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Higher integrability for weak solutions to a degenerate parabolic system with singular coefficients

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Abstract

In this paper, we study the degenerate parabolic system

$$u_t^i + X_\alpha^*\big(a_{ij}^{\alpha\beta}(z)X_\beta u^j\big) = g_i(z,u,Xu) + X_\alpha^* f_i^\alpha(z,u,Xu),$$

where $X = \{X_1, \ldots, X_m\}$ is a system of smooth real vector fields satisfying Hörmander's condition and the coefficients $a_{ij}^{\alpha\beta}$ are measurable functions and their skew-symmetric part can be unbounded. After proving the L^2 estimates for the weak solutions, the higher integrability is proved by establishing a reverse Hölder inequality for weak solutions.

MSC: Primary 35K65; secondary 35K40; 35B65

Keywords: Degenerate parabolic system; Hörmander's vector fields; L^2 estimates; Higher integrability

1 Introduction

Let $\{X_1, \ldots, X_q\}$ be a system of smooth real vector fields in a neighborhood $\widetilde{\Omega}$ of some bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq q)$, satisfying Hörmander's rank condition up to the order s and free up to the order s. The main purpose of this paper is to study higher integrability for weak solutions to nondiagonal quasilinear degenerate parabolic system

$$u_t^i + X_\alpha^*\big(a_{ij}^{\alpha\beta}(z)X_\beta u^j\big) = g_i(z,u,Xu) + X_\alpha^* f_i^\alpha(z,u,Xu), \tag{1.1}$$

where $i,j = 1,2,\ldots,N$; $\alpha,\beta = 1,2,\ldots,q$; $z = (x,t) \in Q_T = \Omega \times (0,T)$; $X_\alpha^* = -X_\alpha + c_\alpha$ ($c_\alpha = \sum_{k=1}^n b_{\alpha k}(x)\frac{\partial}{\partial x_k} \in C^\infty$) is the transposed vector field of X_α. The assumptions on functions g_i, f_i^α and the coefficients will be specified later.

A function $u \in W_2^{1,1}(Q_T, \mathbb{R}^N)$ is called a weak solution to (1.1) if

$$\iint_{Q_T} \big[u_t^i \psi^i + a_{ij}^{\alpha\beta} X_\alpha \psi^i X_\beta u^j\big]\,dz = \iint_{Q_T} \big[g_i \psi^i + f_i^\alpha X_\alpha \psi^i\big]\,dz,$$

for all $\psi \in C_0^\infty(Q_T, \mathbb{R}^N)$.

In the Euclidean space, regularity to elliptic and parabolic equations and systems has been studied by many authors (see [1–8] and the references therein). Giaquinta in [5] proved the reverse Hölder estimates for weak solutions to diagonal elliptic systems with Hölder continuous coefficients and obtained the higher integrability of weak solutions. Giaquinta and Struwe in [2] treated partial regularity for weak solutions to diagonal quasilinear parabolic systems with the natural growth conditions and got Hölder continuity. Wiegner in [9] derived Hölder continuity of weak solutions to nondiagonal elliptic systems with VMO coefficients and natural growth conditions. Recently, Földes and Phan [10] got the higher integrability for gradients of weak solutions to a linear elliptic equation having the skew-symmetric part of coefficients unbounded.

Based on Hömander's fundamental work [11], there has been tremendous work on degenerate PDEs arising from non-commuting vector fields; see, for example, [12–21]. Di Fazio and Fanciullo in [14] obtained gradient estimates for weak solutions to linear diagonal elliptic systems with bounded VMO coefficients. Dong and Niu [17] got the higher L^p estimates for the gradient of weak solutions to nondiagonal quasilinear degenerate elliptic systems. In [16, 18], Dong and her collaborators studied Morrey and Hölder regularity for weak solutions to diagonal and nondiagonal parabolic systems with bounded VMO coefficients.

However, as far as we know, there is no relevant research about quasilinear degenerate parabolic systems with skew-symmetric coefficients. In this paper, we try to generalize the results in [10] to quasilinear degenerate parabolic systems constructed by Hörmander's vector fields. The aim of this paper is to get the higher integrability for weak solutions to (1.1). In order to state our results, we make the following hypotheses:

(H1) The coefficients $a_{ij}^{\alpha\beta}(z) = A_{ij}^{\alpha\beta}(z) + B_{ij}^{\alpha\beta}(z)$, where $A_{ij}^{\alpha\beta}$ are symmetric ($A_{ij}^{\alpha\beta} = A_{ji}^{\alpha\beta}$), bounded, and satisfy the uniform ellipticity condition that for some $\Lambda > 0$,

$$A_{ij}^{\alpha\beta}(z)\xi_i^\alpha\xi_j^\beta \geq \Lambda|\xi|^2, \qquad \left|A_{ij}^{\alpha\beta}(z)\right| \leq \Lambda^{-1}, \quad \text{a.e. } z \in Q_T, \forall\xi \in \mathbb{R}^{qN};$$

$B_{ij}^{\alpha\beta}(z)$ are skew-symmetric ($B_{ij}^{\alpha\beta} = -B_{ji}^{\alpha\beta}$) and belong to BMO space (therefore they can be unbounded).

(H2) For any $(z, u, \xi) \in Q_T \times \mathbb{R}^N \times \mathbb{R}^{qN}$,

$$\left|g_i(z, u, \xi)\right| \leq g^i(z) + L|\xi|^{\gamma_0},$$
$$\left|f_i^\alpha(z, u, \xi)\right| \leq g_i^\alpha(z) + L|\xi|,$$

where $1 \leq \gamma_0 < 1/q_0$, $q_0 = \frac{Q+2}{Q+4}$, L is a positive constant satisfying $L < \Lambda$, and

$$g^i(z) \in L^{pq_0}(Q_T), \qquad g_i^\alpha \in L^p(Q_T), \quad p \geq 2.$$

Here Q is the homogeneous dimension relative to Ω, and in the sequel we set $\tilde{g} = (g^i)$, $\tilde{\tilde{g}} = (g_i^\alpha)$, $\tilde{q} = 2q_0$.

Now we state our main result.

