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Uniqueness result for a fractional diffusion coefficient identification problem

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Abstract

In this paper, we establish an identifiability result for the coefficient identification problem in a fractional diffusion equation in a bounded domain from the observation of the Cauchy data on particular subsets of the boundary.

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1 Introduction

Let Ω be a smooth bounded domain of \mathbb{R}^d ($d \geq 3$) with boundary $\partial\Omega \in C^\infty$, and let $T > 0$ be a fixed real number. Our inverse problem consists in determining the potential $q := q(x)$ via the fractional diffusion equation from given Cauchy data on particular subsets of the boundary. This problem is presented by the following equations:

$$(P_{qf}^\alpha) \begin{cases} \partial_t^\alpha u - \Delta_x u + q(x)u = 0 & \text{in } \Omega_T := \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ u = f & \text{on } \Sigma_T := \partial\Omega \times (0, T), \end{cases}$$

where $\partial_t^\alpha u$ represents the fractional Caputo time derivative of order $0 < \alpha < 1$ defined by equation (3.7). We assume that the coefficient $q \in L^\infty(\bar{\Omega})$, $f \in C^1([0; T]; H^{\frac{3}{2}}(\partial\Omega))$, and $u_0 \in L^2(\Omega)$.

The classical diffusion–advection equation does not often describe the abnormal diffusion (e.g., for the problem of the soil scatter field data [1]) and that the fractional diffusion equation is used as a model equation for this problem and for many physical phenomena such as the diffusion of material in heterogeneous media, the diffusion of fluid flows in inhomogeneous anisotropic porous media, turbulent plasma, the diffusion of carriers in amorphous photoconductors, the diffusion in a turbulent medium flow, a percolation model in porous media, various biological phenomena, and finance problems (see [9]).

The inverse coefficients problems associated with the system (P_{qf}^α) were investigated by many authors for the parabolic case ([8, 12, 13], ...) when $\alpha = 1$ and for the hyperbolic case ([3–6, 23], ...) when $\alpha = 2$. Few works exist dealing with this type of problem in the fractional case ($0 < \alpha < 1$). In [11] the authors, in the one-dimensional case, by Dirichlet boundary measurements determined the fractional order α and a time-independent

coefficient. Using pointwise measurements of the solution over the entire time span, the authors in [16] identified the fractional order α for the case $d \geq 2$. Using a specifically designed Carleman estimate, the authors in [10, 25] derived a stability estimate of a zero-order time-independent coefficient, with respect to partial internal observation of the solution in the particular case where $d = 1$ and $\alpha = \frac{1}{2}$. In [14] the authors proved a unique determination of a time-dependent parameter appearing in the source term or in the zero-order coefficient from pointwise measurements of the solution over the whole time interval. By performing a single measurement of the Cauchy data on the accessible boundary, the authors in [15] gave identifiability and local Lipschitz stability results to solve the inverse problem of identifying fractional sources. The authors in [19] proved a uniqueness result for the time-independent coefficients using the Dirichlet-to-Neumann operator obtained by probing the system with inhomogeneous Dirichlet boundary conditions of the form $\lambda(t)g(x)$, where λ was a fixed real-analytic positive function of the time variable. In [17] the authors proved the uniqueness in the inverse problem of determining the smooth manifold (up to an isometry) and various time-independent smooth coefficients appearing in their equation from measurements of the solution on a subset of the boundary at fixed time.

In this work, we establish a uniqueness result of the coefficient identification problem using Carleman estimates and particular complex geometrical solutions of the fractional diffusion equation. These techniques are inspired by [7], where the authors proved the uniqueness of a coefficient identification problem for the Schrödinger equation.

In Sect. 2, we state the forward problem. In Sect. 3, we recall some definitions and properties of the fractional derivatives, and in Sect. 4, we give the proof of our main result.

