that* $H(u_n) = c_n < 0$. *Moreover, the sequence of the critical values* $(c_n)_n$ *tends to* $c = \inf_X H$.

Theorem 3.12 is derived by searching solutions as critical values of the functional H by means of a symmetric version of the mountain pass theorem.

Many versions of the mountain pass theorem stated according to the geometry of the problem under consideration exist in the literature (see [3, 31, 33, 34]). We state below an appropriate version to our situation.

Theorem 3.13 (A symmetric mountain pass theorem) *Suppose that X is an infinite dimensional real Banach space. Let $H \in C^1(X,\mathbb{R})$ be even and satisfy the* (PS) *condition and $H(0) = 0$. Assume that*

(i) *there are some constants $\rho, \alpha > 0$ such that $H(u) \geq \alpha$ for all $u \in X$ with $\|u\| \geq \rho$;*
(ii) *for each finite dimensional subspace F of X, there is a constant $R > 0$ such that $H(u) < 0$ for all $u \in F$ with $\|u\| \leq R$.*

Then H has a bounded sequence of negative critical values (c_n) tending to $\inf_X H$.

The proof of Theorem 3.13 can be adapted from some works in the literature. We give it as an appendix to this work for the sake of completeness.

4 Proof of Theorem 3.12 and auxiliary results

The proof will consist in showing that H satisfies the geometry required to apply the symmetric mountain pass theorem.

Lemma 4.1 *Under assumptions* (A1), (A2), *and* (A3), *there exist some constants $\rho, \alpha > 0$ such that $H|_{\{u\in X:\|u\|\geq\rho\}} \geq \alpha$.*

Proof Suppose on the contrary that, for any $n \in \mathbb{N}^*$, there exists a sequence $(u_n) \in X$ such that $H(u_n) \leq \frac{1}{n}$ for $\|u_n\| \geq n$. Recalling (3.9), we have

$$1 \leq C_1 \frac{1}{n} + C_2' \|u_n\|^{q^- - p^-} + C_3' \|u_n\|^{\beta^- - p^-}$$

with $q^- - p^- < 0$, $\beta^- - p^- < 0$ and passing to the limit yields a contradiction. Then Lemma 4.1 holds. □

Lemma 4.2 *Suppose that assumptions* (A1) *and* (A3) *are satisfied. Then, for each finite dimensional subspace F, there is a constant $R > 0$ such that $H(u) \leq 0$ for all $u \in F$ with $\|u\| \leq R$.*

Proof By assumption (A3),

$$F(x,t) \geq \gamma(x)|t|^{\theta} \quad \text{for all } x \in \Omega \text{ and for any } t \in [0,1[. \tag{4.1}$$

On the other hand, it is easy to see that the functional $\|\cdot\|_{\theta} : X \to \mathbb{R}$ defined by

$$\|u\|_{\theta} = \left(\int_{\Omega} \gamma(x) \left| u(x) \right|^{\theta} dx \right)^{\frac{1}{\theta}} \tag{4.2}$$

is a norm on the space X. Then on the finite dimensional subspace F the norms $\|\cdot\|$ and $\|\cdot\|_{\theta}$ are equivalent, so there exists a constant $\nu > 0$ such that

$$\nu \|u\| \leq \|u\|_{\theta} \quad \text{for all } u \in F. \tag{4.3}$$

Now, let $v \in F$ and write $v = su$ with $\|u\| = 1$, and $0 < s < 1$,

$$\begin{aligned} H(v) &= \int_{\Omega} \frac{1}{p(x)} s^{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{V(x)}{q(x)} s^{q(x)} |u|^{q(x)} dx - \int_{\Omega} F(x,v)\, dx \\ &\leq \frac{s^{p^-}}{p^-} \|u\|^{p^+} + \frac{2C^{q^+}}{q^-} |V|_{s(x)} s^{q^-} \|u\|^{q^+} - s^{\theta} \|u\|_{\theta}^{\theta}, \end{aligned} \tag{4.4}$$

where C is a constant deriving from the Sobolev embedding.