Theorem 1.1 *Suppose that* (H1) *and* (H2) *hold. Let* $u \in W_2^{1,1}(Q_T, \mathbb{R}^N)$ *be a weak solutions to* (1.1), *then there exists a constant* $\varepsilon_0 >$ *such that for any* $p \in [2, 2 + \tilde{q}\varepsilon_0)$, *we have* $Xu \in$

$L^p_{\mathrm{loc}}(Q_T,\mathbb{R}^N)$, *and for every* $Q'_T \subset\subset Q_T$, *there exists a constant* $C > 0$ *such that*

$$\|Xu\|_{L^p(Q'_T)} \le C\big(\|\tilde{g}\|^{q_0}_{L^{pq_0}(Q_T)} + \|\tilde{\tilde{g}}\|_{L^p(Q_T)}\big). \tag{1.2}$$

The main difficulty in the proof is establishing the reverse Hölder inequality for gradients of weak solutions. We first establish the L^2 estimates of weak solutions by constructing suitable test functions. Then the reverse Hölder inequality of gradients is obtained by the L^2 estimates and the Gehring lemma on a metric measure space.

The paper is organized as follows. In Sect. 2, we introduce some concepts and results related to Hörmander's vector fields that will be used in our proof. Section 3 is devoted to establishing the reverse Hölder inequality for gradients of weak solutions to (1.1) and giving the proof of Theorem 1.1.

2 Preliminaries

Let

$$X_\alpha = \sum_{k=1}^{n} b_{\alpha k}\frac{\partial}{\partial x_k}, \quad b_{\alpha k} \in C^\infty, \alpha = 1,2,\dots,q$$

be a family of vector fields in a neighborhood $\widetilde{\Omega}$ of some bounded domain $\Omega \subset \mathbb{R}^n$. For a multiindex $\alpha = (i_1,\dots,i_k)$, denote by $X_\beta = [X_{i_1},[X_{i_2},\dots,[X_{i_{k-1}},X_{i_k}]]\dots]$ the commutator of vector fields $X_1,\dots,X_q$ with length $k = |\beta|$. We say that the vector fields $X_1,\dots,X_q$ satisfy Hörmander's condition up to the order s (see [11]) provided there exists $s > 0$ such that $\{X_\beta\}_{|\beta|\le s}$ span the tangent space at each point in $\mathbb{R}^n$.

We denote by $Xu = (X_1u,\dots,X_qu)$ the gradient of u with respect to the system $X = \{X_1,\dots,X_q\}$ and hence

$$\big|Xu(x)\big| = \left(\sum_{\alpha=1}^{q}\big|X_\alpha u(x)\big|^2\right)^{\frac{1}{2}}.$$

An absolutely continuous curve $\gamma : [a,b] \to \tilde{\Omega}$ is said to be admissible for the family X, if there exist functions $c_\alpha(t)$, $a \le t \le b$, satisfying

$$\sum_{\alpha=1}^{q} c_\alpha(t)^2 \le 1 \quad \text{and} \quad \gamma'(t) = \sum_{\alpha=1}^{q} c_\alpha(t)X_\alpha\big(\gamma(t)\big), \quad \text{a.e. } t \in [a,b].$$

The Carnot–Carathéodory distance induced by X is defined by

$$d_X(x,y) = \inf\big\{T > 0 : \text{there is an admissible curve } \gamma, \gamma(0) = x, \gamma(T) = y\big\}.$$

Then d_X is a local metric on $\tilde{\Omega}$. The metric ball is denoted by

$$B_R(x) = B(x,R) = \big\{y \in \Omega : d_X(x,y) < R\big\}.$$

If one does not need to consider the center of the ball, then we also write B_R instead of $B(x,R)$.

It is well known that the doubling property for metric balls holds true (see [22, 23]): there exist positive constants $R_d > 0$ and $C_d \geq 1$ such that for any $x \in \Omega$ and $0 < 2R \leq R_d$,

$$\left|B(x,2R)\right| \leq C_d\left|B(x,R)\right|.$$

Here, $|B(x,R)|$ denotes the Lebesgue measure of $B(x,R)$. The number $Q = \log_2 C_d$ is called the homogeneous dimension relative to Ω. Clearly, $Q \geq n$. From the doubling property, we can see that

$$|B_{tR}| \geq Ct^Q|B_R|, \quad \forall R \leq R_d, t \in (0,1),$$

where $C = C_d^{-2}$. In particular, if the vector fields $X_1, \ldots, X_q$ are free up to the order s, there exist two positive constants C_1 and C_2 such that ([24])

$$C_1R^Q \leq \left|B(x,R)\right| \leq C_2R^Q.$$

For $z_0 = (x_0, t_0) \in Q_T \subset \mathbb{R}^{n+1}$, the parabolic cylinder with vertex at z_0 is defined by

$$Q_R(z_0) = B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}\right].$$

Let $I_R(t_0) = (t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}]$, and the parabolic boundary of $Q_R(z_0)$ be denoted by

$$\partial_p Q_R(z_0) = \left(\partial B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}\right]\right) \cup \left(B_R(x_0) \times \left\{t_0 - \frac{R^2}{2}\right\}\right).$$

For any $(x,t), (y,s) \in Q_T$, the parabolic distance in Q_T is defined by

$$d_p\big((x,t),(y,s)\big) = \sqrt{d_X(x,y)^2 + |t-s|},$$

and the parabolic ball is defined by

$$B_p(z_0,R) = \left\{(x,t) \in Q_T : d_p\big((x_0,t_0),(x,t)\big) < R\right\}.$$

To simplify the notations, in the sequel, $Q_R(z_0)$, $B_R(x_0)$, and $I_R(t_0)$ are written as Q_R, B_R, and I_R, respectively. Furthermore, if E is a Lebesgue measurable set with Lebesgue measure $|E|$, we set $u_E = \fint_E u\,dx$ to be the integral average of u on E.

We define the parabolic Sobolev space by

$$W_p^{1,1}(Q_T) = \left\{u \in L^p(Q_T) : X_\alpha u, \partial_t u \in L^p(Q_T), \alpha = 1, 2, \ldots, q\right\},$$

with the norm

$$\|u\|_{W_p^{1,1}(Q_T)} = \|u\|_{L^p(Q_T)} + \|\partial_t u\|_{L^p(Q_T)} + \sum_{\alpha=1}^{q} \|X_\alpha u\|_{L^p(Q_T)}.$$

For any $f \in L^1_{\mathrm{loc}}(Q_T)$, if

$$\|f\|_{\mathrm{BMO}} = \sup_{z_0 \in Q_T, \rho > 0} \frac{1}{|Q_T \cap Q_\rho(z_0)|} \iint_{Q_T \cap Q_\rho(z_0)} |f - f_{Q_T \cap Q_\rho(z_0)}| \, dz < \infty,$$

we say that $f \in \mathrm{BMO}(Q_T)$ (i.e., f has bounded mean oscillation).