2 Forward problem

In this work, our fundamental question is to prove the uniqueness of the potential q via the fractional diffusion equation from the knowledge of the Cauchy data. The forward problem is to find the solution u from the following system of equations:

$$\begin{cases} \partial_t^\alpha u - \Delta_x u + q(x)u = 0 & \text{in } \Omega_T := \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ u = f & \text{on } \Sigma_T := \partial\Omega \times (0, T), \end{cases} \tag{2.1}$$

where the coefficient $q \in L^\infty(\bar{\Omega})$, $f \in C^1([0; T]; H^{\frac{3}{2}}(\partial\Omega))$, and $u_0 \in L^2(\Omega)$ are given. Referring to [17, 20], we can choose $G \in C^1([0; T]; H^2(\Omega))$ satisfying $G = f$ on Σ_T , and by setting $w = u - G$, w is a solution of

$$\begin{cases} \partial_t^\alpha w - \Delta_x w + q(x)w = F & \text{in } \Omega_T, \\ w(x, 0) = a(x) & \text{in } \Omega, \\ w = 0 & \text{on } \Sigma_T, \end{cases} \tag{2.2}$$

where $F = -(\partial_t^\alpha G - \Delta_x G + q(x)G)$ and $a(x) = u_0(x) - G(x, 0)$. The study of the existence and uniqueness of the solution of (2.1) is reduced to the existence and uniqueness of problem (2.2).

Theorem 2.1 *Let $a \in L^2(\Omega)$ and $F \in L^\infty(0, T, L^2(\Omega))$. Then problem (2.2) has a unique weak solution*

$$w \in C([0, T]; L^2(\Omega)) \cap C((0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2([0, T]; H^2(\Omega) \cap H_0^1(\Omega)).$$

Proof We split problem (2.2) into the following two problems by taking $w = w_1 + w_2$, where w_1 is the solution of

$$\begin{cases} \partial_t^\alpha u - \Delta_x u + q(x)u = 0 & \text{in } \Omega_T, \\ u(x, 0) = a & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma_T, \end{cases} \tag{2.3}$$

and w_2 is the solution of

$$\begin{cases} \partial_t^\alpha u - \Delta_x u + q(x)u = F & \text{in } \Omega_T, \\ u(x, 0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma_T. \end{cases} \tag{2.4}$$

From Sakamoto et al. [22] we have

1. Problem (2.3) has a unique weak solution

$$w_1 \in C([0, T]; L^2(\Omega)) \cap C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$$

satisfying

$$\|u(\cdot, t)\|_{H^2(\Omega)} + \|\partial_t^\alpha u(\cdot, t)\|_{L^2(\Omega)} \leq C_1 t^{-\alpha} \|a\|_{L^2(\Omega)}. \tag{2.5}$$

2. Problem (2.4) has a unique weak solution $w_2 \in L^2([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ satisfying

$$\|u\|_{L^2([0, T]; H^2(\Omega))} + \|\partial_t^\alpha u\|_{L^2(\Omega_T)} \leq C_2 \|F\|_{L^2(\Omega_T)}. \tag{2.6}$$

□

3 Preliminaries

We start this section by giving some definitions and fundamental properties of fractional integrals and fractional derivatives, which can be found in [18, 21].

Let $\alpha > 0$, let n be the integer satisfying $n - 1 \leq \alpha < n$, and let $a, b \in \mathbb{R}$.

Definition 3.1 Let $g : [a, b] \rightarrow \mathbb{R}$ be a function, and let Γ be the Euler gamma function.

1. The left and right Riemann–Liouville fractional integrals of order α are defined respectively by

$${}_a I_t^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} g(s) ds \tag{3.1}$$

and

$${}_t I_b^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s - t)^{\alpha-1} g(s) ds. \tag{3.2}$$

2. The left and right Riemann–Liouville fractional derivatives of order α are defined respectively by

$${}_a D_t^\alpha g(t) := \frac{d^n}{dt^n} {}_a I_t^{n-\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} g(s) ds \tag{3.3}$$

and

$${}_t D_b^\alpha g(t) := (-1)^n \frac{d^n}{dt^n} {}_t I_b^{n-\alpha} g(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} g(s) ds. \tag{3.4}$$

In particular, if $\alpha = 0$, then

$${}_a D_t^0 g(t) = {}_t D_b^0 g(t) = g(t),$$

and if $\alpha = k \in \mathbb{N}$, then

$${}_a D_t^k g(t) = {}_t D_b^k g(t) = g^{(k)}(t).$$

3. The left and right Caputo fractional derivatives of order α are defined by

$${}^c D_t^\alpha g(t) := {}_a I_t^{n-\alpha} g^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} g^{(n)}(s) ds \tag{3.5}$$

and

$${}^c D_b^\alpha g(t) := (-1)^n {}_t I_b^{n-\alpha} g^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} g^{(n)}(s) ds. \tag{3.6}$$

