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Least energy sign-changing solutions for Kirchhoff–Poisson systems

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Abstract

The paper deals with the following Kirchhoff–Poisson systems:

$$\begin{cases} -(1 + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx)\Delta u + u + k(x)\phi u + \lambda |u|^{p-2}u = h(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{0.1}$$

where the functions k and h are nonnegative, $0 \le \lambda, b$; $2 \le p \le 4 < q < 6$. Via a constraint variational method combined with a quantitative lemma, some existence results on one least energy sign-changing solution with two nodal domains to the above systems are obtained. Moreover, the convergence property of u_b as $b \searrow 0$ is established.

Keywords: Kirchhoff–Poisson systems; Least energy sign-changing solutions; Constraint variational method; Nodal domains

1 Introduction

Consider the following Kirchhoff–Poisson systems:

$$\begin{cases} -(1 + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx)\Delta u + u + k(x)\phi u + \lambda |u|^{p-2}u = h(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where k and h are nonnegative functions, $0 \le \lambda, b$; $2 \le p \le 4 < q < 6$.

When $b = 0$, systems (1.1) reduce to the following Schrödinger–Poisson systems:

$$\begin{cases} -\Delta u + u + k(x)\phi u + \lambda |u|^{p-2}u = h(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

which stem from quantum mechanics and have important applications in the semiconductor. From the physical viewpoint, the above systems have been introduced as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanics. The unknowns u and ϕ represent the wave functions associated to the particle and electric potential. For more details, one can refer to [1–3] and the references therein.

In [4], Cerami and Vaira studied the following Schrödinger–Poisson systems:

$$\begin{cases} -\Delta u + u + k(x)\phi(x)u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

with $p \in (3,5)$. Under some suitable conditions, some existence results on positive solutions were obtained. Recently, Zhong and Tang [5] investigated the following Schrödinger–Poisson systems:

$$\begin{cases} -\Delta u + u + k(x)\phi(x)u = \lambda f(x) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta \phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where the functions k and f are nonnegative, $0 < \lambda < \lambda_1$ and λ_1 is the eigenvalue of the problem $-\Delta u + u = \lambda f(x)u$ in $H^1(\mathbb{R}^3)$. Via the variational method, the authors obtained the existence results on the ground state sign-changing solution for $0 < \lambda < \lambda_1$. Replacing u with $V(x)u$ in (1.3), Batista and Furtado in [6] studied the following systems:

$$\begin{cases} -\Delta u + V(x)u + k(x)\phi u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $p \in (3,5)$ and $a(x)$ satisfies some mild conditions, especially, the potential function V can be nonconstant and indefinite in sign. By a variational approach, they also get some results of the existence of one nonnegative solution and one sign-changing solution. For the related research on this problem, the reader can refer to the literature [7–14].

On the other hand, if $k = 0$ in (1.1), then (1.1) reduce to the following Kirchhoff-type problem:

$$-\left(1 + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + u + \lambda|u|^{p-2}u = h(x)|u|^{q-2}u, \quad x \in \mathbb{R}^3. \tag{1.4}$$

Associated with the above problem, we have to mention the following Kirchhoff Dirichlet problem:

$$\begin{cases} -(a + b\int_{\Omega} |\nabla u|^2\,dx)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

which stems from the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0$$

proposed by Kirchhoff regarded as an extension of the classical D'Alembert wave equation on free vibrations of elastic strings. Due to its importance on the physical background, the Kirchhoff boundary problem received increasingly more attention. Recently, with the help of the variational methods, a number of results on the existence and multiplicity of solutions for the Kirchhoff problem

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx)\Delta u = f(u), & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.5}$$

have been established under various suitable conditions, where $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and f satisfies various suitable conditions; see, for example [15–21] and the references therein. Recently, Baraket and Molica Bisci in [22] studied the following Kirchhoff-type problem:

$$\begin{cases} -(a+b\int_{\Omega}|\Delta u|^2)\Delta u = \lambda f(x,u)+\mu g(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded open subset. Under some suitable conditions, the authors obtained multiplicity results via applying the three critical points theorem. Very recently, Xu and Chen in [23] investigated the following Kirchhoff-type problem:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\Delta u|^2)\Delta u + Vu = f(u) & \text{in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \end{cases}$$

and established the existence of a ground state solution by applying a critical point theorem similar to the mountain pass lemma (see [24]). Moreover, for the related research on fractional Kirchhoff-type and Schrödinger-type problems, the reader can refer to [25, 26] and to the references therein and the monograph [27] published recently. However, regarding the existence of sign-changing solutions for the Kirchhoff problem, there are very few results in the literature. Recently, Shuai in [28] studied the existence of the least energy sign-changing solution of problem (1.5) and its convergence property on $\{u_n\}$ as $b\searrow 0$. Later, under conditions different from [28], with the help of some analytical techniques and a non-Nehari manifold method, Tang and Cheng in [29] further studied problem (1.5) and obtained some existence results on a ground state sign-changing solution u_b as well as its convergence property.

When $b\neq 0$, and $k\neq 0$, the systems (1.1) stand for Kirchhoff–Poisson systems. Because the nonlocal terms $b\int_{\mathbb{R}^3}(|\nabla u|^2\,dx)\Delta u$ and ϕ_u are involved in the equation, the problem is totally different from the case $b=0$ and $k=0$. In [30], Zhang considered the following general singular Kirchhoff–Poisson systems:

$$\begin{cases} -(a+b\int_{\Omega}|\Delta u|^2)\Delta u + \phi u = \lambda h f(u)+g(u), & \text{in } \Omega, \\ -\Delta\phi = u^2, & \text{in } \Omega, \\ u>0, & \text{in } \Omega, \\ u=\phi=0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, constants $a>0$, $b\geq 0$ and $\lambda>0$ is a parameter, functions f, g, h satisfy some conditions. By combining the variational method with a perturbation method, the author obtained the existence of two positive solutions if the parameter λ is small enough. In [31], Liu and Wang investigated the following Kirchhoff–Poisson systems:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\Delta u|^2)\Delta u + u - qK(x)\phi u = f(x)|u|^{p-1}u, & x\in\mathbb{R}^3, \\ -\Delta\phi = qK(x)u^2, & x\in\mathbb{R}^3, \end{cases}$$

where the constants $a>0$, $b\geq 0$, $1<p<5$, $q>0$ and the functions $f,K:\mathbb{R}^3\to\mathbb{R}$ are nonnegative. Applying the critical point theorem with parameter λ (see [24]), the authors obtained the existence of a positive solution as well as a ground state solution with $q=1$ corresponding to its limit problem. Moreover, very recently, Wang, Rădulescu and Zhang in [32] studied a kind of fractional Kirchhoff–Poisson systems as follows:

$$\begin{cases} M([u]_s^2)(-\Delta)^s u + V(x)u + \phi(x)u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t\phi(x) = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $s,t\in(0,1)$ with $2t+4s>3$, $M:\mathbb{R}_0^+\to\mathbb{R}^+$ is a continuous function satisfying certain assumptions, the potential function $V:\mathbb{R}^3\to\mathbb{R}^+$ is continuous, f satisfies a Carathéodory condition, λ is a positive parameter. By applying the fountain theorem for the subcritical case and the symmetric mountain pass theorem for the critical case, respectively, the authors obtained infinitely many solutions for the system. Different from the works mentioned above, in the present paper, we shall combine a constraint variational method with quantitative deformation properties to establish the existence results as regards one least energy sigh-changing solution with two nodal domains to problem (1.1). Moreover, we also study the convergence property on u_b as $b\searrow 0$.

2 Preliminaries

Throughout this paper, we always assume the following conditions hold.

(l) $0\leq\lambda,b$; $2\leq p\leq 4<q<6$.

(k) $k\in L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)\backslash\{0\}$ and $k(x)\geq 0$ for a.e. $x\in\mathbb{R}^3$.

(h) $h(x)>0$ for a.e. $x\in\mathbb{R}^3$ and there exists $q_1\in(q,6)$ such that $h\in L^{\frac{6}{6-q_1}}$.

In addition, $\mathbb{R}_+=[0,\infty)$, $\mathbb{R}_+^0=(0,\infty)$, $D^{1,2}(\mathbb{R}^3)$ is the Sobolev space equipped with the norm $\|u\|_{D^{1,2}}=(\int_{\mathbb{R}^3}|\nabla u|^2\,dx)^{\frac{1}{2}}$ and L^s is the Lebesgue space with norm $|u|_s=(\int_{\mathbb{R}^3}|u|^s\,dx)^{\frac{1}{s}}$ for $s\geq 1$. Also, $H^1(\mathbb{R}^3)$ is the Sobolev space equipped with the norm

$$\|u\|=\left(\int_{\mathbb{R}^3}\left(|\nabla u|^2+u^2\right)dx\right)^{\frac{1}{2}}.$$

C is for various positive constants, which can be different from one line to another line in the text.

Let $\bar{S}$ be the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$. That is,

$$\bar{S}=\inf_{u\in D^{1,2}(\mathbb{R}^3)\backslash\{0\}}\frac{\|u\|_{D^{1,2}}}{|u|_6}. \tag{2.1}$$

Similarly,

$$S_r=\inf_{u\in H^1(\mathbb{R}^3)\backslash\{0\}}\frac{(\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx)^{\frac{1}{2}}}{|u|_r},\quad r\in[1,6]. \tag{2.2}$$

For any fixed $u \in H^1(\mathbb{R}^3)$, from the Lax–Milgram theorem it follows that there exists an unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ that satisfies $-\Delta\phi = k(x)u^2$ weakly, that is, for any $v \in D^{1,2}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v\,dx = \int_{\mathbb{R}^3} k(x)u^2 v\,dx.$$

Moreover,

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{k(y)u^2(y)}{|x-y|}\,dy. \tag{2.3}$$

Let

$$L_{\phi_u}(v) = \int_{\mathbb{R}^3} k(x)\phi_u v^2\,dx, \quad v \in H^1\bigl(\mathbb{R}^3\bigr),$$

then

$$L_{\phi_u}(v) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{k(x)k(y)u^2(x)v^2(y)}{|x-y|}\,dx\,dy. \tag{2.4}$$

Clearly, the energy functional associated with (1.1) can be expressed by

$$\begin{aligned} J_b(u) = {}& \frac{1}{2}\|u\|^2 + \frac{b}{4}|\nabla u|_2^4 + \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_u(x)u^2\,dx \\ & + \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p\,dx - \frac{1}{q}\int_{\mathbb{R}^3} h(x)|u|^q\,dx. \end{aligned} \tag{2.5}$$

Lemma 2.1 ([5]) *Suppose that $k \in L^\infty(\mathbb{R}^3)$. Then, for any $u \in H^1(\mathbb{R}^3)$, there exists $C > 0$ such that*

$$L_{\phi_u}(u) = \int_{\mathbb{R}^3} k(x)\phi_u u^2\,dx = \int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx \le C\|u\|^4.$$

Lemma 2.2 *Assume that condition (k) holds. Then we have*

(i) $\phi_u \ge 0$, *for any $u \in H^1(\mathbb{R}^3)$;*
(ii) *for any $t \in R$, $\phi_{tu} = t^2\phi_u$;*
(iii) $\|\phi_u\|_{D^{1,2}} \le \bar{S}^{-1}S_6^{-2}|k|_2\|u\|^2$;
(iv) $|\phi_u|_6 \le \bar{S}^{-1}\|\phi_u\|_{D^{1,2}}$.