Using the fact that $\|u\| = 1$ and (4.3), we obtain

$$\begin{aligned} H(v) &\leq \frac{s^{p^-}}{p^-} + \frac{2C^{q^+}}{q^-} |V|_{s(x)} s^{q^-} - \nu^{\theta} s^{\theta} \\ &\leq s^{p^-} \left[\frac{1}{p^-} + \frac{1}{s^{p^- - q^-}} \left(\frac{2C^{q^+}}{q^-} |V|_{s(x)} - \nu^{\theta} \frac{1}{s^{p^- - \theta}} \right) \right]. \end{aligned} \tag{4.5}$$

Since $p^- > q^-$ and $p^- > \theta$, we have for $s < 1$ small enough that H is nonpositive, that is, there is η such that

$$H(v) < 0 \quad \text{for } 0 < s \leq \eta < 1.$$

Take $R = \eta$ and then Lemma 4.2 is proved. □

To conclude with the proof of Theorem 3.12, we notice from (A4) that $H(0) = 0$ and H is even, and from Lemmas 4.1, 4.2, H satisfies the conditions required in Theorem 3.12, and then the result is achieved.

An auxiliary result in terms of multiple solutions in the same spirit of the works in [1, 6], and [22] can be obtained in our context by means of the fountain theorem. However, the sequence of solutions obtained either by the mountain pass theorem or the fountain theorem are quite different. Of course, in our context, the sequence of critical values obtained via the mountain pass theorem converges to a nonzero limit, while the use of the fountain theorem gives rise to critical values sequence converging to 0 as stated in Theorem 4.3 below.

Theorem 4.3 (Dual fountain theorem) *Suppose that X is an infinite dimensional reflexive separable Banach space. Let $H \in C^1(X,\mathbb{R})$ be even and satisfy the* (PS) *condition and* $H(0) = 0$.

Write $X = \overline{\operatorname{span}\{e_n, e_n \in X\ \forall n \geq 1\}}$, $X^* = \overline{\operatorname{span}\{e_n^*, e_n^* \in X^*\ \forall n \geq 1\}}$ with $e_n^*(e_m) = 1$ if $m = n$ and $e_n^*(e_m) = 0$ if $m \neq n$, and define the subspaces

$$X_k = \operatorname{span}\{e_k, k \geq 1\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Assume that there are some constants $\rho_k > r_k > 0$ such that

(i) $H(u) \geq 0$ *for all* $u \in Z_k$ *with* $\|u\| = \rho_k$,

(ii) $H(u) < 0$ *for all* $u \in Y_k$ *with* $\|u\| = r_k$,

(iii) $d_k = \inf_{\{u \in Z_k, \|u\| \leq \rho_k\}} H(u) \to 0$ *as* $k \to +\infty$.

Then H has a sequence of negative critical values (c_n) tending to 0.

Sketch of the proof H satisfies of course (PS), and by means of Lemmas 4.1 and 4.2, (i), (ii) are also satisfied. Thus, to prove Theorem 4.3, we need only to show that (iii) is satisfied. Accordingly, we need the following lemma whose proof can be pointed out similarly as in [1, 6], and [22].

Lemma 4.4

$$\beta_k = \sup_{\{u \in Z_k, \|u\| = 1\}} \int_\Omega \frac{|V|}{q(x)} |u_k|^p \, dx \to 0$$

and

$$\beta_k' = \sup_{\{u \in Z_k, \|u\| = 1\}} \int_\Omega \bigl|F(x, u_k)\bigr| \, dx \to 0 \quad \text{as } k \to +\infty.$$

To conclude the proof of Theorem 4.3, we notice that

$$\begin{aligned} H(u) &\geq \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx - \int_\Omega \frac{|V(x)|}{q(x)} |u|^{q(x)} \, dx - \int_\Omega \bigl|F(x,u)\bigr| \, dx. \\ &\geq -\int_\Omega \frac{|V(x)|}{q(x)} |u|^{q(x)} \, dx - \int_\Omega \bigl|F(x,u)\bigr| \, dx. \end{aligned}$$