Lemma 2.1 (Sobolev inequality, see [12, 23]) *For every compact set $K \subset \Omega$, there exist constants $C > 0$ and $\bar{R} > 0$ such that for any metric ball $B = B(x_0, R)$ with $x_0 \in K$ and $0 < R \le \bar{R}$, it holds that for any $f \in C^\infty(\overline{B_R})$,*

$$\left(\fint_{B_R} |f - f_R|^{\kappa p} \, dx \right)^{\frac{1}{\kappa p}} \le CR \left(\fint_{B_R} |Xf|^p \, dx \right)^{\frac{1}{p}},$$

where $f_R = \fint_{B_R} f \, dx$ is the integral average of f on B_R, and $1 \le \kappa \le Q/(Q-p)$, if $1 \le p < Q$; $1 \le \kappa < \infty$, if $p \ge Q$. Moreover,

$$\left(\fint_{B_R} |f|^{\kappa p} \, dx \right)^{\frac{1}{\kappa p}} \le CR \left(\fint_{B_R} |Xf|^p \, dx \right)^{\frac{1}{p}},$$

whenever $f \in C_0^\infty(\overline{B_R})$.

Lemma 2.2 (Iterative lemma, see [25]) *Let $\varphi(t)$ be a bounded nonnegative function on $[T_0, T_1]$, where $T_1 > T_0 \ge 0$. Suppose that for any t and s, $T_0 \le t < s \le T_1$, $\varphi(t)$ satisfies*

$$\varphi(t) \le \theta \varphi(s) + \frac{A}{(s-t)^\alpha} + B,$$

where θ, A, B, and α are nonnegative constants with $\theta < 1$. Then for any $T_0 \le \rho < R \le T_1$, one has

$$\varphi(\rho) \le c \left[\frac{A}{(R-\rho)^\alpha} + B \right],$$

where c depends only on α and θ.

The following Gehring lemma on the metric measure space (Y, d, μ) (d is a metric and μ is a doubling measure) can be found in [13, 26].

Lemma 2.3 *Let $q \in [q_0, 2Q]$, where $q_0 > 1$ is fixed. Assume that functions f, g are nonnegative and $g \in L^q_{\mathrm{loc}}(Y, \mu)$, $f \in L^{r_0}_{\mathrm{loc}}(Y, \mu)$, for some $r_0 > q$. If there exist constants $b > 1$ and θ such that for every ball $B \subset \sigma B \subset Y$ the following inequality holds*:

$$\fint_B g^q \, d\mu \le b \left[\left(\fint_{\sigma B} g \, d\mu \right)^q + \fint_{\sigma B} f^q \, d\mu \right] + \theta \fint_{\sigma B} g^q \, d\mu,$$

then there exist nonnegative constants $\theta_0 = \theta_0(q_0, Q, C_d, \sigma)$ *and* $\varepsilon_0 = \varepsilon_0(b, q_0, Q, C_d, \sigma)$ *such that if* $0 < \theta < \theta_0$ *then* $g \in L^p_{\mathrm{loc}}(Y, \mu)$ *for* $p \in [q, q + \varepsilon_0)$ *and moreover*

$$\left(\fint_B g^p \, d\mu\right)^{\frac{1}{p}} \le C\left[\left(\fint_{\sigma B} g^q \, d\mu\right)^{\frac{1}{q}} + \left(\fint_{\sigma B} f^p \, d\mu\right)^{\frac{1}{p}}\right]$$

for some positive constant $C = C(q_0, Q, C_d, \sigma)$.

3 Higher integrability

We first introduce two cutoff functions $\xi(x)$ and $\eta(t)$ (see to [4]) such that for any $0 < \rho < R, B_\rho \subset B_R \subset \Omega$,

$$\xi(x) \in C_0^\infty(B_R), \quad 0 \le \xi \le 1, \qquad |X\xi| \le \frac{C}{R - \rho} \quad \text{and} \quad \xi = 1 \text{ in } B_\rho;$$

$$\eta(t) = \begin{cases} \frac{2t - 2(t_0 - \frac{R^2}{2})}{R^2 - \rho^2}, & t \in (t_0 - \frac{R^2}{2}, t_0 - \frac{\rho^2}{2}), \\ 1, & t \in [t_0 - \frac{\rho^2}{2}, t_0 + \frac{R^2}{2}]. \end{cases}$$

Setting $N_1 = \fint_{B_R} \xi^2(x)\, dx$, we denote the average of $u(x,t)$ on B_R by

$$\bar{u}(t) = \left(\int_{B_R} \xi^2 \, dx\right)^{-1} \int_{B_R} u\xi^2 \, dx = \frac{1}{N_1 |B_R|} \int_{B_R} u\xi^2 \, dx.$$

Lemma 3.1 *Let* $u \in W_2^{1,1}(\Omega_T, \mathbb{R}^N)$ *be a weak solution to* (1.1). *Then for any* $Q_R \subset\subset \Omega_T$, *we have*

$$\int_{B_R} \left|u - \bar{u}(t)\right|^2 dx + \iint_{Q_R} |Xu|^2 \, dz \le c \iint_{Q_R} |\tilde{g}|^{\tilde{q}} \, dz + c \iint_{Q_R} \left|\tilde{\tilde{g}}\right|^2 dz. \tag{3.1}$$

Proof Multiplying both sides of (1.1) by the test function $u - \bar{u}(t)$ and integrating on Q_R, we get

$$\iint_{Q_R} \left[u_t^i + X_\alpha{}^*\left(a_{ij}^{\alpha\beta} X_\beta u^j\right)\right]\left(u^i - \bar{u}(t)\right) dz = \iint_{Q_R} \left[g_i + X_\alpha{}^* f_i^\alpha\right]\left(u^i - \bar{u}(t)\right) dz. \tag{3.2}$$

So we have

$$\iint_{Q_R} \left[u_t^i\left(u^i - \bar{u}(t)\right) + a_{ij}^{\alpha\beta} X_\alpha u^i X_\beta u^j\right] dz = \iint_{Q_R} \left[g_i\left(u^i - \bar{u}(t)\right) + f_i^\alpha X_\alpha u^i\right] dz.$$