Proof Under the condition (k), the conclusions (i) and (ii) directly follow from Eq. (2.3). the conclusions (iii) and (iv) directly follow from (2.4) in [4]. □

For $R > 0$, let $\Omega_R = \{x \in \mathbb{R}^3 : |x| \le R\}$, $\Omega_R^C = \{x \in \mathbb{R}^3 : |x| > R\}$. Denote

$$u^+(x) = \max\bigl\{u(x), 0\bigr\}, u^-(x) = \min\bigl\{u(x), 0\bigr\}.$$

Lemma 2.3 *Assume that conditions* (l), (k) *and* (h) *hold. Then, for any* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *with* $u_n^{\pm} \rightharpoonup u^{\pm}$ *weakly in* $H^1(\mathbb{R}^3)$ *and* $u_n^{\pm}(x) \longrightarrow u^{\pm}(x)$ *for a.e.* $x \in \mathbb{R}^3$, *we have*

(i) (a) $$\int_{\mathbb{R}^3} k(x)\phi_{u_n} u_n^2\,dx \to \int_{\mathbb{R}^3} k(x)\phi_u u^2\,dx,$$
$$\int_{\mathbb{R}^3} k(x)\phi_{u_n^{\pm}} \left(u_n^{\pm}\right)^2 dx \to \int_{\mathbb{R}^3} k(x)\phi_{u^{\pm}} \left(u^{\pm}\right)^2 dx,$$

(b) $$\int_{\mathbb{R}^3} k(x)\phi_{u_n} \left(u_n^{\pm}\right)^2 dx \to \int_{\mathbb{R}^3} k(x)\phi_u \left(u^{\pm}\right)^2 dx,$$

(c) $$\int_{\mathbb{R}^3} k(x)\phi_{u_n^+} \left(u_n^-\right)^2 dx \to \int_{\mathbb{R}^3} k(x)\phi_{u^+} \left(u^-\right)^2 dx,$$

(d) $$\int_{\mathbb{R}^3} k(x)\phi_{u_n} u_n\varphi\,dx \to \int_{\mathbb{R}^3} k(x)\phi_u u\varphi\,dx, \quad \forall \varphi \in H^1\left(\mathbb{R}^3\right).$$

(ii) (a) $$\int_{\mathbb{R}^3} h(x)|u_n|^q\,dx \to \int_{\mathbb{R}^3} h(x)|u|^q\,dx,$$

(b) $$\int_{\mathbb{R}^3} h(x)|u_n|^{q-2}u_n\varphi\,dx \to \int_{\mathbb{R}^3} h(x)|u|^{q-2}u\varphi\,dx, \quad \forall \varphi \in H^1\left(\mathbb{R}^3\right),$$

(c) $$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n\varphi\,dx \to \int_{\mathbb{R}^3} |u|^{p-2}u\varphi\,dx, \quad \forall \varphi \in H^1\left(\mathbb{R}^3\right).$$

Proof Item (i) Conclusions (a) and (d) follow from Lemma 6 in [5] and Lemma 2.1 in [6] and Lemma 2.1 in [4]. Conclusion (b) and (c) can be similarly proved, we omit it.

Item (ii). We only prove that $\int_{\mathbb{R}^3} h(x)|u_n|^q\,dx \to \int_{\mathbb{R}^3} h(x)|u|^q\,dx$. The other relations can be obtained similarly.

In fact, the condition $q < q_1 < 6$ implies that $4 < \frac{6q}{q_1} < 6$. Let $r = \frac{6q}{q_1}$. The sequence $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ shows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Therefore, by (2.2) there exists $M > 0$ such that $|u_n|_r^q \le M$, $|u|_r^q \le M$. Thus

$$\begin{aligned}\left|\int_{\Omega_R^C} h(x)\left(|u_n|^q - |u|^q\right)dx\right| &\le \int_{\Omega_R^C} h(x)\left(|u_n|^q + |u|^q\right)dx \\ &\le \left(\int_{\Omega_R^C} h^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}} \left(|u_n|_r^q + |u|_r^q\right) \\ &\le 2M\left(\int_{\Omega_R^C} h^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}}.\end{aligned}$$

Because $h \in L_{\frac{6}{6-q_1}}$, we can choose $R > 0$ large enough so that $2M(\int_{\Omega_R^C} h^{\frac{6}{6-q_1}}\,dx)^{\frac{6-q_1}{6}} < \varepsilon$, and therefore,

$$\left|\int_{\Omega_R^C} h(x)\left(|u_n|^q - |u|^q\right)dx\right| < \varepsilon.$$

On the other hand, by the absolute continuity on integral together with $h \in L_{\frac{6}{6-q_1}}$, for any $\eta > 0$, there is $\delta > 0$ such that $(\int_G |h(x)|^{\frac{6}{6-q_1}}\,dx)^{\frac{6-q_1}{6}} < \eta$ for each $G \subset \Omega_R$ with mes $G < \delta$.

Then

$$\int_G h(x)|u_n|^q\,dx \le |u_n|_r^q \left(\int_G |h(x)|^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}} < M\eta.$$

So, we can apply the Vitali theorem to obtain

$$\lim_{n\to\infty}\int_{\Omega_R} h(x)|u_n|^q\,dx = \int_{\Omega_R} h(x)|u|^q\,dx.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^3} h(x)|u_n|^q\,dx &= \lim_{n\to\infty}\int_{\Omega_R^c} h(x)|u_n|^q\,dx + \lim_{n\to\infty}\int_{\Omega_R} h(x)|u_n|^q\,dx\\ &= \int_{\Omega_R^c} h(x)|u|^q\,dx + \int_{\Omega_R} h(x)|u|^q\,dx = \int_{\mathbb{R}^3} h(x)|u|^q\,dx.\end{aligned}$$

The proof is complete. □

In terms of Lemmas 2.1–2.3, it can be verified that J_b is well defined on $H^1(\mathbb{R}^3)$ and is of C^1 as well as

$$\begin{aligned}\langle J_b'(u), v\rangle &= \int_{\mathbb{R}^3}(\nabla u\cdot\nabla v + uv)\,dx + b|\nabla u|_2^2\left(\int_{\mathbb{R}^3}\nabla u\cdot\nabla v\,dx\right)\\ &\quad+\int_{\mathbb{R}^3} k(x)\phi_u uv\,dx + \lambda\int_{\mathbb{R}^3}|u|^{p-2}uv\,dx - \int_{\mathbb{R}^3} h(x)|u|^{q-2}uv\,dx.\end{aligned}\tag{2.6}$$

Obviously, $u \in H^1(\mathbb{R}^3)$ is a critical point of J_b if and only if (u, ϕ_u) is a solution of systems (1.1). Noting that ϕ_u is nonnegative for any $u \in H^1(\mathbb{R}^3)$, (u, ϕ_u) is a sign-changing solution of system (1.1) if and only if u is a critical point of J_b with $u^\pm \neq 0$.

By (2.3)–(2.4) and the Fubini theorem, we know that

$$L_{\phi_{u^+}}(u^-) = \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx = \int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^+)^2\,dx = L_{\phi_{u^-}}(u^+)\tag{2.7}$$

and

$$J_b(u) = J_b(u^+) + J_b(u^-) + \frac{b}{2}|\nabla u^+|_2^2|\nabla u^-|_2^2 + \frac{1}{2}L_{\phi_{u^+}}(u^-),\tag{2.8}$$

$$\langle J_b'(u), u^+\rangle = \langle J_b'(u^+), u^+\rangle + b|\nabla u^+|_2^2|\nabla u^-|_2^2 + L_{\phi_{u^+}}(u^-),\tag{2.9}$$

$$\langle J_b'(u), u^-\rangle = \langle J_b'(u^-), u^-\rangle + b|\nabla u^+|_2^2|\nabla u^-|_2^2 + L_{\phi_{u^+}}(u^-).\tag{2.10}$$

Let

$$M_b = \{u \in H^1(\mathbb{R}^3) : u^\pm \neq 0, \langle J_b'(u), u^+\rangle = \langle J_b'(u), u^-\rangle = 0\}.$$

In the following, we will look for the minimum point of the functional J_b on M_b, which is the sigh-changing solution of systems (1.1).

The following lemma is crucial and plays an important role in obtaining our main results later.

Lemma 2.4 *Assume $u \in H^1(\mathbb{R}^3)$ with $u^\pm \neq 0$. Then there is an unique pair $(s_u, t_u) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$ satisfying that $s_u u^+ + t_u u^- \in M_b$.*

Proof For any $u \in H^1(\mathbb{R}^3)$ with $u^\pm \neq 0$, clearly, by (2.6)–(2.10) we know that $su^+ + tu^- \in M_b$ with $(s,t) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$ if and only if the pair (s,t) satisfies the following systems:

$$\begin{cases} s^2\|u^+\|^2 + bs^4|\nabla u^+|_2^4 + s^4\int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^+)^2\,dx + \lambda s^p|u^+|_p^p \\ \quad + s^2t^2[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx] - s^q\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx = 0, \\ t^2\|u^-\|^2 + bt^4|\nabla u^-|_2^4 + t^4\int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx + \lambda t^p|u^-|_p^p \\ \quad + s^2t^2[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx] - t^q\int_{\mathbb{R}^3} h(x)|u^-|^q\,dx = 0. \end{cases} \tag{2.11}$$

To study the solvability of systems (2.11), we investigate the following auxiliary systems with a parameter $\eta \in [0,1]$:

$$\begin{cases} s^2\|u^+\|^2 + bs^4|\nabla u^+|_2^4 + s^4\int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^+)^2\,dx + \lambda s^p|u^+|_p^p \\ \quad + \eta s^2t^2[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx] - s^q\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx = 0, \\ t^2\|u^-\|^2 + bt^4|\nabla u^-|_2^4 + t^4\int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx + \lambda t^p|u^-|_p^p \\ \quad + \eta s^2t^2[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx] - t^q\int_{\mathbb{R}^3} h(x)|u^-|^q\,dx = 0. \end{cases} \tag{2.12}$$

Let

$$E = \{\eta | 0 \le \eta \le 1 \text{ such that systems (2.12) have an unique solution in } \mathbb{R}_+^0 \times \mathbb{R}_+^0\}.$$

Put

$$\begin{cases} \varphi_\eta(s,t) = s^2\|u^+\|^2 + bs^4|\nabla u^+|_2^4 + s^4\int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^+)^2\,dx + \lambda s^p|u^+|_p^p \\ \qquad + \eta s^2t^2[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx] - s^q\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx, \\ \psi_\eta(s,t) = t^2\|u^-\|^2 + bt^4|\nabla u^-|_2^4 + t^4\int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx + \lambda t^p|u^-|_p^p \\ \qquad + \eta s^2t^2[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx] - t^q\int_{\mathbb{R}^3} h(x)|u^-|^q\,dx. \end{cases} \tag{2.13}$$

(1) In this part, we show that $0 \in E$.