In particular, if $0 < \alpha < 1$, then we denote

$$\partial_t^\alpha g(t) := {}^c D_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} g'(s) ds. \tag{3.7}$$

Lemma 3.2 ([18]) *If $g \in AC^n[a, b]$, then the Riemann–Liouville fraction derivative and the Caputo fractional derivative are connected with each other by the following relations:*

$${}_a D_t^\alpha g(t) = {}^c D_t^\alpha g(t) + \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{\Gamma(1+k-\alpha)} (t-a)^{k-\alpha} \tag{3.8}$$

and

$${}_t D_b^\alpha g(t) = {}^c D_b^\alpha g(t) + \sum_{k=0}^{n-1} \frac{(-1)^k g^{(k)}(b)}{\Gamma(1+k-\alpha)} (b-t)^{k-\alpha}. \tag{3.9}$$

Here

$$AC^n[a, b] = \left\{ g : [a, b] \rightarrow \mathbb{R} \text{ such that } \frac{d^{n-1}}{dx^{n-1}}(g) \in AC[a, b] \right\},$$

$$g \in AC[a, b] \iff \text{there exists } \varphi \in L(a, b) \text{ such that } g(x) = c + \int_a^b \varphi(t) dt, \quad c \in \mathbb{R},$$

and $L(a, b)$ is the set of Lebesgue-measurable complex-valued functions on $[a, b]$.

Remark 3.3 If $0 < \alpha < 1$, then

$${}_0D_t^\alpha u(x, t) = {}^c_0D_t^\alpha u(x, t) + \frac{u(x, 0)}{\Gamma(1 - \alpha)} t^{-\alpha}. \tag{3.10}$$

In addition, if $u(x, 0) = 0$, then ${}_0D_t^\alpha u(x, t) = {}^c_0D_t^\alpha u(x, t)$.

Definition 3.4 ([18])

1. The Mittag-Leffler function is

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \tag{3.11}$$

where $\alpha > 0$ and β are arbitrary constants.

2. The α -exponential function is defined by

$$e_{\alpha}^{\lambda z} = z^{\alpha-1} E_{\alpha, \alpha}(\lambda z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{3.12}$$

where $\alpha > 0$ and $\lambda \in \mathbb{C}$.

Lemma 3.5 ([18])

1. For $0 < \alpha < 1$ and $\lambda > 0$, we have

$$\partial_t^\alpha E_{\alpha, 1}(-\lambda t^\alpha) = -\lambda E_{\alpha, 1}(-\lambda t^\alpha), \quad t > 0.$$

2. For $0 < \alpha < 1$ and $\lambda \in \mathbb{C}$, we have

$${}_0D_t^\alpha e_{\alpha}^{\lambda t} = \lambda e_{\alpha}^{\lambda t}, \quad t > 0.$$

Lemma 3.6 ([21]) For $s > |\lambda|^{\frac{1}{\alpha}}$, we have

$$\int_0^\infty e^{-st} t^{\alpha k + \beta - 1} E_{\alpha, \beta}^{(k)}(-\lambda t^\alpha) dt = \frac{k! s^{\alpha - \beta}}{(s^\alpha - \lambda)^{k+1}}.$$

4 The main result

In this work, we focus on the uniqueness of the solution q of the inverse problem.

Let

$$H_\Delta(\Omega_T) = \{u \in \mathcal{D}'(\Omega_T) / u \in L^2(\Omega_T), \Delta u \in L^2(\Omega_T)\}.$$

For $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d) \in \mathbb{C}^d$, $\langle x, y \rangle := \sum_{i=1}^d x_i y_i$.