By (H1), the above can be written as

$$\begin{aligned} &\iint_{Q_R} \left(\frac{1}{2}\left|u^i - \bar{u}(t)\right|^2\right)_t dz + \iint_{Q_R} A_{ij}^{\alpha\beta} X_\alpha u^i X_\beta u^j \, dz \\ &\quad = -\iint_{Q_R} B_{ij}^{\alpha\beta} X_\alpha u^i X_\beta u^j \, dz + \iint_{Q_R} \left[g_i\left(u^i - \bar{u}(t)\right) + f_i^\alpha X_\alpha u^i\right] dz. \end{aligned} \tag{3.3}$$

Due to the skew-symmetry of $B_{ij}^{\alpha\beta}$,

$$\iint_{Q_R} B_{ij}^{\alpha\beta} X_\alpha u^i X_\beta u^j \, dz = 0. \tag{3.4}$$

By (H2), Hölder's, Sobolev's, and Young's inequalities, we have

$$
\begin{aligned}
&\iint_{Q_R} g_i\bigl(u^i - \bar{u}(t)\bigr)\,dz \\
&\quad\le \iint_{Q_R} \bigl(g^i(z) + L|Xu|^{\gamma_0}\bigr)\bigl(u^i - \bar{u}(t)\bigr)\,dz \\
&\quad\le \int_{I_R}\biggl[\biggl(\int_{B_R} |\tilde{g}|^{\tilde{q}}\,dx\biggr)^{\frac{1}{\tilde{q}}}\biggl(\int_{B_R} \bigl|u - \bar{u}(t)\bigr|^{\frac{2(Q+2)}{Q}}\,dx\biggr)^{\frac{Q}{2(Q+2)}}\biggr]\,dt \\
&\qquad + L\int_{I_R}\biggl[\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\frac{\gamma_0}{2}}\biggl(\int_{B_R} \bigl|u - \bar{u}(t)\bigr|^{\frac{2}{2-\gamma_0}}\,dx\biggr)^{\frac{2-\gamma_0}{2}}\biggr]\,dt \\
&\quad\le \int_{I_R}\biggl[\biggl(\int_{B_R} |\tilde{g}|^{\tilde{q}}\,dx\biggr)^{\frac{1}{\tilde{q}}} cR^{\frac{2}{Q+2}}\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\frac{1}{2}}\biggr]\,dt \\
&\qquad + \int_{I_R}\biggl[\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\frac{\gamma_0}{2}} cR^{\frac{Q+2-Q\gamma_0}{2}}\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\frac{1}{2}}\biggr]\,dt \\
&\quad\le c_\varepsilon \iint_{Q_R} |\tilde{g}|^{\tilde{q}}\,dz + \varepsilon R^{\frac{4}{Q}}\int_{I_R}\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\frac{Q+2}{Q}}\,dt \\
&\qquad + cR^{\frac{Q+2-Q\gamma_0}{2}}\sup_{I_R}\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\frac{\gamma_0-1}{2}}\iint_{Q_R} |Xu|^2\,dz \\
&\quad\le c_\varepsilon \iint_{Q_R} |\tilde{g}|^{\tilde{q}}\,dz + \varepsilon R^{\frac{4}{Q}}\sup_{I_R}\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\frac{2}{Q}}\iint_{Q_R} |Xu|^2\,dz \\
&\qquad + cR^{\frac{Q+2-Q\gamma_0}{2}}\sup_{I_R}\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\frac{\gamma_0-1}{2}}\iint_{Q_R} |Xu|^2\,dz,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
\iint_{Q_R} f_i^\alpha X_\alpha u^i\,dz &\le \iint_{Q_R} \bigl|g_i^\alpha(z)\bigr||Xu|\,dz + L\iint_{Q_R} |Xu|^2\,dz \\
&\le c_\varepsilon \iint_{Q_R} |\tilde{\tilde{g}}|^2\,dz + (\varepsilon + L)\iint_{Q_R} |Xu|^2\,dz.
\end{aligned}
\tag{3.6}
$$

Inserting (3.4), (3.5), and (3.6) into (3.3), and by (H1), we get

$$
\begin{aligned}
&\int_{B_R} \frac{1}{2}\bigl|u - \bar{u}(t)\bigr|^2\,dx + \Lambda\iint_{Q_R} |Xu|^2\,dz \\
&\quad\le c_\varepsilon \iint_{Q_R} |\tilde{g}|^{\tilde{q}}\,dz + c_\varepsilon \iint_{Q_R} |\tilde{\tilde{g}}|^2\,dz + \theta\iint_{Q_R} |Xu|^2\,dz,
\end{aligned}
$$

where $\theta = \varepsilon R^{\frac{4}{Q}}\sup_{I_R}(\int_{B_R} |Xu|^2\,dx)^{\frac{2}{Q}} + cR^{\frac{Q+2-Q\gamma_0}{2}}\sup_{I_R}(\int_{B_R} |Xu|^2\,dx)^{\frac{\gamma_0-1}{2}} + \varepsilon + L$. Because $L < \Lambda$, by choosing ε, R small enough we can get that $\theta < \Lambda$. So using Lemma 2.2, we complete the proof. □

Lemma 3.2 *Let $u \in W_2^{1,1}(\Omega_T, \mathbb{R}^N)$ be a weak solution of (1.1). Then for any $0 < \rho < R$, $Q_R \subset\subset \Omega_T$, we have*

$$\sup_{I_\rho} \int_{B_\rho} \left|u - \bar{u}(t)\right|^2 dx + \iint_{Q_\rho} |Xu|^2 dz \\ \leq \frac{c}{(R-\rho)^2} \iint_{Q_R} \left|u - \bar{u}(t)\right|^2 dz + c\left(\frac{R^3}{(R-\rho)^2} + 1\right) \iint_{Q_R} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right) dz. \tag{3.7}$$

Proof Let $B_\rho \subset B_R \subset \Omega$. Multiplying both sides of (1.1) by the test function $(u - \bar{u}(t)) \times \xi^2(x)\eta(t)$ and integrating on $Q'_R = B_R(x_0) \times (t_0 - \frac{R^2}{2}, s]$ $(s \leq t_0 + \frac{R^2}{2})$, we get

$$\iint_{Q'_R} \left[u_t^i + X_\alpha{}^*\left(a_{ij}^{\alpha\beta} X_\beta u^j\right)\right]\left(u^i - \bar{u}(t)\right)\xi^2\eta\, dz \\ = \iint_{Q'_R} \left[g_i + X_\alpha{}^* f_i^\alpha\right]\left(u^i - \bar{u}(t)\right)\xi^2\eta\, dz. \tag{3.8}$$