Since $\varphi_0(s,t) = \varphi_0(s,0)$, $\psi_0(s,t) = \psi_0(0,t)$, the solvability on $\varphi_0(s,t) = 0$ and $\psi_0(s,t) = 0$ is the same. So, we only show that there an unique $\bar{t} \in \mathbb{R}_+^0$ such that $\psi_0(0,\bar{t}) = 0$.

In fact, owing to the fact that $2 \le p \le 4 < q < 6$, $\lambda \ge 0$, and $h(x) > 0$, *a.e.* $x \in \mathbb{R}$, we know that $\psi_0(0,t) > 0$ as $t > 0$ is small enough and $\psi_0(0,t) < 0$ as $t > 0$ is large enough. Hence, there is a $\bar{t} \in \mathbb{R}_+^0$ such that $\psi_0(0,\bar{t}) = 0$.

Now, we prove that such a number $\bar{t} \in \mathbb{R}_+^0$ is unique. To this end, let $g(t) = t^{-2}\psi_0(0,t)$, $t \ge 0$.

By

$$g(t) = \|u^-\|^2 + bt^2|\nabla u^-|_2^4 + t^2\int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx + \lambda t^{p-2}|u^-|_p^p \\ - t^{q-2}\int_{\mathbb{R}^3} h(x)|u^-|^q\,dx, \quad t \ge 0,$$

we have $g(0) > 0$ and $g(\bar{t}) = 0$. If there exists another number $t_0 > 0$ with $t_0 \neq \bar{t}$ such that $\psi_0(0, t_0) = 0$, that is, $g(t_0) = 0$, then we will obtain a contradiction via the following argument:

Case 1. If $t_0 < \bar{t}$, then

$$\begin{aligned} t_0 g'(t_0) &= 2bt_0^2 |u^-|_2^4 + 2t_0^2 \int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx + \lambda(p-2)t_0^{p-2}|u^-|_p^p \\ &\quad - (q-2)t_0^{q-2} \int_{\mathbb{R}^3} h(x)|u^-|^q\,dx. \end{aligned} \tag{2.14}$$

On the other hand, the fact that $g(t_0) = 0$ implies that

$$\begin{aligned} &bt_0^2 |\nabla u^-|_2^4 + t_0^2 \int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx \\ &\quad = -\|u^-\|^2 - \lambda t_0^{p-2}|u^-|_p^p + t_0^{q-2} \int_{\mathbb{R}^3} h(x)|u^-|^q\,dx. \end{aligned} \tag{2.15}$$

Thus, substituting (2.15) into (2.14) and taking into account that $0 < t_0$, $u^- \neq 0$, $2 \leq p \leq 4 < q < 6$, $\lambda \geq 0$ and $h(x) > 0$, *a.e.* $x \in \mathbb{R}^3$, we get

$$t_0 g'(t_0) = (p-4)\lambda t_0^{p-2}|u^-|_p^p + (4-q)t_0^{q-2} \int_{\mathbb{R}^3} h(x)|u^-|^q\,dx - 2\|u^-\|^2 < 0,$$

which together with $g(t_0) = 0$ implies that $g(t_0 + \delta) < 0$ as $0 < \delta$ is small enough. Hence, there is a $t_* \in (t_0, \bar{t})$ satisfying

$$g(t_*) = \min_{t \in [t_0, \bar{t}]} g(t) < 0, \qquad g''(t_*) \geq 0.$$

However, observing that

$$\begin{aligned} t_*^2 g''(t_*) &= 2bt_*^2 |\nabla u^-|_2^4 + 2t_*^2 \int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx + \lambda(p-2)(p-3)t_*^{p-2}|u^-|_p^p \\ &\quad - (q-2)(q-3)t_*^{q-2} \int_{\mathbb{R}^3} h(x)|u^-|^q\,dx \end{aligned}$$

and

$$bt_*^2 |\nabla u^-|_2^4 + t_*^2 \int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx < -\lambda t_*^{p-2}|u^-|_p^p + t_*^{q-2} \int_{\mathbb{R}^3} h(x)|u^-|^q\,dx - \|u^-\|^2$$

(following from $g(t_*) < 0$), we have

$$t_*^2 g''(t_*) < \lambda(p-1)(p-4)t_*^{p-2}|u^-|_p^p - (q-1)(q-4)t_*^{q-2} \int_{\mathbb{R}^3} h(x)|u^-|^q\,dx - 2\|u^-\|^2 < 0$$

(noting that $2 \leq p \leq 4 < q < 6$, $\lambda \geq 0$ and $h(x) > 0$, *a.e.* $x \in \mathbb{R}^3$), which contradicts the fact that $g''(t_*) \geq 0$.

Case 2. If $\bar{t} < t_0$, the proof is the same. In fact, by only replacing t_0 with $\bar{t}$ in the above argument on case 1, we also obtain a contradiction.

Hence, we have proved that $0 \in E$.

(2) In this part, we show that the set E is open and closed in $[0, 1]$.

(i) E is an open set in $[0,1]$.

For any fixed $\eta_0 \in E$ and $(s_0,t_0) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$ is an unique solution of (2.12) associated with $\eta = \eta_0$. By calculation, from (2.13) we know that

$$\begin{aligned}\frac{\partial \varphi_{\eta_0}}{\partial s}\bigg|_{(s_0,t_0)} &= 2s_0\|u^+\|^2 + 4bs_0^3|\nabla u^+|_2^4 + 4s_0^3\int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^+)^2\,dx \\ &\quad + 2\eta_0 s_0 t_0^2\left[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\right] \\ &\quad + \lambda p s_0^{p-1}|u^+|_p^p - q s_0^{q-1}\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx, \qquad (2.16)\\ \frac{\partial \varphi_{\eta_0}}{\partial t}\bigg|_{(s_0,t_0)} &= 2\eta_0 s_0^2 t_0\left[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\right], \qquad (2.17)\\ \frac{\partial \psi_{\eta_0}}{\partial s}\bigg|_{(s_0,t_0)} &= 2\eta_0 s_0 t_0^2\left[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\right], \qquad (2.18)\\ \frac{\partial \psi_{\eta_0}}{\partial t}\bigg|_{(s_0,t_0)} &= 2t_0\|u^-\|^2 + 4bt_0^3|\nabla u^-|_2^4 + 4t_0^3\int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx \\ &\quad + 2\eta_0 s_0^2 t_0\left[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\right] \\ &\quad + \lambda p t_0^{p-1}|u^-|_p^p - q t_0^{q-1}\int_{\mathbb{R}^3} h(x)|u^-|^q\,dx. \qquad (2.19)\end{aligned}$$

Again, by $\varphi_{\eta_0}(s_0,t_0) = 0$, from (2.13) we have

$$\begin{aligned}& bs_0^3|\nabla u^+|_2^4 + s_0^3\int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^+)^2\,dx \\ &\quad = -\eta_0 s_0 t_0^2\left[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\right] \\ &\qquad - \lambda s_0^{p-1}|u^+|_p^p + s_0^{q-1}\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx - s_0\|u^+\|^2. \qquad (2.20)\end{aligned}$$

By (2.16) combined with (2.20), we get

$$\begin{aligned}\frac{\partial \varphi_{\eta_0}}{\partial s}\bigg|_{(s_0,t_0)} &= -2s_0\|u^+\|^2 - 2\eta_0 s_0 t_0^2\left[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\right] \\ &\quad - \lambda(4-p)s_0^{p-1}|u^+|_p^p - (q-4)s_0^{q-1}\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx. \qquad (2.21)\end{aligned}$$

Similarly, by $\psi_{\eta_0}(s_0,t_0) = 0$, we can deduce that

$$\begin{aligned}\frac{\partial \psi_{\eta_0}}{\partial t}\bigg|_{(s_0,t_0)} &= -2t_0\|u^-\|^2 - 2\eta_0 s_0^2 t_0\left[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\right] \\ &\quad - \lambda(4-p)t_0^{p-1}|u^-|_p^p - (q-4)t_0^{q-1}\int_{\mathbb{R}^3} h(x)|u^-|^q\,dx. \qquad (2.22)\end{aligned}$$

Thus, by (2.17)–(2.18) and (2.21)–(2.22) as well as $\phi_{u^+}, \phi_{u^-} \geq 0$, $\lambda \geq 0$, $2 \leq p \leq 4 < q$, $h(x) > 0$, *a.e.* $x \in \mathbb{R}^3$, we have the determinant

$$M = \frac{\partial \varphi_{\eta_0}(s_0,t_0)}{\partial s} \cdot \frac{\partial \psi_{\eta_0}(s_0,t_0)}{\partial t} - \frac{\partial \varphi_{\eta_0}(s_0,t_0)}{\partial t} \cdot \frac{\partial \psi_{\eta_0}(s_0,t_0)}{\partial s}$$
$$> AB - 4\eta_0^2 s_0^3 t_0^3 \left[b\left|\nabla u^+\right|_2^2 \left|\nabla u^-\right|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}\left(u^-\right)^2 dx \right]^2 > 0,$$

where

$$A = 2s_0\left\|u^+\right\|^2 + 2\eta_0 s_0 t_0^2 \left(b\left|\nabla u^+\right|_2^2 \left|\nabla u^-\right|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}\left(u^-\right)^2 dx \right),$$
$$B = 2t_0\left\|u^-\right\|^2 + 2\eta_0 s_0^2 t_0 \left(b\left|\nabla u^+\right|_2^2 \left|\nabla u^-\right|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}\left(u^-\right)^2 dx \right),$$
$$M = \begin{vmatrix} \frac{\partial \varphi_{\eta_0}(s_0,t_0)}{\partial s} & \frac{\partial \varphi_{\eta_0}(s_0,t_0)}{\partial t} \\ \frac{\partial \psi_{\eta_0}(s_0,t_0)}{\partial s} & \frac{\partial \psi_{\eta_0}(s_0,t_0)}{\partial t} \end{vmatrix}.$$

Hence, by the implicit function theorem, there exist an open neighborhood V_0 of η_0 and $\wedge_0 \subset \mathbb{R}_+^0 \times \mathbb{R}_+^0$ of (s_0,t_0) such that the implicit function $s = s(\eta)$, $t = t(\eta)$ satisfies system (2.12) on $V_0 \times \wedge_0$.

Now, we show that, for any $\eta \in V_0$, the system (2.12) has no solution in $(\mathbb{R}_+^0 \times \mathbb{R}_+^0)\backslash\wedge_0$.

Suppose by contradiction that there exists $\eta_1 \in V_0$ such that system (2.12) has another solution $(\bar{s},\bar{t})$ in $(\mathbb{R}_+^0 \times \mathbb{R}_+^0)\backslash\wedge_0$ associated with η_1 apart from the solution (s,t) in $\wedge_0$ associated with η_1. Then, by the implicit function theorem again, we can find a solution function $\bar{s} = \bar{s}(\eta)$, $\bar{t} = \bar{t}(\eta)$ in $(\eta_1 - \varepsilon, \eta_1 + \varepsilon)$ for some $\varepsilon > 0$, which satisfies (2.12) and goes through $(\eta_1,(\bar{s},\bar{t}))$.