Fix $\xi \in S^{d-1}$, where $S^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$, and for $\varepsilon > 0$, denote

$$\partial\Omega_+(\xi) = \{x \in \partial\Omega, \langle v(x), \xi \rangle > 0\},$$

$$\begin{aligned} \partial\Omega_-(\xi) &= \{x \in \partial\Omega, \langle v(x), \xi \rangle < 0\}, \\ \partial\Omega_{+,\varepsilon}(\xi) &= \{x \in \partial\Omega, \langle v(x), \xi \rangle > \varepsilon\}, \\ \partial\Omega_{-,\varepsilon}(\xi) &= \{x \in \partial\Omega, \langle v(x), \xi \rangle < \varepsilon\}, \end{aligned}$$

where v represents the unit outer normal to the boundary $\partial\Omega$,

$$C_{q,\varepsilon} = \{(u(\cdot, 0), u|_{\Sigma_T}, \partial_\nu u|_{\partial\Omega_{-,\varepsilon}(\xi) \times (0,T)}) / u \in H_\Delta(\Omega_T), (\partial_t^\alpha - \Delta + q)u = 0 \text{ in } \Omega_T\}.$$

In the following theorem, we give a global uniqueness result for the fractional diffusion equation in which the Cauchy data are given only on part of the boundary.

Theorem 4.1 *Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Given $\xi \in S^{d-1}$ and $\varepsilon > 0$, assume that $C_{q_1,\varepsilon} = C_{q_2,\varepsilon}$. Then $q_1 = q_2$.*

To prove this result, we need the following results.

Lemma 4.2 ([7]) *Let $\rho \in \mathbb{C}^d$ be such that $\langle \rho, \rho \rangle = 0$, where $\rho = \tau(\xi + i\eta)$ with $\xi, \eta \in S^{d-1}$. Suppose that $f(\cdot, \rho/|\rho|) \in W^{2,\infty}(\Omega)$ satisfies $\partial_\xi f = \partial_\eta f = 0$, where ∂_ξ denotes the directional derivative in the direction ξ . Then there is a solution to $\Delta u - qu = 0$ in Ω of the form*

$$u(x, \rho) = e^{(x,\rho)} (f(x, \rho/|\rho|) + \psi(x, \rho)),$$

where

$$\psi|_{\partial\Omega_-(\xi)} = 0,$$

and

$$\|\psi(\cdot, \rho)\|_{L^2(\Omega)} \leq \frac{M}{\tau} \tag{4.1}$$

for some $M > 0$ and $\tau \geq \tau_0 > 0$.

Corollary 4.3 *Let $\rho \in \mathbb{C}^d$ be such that $\langle \rho, \rho \rangle = 0$, where $\rho = \tau(\xi + i\eta)$ with $\xi, \eta \in S^{d-1}$, and $\lambda > 0$.*

1. *There is a solution w to $-\Delta w + (q - \lambda)w = 0$ in Ω of the form*

$$u_{q,\lambda}(x, \rho) := e^{(x,\rho)} (1 + \psi_q(x, \rho)), \tag{4.2}$$

where

$$\psi_q|_{\partial\Omega_-(\xi)} = 0$$

and

$$\|\psi_q(\cdot, \rho)\|_{L^2(\Omega)} \leq \frac{M}{\tau}, \quad \tau \geq \tau_0,$$

for some $M > 0$ and $\tau_0 > 0$.

- 2. The function $(x, t) \mapsto E_{\alpha,1}(-\lambda t^\alpha)u_{q,\lambda}(x, \rho)$ is a solution of $(\partial_t^\alpha - \Delta + q)w = 0$ in Ω_T .
- 3. The function $(x, t) \mapsto e^{-\lambda t}u_{q,\lambda}(x, \rho)$ is a solution of $({}_0D_t^\alpha - \Delta + q)w = 0$ in Ω_T .

Proof 1. Applying Lemma 4.2 for $f = 1$ and $q := q - \lambda$, we obtain the result.

- 2. Using Proposition 3.5 and item 1, it is easy to prove items 2 and 3. □

Lemma 4.4 (Carleman estimates [7]) *For $q \in L^\infty(\Omega)$, there exist $\tau_0 > 0$ and $C > 0$ such that for all $u \in C^2(\Omega)$, $u|_{\partial\Omega} = 0$, and $\tau \geq \tau_0$, we have the estimate*

$$\begin{aligned}
 & C\tau^2 \int_{\Omega} |e^{-\tau(x,\xi)} u|^2 dx + \tau \int_{\partial\Omega_+} \langle v, \xi \rangle |e^{-\tau(x,\xi)} \partial_v u|^2 dS \\
 & \leq \int_{\Omega} |e^{-\tau(x,\xi)} (\Delta - q)u|^2 dx - \tau \int_{\partial\Omega_-} \langle v, \xi \rangle |e^{-\tau(x,\xi)} \partial_v u|^2 dS.
 \end{aligned} \tag{4.3}$$