Choosing $u = tv$ with $0 < t \leq \rho_k$ and $\|v\| = 1$, we have

$$0 = H(0) \geq d_k = \inf_{\{u \in Z_k, \|u\| \leq \rho_k\}} H(u) \geq -\beta_k - \beta_k' \quad \text{for } k \text{ large enough.}$$

Consequently,

$$\inf_{\{u \in Z_k, \|u\| \leq \rho_k\}} H(u) \to 0 \quad \text{as } k \to +\infty.$$

□

Appendix: A symmetric mountain pass theorem version

Theorem A.1 (A symmetric mountain pass theorem) *Suppose that X is an infinite dimensional real Banach space. Let $H \in C^1(X,\mathbb{R})$ be even, satisfy the* (PS) *condition, and* $H(0) = 0$. *Assume that*

(i) *there are constants $\rho, \alpha > 0$ such that $H(u) \geq \alpha$ for all $u \in X$ with $\|u\| \geq \rho$;*

(ii) *for each finite dimensional subspace F of X, there is a constant $R > 0$ such that $H(u) < 0$ for all $u \in F$ with $\|u\| \leq R$.*

Then H has a bounded sequence of negative critical values (c_n) tending to $\inf_X H$.

The proof of Theorem A.1 is based on the notions of genus and an appropriate deformation lemma [3, 33]. We state them below. Let us denote respectively by $C(X, Y)$ and $\Sigma(X)$ the space of continuous maps from X into Y and the family of closed in X subsets of $X \setminus \{0\}$, symmetric with respect to the origin.

Definition A.2 A in $\Sigma(X)$ has the genus n (denoted by $\gamma(A) = n$) if n is the smallest integer for which there exists a map $\phi \in C(A, \mathbb{R}^n \setminus \{0\})$

$\gamma(A) = \infty$ if there exists no finite such n and $\gamma(\emptyset) = 0$.

Let us denote

$$N_\delta(A) = \{u \in X, d(u,A) \leq \delta\}.$$

Below, we state some useful properties in the sequel for γ and the deformation lemma.

Proposition A.3 *Let $A, B \in \Sigma(X)$ and $U \subset \mathbb{R}^k$ be an open, bounded, symmetric neighborhood of the origin.*

1. *If there is odd $\phi \in C(A,B)$, then $\gamma(A) \leq \gamma(B)$.*
2. *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
3. $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
4. *If $\gamma(B) < \infty$, then $\gamma(A \setminus B) \geq \gamma(A) - \gamma(B)$.*
5. *If A is compact, then $\gamma(A) < \infty$ and $\gamma(N_\delta(A)) = \gamma(A)$ for all sufficiently small δ.*
6. *$\gamma(\partial U) = k$ and in particular $\gamma(S^{k-1}) = k$.*
7. *If A is homeomorphic to ∂U by an odd homeomorphism, then $\gamma(A) = k$.*

In order to state the deformation lemma, let us define for $H \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$, the sets

$$K_c = \{u \in X, H(u) = c \text{ and } H'(u) = 0\} \quad \text{and} \quad \tilde{H}_c = \{u \in X, H(u) \geq c\}.$$

Clearly K_c is a compact set because of the (PS) condition. Then we have the following variant of the deformation lemma.

Proposition A.4 (Deformation lemma) *Suppose that X is a real Banach space and $H \in C^1(X,\mathbb{R})$ satisfies the (PS) condition. If $c \in \mathbb{R}$, $\bar{\epsilon} > 0$, and $\mathcal{U}$ is a neighborhood of K_C, then there exist $\epsilon \in (0,\bar{\epsilon})$ and $\eta : [0,1] \times X \to X$ such that*

1. *$\eta(0,u) = u$ $\forall u \in X$.*
2. *$\eta(t,u) = u$ $\forall t \in [0,1]$ and $u \in X$ with $|H(u) - c| \geq \bar{\epsilon}$.*

3. $H(\eta(t,u)) \geq H(u)\ \forall t \in [0,1]$, $u \in X$.
4. $\eta(1, \tilde{H}_{c-\epsilon} \setminus \mathcal{U}) \subset \tilde{H}_{c+\epsilon}$.
5. *If H is even, $\eta(t,u)$ is odd in u.*

Remark A.5 The proof of Proposition A.4 here is a slight modification of the proof encountered in many works in the literature for the standard deformation lemma (cf. [3, 12, 13, 28, 31, 33]). The method is technical and is based on the construction of a positive modified gradient flow for H rather than the negative gradient flow one in the standard deformation lemma.