By (H1), one has

$$\begin{aligned}
&\iint_{Q'_R} \left[u_t^i + X_\alpha^*\left(a_{ij}^{\alpha\beta} X_\beta u^j\right)\right]\left(u^i - \bar{u}(t)\right)\xi^2\eta\, dz \\
&= \iint_{Q'_R} \left[u_t^i\left(u^i - \bar{u}(t)\right)\xi^2\eta + a_{ij}^{\alpha\beta} X_\beta u^j X_\alpha\left(\left(u^i - \bar{u}(t)\right)\xi^2\eta\right)\right] dz \\
&= \iint_{Q'_R} \left[u_t^i\left(u^i - \bar{u}(t)\right)\xi^2\eta + a_{ij}^{\alpha\beta}\xi^2\eta X_\alpha u^i X_\beta u^j + 2a_{ij}^{\alpha\beta}\left(u^i - \bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\right] dz \\
&= \iint_{Q'_R} \left[\left(\frac{1}{2}\left|u^i - \bar{u}(t)\right|^2\eta\right)_t \xi^2 - \frac{1}{2}\left|u^i - \bar{u}(t)\right|^2\xi^2\eta_t + A_{ij}^{\alpha\beta}\xi^2\eta X_\alpha u^i X_\beta u^j\right] dz \\
&\quad + \iint_{Q'_R} B_{ij}^{\alpha\beta}\xi^2\eta X_\alpha u^i X_\beta u^j + 2a_{ij}^{\alpha\beta}\left(u^i - \bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\, dz,
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{Q'_R} \left[g_i + X_\alpha{}^* f_i^\alpha\right]\left(u^i - \bar{u}(t)\right)\xi^2\eta\, dz \\
&= \iint_{Q'_R} \left[g_i\left(u^i - \bar{u}(t)\right)\xi^2\eta + f_i^\alpha X_\alpha\left(\left(u^i - \bar{u}(t)\right)\xi^2\eta\right)\right] dz \\
&= \iint_{Q'_R} \left[g_i\left(u^i - \bar{u}(t)\right)\xi^2\eta + f_i^\alpha\xi^2\eta X_\alpha u^i + 2\xi\eta\left(u^i - \bar{u}(t)\right)f_i^\alpha X_\alpha\xi\right] dz.
\end{aligned}$$

By the above, (3.8) can be written as

$$\begin{aligned}
&\iint_{Q'_R} \left(\frac{1}{2}\left|u^i - \bar{u}(t)\right|^2\eta\right)_t \xi^2\, dz + \iint_{Q'_R} A_{ij}^{\alpha\beta}\xi^2\eta X_\alpha u^i X_\beta u^j\, dz \\
&= \iint_{Q'_R} \left[\frac{1}{2}\left|u^i - \bar{u}(t)\right|^2\xi^2\eta_t - B_{ij}^{\alpha\beta}\xi^2\eta X_\alpha u^i X_\beta u^j\right] dz \\
&\quad - 2\iint_{Q'_R} a_{ij}^{\alpha\beta}\left(u^i - \bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\, dz
\end{aligned}$$

$$+\iint_{Q'_R}\left[g_i\left(u^i-\bar{u}(t)\right)\xi^2\eta+2\xi\eta\left(u^i-\bar{u}(t)\right)f_i^\alpha X_\alpha\xi+f_i^\alpha\xi^2\eta X_\alpha u^i\right]dz. \tag{3.9}$$

Due to the skew-symmetry of $B_{ij}^{\alpha\beta}$,

$$\iint_{Q'_R}\left(B_{ij}^{\alpha\beta}\right)_R\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\,dz=0. \tag{3.10}$$

By (H1), (3.10) and Young's inequality, we have

$$\begin{aligned}
&\iint_{Q'_R}a_{ij}^{\alpha\beta}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\,dz\\
&=\iint_{Q'_R}A_{ij}^{\alpha\beta}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\,dz+\iint_{Q'_R}B_{ij}^{\alpha\beta}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\,dz\\
&=\iint_{Q'_R}A_{ij}^{\alpha\beta}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\,dz\\
&\quad+\iint_{Q'_R}\left(B_{ij}^{\alpha\beta}-\left(B_{ij}^{\alpha\beta}\right)_R\right)\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\,dz\\
&\le\Lambda^{-1}\iint_{Q'_R}\left|u-\bar{u}(t)\right||X\xi||Xu|\xi\eta\,dz\\
&\quad+\iint_{Q'_R}\left|B_{ij}^{\alpha\beta}-\left(B_{ij}^{\alpha\beta}\right)_R\right|\left|u-\bar{u}(t)\right||X\xi||Xu|\xi\eta\,dz\\
&\le c_\varepsilon\iint_{Q'_R}\left|u-\bar{u}(t)\right|^2|X\xi|^2\eta\,dz+2\varepsilon\iint_{Q'_R}|Xu|^2\xi^2\eta\,dz\\
&\quad+c_\varepsilon\iint_{Q'_R}\left|B_{ij}^{\alpha\beta}-\left(B_{ij}^{\alpha\beta}\right)_R\right|^2\left|u-\bar{u}(t)\right|^2|X\xi|^2\eta\,dz.
\end{aligned} \tag{3.11}$$

By Hölder's and Sobolev's inequalities, we have

$$\begin{aligned}
&\iint_{Q'_R}\left|B_{ij}^{\alpha\beta}-\left(B_{ij}^{\alpha\beta}\right)_R\right|^2\left|u-\bar{u}(t)\right|^2dz\\
&\le\left(\iint_{Q'_R}\left|B_{ij}^{\alpha\beta}-\left(B_{ij}^{\alpha\beta}\right)_R\right|^Q dz\right)^{\frac{2}{Q}}\left(\iint_{Q'_R}\left|u-\bar{u}(t)\right|^{\frac{2Q}{Q-2}}dz\right)^{\frac{Q-2}{Q}}\\
&\le c|Q_R|^{\frac{2}{Q}}\cdot\|B\|_{\mathrm{BMO}}^2\left(\int_{I_R}\left(\int_{B_R}|Xu|^2\,dx\right)^{\frac{Q}{Q-2}}dt\right)^{\frac{Q-2}{Q}}\\
&\le c\|B\|_{\mathrm{BMO}}^2R^3\left(\int_{I_R}\left(\int_{B_R}|Xu|^2\,dx\right)^2dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.12}$$