Lemma 4.5 *If $u \in H_\Delta(\Omega_T)$ and $u|_{\Sigma_T} = 0$, then $u \in L^2(0, T; H^2(\Omega))$.*

Proof Suppose first that $u \in C^2(\bar{\Omega}_T)$. From the first Green formula we obtain

$$2 \int_{\Omega_T} |\nabla u|^2 dx = -2 \int_{\Omega_T} u \Delta u dx \leq \int_{\Omega_T} (|u|^2 + |\Delta u|^2) dx = \|u\|_{H_\Delta(\Omega_T)}^2 < \infty.$$

From the formula of Kadlec [24],

$$\sum_{ij} \int_{\Omega_T} |\partial_i \partial_j u|^2 dx \leq \int_{\Omega_T} |\Delta u|^2 dx + C_0 \int_{\Sigma_T} |\partial_v u|^2 dx,$$

since $\int_{\Sigma_T} |\partial_v u|^2 dx \leq C_1 \int_{\Omega_T} |\Delta u|^2 dx$, we get

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leq C_2 \|u\|_{H_\Delta(\Omega_T)}.$$

Using standard density arguments, we obtain

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leq C_2 \|u\|_{H_\Delta(\Omega_T)}, \quad \forall u \in H_\Delta(\Omega_T), \text{ and } u|_{\Sigma_T} = 0.$$

Thus $u \in L^2(0, T; H^2(\Omega))$. □

Lemma 4.6 *Let $u_i := u_{q_i f_i}^\alpha$, $i = 1, 2$, be solutions of the fractional diffusion equations $(P_{q_i f_i}^\alpha)$, $q_i \in L^\infty(\Omega)$ and $f_i \in L^2(\Sigma_T)$. Then we have*

$$\partial_t^\alpha u_1 \star u_2 = u_1 \star {}_0D_t^\alpha u_2, \tag{4.4}$$

where \star is the convolution product.

Proof From [2] we have

$$\int_a^b g(s) {}_a^c D_s^\alpha f(s) ds = \int_a^b f(s) {}_s D_b^\alpha g(s) ds + \sum_{j=0}^{n-1} [{}_s D_b^{\alpha+j-n} g(s) \cdot {}_s D_b^{n-1-j} f(s)]_a^b \tag{4.5}$$

for $0 < \alpha < 1$, $a = 0$, and $b = t$, and

$$\int_0^t g(s) \partial_s^\alpha f(s) ds = \int_0^t f(s) {}_s D_t^\alpha g(s) ds + [{}_s D_t^{\alpha-1} g(s) \cdot f(s)]_0^t. \tag{4.6}$$

Then

$$\begin{aligned} \partial_t^\alpha u_1 \star u_2 &= \int_0^t \partial_s^\alpha u_1(x, s) u_2(x, t-s) ds \\ &= \int_0^t u_1(x, s) {}_s D_t^\alpha u_2(x, t-s) ds + [{}_s D_t^{\alpha-1} u_2(x, t-s) u_1(x, s)]_0^t. \end{aligned} \tag{4.7}$$

Since $\alpha - 1 < 0$, we have ${}_s D_t^{\alpha-1} u_2(x, t-s) = {}_s I_t^{1-\alpha} u_2(x, t-s)$ (see [2]) and

$${}_s I_t^{1-\alpha} u_2(x, t-s) = \frac{1}{\Gamma(1-\alpha)} \int_s^t (\tau-s)^{-\alpha} u_2(x, t-\tau) d\tau.$$

Using integration by parts, we get

$${}_s I_t^{1-\alpha} u_2(x, t-s) = \frac{1}{\Gamma(2-\alpha)} \left[(t-s)^{1-\alpha} u_2(x, t-s) + \int_s^t (\tau-s)^{1-\alpha} \frac{\partial u_2}{\partial \tau}(x, t-\tau) d\tau \right].$$