1. If $\eta_0 < \eta_1$, then consider the saturated solution $\bar{s} = \bar{s}(\eta)$, $\bar{t} = \bar{t}(\eta)$ on its saturated interval. Since it cannot be defined at η_0 and cannot enter $V_0 \times \wedge_0$, there exists a point $\eta_2 \in [\eta_0,\eta_1)$ such that the solution $\bar{s} = \bar{s}(\eta)$, $\bar{t} = \bar{t}(\eta)$ in (η_2,η_1) and $\bar{s}^2(\eta) + \bar{t}^2(\eta) \to \infty$ as $\eta \to \eta_2^+$, which contradicts systems (2.12) noting that $2 \leq p \leq 4 < q < 6$, $\lambda \geq 0$, *and* $h(x) > 0$, *a.e.* $x \in \mathbb{R}^3$. Hence $V_0 \subset E$.
2. If $\eta_0 > \eta_1$, the proof is similar.

(ii) E is a closed set in $[0,1]$.

In fact, let $\{\eta_n\} \subset E$ be a sequence with $\eta_n \to \eta_0 \in [0,1]$ and $(s_n,t_n) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$ be the unique solution of (2.12) associated with η_n. Because the sequence $\{\eta_n\}$ is bounded, from (2.12) it follows that $\{(s_n,t_n)\}$ is bounded. Therefore, there exists a subsequence of $\{(s_n,t_n)\}$, still denoted by $\{(s_n,t_n)\}$, such that $(s_n,t_n) \to (s_0,t_0)$. Of course, (s_0,t_0) satisfies systems (2.12) for $\eta = \eta_0$. Furthermore, by (2.12), we have

$$\left\|u^+\right\|^2 \leq s_n^{q-2} \int_{\mathbb{R}^3} h(x)\left|u^+\right|^q dx,$$

which implies that $s_n \geq c_1 > 0$ for some $c_1 > 0$ because $2 \leq p \leq 4 < q < 6$, $h(x) > 0$, *a.e.* $x \in \mathbb{R}^3$ and $u^+ \neq 0$. Similarly, there exists $c_2 > 0$ such that $t_n \geq c_2 > 0$. Thus, $(s_0,t_0) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$ is a solution of (2.12). Also, the implicit function theorem ensures that (s_0,t_0) is the unique solution of (2.12) for $\eta = \eta_0$ again. Hence, E is closed in $[0,1]$.

Summing up the above arguments (i) and (ii), we get $E = [0,1]$, and therefore, the conclusion of Lemma 2.4 is true. □

Lemma 2.5 *Assume that* $u \in H^1(\mathbb{R}^3)$ *with* $u^\pm \not\equiv 0$ *and* $\varphi_1(1,1) \le 0$, $\psi_1(1,1) \le 0$, *where* φ_1, ψ_1 *are given as in* (2.13) *with* $\eta = 1$. *Then the unique pair* $(s_u, t_u) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$ *given in Lemma* 2.4 *satisfies* $0 < s_u, t_u \le 1$.

Proof If $s_u \ge t_u > 0$, then, by $s_u u^+ + t_u u^- \in M_b$ together with (2.11), we have

$$\begin{aligned} &s_u^2\|u^+\|^2 + s^4 b|\nabla u^+|_2^4 + s_u^4\int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^+)^2\,dx \\ &\quad + s_u^4\Big[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\Big] \\ &\ge -\lambda s_u^p|u^+|_p^p + s_u^q\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx. \end{aligned}$$

Thus

$$\begin{aligned} &s_u^{-2}\|u^+\|^2 + b|\nabla u^+|_2^4 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^+)^2\,dx + \Big[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\Big] \\ &\ge -\lambda s_u^{p-4}|u^+|_p^p + s_u^{q-4}\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx. \end{aligned} \tag{2.23}$$

On the other hand, the assumption $\varphi_1(1,1) \le 0$ implies that

$$\begin{aligned} &\|u^+\|^2 + b|\nabla u^+|_2^4 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^+)^2\,dx + \Big[b|\nabla u^+|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\Big] \\ &\le -\lambda|u^+|_p^p + \int_{\mathbb{R}^3} h(x)|u^+|^q\,dx. \end{aligned} \tag{2.24}$$

By (2.23)–(2.24), we have

$$(s_u^{-2} - 1)\|u^+\|^2 + \lambda(s_u^{p-4} - 1)|u^+|_p^p \ge (s_u^{q-4} - 1)\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx.$$

Because $u^+ \not\equiv 0$, $0 \le \lambda$, $2 \le p \le 4 < q < 6$ and $h(x) > 0$, *for a.e.* $x \in \mathbb{R}^3$, the above inequality implies that $0 < s_u \le 1$, and so, $0 < t_u \le 1$.

For the case $0 < t_u \le s_u$, the proof is similar. □

Let $m_b := \inf\{J_b(u) | u \in M_b\}$. We have the following lemma.

Lemma 2.6 *Assume that* (*l*), (*k*), (*h*) *hold. Then* $m_b > 0$ *can be achieved at some point* $u_b \in M_b$.

Proof For any $u \in M_b$, by $\langle J_b'(u), u\rangle = 0$ and (2.2), we have

$$\begin{aligned} \|u\|^2 &\le \|u\|^2 + b|\nabla u|_2^4 + \int_{\mathbb{R}^3} k(x)\phi_u u^2\,dx + \lambda\int_{\mathbb{R}^3}|u|^p\,dx \\ &= \int_{\mathbb{R}^3} h(x)|u|^q\,dx \le |h|_{\frac{6}{6-q_1}}|u|_r^q \le |h|_{\frac{6}{6-q_1}} S_r^{-q}\|u\|^q, \end{aligned}$$

where $r = \frac{6q}{q_1}$ satisfying $4 < r < 6$. Hence, $\|u\| \geq c_0 > 0$, where $c_0 = (\frac{S_r^q}{|h|_{\frac{6}{6-q_1}}})^{\frac{1}{q-2}}$. Also, by

$$\begin{aligned}
J_b(u) &= J_b(u) - \frac{1}{q}\langle J_b'(u), u\rangle \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right)b|\nabla u|_2^4 + \left(\frac{1}{4} - \frac{1}{q}\right)\int_{\mathbb{R}^3} k(x)\phi_u u^2\,dx \\
&\quad + \lambda\left(\frac{1}{p} - \frac{1}{q}\right)|u|_p^p \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{q}\right)c_0^2 > 0,
\end{aligned}$$

we have $m_b \geq (\frac{1}{2} - \frac{1}{q})c_0^2 > 0$.

Let $\{u_n\} \subset M_b$ with $J_b(u_n) \to m_b$ as $n \to \infty$. By

$$J_b(u_n) = J_b(u_n) - \frac{1}{q}\langle J_b'(u_n), u_n\rangle \geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2,$$

and $J_b(u_n) \to m_b$, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, which implies that there exist $u_b \in H^1(\mathbb{R}^3)$ and a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u_b$ as well as $u_n^{\pm} \rightharpoonup u_b^{\pm}$ weakly in $H^1(\mathbb{R}^3)$.

By $\{u_n\} \subset M_b$, it follows from $\langle J_b'(u_n), u_n^{\pm}\rangle = 0$ that

$$\begin{aligned}
\|u_n^{\pm}\|^2 &\leq \|u_n^{\pm}\|^2 + b|\nabla u_n|_2^2|\nabla u_n^{\pm}|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u_n}(u_n^{\pm})^2\,dx + \lambda|u_n^{\pm}|_p^p \\
&= \int_{\mathbb{R}^3} h(x)|u_n^{\pm}|^q\,dx \leq |h|_{\frac{6}{6-q_1}} S_r^{-q}\|u_n^{\pm}\|^q,
\end{aligned} \tag{2.25}$$

where $r = \frac{6q}{q_1}$. Therefore,

$$\|u_n^{\pm}\| \geq c_0 > 0. \tag{2.26}$$

Thus, by (2.25)–(2.26) and Lemma 2.3, we obtain $0 < c_0^2 \leq \lim_{n\to\infty}\int_{\mathbb{R}^3} h(x)|u_n^{\pm}|^q\,dx = \int_{\mathbb{R}^3} h(x)|u_b^{\pm}|^q\,dx$, which implies that $u_b^{\pm} \neq 0$ taking into account the assumption $h(x) > 0$, *a.e.* $x \in \mathbb{R}^3$.

Again, by the weak semi-continuity of the norm and Lemma 2.3, we get

$$\begin{aligned}
&\|u_b^{\pm}\|^2 + b|\nabla u_b|_2^2|\nabla u_b^{\pm}|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u_b}(u_b^{\pm})^2\,dx + \lambda|u_b^{\pm}|_p^p \\
&\quad\leq \liminf_{n\to\infty}\left\{\|u_n^{\pm}\|^2 + b|\nabla u_n|_2^2|\nabla u_n^{\pm}|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u_n}(u_n^{\pm})^2\,dx + \lambda|u_n^{\pm}|_p^p\right\} \\
&\quad= \liminf_{n\to\infty}\left\{\int_{\mathbb{R}^3} h(x)|u_n^{\pm}|^q\,dx\right\} = \int_{\mathbb{R}^3} h(x)|u_b^{\pm}|^q\,dx,
\end{aligned}$$

which means that $\varphi_1(1,1) \leq 0$ and $\psi_1(1,1) \leq 0$ in (2.13) corresponding to u_b. Hence, by Lemma 2.5, there exists an unique $(s,t) \in (0,1] \times (0,1]$ with $u_b^* = su_b^+ + tu_b^- \in M_b$. Moreover, by the weak semi-continuity of the norm and Lemmas 2.2–2.3 and (2.7), we have