Proof of Theorem A.1 Let us choose ρ as in Theorem 3.13 and call B_ρ the ball centered at the origin O. Let F_m be a finite dimensional subspace of X with $\dim F_m = m$, F_m is spanned by $\{e_1, e_2, \dots, e_m\}$ such that $B_\rho \cap F_m \neq \varnothing$. Denote $D_\rho^m = \overline{B_\rho} \cap F_m$ and

$$G_m = \bigl\{h \in C\bigl(D_\rho^m, X\bigr), h \text{ is odd, and } h = id \text{ on } \partial\bar{B}_\rho \cap F_m\bigr\}.$$

Since $id \in G_m$, $G_m \neq \varnothing$: Next, we set

$$\Gamma_j = \bigl\{h\overline{\bigl(D_\rho^m \setminus Y\bigr)}, h \in G_m, Y \subset \Sigma(X), \gamma(Y) \leq m-j, j \leq m\bigr\},$$

where $\gamma(Y)$ is the genus of Y. $h\overline{(D_\rho^m \setminus Y)} = \overline{h(D_\rho^m \setminus Y)}$ and is compact and the family of the sets $(\Gamma_j)_j$ satisfies the following properties:

(i) $(\Gamma_j) \neq \varnothing$ for all j.
(ii) $\Gamma_{j+1} \subset \Gamma_j$. (Monotonicity).
(iii) If $B \in \Gamma_j$ and $Y \in \Sigma(X)$ with $\gamma(Y) \leq s < j$, then $\overline{(B \setminus Y)} \in \Gamma_{j-s}$ for each j. (Excision)
(iv) If $\phi \in C(X,X)$ is odd and $\phi = id$ on $\partial\bar{B}_\rho \cap F_m$, then $\phi : \Gamma_j \to \Gamma_j$ for each $j \in \mathbb{N}$. (Invariance)

Indeed, (i) is satisfied since the set $\{Y \in \Sigma(X)/\gamma(Y) \leq m-j\}$ is not empty as a consequence of (7) of Proposition A.3. (ii) is a trivial consequence of the construction of Γ_j. (iii) is derived from (4), Proposition A.3. For (iv), we just have to notice that $\phi \circ h \in G_m$.

Note

$$c_j = \sup_{B\in\Gamma_j} \min_{u\in B} H(u).$$

Since B is compact $c_j < \infty$ for any j and, moreover, $c_{j+1} \leq c_j$ because $\Gamma_{j+1} \subset \Gamma_j$.

Next, we claim that $c_1 < 0$.

Indeed, let $B \in \Gamma_1$, then $B = h\overline{(D_\rho^m \setminus Y)}$ with $Y \in \Sigma(X)$ and $\gamma(Y) \leq m-1$.

We have that $H(u) \geq \alpha$ on ∂B_ρ and $H(u) < 0$ for $u \in \overline{B_R}$, where R is provided by (ii) of Theorem 3.13. Clearly, ρ is chosen such that $\rho > R$.

Let $\tilde{\Omega} = \{u \in D_\rho^m / h(u) \in B_R\}$ and denote by Ω the component of $\tilde{\Omega}$ containing the origin O. Since h is odd and $h = id$ on $\partial B_\rho \cap F_m$, Ω is a symmetric bounded open neighborhood of O in F_m, and hence $\gamma(\partial\Omega) = m$.