We deduce that $\lim_{s \rightarrow t} {}_s D_t^{\alpha-1} u_2(x, t-s) = 0$ and

$$\int_0^t \partial_s^\alpha u_1(x, s) u_2(x, t-s) ds = \int_0^t u_1(x, s) {}_s D_t^\alpha u_2(x, t-s) ds.$$

By definition

$${}_s D_t^\alpha u_2(x, t-s) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_s^t (t-\tau)^{-\alpha} u_2(x, \tau-s) d\tau.$$

By change of variable $\xi = \tau - s$ we obtain

$$\begin{aligned} {}_s D_t^\alpha u_2(x, t-s) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^{t-s} (t-s-\xi)^{-\alpha} u_2(x, \xi) d\xi \\ &= {}_0 D_{t-s}^\alpha u_2(x, t-s), \end{aligned} \tag{4.8}$$

which implies

$$\int_0^t \partial_s^\alpha u_1(x, s) u_2(x, t-s) ds = \int_0^t u_1(x, s) {}_0 D_{t-s}^\alpha u_2(x, t-s) ds$$

and

$$\partial_t^\alpha u_1 \star u_2 = u_1 \star_0 D_t^\alpha u_2. \tag{□}$$

Remark 4.7 If $u_2(x, 0) = 0$, then from Remark 3.3 we obtain

$$\partial_t^\alpha u_1 \star u_2 = u_1 \star \partial_t^\alpha u_2.$$

Proof of Theorem 4.1 Let $\xi \in S^{d-1}$. Fix $k \in \mathbb{R}^d$ such that $\langle k, \xi \rangle = 0$. By Corollary 4.3 there exists $u_2 \in H_\Delta(\Omega_T)$ satisfying $(\partial_t^\alpha - \Delta + q_2)u_2 = 0$ in Ω_T of the form

$$u_2(x, t) = E_\alpha(-\lambda t^\alpha)u_{q_2, \lambda}(x, \rho_2),$$

where $u_{q_2, \lambda}$ is defined by (4.2), $\rho_2 = \tau\xi - i\frac{k+\ell}{2}$, $\langle k, \ell \rangle = \langle \xi, \ell \rangle = 0$, and $|k + \ell|^2 = 4\tau^2$ (under these conditions, $\langle \rho_2, \rho_2 \rangle = 0$). In dimension $d \geq 3$, we can always choose such a vector ℓ .

Since $C_{q_1, \varepsilon} = C_{q_2, \varepsilon}$, there exists $u_1 \in H_\Delta(\Omega_T)$ such that

$$\begin{cases} (\partial_t^\alpha - \Delta + q_1)u_1 = 0 & \text{in } \Omega_T, \\ u_1(x, 0) = u_2(x, 0) & \text{in } \Omega, \\ u_1 = u_2 & \text{on } \Sigma_T, \\ \partial_\nu u_1 = \partial_\nu u_2 & \text{on } \partial\Omega_{-\varepsilon}(\xi) \times (0, T). \end{cases}$$

Setting $u = u_1 - u_2$ and $q = q_1 - q_2$, we have

$$\begin{cases} (\partial_t^\alpha - \Delta + q_1)u = -qu_2 & \text{in } \Omega_T, \\ u(x, 0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma_T, \\ \partial_\nu u = 0 & \text{on } \partial\Omega_{-\varepsilon}(\xi) \times (0, T). \end{cases}$$

By Lemma 4.5, since $u \in H_\Delta(\Omega_T)$ and $u|_{\Sigma_T} = 0$, it follows that $u \in L^2(0, T; H^2(\Omega))$.

From Green’s formula and Lemma 4.6, for $v \in H_\Delta(\Omega_T)$, we obtain

$$\begin{aligned} \int_\Omega (\partial_t^\alpha - \Delta + q_1)u \star \bar{v} \, dx &= - \int_\Omega qu_2 \star \bar{v} \, dx \\ &= \int_\Omega u \star ({}_0D_t^\alpha - \Delta + q_1)\bar{v} \, dx + \int_{\partial\Omega} \partial_\nu u \star \bar{v} \, ds. \end{aligned} \tag{4.9}$$