$$m_b \leq J_b(u_b^*) - \frac{1}{q}\langle J_b'(u_b^*), u_b^*\rangle$$

$$
\begin{aligned}
&= \left(\frac{1}{2}-\frac{1}{q}\right)\|u_b^*\|^2 + \left(\frac{1}{4}-\frac{1}{q}\right)b|\nabla u_b^*|_2^4 + \left(\frac{1}{4}-\frac{1}{q}\right)\int_{\mathbb{R}^3} k(x)\phi_{u_b^*}\big(u_b^*\big)^2\,dx \\
&\quad + \lambda\left(\frac{1}{p}-\frac{1}{q}\right)|u_b^*|_p^p \\
&= \left(\frac{1}{2}-\frac{1}{q}\right)\big[s^2\|u_b^+\|^2 + t^2\|u_b^-\|^2\big] \\
&\quad + \left(\frac{1}{4}-\frac{1}{q}\right)b\big[s^4|\nabla u_b^+|_2^4 + t^4|\nabla u_b^-|_2^4 + 2s^2t^2|\nabla u_b^+|_2^4|\nabla u_b^-|_2^4\big] \\
&\quad + \left(\frac{1}{4}-\frac{1}{q}\right)\bigg[s^4\int_{\mathbb{R}^3} k(x)\phi_{u_b^+}\big(u_b^+\big)^2\,dx + t^4\int_{\mathbb{R}^3} k(x)\phi_{u_b^-}\big(u_b^-\big)^2\,dx \\
&\quad + 2s^2t^2\int_{\mathbb{R}^3} k(x)\phi_{u_b^+}\big(u_b^-\big)^2\,dx\bigg] \\
&\quad + \lambda\left(\frac{1}{p}-\frac{1}{q}\right)\big(s^p|u_b^+|_p^p + t^p|u_b^-|_p^p\big) \\
&\le \left(\frac{1}{2}-\frac{1}{q}\right)\big(\|u_b^+\|^2 + \|u_b^-\|^2\big) + \left(\frac{1}{4}-\frac{1}{q}\right)b\big[|\nabla u_b^+|_2^4 + |\nabla u_b^-|_2^4 + 2|\nabla u_b^+|_2^2|\nabla u_b^-|_2^2\big] \\
&\quad + \left(\frac{1}{4}-\frac{1}{q}\right)\bigg[\int_{\mathbb{R}^3} k(x)\phi_{u_b^+}\big(u_b^+\big)^2\,dx + \int_{\mathbb{R}^3} k(x)\phi_{u_b^-}\big(u_b^-\big)^2\,dx + 2\int_{\mathbb{R}^3} k(x)\phi_{u_b^+}\big(u_b^-\big)^2\,dx\bigg] \\
&\quad + \lambda\left(\frac{1}{p}-\frac{1}{q}\right)\big(|u_b^+|_p^p + |u_b^-|_p^p\big) \\
&\le \liminf_{n\to\infty}\bigg\{\left(\frac{1}{2}-\frac{1}{q}\right)\big(\|u_n^+\|^2 + \|u_n^-\|^2\big) \\
&\quad + \left(\frac{1}{4}-\frac{1}{q}\right)b\big[|\nabla u_n^+|_2^4 + |\nabla u_n^-|_2^4 + 2|\nabla u_n^+|_2^2|\nabla u_n^-|_2^2\big] \\
&\quad + \left(\frac{1}{4}-\frac{1}{q}\right)\bigg[\int_{\mathbb{R}^3} k(x)\phi_{u_n^+}\big(u_n^+\big)^2\,dx + \int_{\mathbb{R}^3} k(x)\phi_{u_n^-}\big(u_n^-\big)^2\,dx + 2\int_{\mathbb{R}^3} k(x)\phi_{u_n^+}\big(u_n^-\big)^2\,dx\bigg] \\
&\quad + \lambda\left(\frac{1}{p}-\frac{1}{q}\right)\big(|u_n^+|_p^p + |u_n^-|_p^p\big)\bigg\} \\
&= \liminf_{n\to\infty}\bigg[J_b(u_n) - \frac{1}{q}\langle J_b'(u_n), u_n\rangle\bigg] = \liminf_{n\to\infty}\big[J_b(u_n)\big] = m_b.
\end{aligned}
$$

Thus, $s = t = 1$, and therefore, $u_b \in M_b$, $J_b(u_b) = m_b$. □

Lemma 2.7 *For any given $u \in H^1(\mathbb{R}^3)$ with $u^\pm \ne 0$. Let $g(s,t) = J_b(su^+ + tu^-)$, $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, then there exists an unique pair $(s_u, t_u) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$ such that g attains its global maximum on $\mathbb{R}_+ \times \mathbb{R}_+$ at the point (s_u, t_u), which is exactly obtained in Lemma 2.4.*

Proof Because $2 \le p \le 4 < q < 6$, $\lambda \ge 0$ and $h(x) > 0$, *a.e.* $x \in \mathbb{R}^3$, we can know that $g(s,t) \to -\infty$ as $|(s,t)| \to \infty$. Hence, we only need to prove that g cannot achieve its maximum on the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$. In fact, if not—if g achieved its maximum at point $(0,\bar{t})$, $\bar{t} > 0$, then, by

$$
\begin{aligned}
g(s,\bar{t}) &= J_b\big(su^+ + \bar{t}u^-\big) \\
&= \frac{s^2}{2}\|u^+\|^2 + \frac{s^4}{4}b|\nabla u^+|_2^4 + \frac{s^4}{4}\int_{\mathbb{R}^3} k(x)\phi_{u^+}\big(u^+\big)^2\,dx + \frac{\lambda}{p}s^p|u^+|_p^p
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{s^q}{q}\int_{\mathbb{R}^3} h(x)|u^+|^q\,dx \\
&+\frac{\bar{t}^2}{2}\|u^-\|^2+\frac{\bar{t}^4}{4}b|\nabla u^-|_2^4+\frac{\bar{t}^4}{4}\int_{\mathbb{R}^3} k(x)\phi_{u^-}(u^-)^2\,dx+\frac{\lambda}{p}\bar{t}^p|u^-|_p^p \\
&-\frac{\bar{t}^q}{q}\int_{\mathbb{R}^3} h(x)|u^-|^q\,dx \\
&+2s^2\bar{t}^2\left(b|\nabla u^+|_2^4|\nabla u^-|_2^4+\int_{\mathbb{R}^3} k(x)\phi_{u^+}(u^-)^2\,dx\right)\geq g(s,0)+g(0,\bar{t}),
\end{aligned}
$$

and the fact that $g(s,0)>0$ as $s>0$ is small enough, we know that $g(s,\bar{t})>g(0,\bar{t})$ as $s>0$ is small enough, which contradicts the assumption that g achieved its maximum value at $(0,\bar{t})$ on $\mathbb{R}_+\times\mathbb{R}_+$. So, g achieved its maximum in some point $(s_u,t_u)\in\mathbb{R}_+^0\times\mathbb{R}_+^0$ on $\mathbb{R}_+\times\mathbb{R}_+$. Of course, (s_u,t_u) is a critical point of g on $\mathbb{R}_+\times\mathbb{R}_+$, and therefore it follows from the proof of Lemma 2.4 that the pair (s_u,t_u) is the unique solution of systems (2.11) in $\mathbb{R}_+^0\times\mathbb{R}_+^0$. □

3 Main results

We are now in a position to give our main results in this paper.

Theorem 3.1 *Assume that conditions* (*l*), (*k*), (*h*) *hold, then problem* (1.1) *possesses one least energy sign-changing solution* u_b, *which has exactly two nodal domains, where* u_b *is given in Lemma* 2.6.

Proof We apply the quantitative deformation lemma to prove that $J_b'(u_b)=0$.

Owing to the fact that $u_b\in M_b$ and $J_b(u_b)=m_b$, in terms of Lemma 2.7, for any $(s,t)\in\mathbb{R}_+^0\times\mathbb{R}_+^0$ with $(s,t)\neq(1,1)$, we immediately obtain

$$J_b(su_b^++tu_b^-)<J_b(u_b^++u_b^-)=m_b. \tag{3.1}$$

If $J_b'(u_b)\neq 0$, then there exist $r>0$ and $\tau>0$ satisfying

$$\|J_b'(u)\|>\tau \quad \text{as } \|u-u_b\|<3r.$$

Set $U=(1-\sigma_0,1+\sigma_0)\times(1-\sigma_0,1+\sigma_0)$, where $0<\sigma_0<\frac{1}{2}$, and $h(s,t)=su_b^++tu_b^-$, $(s,t)\in\bar{U}$. It follows from (3.1) that

$$m^*:=\max_{\partial U} J_b\circ h=\max_{\partial U} J_b(su_b^++tu_b^-)<m_b. \tag{3.2}$$

Take $0<\varepsilon<\min\{(m_b-m^*)/2,\tau r/8\}$, $S=\{u\in H^1(\mathbb{R}^3):\|u-u_b\|<r\}$.

Let $S_{2r}=\{u\in H^1(\mathbb{R}^3):\operatorname{dist}(u,S)\leq 2r\}$. Applying Lemma 2.3 in [33], there exists a deformation η satisfying that

(i) $\eta(1,u)=u$, if $u\notin J_b^{-1}(m_b-2\varepsilon,m_b+2\varepsilon)\cap S_{2r}$;
(ii) $\eta(1,J_b^{m_b+\varepsilon}\cap S)\subset J_b^{m_b-\varepsilon}$;
(iii) $J_b(\eta(1,u))\leq J_b(u)$ for any $u\in H^1(\mathbb{R}^3)$.

Now, we show that

$$\eta(1,h(U))\cap M_b\neq\emptyset,$$

which will contradict the definition of m_b.

First, we claim that

$$\max_{(s,t)\in\bar{U}} J_b\big(\eta\big(1,h(s,t)\big)\big) < m_b. \tag{3.3}$$

In fact, for any $(s,t)\in\bar{U}$, we consider the following two cases:

(a) If $(s,t)\neq(1,1)$, then, by (3.3) together with (iii) above, we have

$$J_b\big(\eta\big(1,h(s,t)\big)\big) \le J_b\big(h(s,t)\big) = J_b\big(su_b^+ + tu_b^-\big) < m_b.$$

(b) If $(s,t)=(1,1)$, then, by $J_b(h(1,1)) = J_b(u_b) = m_b < m_b+\varepsilon$ and $h(1,1)=u_b\in S$, it follows from (ii) that

$$J_b\big(\eta\big(1,h(1,1)\big)\big) \le m_b - \varepsilon < m_b.$$

Hence, from the above arguments (a) and (b), we know that (3.3) is true.

Secondly, for $(s,t)\in\bar{U}$, let $\varphi(s,t)=\eta(1,h(s,t))$ and

$$\phi(s,t) = \big(\phi_1(s,t),\phi_2(s,t)\big), \qquad \psi(s,t) = \big(\psi_1(s,t),\psi_2(s,t)\big),$$

where

$$\phi_1(s,t) = \big\langle J_b'\big(h(s,t)\big),u_b^+\big\rangle, \qquad \phi_2(s,t) = \big\langle J_b'\big(h(s,t)\big),u_b^-\big\rangle,$$
$$\psi_1(s,t) = \frac{1}{s}\big\langle J_b'\big(\varphi(s,t)\big),\varphi^+(s,t)\big\rangle, \qquad \psi_2(s,t) = \frac{1}{t}\big\langle J_b'\big(\varphi(s,t)\big),\varphi^-(s,t)\big\rangle.$$

Since $u_b\in M_b$, by Lemma 2.4, $\phi(s,t)=0 \Leftrightarrow (s,t)=(1,1)$. Again, by the proof of Lemma 2.4, we know that $\frac{\partial(\phi_1,\phi_2)}{\partial(s,t)}|_{(1,1)}>0$, and therefore, the degree theory yields

$$\deg(\phi,U,\theta)=1,$$

where $\theta=(0,0)$. On the other hand, for any $(s,t)\in\partial U$, by (3.2) and $0<\varepsilon<\frac{m_b-m^*}{2}$, we have

$$J_b\big(h(s,t)\big) \le m^* < m_b - 2\varepsilon.$$

Thus, $h(s,t)\notin J_b^{-1}(m_b-2\varepsilon,m_b+2\varepsilon)\cap S_{2r}$. So, the conclusion (i) above implies that

$$\varphi(s,t) = \eta\big(1,h(s,t)\big) = h(s,t), \quad \text{for any } (s,t)\in\partial U,$$

which means that $\phi(s,t)=\psi(s,t)$, for any $(s,t)\in\partial U$. Hence, $\deg(\psi,U,\theta)=\deg(\phi,U,\theta)=1$. Therefore, there exists $(s_0,t_0)\in U$ such that $\psi(s_0,t_0)=0$. That means that $\eta(1,h(s_0,t_0)) = \varphi(s_0,t_0)\in M_b$. Hence, $J_b(1,h(s_0,t_0))\ge m_b$, which contradicts (3.3). Thus, we have deduced that $J_b'(u_b)=0$, $u_b\in M_b$, and u_b is a least energy sign-changing solution of problem (1.1).