Moreover, $h(u) \in \partial B_R \cap F_m$ if $u \in \partial\Omega$. Indeed, suppose $h(u) \in B_R \cap F_m$ if $u \in \partial\Omega$. If $u \in D_\rho^m$, then one can find a neighborhood $\mathcal{V}(u)$ of u such that $h(u) \in B_R$, and hence $u \notin \partial\Omega$. If, on the other hand, $u \in \partial D_\rho^m$, one has $h(u) = u$ and then $\|h(u)\| = \|u\| = \rho > R$. This is a contradiction, so $h(u) \in \partial B_R \cap F_m$ if $u \in \partial\Omega$ must hold.

Let

$$W = \{u \in D_\rho^m / h(u) \in \partial B_R \cap F_m\}.$$

Since $\partial\Omega \subset W$, $\gamma(W) \geq \gamma(\partial\Omega) = m$. Thus $\gamma(\overline{W \setminus Y}) \geq m - (m-1) = 1$. Consequently, $\overline{W \setminus Y} \neq \emptyset$ and then there exists $\bar{u} \in W \setminus Y$ such that $h(\bar{u}) \in B \cap \partial B_R$. Hence

$$\min_B H(u) \leq H\big(h(\bar{u})\big) \leq \inf_{\partial B_R} H(u) < 0. \tag{A.1}$$

Since A.1 is valid for any $B \in \Gamma_1$, we have $c_1 < 0$.

Given j, it may happen that the multiplicity of c_j occurs, that is,

$$c_j = \cdots = c_{j+k} = c \quad \text{for some } k \geq 0. \tag{A.2}$$

We shall prove in this case that $\gamma(K_c) \geq k+1$ in order to conclude that K_c contains no less than $k+1$ points. $0 \notin K_c$ since $H(0) = 0$; moreover, H is even and accordingly $K_c \in \Sigma(X)$ and $\gamma(K_c) < \infty$.

If $\gamma(K_c) < k+1$, then there exists $\delta > 0$ such that $\gamma(N_\delta(K_c)) < k+1$. Denote $\mathcal{U} = N_\delta(K_c)$. For any $\bar{\epsilon} > 0$, by Proposition A.4 (the deformation lemma) and Remark A.5, there are $\epsilon \in (0, \bar{\epsilon})$ and $\eta \in C([0,1] \times X, X)$ such that

$$\eta(1, \tilde{H}_{c-\epsilon} \setminus \mathcal{U}) \subset \tilde{H}_{c+\epsilon}. \tag{A.3}$$

Choose any $B \in \Gamma_{j+k}$ such that $\min_B H(u) \geq c - \epsilon$.

By A.3 we get

$$\min_{\eta(1, B\setminus\mathcal{U})} H(u) \geq c + \epsilon. \tag{A.4}$$

Recalling the excision property (iii) above, we have $\overline{B \setminus \mathcal{U}} \in \Gamma_j$, and by choosing $\bar{\epsilon} = -c/2$ in Proposition A.4, points (2) and (5) of the proposition enable us to conclude that $\eta(1,\cdot)$ is odd and $H_{|\partial B_\rho} \geq \alpha > c - c/2 = c/2$, and hence $\eta(1,\cdot) \in G_m$ for all $m \in \mathbb{N}$. Consequently, $\eta(1, \overline{B \setminus \mathcal{U}}) \in \Gamma_j$ and the definition of c_j gives

$$\min_{\eta(1, B\setminus\mathcal{U})} H(u) \leq c,$$

which is in contradiction with (A.4). Thus $\gamma(K_c) \geq k+1$ and hence K_C contains no less than $k+1$ points.

Let us show that each c_j is a critical value of H for any j. In order to prove it, assume that c_j (for fixed j) is not a critical value, that is, $K_{c_j} = \emptyset$. Then, choosing $\mathcal{U} = \emptyset$, by Proposition A.4 (the deformation lemma) and Remark A.5, there are $\epsilon \in (0, \bar{\epsilon})$ and $\eta \in C([0,1] \times X, X)$ such that

$$\eta(1, \tilde{H}_{c-\epsilon}) \subset \tilde{H}_{c+\epsilon} \quad \text{for } c - \epsilon > 0. \tag{A.5}$$

Choosing any $B \in \Gamma_j$ such that $\min_B H(u) \geq c - \epsilon$.