We choose

$$\bar{v}(x, t) = e^{-\lambda t} u_{q_1, \lambda}(x, \rho_1),$$

which is a solution of $({}_0D_t^\alpha - \Delta + q_1)\bar{v} = 0$, where $u_{q_1, \lambda}(x, \rho_1) = e^{(x, \rho_1)}(1 + \psi_{q_1}(x, \rho_1))$,

$$\psi_{q_1}|_{\partial\Omega_{-\varepsilon}(\xi)} = 0,$$

and

$$\|\psi_{q_1}(\cdot, \rho_1)\|_{L^2(\Omega)} \leq \frac{\tilde{M}}{\tau}, \quad \tau \geq \tau_0,$$

where $\rho_1 = -\tau\xi - i\frac{k-\ell}{2}$. We have $\rho_1 + \rho_2 = -ik$ and $|\rho_1|^2 = 2\tau^2$.

Then equation (4.9) becomes

$$- \int_\Omega qu_2 \star \bar{v} \, dx = \int_{\partial\Omega} \partial_\nu u \star \bar{v} \, ds. \tag{4.10}$$

Let $\lambda > 0$ and $s > \lambda^{\frac{1}{\alpha}}$, extending u by 0 on (T, ∞) . Applying the Laplace transform to (4.10), we obtain

$$-\int_{\Omega} q\tilde{u}_2\tilde{v} \, dx = \int_{\partial\Omega} \partial_\nu \tilde{u}\tilde{v} \, dS, \tag{4.11}$$

where $\tilde{u} := \tilde{u}(\cdot, s)$ is a solution of the following problem:

$$\begin{cases} \Delta \tilde{u} - (q_1 + s^\alpha)\tilde{u} = q\tilde{u}_2 & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega, \\ \partial_\nu \tilde{u} = 0 & \text{on } \partial\Omega_{-, \varepsilon}(\xi), \end{cases}$$

and the Laplace transform \tilde{g} of a function g is defined by

$$\tilde{g}(s) = \int_0^\infty e^{-st} g(t) \, dt.$$

Using Lemma 3.6, we obtain

$$\begin{aligned} \tilde{u}_2(x, s) &= \frac{s^{\alpha-1}}{s^\alpha - \lambda} u_{q_2, \lambda}(x, \rho_2), \\ \tilde{v}(x, s) &= \frac{1}{s^\alpha - \lambda} u_{q_1, \lambda}(x, \rho_1). \end{aligned}$$

Equation (4.11) becomes

$$\int_{\Omega} qu_{q_1, \lambda}(x, \rho_1)u_{q_2, \lambda}(x, \rho_2) \, dx = \frac{(s^\alpha - \lambda)}{s^{\alpha-1}} \int_{\partial\Omega} \partial_\nu \tilde{u}u_{q_1, \lambda}(x, \rho_1) \, dS.$$

In the following step, we show that $\lim_{\tau \rightarrow \infty} \int_{\partial\Omega} \partial_\nu \tilde{u}u_{q_1, \lambda}(x, \rho_1) \, dS = 0$. Indeed, since $\partial_\nu \tilde{u} = 0$ on $\partial\Omega_{-, \varepsilon}(\xi)$, we have

$$\begin{aligned} &\left| \int_{\partial\Omega} \partial_\nu \tilde{u}u_{q_1, \lambda}(x, \rho_1) \, dS \right| \\ &= \left| \int_{\partial\Omega_{+, \varepsilon}(\xi)} \partial_\nu \tilde{u}e^{(x, \rho_1)} (1 + \psi_{q_1}(x, \rho_1)) \, dS \right| \\ &\leq \int_{\partial\Omega_{+, \varepsilon}(\xi)} |\partial_\nu \tilde{u}e^{-\tau(x, \xi)}| |1 + \psi_{q_1}(x, \rho_1)| \, dS \\ &\leq \left(\int_{\partial\Omega_{+, \varepsilon}(\xi)} |\partial_\nu \tilde{u}e^{-\tau(x, \xi)}|^2 \, dS \right)^{\frac{1}{2}} \left(\int_{\partial\Omega_{+, \varepsilon}(\xi)} |1 + \psi_{q_1}(x, \rho_1)|^2 \, dS \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\partial\Omega_{+, \varepsilon}(\xi)} |\partial_\nu \tilde{u}e^{-\tau(x, \xi)}|^2 \, dS \right)^{\frac{1}{2}} \left(2 \int_{\partial\Omega_{+, \varepsilon}(\xi)} (1 + |\psi_{q_1}(x, \rho_1)|^2) \, dS \right)^{\frac{1}{2}} \\ &\leq \sqrt{2} \left(\int_{\partial\Omega_{+, \varepsilon}(\xi)} |\partial_\nu \tilde{u}e^{-\tau(x, \xi)}|^2 \, dS \right)^{\frac{1}{2}} (\text{vol}(\partial\Omega) + \|\psi_{q_1}\|_{L^2(\partial\Omega)}^2)^{\frac{1}{2}}. \end{aligned}$$