Putting (3.12) into (3.11), we get

$$\begin{aligned}
&\iint_{Q'_R}a_{ij}^{\alpha\beta}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\,dz\\
&\le c_\varepsilon\iint_{Q'_R}\left|u-\bar{u}(t)\right|^2|X\xi|^2\eta\,dz+2\varepsilon\iint_{Q'_R}|Xu|^2\xi^2\eta\,dz\\
&\quad+\frac{c_\varepsilon\|B\|_{\mathrm{BMO}}^2R^3}{(R-\rho)^2}\left(\int_{I_R}\left(\int_{B_R}|Xu|^2\,dx\right)^2dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.13}$$

Using properties of $\xi(x), \eta(t)$ and (3.5),

$$\begin{aligned}
&\iint_{Q'_R} g_i\big(u^i - \bar{u}(t)\big)\xi^2\eta\,dz \\
&\quad\le \iint_{Q_R} g_i\big(u^i - \bar{u}(t)\big)\,dz \\
&\quad\le c_\varepsilon \iint_{Q_R} |\tilde{g}|^{\tilde{q}}\,dz + \varepsilon R^{\frac{4}{Q}} \sup_{I_R}\left(\int_{B_R} |Xu|^2\,dx\right)^{\frac{2}{Q}} \iint_{Q_R} |Xu|^2\,dz \\
&\qquad + cR^{\frac{Q+2-Q\gamma_0}{2}} \sup_{I_R}\left(\int_{B_R} |Xu|^2\,dx\right)^{\frac{\gamma_0-1}{2}} \iint_{Q_R} |Xu|^2\,dz.
\end{aligned} \tag{3.14}$$

By (H2), Hölder's and Young's inequalities,

$$\begin{aligned}
&\iint_{Q'_R} \big[2\xi\eta\big(u^i - \bar{u}(t)\big)f_i^\alpha X_\alpha\xi + f_i^\alpha\xi^2\eta X_\alpha u^i\big]\,dz \\
&\quad\le 2\iint_{Q'_R} \big|u - \bar{u}(t)\big|\big|g_i^\alpha(z)\big||X\xi|\xi\eta\,dz + 2L\iint_{Q'_R} \big|u - \bar{u}(t)\big||Xu||X\xi|\xi\eta\,dz \\
&\qquad + \iint_{Q'_R} \big|g_i^\alpha(z)\big||Xu|\xi^2\eta\,dz + L\iint_{Q'_R} |Xu|^2\xi^2\eta\,dz \\
&\quad\le 2c_\varepsilon \iint_{Q'_R} \big|u - \bar{u}(t)\big|^2|X\xi|^2\eta\,dz + c_\varepsilon \iint_{Q'_R} |\tilde{\tilde{g}}|^2\xi^2\eta\,dz \\
&\qquad + (2\varepsilon + L)\iint_{Q'_R} |Xu|^2\xi^2\eta\,dz.
\end{aligned} \tag{3.15}$$

Inserting (3.13), (3.14), and (3.15) into (3.9), and by (H1), (3.3), (3.4), and Young's inequality, we get

$$\begin{aligned}
&\int_{B_R} \frac{1}{2}\big|u - \bar{u}(t)\big|^2\xi^2\eta\,dx + \Lambda\iint_{Q'_R} |Xu|^2\xi^2\eta\,dz \\
&\quad\le \iint_{Q'_R} \frac{1}{2}\big|u - \bar{u}(t)\big|^2\xi^2\eta_t\,dz + 3c_\varepsilon \iint_{Q'_R} \big|u - \bar{u}(t)\big|^2|X\xi|^2\eta\,dz \\
&\qquad + \frac{c_\varepsilon\|B\|^2_{\mathrm{BMO}}R^3}{(R-\rho)^2}\left(\int_{I_R}\left(\int_{B_R} |Xu|^2\,dx\right)^2 dt\right)^{\frac{1}{2}} + c_\varepsilon \iint_{Q_R} |\tilde{g}|^{\tilde{q}}\,dz \\
&\qquad + c_\varepsilon \iint_{Q'_R} |\tilde{\tilde{g}}|^2\xi^2\eta\,dz + \theta_1 \iint_{Q_R} |Xu|^2\,dz,
\end{aligned}$$

where $\theta_1 = \varepsilon R^{\frac{4}{Q}} \sup_{I_R}(\int_{B_R} |Xu|^2\,dx)^{\frac{2}{Q}} + cR^{\frac{Q+2-Q\gamma_0}{2}} \sup_{I_R}(\int_{B_R} |Xu|^2\,dx)^{\frac{\gamma_0-1}{2}} + 4\varepsilon + L$. Employing properties of $\xi(x), \eta(t)$, (H1), and since $\frac{1}{R^2-\rho^2} \le \frac{C}{(R-\rho)^2}$, we have

$$\begin{aligned}
&\frac{1}{2}\sup_{I_\rho} \int_{B_\rho} \big|u - \bar{u}(t)\big|^2\,dx + \Lambda\iint_{Q_\rho} |Xu|^2\,dz \\
&\quad\le \frac{c}{(R-\rho)^2}\iint_{Q_R} \big|u - \bar{u}(t)\big|^2\,dz + \frac{c_\varepsilon\|B\|^2_{\mathrm{BMO}}R^3}{(R-\rho)^2}\left(\int_{I_R}\left(\int_{B_R} |Xu|^2\,dx\right)^2 dt\right)^{\frac{1}{2}} \\
&\qquad + c_\varepsilon \iint_{Q_R} |\tilde{g}|^{\tilde{q}}\,dz + c_\varepsilon \iint_{Q_R} |\tilde{\tilde{g}}|^2\,dz + \theta_1 \iint_{Q_R} |Xu|^2\,dz.
\end{aligned}$$

Because $L < \Lambda$, by choosing ε, R small enough we can get that $\theta_1 < \Lambda$, so Lemma 2.2 yields

$$\begin{aligned}
&\sup_{I_\rho} \int_{B_\rho} \left|u - \bar{u}(t)\right|^2 dx + \iint_{Q_\rho} |Xu|^2 \, dz \\
&\quad\le \frac{c}{(R-\rho)^2} \iint_{Q_R} \left|u - \bar{u}(t)\right|^2 dz + c \iint_{Q_R} |\tilde{g}|^{\tilde{q}} \, dz + c \iint_{Q_R} \left|\tilde{\tilde{g}}\right|^2 dz \\
&\qquad + \frac{c\|B\|_{\mathrm{BMO}}^2 R^3}{(R-\rho)^2} \left(\sup_{I_R} \int_{B_R} |Xu|^2 \, dx\right)^{\frac{1}{2}} \left(\iint_{Q_R} |Xu|^2 \, dz\right)^{\frac{1}{2}}.
\end{aligned}$$