By (A.5) we get

$$\min_{\eta(1,B)} H(u) \geq c + \epsilon. \tag{A.6}$$

Choosing $\bar{\epsilon} = -c/2$ in Proposition A.4 and arguing around points (2) and (5) of the proposition as above, we have that $\eta(1,\cdot) \in G_m$ for all $m \in \mathbb{N}$. Consequently, $\eta(1,\overline{B}) \in \Gamma_j$ and the definition of c_j gives

$$\min_{\eta(1,B)} H(u) \leq c.$$

Contradiction with (A.6), so c_j is a critical value.

Let us show that c_j tends to $\inf_X H = c$ as $j \to +\infty$. Obviously, $c = \inf_X H < \infty$ since H is bounded below on X. The critical values sequence $(c_j)_j$ is monotone nonincreasing, so there is $\bar{\alpha} \geq c$ such that $c_j \to \bar{\alpha}$ as $j \to +\infty$. $\bar{\alpha} < c_j$ for all j, otherwise $\gamma(K_{\bar{\alpha}}) = \infty$ according to (A.2); but $K_{\bar{\alpha}} \in \Sigma(X)$ and is compact, so $\gamma(K_{\bar{\alpha}}) < \infty$, contradiction. So $\bar{\alpha} < c_j$ for all j. Suppose that c_j does not tend to c, that is, $\bar{\alpha} > c$, and denote

$$\mathcal{H} = \bigl\{u \in X, c \leq H(u) \leq \bar{\alpha}, \text{ and } H'(u) = 0\bigr\}.$$

$\mathcal{H}$ is compact by the (PS). Besides, $\mathcal{H} \in \Sigma(X)$ and there exists $\delta > 0$ such that $\gamma(N_\delta(\mathcal{H})) = \gamma(\mathcal{H}) < \infty$. Suppose $\gamma(\mathcal{H}) = n$. Recalling Proposition A.4 and Remark A.5, with $\bar{\epsilon} = \bar{\alpha} - c$ and $\mathcal{O} = N_\delta(H)$, there are $\epsilon \in (0,\bar{\epsilon})$ and $\eta \in C([0,1] \times X, X)$ with $\eta(1,\cdot)$ odd such that

$$\eta(1, \tilde{H}_{\bar{c}-\epsilon} \setminus \mathcal{O}) \subset \tilde{H}_{\bar{c}+\epsilon}. \tag{A.7}$$

Choose the smallest value $\bar{m}$ of the dimensions of the finite dimensional subspaces of X such that $c_{\bar{m}} < \bar{\alpha} + \epsilon$ and choose $B \in \Gamma_{\bar{m}+n}$ such that

$$\min_B H(u) \geq \bar{\alpha} - \epsilon.$$

By (A.7) we get

$$\min_{\eta(1,\overline{B \setminus \mathcal{O}})} H(u) \geq \bar{\alpha} + \epsilon. \tag{A.8}$$

Since $H_{|\partial B_\rho} \geq \alpha > 0$, we have $|H(u) - c| \geq \alpha - c > \bar{\alpha} - c = \bar{\epsilon}$ for $u \in \partial B_\rho \cap F_{\bar{m}}$, and recalling points (2) and (5) of Proposition A.4, we have $\eta(1,\cdot) \in G_{\bar{m}}$ and then $\overline{B \setminus \mathcal{O}}$ and $\eta(1, \overline{B \setminus \mathcal{O}}) \in \Gamma_{\bar{m}}$. Consequently,

$$\bar{\alpha} + \epsilon > c_{\bar{m}} \geq \min_{\eta(1,\overline{B \setminus \mathcal{O}})} H(u) \geq \bar{\alpha} + \epsilon.$$

Contradiction. So $c_j \to c$ and the proof is complete. □

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