Finally, we show that u_b has exactly two nodal domains. To this end, let $u_b=u_1+u_2+u_3$, satisfying that $u_1(x)\ge 0$, $u_2(x)\le 0$, for any $x\in\mathbb{R}^3$; $u_1(x)=u_2(x)=0$, for any $x\in\mathbb{R}^3\backslash(\Omega_1\cup\Omega_2)$; $u_3(x)=0$, for any $x\in\Omega_1\cup\Omega_2$, where $\Omega_1=\{x\in\mathbb{R}^3|u_1(x)>0\}$, $\Omega_2=\{x\in\mathbb{R}^3|u_2(x)<0\}$ are connected open subsets of Ω.

Set $u = u_1 + u_2$. Then $u^+ = u_1$ and $u^- = u_2$ with $u^\pm \neq 0$.

In the following, we deduce that $u_3 = 0$.

Suppose by contradiction that $u_3 \neq 0$. Then, by $J_b'(u_b)u_1 = 0$, we can deduce that $\varphi_1(1,1) \le 0$. Similarly, by $J_b'(u_b)u_2 = 0$, we can deduce that $\psi_1(1,1) \le 0$, where φ_1, ψ_1 are given by (2.13) with $u^+ = u_1$, $u^- = u_2$. So, by Lemmas 2.4–2.5, there exists an unique pair $(s_u, t_u) \in (0,1] \times (0,1]$ such that $s_u u^+ + t_u u^- \in M_b$. Then

$$m_b \le J_b(s_u u_1 + t_u u_2), \qquad \langle J_b'(s_u u_1 + t_u u_2), s_u u_1 + t_u u_2\rangle = 0.$$

Consequently,

$$\begin{aligned}
m_b &\le J_b(s_u u_1 + t_u u_2) = J_b(s_u u_1 + t_u u_2) - \frac{1}{q}\langle J_b'(s_u u_1 + t_u u_2), s_u u_1 + t_u u_2\rangle \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\left(s_u^2\|u_1\|^2 + t_u^2\|u_2\|^2\right) + \left(\frac{1}{4} - \frac{1}{q}\right)b\left[s_u^2|\nabla u_1|_2^2 + t_u^2|\nabla u_2|_2^2\right]^2 \\
&\quad + \left(\frac{1}{4} - \frac{1}{q}\right)\int_{\mathbb{R}^3} k(x)\left[s_u^4\phi_{u_1}(u_1)^2 + t_u^4\phi_{u_2}(u_2)^2 + 2s_u^2t_u^2\phi_{u_1}(u_2)^2\right]dx \\
&\quad + \lambda\left(\frac{1}{p} - \frac{1}{q}\right)\left(s_u^p|u_1|_p^p + t_u^p|u_2|_p^p\right) \\
&\le \left(\frac{1}{2} - \frac{1}{q}\right)\left(\|u_1\|^2 + \|u_2\|^2\right) + \left(\frac{1}{4} - \frac{1}{q}\right)b\left[|\nabla u_1|_2^2 + |\nabla u_2|_2^2\right]^2 \\
&\quad + \left(\frac{1}{4} - \frac{1}{q}\right)\int_{\mathbb{R}^3} k(x)\left[\phi_{u_1}(u_1)^2 + \phi_{u_2}(u_2)^2 + 2\phi_{u_1}(u_2)^2\right]dx \\
&\quad + \lambda\left(\frac{1}{p} - \frac{1}{q}\right)\left(|u_1|_p^p + |u_2|_p^p\right) \\
&= J_b(u_1 + u_2) - \frac{1}{q}\langle J_b'(u_1 + u_2), u_1 + u_2\rangle \\
&= J_b(u_b) - \frac{1}{q}\langle J_b'(u_b), u_1 + u_2\rangle \\
&\quad - \left[J_b(u_3) + \left(\frac{1}{2} - \frac{1}{q}\right)\left(b\left|\nabla(u_1 + u_2)\right|_2^2|\nabla u_3|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u_1+u_2}(u_3)^2\,dx\right)\right] \\
&= J_b(u_b) - \left[J_b(u_3) + \left(\frac{1}{2} - \frac{1}{q}\right)\left(b\left|\nabla(u_1 + u_2)\right|_2^2|\nabla u_3|_2^2\right.\right. \\
&\quad \left.\left. + \int_{\mathbb{R}^3} k(x)\phi_{u_1+u_2}(u_3)^2\,dx\right)\right].
\end{aligned} \tag{3.4}$$

On the other hand, by the fact that $u_3 \neq 0$, $2 \le p \le 4 < q < 6$, $\lambda \ge 0$, we have

$$\begin{aligned}
J_b(u_3) &= J_b(u_3) - \frac{1}{q}\langle J_b'(u_b), u_3\rangle \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_3\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right)b|\nabla u_3|_2^4 + \left(\frac{1}{4} - \frac{1}{q}\right)\int_{\mathbb{R}^3} k(x)\phi_{u_3}(u_3)^2\,dx \\
&\quad + \lambda\left(\frac{1}{p} - \frac{1}{q}\right)|u_3|_p^p - \frac{b}{q}\left|\nabla(u_1 + u_2)\right|_2^2|\nabla u_3|_2^2 - \frac{1}{q}\int_{\mathbb{R}^3} k(x)\phi_{u_1+u_2}(u_3)^2\,dx \\
&> -\frac{b}{q}\left|\nabla(u_1 + u_2)\right|_2^2|\nabla u_3|_2^2 - \frac{1}{q}\int_{\mathbb{R}^3} k(x)\phi_{u_1+u_2}(u_3)^2\,dx.
\end{aligned}$$

Thus,

$$J_b(u_3) + \frac{1}{q}\left(b\left|\nabla(u_1+u_2)\right|_2^2|\nabla u_3|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u_1+u_2}(u_3)^2\,dx\right) > 0.$$

The condition $q > 4$ shows that $\frac{1}{2} - \frac{1}{q} > \frac{1}{q}$, and therefore, it follows from the above inequality that

$$J_b(u_3) + \left(\frac{1}{2} - \frac{1}{q}\right)\left(b\left|\nabla(u_1+u_2)\right|_2^2|\nabla u_3|_2^2 + \int_{\mathbb{R}^3} k(x)\phi_{u_1+u_2}(u_3)^2\,dx\right) > 0. \tag{3.5}$$

By (3.4)–(3.5), we have

$$m_b < J_b(u_b) = m_b,$$

which is a contradiction. Hence, $u_3 = 0$, and therefore, u_b has exactly two nodal domains. □

In the following, we always assume that $b > 0$ in problem (1.1). We will investigate the convergence of u_b as $b \searrow 0$.

Theorem 3.2 *Assume that the conditions* (l), (k), (h) *hold. Then, for any sequence* $\{b_n\}$ *with* $b_n \searrow 0$ *as* $n \to \infty$, *there exists a subsequence, still denoted by* $\{b_n\}$ *such that* $u_{b_n} \to u_0 \in H^1(\mathbb{R}^3)$ *in* $H^1(\mathbb{R}^3)$ *as* $n \to \infty$. *Moreover,* u_0 *is a least energy sign-changing solution of problem* (1.2).

Proof For any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \to \infty$, u_{b_n} is one least energy sign-changing solution of problem (1.1) corresponding to $b = b_n$.

(1) Firstly, we show that $\{u_{b_n}\}$ is bounded in $H^1(\mathbb{R}^3)$.

Take a nonzero function $g \in C_0^\infty(\mathbb{R}^3)$ with $g^\pm \not\equiv 0$. Because $2 \le p \le 4 < q < 6$, $\lambda \ge 0$, and $h(x) > 0$, *a.e.* $x \in \mathbb{R}^3$, we can choose an appropriate positive number $\tau > 0$ such that

$$\varphi(1,1) \le 0, \qquad \psi(1,1) \le 0$$

holds for all $b \in [0,1]$ corresponding to $u^+ = \tau g^+$, $u^- = \tau g^-$ in (2.13). Thus, by Lemma 2.5, for each $b \in (0,1]$, there exists a pair $(s_b, t_b) \in (0,1] \times (0,1]$ such that $s_b u^+ + t_b u^- \in M_b$. Let $\bar{u} := s_b u^+ + t_b u^-$ and u_b be a least energy sign-changing solution of problem (1.1), then

$$\begin{aligned}
J_b(u_b) &\le J_b(\bar{u}) = J_b(\bar{u}) - \frac{1}{q}\langle J_b'(\bar{u}), \bar{u}\rangle \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\left(s_b\|u^+\|^2 + t_b\|u^-\|^2\right) + \left(\frac{1}{4} - \frac{1}{q}\right)b\left[s_b^2|\nabla u^+|_2^2 + t_b^2|\nabla u^-|_2^2\right]^2 \\
&\quad + \left(\frac{1}{4} - \frac{1}{q}\right)\int_{\mathbb{R}^3} k(x)\left[s_b^4\phi_{u^+}(u^+)^2 + t_b^4\phi_{u^-}(u^-)^2 + 2s_b^2t_b^2\phi_{u^+}(u^-)^2\right]dx \\
&\quad + \lambda\left(\frac{1}{p} - \frac{1}{q}\right)\left(s_b^p|u^+|_p^p + t_b^p|u^-|_p^p\right) \\
&\le \left(\frac{1}{2} - \frac{1}{q}\right)\left(\|u^+\|^2 + \|u^-\|^2\right) + \left(\frac{1}{4} - \frac{1}{q}\right)\left[|\nabla u^+|_2^2 + |\nabla u^-|_2^2\right]^2
\end{aligned}$$

$$+\lambda\left(\frac{1}{p}-\frac{1}{q}\right)\left(\left|u^{+}\right|_{p}^{p}+\left|u^{-}\right|_{p}^{p}\right)$$
$$+\left(\frac{1}{4}-\frac{1}{q}\right)\left[\int_{\mathbb{R}^3}k(x)\left(\phi_{u^{+}}\left(u^{+}\right)^{2}+\phi_{u^{-}}\left(u^{-}\right)^{2}+2\phi_{u^{+}}\left(u^{-}\right)^{2}\right)dx\right]:=M_0.$$

Thus, $J_b(u_b)\le M_0$ for all $b\in[0,1]$.

Owing to the fact that $b_n\searrow 0$, we can assume $b_n\in(0,1]$, and therefore, $J_{b_n}(u_{b_n})\le M_0$ for all $n\ge 1$. On the other hand, for all $n\ge 1$,

$$M_0\ge J_{b_n}(u_{b_n})=J_{b_n}(u_{b_n})-\frac{1}{q}\langle J'_{b_n}(u_{b_n}),u_{b_n}\rangle\ge\left(\frac{1}{2}-\frac{1}{q}\right)\|u_{b_n}\|^2.$$

Hence, $\{u_{b_n}\}$ is bounded in $H^1(\mathbb{R}^3)$.