By (3.1),

$$\iint_{Q_R} |Xu|^2 \, dz \le c \iint_{Q_R} |\tilde{g}|^{\tilde{q}} \, dz + c \iint_{Q_R} \left|\tilde{\tilde{g}}\right|^2 dz.$$

Then

$$\begin{aligned}
&\sup_{I_\rho} \int_{B_\rho} \left|u - \bar{u}(t)\right|^2 dx + \iint_{Q_\rho} |Xu|^2 \, dz \\
&\quad\le \frac{c}{(R-\rho)^2} \iint_{Q_R} \left|u - \bar{u}(t)\right|^2 dz + c \iint_{Q_R} |\tilde{g}|^{\tilde{q}} \, dz + c \iint_{Q_R} \left|\tilde{\tilde{g}}\right|^2 dz \\
&\qquad + \frac{c\|B\|_{\mathrm{BMO}}^2 R^3 \sup_{I_R} \int_{B_R} |Xu|^2 \, dx}{(R-\rho)^2} \left(\iint_{Q_R} \left(|\tilde{g}|^{\tilde{q}} + \left|\tilde{\tilde{g}}\right|^2\right) dz\right)^{\frac{1}{2}} \\
&\quad\le \frac{c}{(R-\rho)^2} \iint_{Q_R} \left|u - \bar{u}(t)\right|^2 dz + c \iint_{Q_R} |\tilde{g}|^{\tilde{q}} \, dz + c \iint_{Q_R} \left|\tilde{\tilde{g}}\right|^2 dz \\
&\qquad + \frac{c\|B\|_{\mathrm{BMO}}^2 R^3 \sup_{I_R} \int_{B_R} |Xu|^2 \, dx}{(R-\rho)^2 (\|\tilde{g}\|_{L^{\tilde{q}}}^{\tilde{q}} + \|\tilde{\tilde{g}}\|_{L^2}^2)^{\frac{1}{2}}} \iint_{Q_R} \left(|\tilde{g}|^{\tilde{q}} + \left|\tilde{\tilde{g}}\right|^2\right) dz \\
&\quad\le \frac{c}{(R-\rho)^2} \iint_{Q_R} \left|u - \bar{u}(t)\right|^2 dz + c\left(\frac{R^3}{(R-\rho)^2} + 1\right) \iint_{Q_R} \left(|\tilde{g}|^{\tilde{q}} + \left|\tilde{\tilde{g}}\right|^2\right) dz.
\end{aligned}$$

The proof is completed. □

Lemma 3.3 *Let $u \in W_2^{1,1}(\Omega_T, \mathbb{R}^N)$ be a weak solution of (1.1). Then there exists a positive constant ε_0 such that for any $p \in [2, 2 + \tilde{q}\varepsilon_0)$, we have $u \in L_{\mathrm{loc}}^{\frac{p\gamma}{2}}(Q_T), Xu \in L_{\mathrm{loc}}^p(Q_T)$, and for any $Q_{2R} \subset\subset Q_T$,*

$$\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p \, dz \le c\left[\left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} |Xu|^2 \, dz\right)^{\frac{p}{2}} + \frac{1}{|Q_{2R}|} \iint_{Q_{2R}} (|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2)^{\frac{p}{2}} \, dz\right].$$

Proof By (3.7) and Sobolev's inequality,

$$\begin{aligned}
&\sup_{I_{4R/5}} \left(\int_{B_{4R/5}} \left|u - \bar{u}(t)\right|^2 dx\right)^{\frac{1}{2}} \\
&\quad\le \left(\frac{c}{R^2} \iint_{Q_R} \left|u - \bar{u}(t)\right|^2 dz\right)^{\frac{1}{2}} + c\left(\iint_{Q_R} \left(|\tilde{g}|^{\tilde{q}} + \left|\tilde{\tilde{g}}\right|^2\right) dz\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq c\left(\iint_{Q_R}|Xu|^2\,dz\right)^{\frac{1}{2}}+c\left(\iint_{Q_R}\left(|\tilde{g}|^{\tilde{q}}+|\tilde{\tilde{g}}|^2\right)dz\right)^{\frac{1}{2}}. \tag{3.16}$$

By Hölder's and Sobolev's inequalities, it follows

$$\begin{aligned}
&\int_{I_{4R/5}}\left(\int_{B_{4R/5}}\left|u-\bar{u}(t)\right|^2dx\right)^{\frac{1}{2}}dt\\
&\quad\leq\int_{I_R}\left(\int_{B_R}\left|u-\bar{u}(t)\right|^{\tilde{q}}dx\right)^{\frac{1}{2\tilde{q}}}\left(\int_{B_R}\left|u-\bar{u}(t)\right|^{\gamma}dx\right)^{\frac{1}{2\gamma}}dt\\
&\quad\leq cR^{\frac{1}{\tilde{q}}}\int_{I_R}\left(\int_{B_R}|Xu|^{\tilde{q}}\,dx\right)^{\frac{1}{2\tilde{q}}}\left(\int_{B_R}|Xu|^2\,dx\right)^{\frac{1}{4}}dt\\
&\quad\leq cR^{\frac{1}{\tilde{q}}}\left(\iint_{Q_R}|Xu|^{\tilde{q}}\,dz\right)^{\frac{1}{2\tilde{q}}}\left(\int_{I_R}\left(\int_{B_R}|Xu|^2\,dx\right)^{\frac{1}{2}\frac{\tilde{q}}{2\tilde{q}-1}}dt\right)^{\frac{2\tilde{q}-1}{2\tilde{q}}}\\
&\quad\leq cR^{\frac{3}{2}}\left(\iint_{Q_R}|Xu|^{\tilde{q}}\,dz\right)^{\frac{1}{2\tilde{q}}}\left(\iint_{Q_R}|Xu|^2\,dz\right)^{\frac{1}{4}},
\end{aligned} \tag{3.17}$$