From [7, p. 666] we have

$$\|\psi_{q_1}\|_{C(\partial\Omega)} \leq C\tau^{\frac{1}{4}},$$

and since

$$\|\psi_{q_1}\|_{L^2(\partial\Omega)} \leq \text{vol}(\partial\Omega) \|\psi_{q_1}\|_{C(\partial\Omega)},$$

we have

$$\left| \int_{\partial\Omega} \partial_\nu \tilde{u}_{q_1, \lambda}(x, \rho_1) dS \right| \leq C\sqrt{2}(\text{vol}(\partial\Omega))^{\frac{1}{2}} (1 + \tau^{\frac{1}{4}}) \left(\int_{\partial\Omega_{+, \varepsilon}(\xi)} |\partial_\nu \tilde{u} e^{-\tau(x, \xi)}|^2 dS \right)^{\frac{1}{2}}.$$

From estimates (4.3) and (4.1) we obtain

$$\begin{aligned} \tau \varepsilon \int_{\partial\Omega_{+, \varepsilon}(\xi)} |\partial_\nu \tilde{u} e^{-\tau(x, \xi)}|^2 ds &\leq \tau \int_{\partial\Omega_{+, \varepsilon}(\xi)} \langle \nu, \xi \rangle |\partial_\nu \tilde{u} e^{-\tau(x, \xi)}|^2 ds \\ &\leq \int_{\Omega} |e^{-\tau(x, \xi)} (\Delta - (q_1 + s^\alpha)) \tilde{u}|^2 dx = \int_{\Omega} |e^{-\tau(x, \xi)} q \tilde{u}|^2 dx \\ &\leq 2(\|q_1\|_\infty + \|q_2\|_\infty)^2 \int_{\Omega} (1 + |\psi_{q_2}(x, \rho_2)|^2) dx \\ &\leq 2(\|q_1\|_\infty + \|q_2\|_\infty)^2 (\text{vol}(\Omega) + \|\psi_{q_2}(x, \rho_2)\|_{L^2(\Omega)}^2) \\ &\leq 2(\|q_1\|_\infty + \|q_2\|_\infty)^2 \left(\text{vol}(\Omega) + \frac{M}{\tau^2} \right). \end{aligned}$$

Hence we have proved that

$$\left| \int_{\partial\Omega} \partial_\nu \tilde{u} \tilde{v} ds \right| \leq \tilde{C} \frac{(1 + \tau^{\frac{1}{4}})}{\sqrt{\tau}} \left(\sqrt{\text{vol}(\Omega)} + \frac{M}{\tau} \right),$$

where

$$\tilde{C} = \frac{2C(\|q_1\|_\infty + \|q_2\|_\infty) \sqrt{\text{vol}(\partial\Omega)}}{\sqrt{\varepsilon}},$$

and then

$$\lim_{\tau \rightarrow \infty} \int_{\partial\Omega} \partial_\nu \tilde{u} \tilde{v} dS = 0.$$

Using equalities (4.1) and (4.11), by tending τ to ∞ we have

$$\int_{\Omega} q(x) e^{-i(x, k)} dx = 0.$$

Changing $\xi \in S^{d-1}$ in a small conic neighborhood and using the fact that $\hat{q}(k)$ is analytic, we get that $q = 0$, and then $q_1 = q_2$. □

5 Conclusion

In this paper, we established a uniqueness result for the coefficient identification problem in a fractional diffusion equation in a bounded domain from the observation of the Cauchy data on particular subsets of the boundary using Carleman estimates and particular complex geometrical solutions of the fractional diffusion equation.

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