(2) Because $\{u_{b_n}\}$ is bounded in $H^1(\mathbb{R}^3)$, there exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, such that $u_{b_n}\rightharpoonup u_0$ as well as $u_{b_n}^{\pm}\rightharpoonup u_0^{\pm}$ weakly in $H^1(\mathbb{R}^3)$. Then $u_{b_n}\to u_0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ for $r\in[1,6)$ and $u_{b_n}(x)\to u_0(x)$ *a.e.* $x\in\mathbb{R}^3$.

Now, we show that $u_{b_n}\to u_0$ in $H^1(\mathbb{R}^3)$.

In fact, from $u_{b_n}\rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ and $J'_{b_n}(u_{b_n})=0$, it follows that

$$\langle J'_{b_n}(u_{b_n})-J'_0(u_0),u_{b_n}-u_0\rangle=-\langle J'_0(u_0),u_{b_n}-u_0\rangle\to 0\quad\text{as } n\to\infty.$$

On the other hand, we have

$$\begin{aligned}\|u_{b_n}-u_0\|^2&=\langle J'_{b_n}(u_{b_n})-J'_0(u_0),u_{b_n}-u_0\rangle-d_n\\&\quad-\int_{\mathbb{R}^3}k(x)\phi_{u_{b_n}}(u_{b_n}-u_0)^2\,dx-\lambda\int_{\mathbb{R}^3}|u_{b_n}|^{p-2}(u_{b_n}-u_0)^2\,dx\\&\le\langle J'_{b_n}(u_{b_n})-J'_0(u_0),u_{b_n}-u_0\rangle-d_n,\end{aligned}$$

where

$$\begin{aligned}d_n&=b_n|\nabla u_{b_n}|_2^2\int_{\mathbb{R}^3}\nabla u_{b_n}\cdot(\nabla u_{b_n}-\nabla u_0)\,dx+\int_{\mathbb{R}^3}k(x)(\phi_{u_{b_n}}-\phi_{u_0})u_0(u_{b_n}-u_0)\,dx\\&\quad+\lambda\int_{\mathbb{R}^3}\left(|u_{b_n}|^{p-2}-|u_0|^{p-2}\right)u_0(u_{b_n}-u_0)\,dx\\&\quad-\int_{\mathbb{R}^3}h(x)|u_{b_n}|^{q-2}u_{b_n}(u_{b_n}-u_0)\,dx+\int_{\mathbb{R}^3}h(x)|u_0|^{q-2}u_0(u_{b_n}-u_0)\,dx.\end{aligned}$$

In the following, we deduce that $d_n\to 0$ as $n\to\infty$.

Owing to the fact that $\{u_{b_n}\}$ is bounded in $H^1(\mathbb{R}^3)$, it is easy to verify that the sequence $\{|\nabla u_{b_n}|_2^2\int_{\mathbb{R}^3}\nabla u_{b_n}\cdot(\nabla u_{b_n}-\nabla u_0)\,dx\}$ is bounded, and therefore

$$b_n|\nabla u_{b_n}|_2^2\int_{\mathbb{R}^3}\nabla u_{b_n}\cdot(\nabla u_{b_n}-\nabla u_0)\,dx\to 0\tag{3.6}$$

as $n\to\infty$ noting that $b_n\to 0$ as $n\to\infty$.

Now, we prove that

$$\int_{\mathbb{R}^3}\left(|u_{b_n}|^{p-2}-|u_0|^{p-2}\right)u_0(u_{b_n}-u_0)\,dx\to 0\quad\text{as } n\to\infty.\tag{3.7}$$

For $R > 0$, let $\Omega_R = \{x \in \mathbb{R}^3 : \|x\| < R\}$, $\Omega_R^C = \{x \in \mathbb{R}^3 : \|x\| \geq R\}$, we have

$$\begin{aligned}
A_n &:= \left|\int_{\Omega_R^C} \left(|u_{b_n}|^{p-2} - |u_0|^{p-2}\right)u_0(u_{b_n} - u_0)\,dx\right| \\
&\leq \int_{\Omega_R^C} \left(|u_{b_n}| + |u_0|\right)^{p-1}|u_0|\,dx \\
&\leq \left(\int_{\Omega_R^C} \left(|u_{b_n}| + |u_0|\right)^{2(p-1)}\,dx\right)^{\frac{1}{2}} \left(\int_{\Omega_R^C} |u_0|^2\,dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^3} \left(|u_{b_n}| + |u_0|\right)^{2(p-1)}\,dx\right)^{\frac{1}{2}} \left(\int_{\Omega_R^C} |u_0|^2\,dx\right)^{\frac{1}{2}} \\
&\leq 4^{p-1}\left(\int_{\mathbb{R}^3} \left(|u_{b_n}|^{2(p-1)} + |u_0|^{2(p-1)}\right)\,dx\right)^{\frac{1}{2}} \left(\int_{\Omega_R^C} |u_0|^2\,dx\right)^{\frac{1}{2}} \\
&= 4^{p-1}\left(|u_{b_n}|_{2(p-1)}^{2(p-1)} + |u_0|_{2(p-1)}^{2(p-1)}\right)^{\frac{1}{2}} \left(\int_{\Omega_R^C} |u_0|^2\,dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Because of the boundedness of $\{u_{b_n}\}$ combined with (2.2), there exists $M_1 > 0$ such that

$$4^{p-1}\left(|u_n|_{2(p-1)}^{2(p-1)} + |u_0|_{2(p-1)}^{2(p-1)}\right)^{\frac{1}{2}} \leq M_1.$$

On the other hand, the fact that $u_0 \in L^2(\mathbb{R}^3)$ implies that $\forall \varepsilon > 0$, we can choose $R > 0$ large enough so that

$$\left(\int_{\Omega_R^C} |u_0|^2\,dx\right)^{\frac{1}{2}} < \frac{\varepsilon}{2M_1}.$$

Then

$$\left|\int_{\Omega_R^C} \left(|u_{b_n}|^{p-2} - |u_0|^{p-2}\right)u_0(u_{b_n} - u_0)\,dx\right| < \frac{\varepsilon}{2}. \tag{3.8}$$

Similarly, for given R above, there exists $M_2 > 0$ such that

$$\begin{aligned}
B_n &:= \left|\int_{\Omega_R} \left(|u_{b_n}|^{p-2} - |u_0|^{p-2}\right)u_0(u_{b_n} - u_0)\,dx\right| \\
&\leq \int_{\Omega_R} \left(|u_{b_n}|^{p-2} + |u_0|^{p-2}\right)|u_0||u_{b_n} - u_0|\,dx \\
&\leq \int_{\Omega_R} \left(|u_{b_n}| + |u_0|\right)^{p-2}|u_0||u_{b_n} - u_0|\,dx \\
&\leq \int_{\Omega_R} \left(|u_{b_n}| + |u_0|\right)^{p-1}|u_{b_n} - u_0|\,dx \\
&\leq \left(\int_{\Omega_R} \left(|u_{b_n}| + |u_0|\right)^{2(p-1)}\,dx\right)^{\frac{1}{2}} \left(\int_{\Omega_R} |u_{b_n} - u_0|^2\,dx\right)^{\frac{1}{2}} \\
&\leq M_2\left(\int_{\Omega_R} |u_{b_n} - u_0|^2\,dx\right)^{\frac{1}{2}}.
\end{aligned}$$

By the fact that $u_{b_n} \to u_0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$, $r \in [1,6)$, there exists $N_1 \geq 1$, such that

$$M_2\left(\int_{\Omega_R} |u_{b_n} - u_0|^2\,dx\right)^{\frac{1}{2}} < \frac{\varepsilon}{2}$$

as $n \geq N_1$. Thus,

$$\left|\int_{\Omega_R} \left(|u_{b_n}|^{p-2} - |u_0|^{p-2}\right)u_0(u_{b_n} - u_0)\,dx\right| < \frac{\varepsilon}{2} \tag{3.9}$$

as $n \geq N_1$.

Hence, (3.8)–(3.9) shows that (3.7) holds.

Now, we show that

$$\int_{\mathbb{R}^3} h(x)|u_{b_n}|^{q-2}u_{b_n}(u_{b_n} - u_0)\,dx - \int_{\mathbb{R}^3} h(x)|u_0|^{q-2}u_0(u_{b_n} - u_0)\,dx \to 0, \quad \text{as } n \to \infty. \tag{3.10}$$

We only prove that $\int_{\mathbb{R}^3} h(x)|u_{b_n}|^{q-2}u_{b_n}(u_{b_n} - u_0)\,dx \to 0$ as $n \to \infty$, because the proof on $\int_{\mathbb{R}^3} h(x)|u_0|^{q-2}u_0(u_{b_n} - u_0)\,dx \to 0$ is similar.

We make an argument similar to (3.7) as follows:

For $R > 0$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} h(x)|u_{b_n}|^{q-2}u_{b_n}(u_{b_n} - u_0)\,dx\right| \\
&\quad\leq \int_{\mathbb{R}^3} h(x)|u_{b_n}|^{q-1}|u_{b_n} - u_0|\,dx \\
&\quad= \int_{\Omega_R} h(x)|u_{b_n}|^{q-1}|u_{b_n} - u_0|\,dx + \int_{\Omega_R^C} h(x)|u_{b_n}|^{q-1}|u_{b_n} - u_0|\,dx.
\end{aligned}$$

Take $r_1 = \frac{6}{q-1}$, $r_2 = \frac{6}{1+(q_1-q)}$. Then the condition (l) ensures that $1 < r_1 < 2 < r_2 < 6$. By the boundedness of $\{u_{b_n}\}$ combined with (2.2) and the fact that $u_{b_n} \to u_0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$, $r \in [1,6)$, there exists $M_1 > 0$ such that

$$\begin{aligned}
A_n &:= \int_{\Omega_R} h(x)|u_{b_n}|^{q-1}|u_{b_n} - u_0|\,dx \\
&\leq \left(\int_{\Omega_R} h^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}} \left(\int_{\Omega_R} |u_{b_n}|^{(q-1)r_1}\,dx\right)^{\frac{1}{r_1}} \left(\int_{\Omega_R} |u_{b_n} - u_0|^{r_2}\,dx\right)^{\frac{1}{r_2}} \\
&\leq |h|_{\frac{6}{6-q_1}} |u_{b_n}|_6^{q-1} \left(\int_{\Omega_R} |u_{b_n} - u_0|^{r_2}\,dx\right)^{\frac{1}{r_2}} \\
&\leq M_1\left(\int_{\Omega_R} |u_{b_n} - u_0|^{r_2}\,dx\right)^{\frac{1}{r_2}} \to 0
\end{aligned} \tag{3.11}$$

as $n \to \infty$.

Similarly, let $r = \frac{6q}{q_1}$, $r_1 = \frac{6}{q-1}$, $r_2 = \frac{6}{1+(q_1-q)}$, then $4 < r < 6$ and there exists $M_2 > 0$ such that

$$\begin{aligned}
B_n &:= \int_{\Omega_R^C} h(x)|u_{b_n}|^{q-1}|u_{b_n} - u_0|\,dx \\
&\le \int_{\Omega_R^C} h(x)\bigl(|u_{b_n}|^q + |u_{b_n}|^{q-1}|u_0|\bigr)\,dx \\
&\le \left(\int_{\Omega_R^C} h^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}} \left(\int_{\Omega_R^C} |u_{b_n}|^r\,dx\right)^{\frac{q}{r}} \\
&\quad + \left(\int_{\Omega_R^C} h^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}} \left(\int_{\Omega_R^C} |u_{b_n}|^{r_1(q-1)}\,dx\right)^{\frac{1}{r_1}} \left(\int_{\Omega_R^C} |u_0|^{r_2}\,dx\right)^{\frac{1}{r_2}} \\
&\le \left(\int_{\Omega_R^C} h^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}} \bigl[|u_{b_n}|_r^q + |u_{b_n}|_6^{q-1}|u_0|^{r_2}\bigr] \le M_2\left(\int_{\Omega_R^C} h^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}}.
\end{aligned}$$

Because $h \in L^{\frac{6}{6-q_1}}(\mathbb{R}^3)$, for any $\varepsilon > 0$, we can choose $R > 0$ large enough so that

$$\left(\int_{\Omega_R^C} h^{\frac{6}{6-q_1}}\,dx\right)^{\frac{6-q_1}{6}} < \frac{\varepsilon}{M_2}$$

and therefore

$$\int_{\Omega_R^C} h(x)|u_{b_n}|^{q-1}|u_{b_n} - u_0|\,dx < \varepsilon. \tag{3.12}$$

Hence, by (3.11)–(3.12), we conclude that $\int_{\mathbb{R}^3} h(x)|u_{b_n}|^{q-2}u_{b_n}(u_{b_n} - u_0)\,dx \to 0$ as $n \to \infty$.

Finally, we show that

$$\int_{\mathbb{R}^3} k(x)(\phi_{u_{b_n}} - \phi_{u_0})u_0(u_{b_n} - u_0)\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

In fact, by Lemma 2.2(iii)–(iv) together with the boundedness of $\{u_{b_n}\}$ in $H^1(\mathbb{R}^3)$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} k(x)(\phi_{u_{b_n}} - \phi_{u_0})u_0(u_{b_n} - u_0)\,dx\right| \\
&\quad\le \int_{\mathbb{R}^3} k(x)|\phi_{u_{b_n}} - \phi_{u_0}|\bigl|u_0(u_{b_n} - u_0)\bigr|\,dx \\
&\quad\le |k|_\infty\left(\int_{\mathbb{R}^3} |\phi_{u_{b_n}} - \phi_{u_0}|^6\right)^{\frac{1}{6}}\left(\int_{\mathbb{R}^3} \bigl|u_0(u_{b_n} - u_0)\bigr|^{\frac{6}{5}}\,dx\right)^{\frac{5}{6}} \\
&\quad\le |k|_\infty\bigl(|\phi_{u_{b_n}}|_6 + |\phi_{u_0}|_6\bigr)\left(\int_{\mathbb{R}^3} \bigl|u_0(u_{b_n} - u_0)\bigr|^{\frac{6}{5}}\,dx\right)^{\frac{5}{6}} \\
&\quad\le |k|_\infty\bar{S}^{-2}S_6^{-2}|k|_2\bigl(\|u_{b_n}\|^2 + \|u_0\|^2\bigr)\left(\int_{\mathbb{R}^3} \bigl|u_0(u_{b_n} - u_0)\bigr|^{\frac{6}{5}}\,dx\right)^{\frac{5}{6}} \\
&\quad\le M_1\left(\int_{\mathbb{R}^3} \bigl|u_0(u_{b_n} - u_0)\bigr|^{\frac{6}{5}}\,dx\right)^{\frac{5}{6}}.
\end{aligned} \tag{3.14}$$

For any $R > 0$, we know that

$$\int_{\mathbb{R}^3} \left|u_0(u_{b_n} - u_0)\right|^{\frac{6}{5}} dx = \int_{\Omega_R} \left|u_0(u_{b_n} - u_0)\right|^{\frac{6}{5}} dx + \int_{\Omega_R^C} \left|u_0(u_{b_n} - u_0)\right|^{\frac{6}{5}} dx.$$

Because

$$\begin{aligned}\left(\int_{\Omega_R^C} \left|u_0(u_{b_n} - u_0)\right|^{\frac{6}{5}} dx\right) &\le \left(\int_{\Omega_R^C} |u_0|^{\frac{3}{2}} dx\right)^{\frac{4}{5}} \left(\int_{\Omega_R^C} |u_{b_n} - u_0\big)|^6 dx\right)^{\frac{1}{5}} \\ &\le \left(\int_{\Omega_R^C} |u_0|^{\frac{3}{2}} dx\right)^{\frac{4}{5}} \left(\int_{\Omega_R^C} \left(|u_{b_n}| + |u_0|\right)^6 dx\right)^{\frac{1}{5}} \\ &\le \left(\int_{\Omega_R^C} |u_0|^{\frac{3}{2}} dx\right)^{\frac{4}{5}} \left(|u_{b_n}|_6 + |u_0|_6\right)^{\frac{6}{5}} \\ &\le M_2\left(\int_{\Omega_R^C} |u_0|^{\frac{3}{2}} dx\right)^{\frac{4}{5}}, \end{aligned} \tag{3.15}$$

and $u_0 \in L^{\frac{3}{2}}(\mathbb{R}^3)$, for any given $\varepsilon > 0$, we can choose $R > 0$ large enough so that

$$M_2\left(\int_{\Omega_R^C} |u_0|^{\frac{3}{2}} dx\right)^{\frac{4}{5}} < \varepsilon/2. \tag{3.16}$$

On the other hand, since $u_{b_n} \to u_0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$, $r \in [1, 6)$, there exists $N > 0$ such that

$$\begin{aligned}\int_{\Omega_R} \left|u_0(u_{b_n} - u_0)\right|^{\frac{6}{5}} dx &= \left(\int_{\Omega_R} |u_{b_n} - u_0\big)|^{\frac{3}{2}} dx\right)^{\frac{4}{5}} \left(\int_{\Omega_R} |u_0|^6 dx\right)^{\frac{1}{5}} \\ &\le \left(\int_{\Omega_R} |u_{b_n} - u_0\big)|^{\frac{3}{2}} dx\right)^{\frac{4}{5}} \left(\int_{\mathbb{R}^3} |u_0|^6 dx\right)^{\frac{1}{5}} < \varepsilon/2 \end{aligned} \tag{3.17}$$

as $n > N$. Thus, by (3.14)–(3.17), we find that (3.13) holds. Consequently, by (3.6), (3.7), (3.10) and (3.13), we obtain $d_n \to 0$ as $n \to \infty$, and therefore, $u_{b_n} \to u_0$ as $n \to \infty$ in $H^1(\mathbb{R}^3)$.

Now, by

$$\begin{aligned}\langle J_0'(u_0), u_{b_n}^{\pm}\rangle &= \langle J_{b_n}'(u_{b_n}), u_{b_n}^{\pm}\rangle - \langle J_{b_n}'(u_{b_n}) - J_0'(u_{b_n}), u_{b_n}^{\pm}\rangle - \langle J_0'(u_{b_n}) - J_0'(u_0), u_{b_n}^{\pm}\rangle \\ &= -\langle J_{b_n}'(u_{b_n}) - J_0'(u_{b_n}), u_{b_n}^{\pm}\rangle - \langle J_0'(u_{b_n}) - J_0'(u_0), u_{b_n}^{\pm}\rangle\end{aligned}$$

combined with the fact that $u_{b_n} \to u_0$, $u_{b_n}^{\pm} \rightharpoonup u_0^{\pm}$ in $H^1(\mathbb{R}^3)$ and $b_n \to 0$ and that J_b is a C^1 functional in $H^1(\mathbb{R}^3)$, we have $J_0'(u_0) = 0$ and

$$\langle J_0'(u_0), u_0^{\pm}\rangle = \lim_{n\to\infty} \langle J_0'(u_0), u_{b_n}^{\pm}\rangle = 0.$$

On the other hand, by an argument similar to (2.26) and a subsequent derivation on $u_b^{\pm} \neq 0$, we know that $u_0^{\pm} \neq 0$. So, $u_0 \in M_0$. That is, u_0 is a sign-changing weak solution of problem (1.2)

Now, we prove that u_0 is also a least energy sign-changing solution of problem (1.2). In fact, assume that $v_0 \in M_0$ is a least energy sign-changing solution of problem (1.2). Then, by Lemma 2.4, there is a sequence of pairs $(s_{b_n}, t_{b_n}) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$ with $s_{b_n} v_0^+ + t_{b_n} v_0^- \in M_{b_n}$. Because $b_n \to 0$ as $n \to \infty$, observing that (2.11) holds corresponding to $b = b_n$, $u = v_0$ and taking into account that $2 \le p \le 4 < q < 6$, $\lambda \ge 0$, and $h(x) > 0$, a.e. $x \in \mathbb{R}^3$, it is easy to check that $\{(s_{b_n}, t_{b_n})\}$ is bounded. Without loss of generality, we can assume that $s_{b_n} \to s_0 \in \mathbb{R}_+$, $t_{b_n} \to t_0 \in \mathbb{R}_+$. Moreover, by an argument similar to that in (2)(ii) of the proof of Lemma 2.4, it can be verified that $s_0 > 0$, $t_0 > 0$. Consequently, we have

$$s_0 v_0^+ + t_0 v_0^- = \lim_{n\to\infty} \left(s_{b_n} v_0^+ + t_{b_n} v_0^- \right) \in M_0.$$

Applying Lemma 2.4 for $b = 0$, it follows from $v_0 \in M_0$ that $s_0 = t_0 = 1$. Thus, we have

$$J_0(v_0) \le J_0(u_0) = \lim_{n\to\infty} J_{b_n}(u_{b_n}) \le \lim_{n\to\infty} J_{b_n}\left(s_{b_n} v_0^+ + t_{b_n} v_0^- \right) = J_0\left(v_0^+ + v_0^- \right) = J_0(v_0).$$

Hence, $J_0(v_0) = J_0(u_0)$. That is, u_0 is also a least energy sign-changing solution of problem (1.2). The proof of Theorem 3.2 is complete. □

4 Conclusion

In this paper, with the help of the constraint variational method combined with a quantitative lemma, Kirchhoff–Poisson systems (1.1) are investigated and the existence result on the least energy sign-changing solution with two nodal domains to the problem is established. Moreover, the convergence property of u_b as $b \searrow 0$ is also obtained. It should be pointed out that, because the nonlocal terms $b \int_{\mathbb{R}^3} (|\nabla u|^2\, dx) \Delta u$ and ϕ_u are involved here, the above Kirchhoff–Poisson systems are totally different from the case $b = 0$ and $k = 0$ and there are more difficulties we need to overcome in the proof.

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