where $\gamma=\frac{2(Q+2)}{Q}$. By (3.16) and (3.17),

$$\begin{aligned}
&\iint_{Q_{4R/5}}\left|u-\bar{u}(t)\right|^2dz\\
&\quad=\int_{I_{4R/5}}\left(\int_{B_{4R/5}}\left|u-\bar{u}(t)\right|^2dx\right)dt\\
&\quad\leq\sup_{I_{4R/5}}\left(\int_{B_{4R/5}}\left|u-\bar{u}(t)\right|^2dx\right)^{\frac{1}{2}}\cdot\left(\int_{I_{4R/5}}\left(\int_{B_{4R/5}}\left|u-\bar{u}(t)\right|^2dx\right)^{\frac{1}{2}}dt\right)\\
&\quad\leq cR^{\frac{3}{2}}\left(\iint_{Q_R}|Xu|^{\tilde{q}}\,dz\right)^{\frac{1}{2\tilde{q}}}\left(\iint_{Q_R}|Xu|^2\,dz\right)^{\frac{3}{4}}\\
&\qquad+cR^{\frac{3}{2}}\left(\iint_{Q_R}|Xu|^{\tilde{q}}\,dz\right)^{\frac{1}{2\tilde{q}}}\left(\iint_{Q_R}|Xu|^2\,dz\right)^{\frac{1}{4}}\left(\iint_{Q_R}\left(|\tilde{g}|^{\tilde{q}}+|\tilde{\tilde{g}}|^2\right)dz\right)^{\frac{1}{2}}\\
&\quad\equiv I_1+I_2.
\end{aligned} \tag{3.18}$$

By Young's inequality,

$$\begin{aligned}
I_1&\leq c_\varepsilon\left(\iint_{Q_R}|Xu|^{\tilde{q}}\,dz\right)^{\frac{2}{\tilde{q}}}+\varepsilon R^2\iint_{Q_R}|Xu|^2\,dz,\\
I_2&\leq\varepsilon R\left(\iint_{Q_R}|Xu|^{\tilde{q}}\,dz\right)^{\frac{1}{\tilde{q}}}\left(\iint_{Q_R}|Xu|^2\,dz\right)^{\frac{1}{2}}+c_\varepsilon R^2\iint_{Q_R}\left(|\tilde{g}|^{\tilde{q}}+|\tilde{\tilde{g}}|^2\right)dz\\
&\leq c_\varepsilon\left(\iint_{Q_R}|Xu|^{\tilde{q}}\,dz\right)^{\frac{2}{\tilde{q}}}+\varepsilon R^2\iint_{Q_R}|Xu|^2\,dz+c_\varepsilon R^2\iint_{Q_R}\left(|\tilde{g}|^{\tilde{q}}+|\tilde{\tilde{g}}|^2\right)dz.
\end{aligned}$$

Inserting the estimates of I_1 and I_2 into (3.18), we get

$$\iint_{Q_{4R/5}}\left|u-\bar{u}(t)\right|^2dz$$

$$\leq c_\varepsilon \left(\iint_{Q_R} |Xu|^{\tilde{q}}\,dz\right)^{\frac{2}{\tilde{q}}} + \varepsilon R^2 \iint_{Q_R} |Xu|^2\,dz + c_\varepsilon R^2 \iint_{Q_R} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right)dz. \tag{3.19}$$

By (3.7) and (3.19),

$$\begin{aligned}
&\frac{1}{|Q_{3R/4}|}\iint_{Q_{3R/4}} |Xu|^2\,dz \\
&\leq \frac{c}{R^2}\frac{1}{|Q_{3R/4}|}\iint_{Q_{4R/5}} \left|u-\bar{u}(t)\right|^2 dz + \frac{c}{|Q_{3R/4}|}\iint_{Q_{4R/5}} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right)dz \\
&\leq \frac{c_\varepsilon |Q_R|^{\frac{2}{\tilde{q}}}}{|Q_{3R/4}|R^2}\left(\frac{1}{|Q_R|}\iint_{Q_R} |Xu|^{\tilde{q}}\,dz\right)^{\frac{2}{\tilde{q}}} + \frac{\varepsilon}{|Q_{3R/4}|}\iint_{Q_R} |Xu|^2\,dz \\
&\quad + \frac{c_\varepsilon}{|Q_{3R/4}|}\iint_{Q_R} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right)dz \\
&\leq c_\varepsilon\left(\frac{1}{|Q_R|}\iint_{Q_R} |Xu|^{\tilde{q}}\,dz\right)^{\frac{2}{\tilde{q}}} + \frac{\varepsilon}{|Q_R|}\iint_{Q_R} |Xu|^2\,dz \\
&\quad + \frac{c_\varepsilon}{|Q_R|}\iint_{Q_R} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right)dz.
\end{aligned} \tag{3.20}$$

Let $\hat{g} = |Xu|^{\tilde{q}}$ ($\hat{q} = \frac{2}{\tilde{q}} = \frac{Q+4}{Q+2} > 1$), $\hat{f} = (|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2)^{\frac{\tilde{q}}{2}}$, then the above can be written as

$$\frac{1}{|Q_{3R/4}|}\iint_{Q_{3R/4}} \hat{g}^{\hat{q}}\,dz \leq c\left[\left(\frac{1}{|Q_R|}\iint_{Q_R} \hat{g}\,dz\right)^{\hat{q}} + \frac{1}{|Q_R|}\iint_{Q_R} \hat{f}^{\hat{q}}\,dz\right] + \frac{\varepsilon}{|Q_R|}\iint_{Q_R} \hat{g}^{\hat{q}}\,dz.$$

By Lemma 2.3, we know that there exists a positive constant ε_0 such that for any $\hat{p} \in [\hat{q}, \hat{q}+\varepsilon_0)$,

$$\begin{aligned}
&\left(\frac{1}{|Q_R|}\iint_{Q_R} |Xu|^{\hat{p}\tilde{q}}\,dz\right)^{\frac{1}{\hat{p}}} \\
&\leq c\left[\left(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}} |Xu|^2\,dz\right)^{\frac{\tilde{q}}{2}} + \left(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right)^{\frac{\hat{p}\tilde{q}}{2}} dz\right)^{\frac{1}{\hat{p}}}\right].
\end{aligned}$$

Letting $p = \hat{p}\tilde{q} \in [2, 2+\tilde{q}\varepsilon_0)$, we finish the proof. □

Proof of Theorem 1.1 By (3.1), Lemma 3.3, and Hölder's inequality, we have

$$\begin{aligned}
&\iint_{Q_R} |Xu|^p\,dz \\
&\leq c|Q_R|\left[\left(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right)dz\right)^{\frac{p}{2}} + \frac{1}{|Q_{2R}|}\iint_{Q_{2R}} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right)^{\frac{p}{2}} dz\right] \\
&\leq c\iint_{Q_{2R}} \left(|\tilde{g}|^{\tilde{q}} + |\tilde{\tilde{g}}|^2\right)^{\frac{p}{2}} dz \leq c\iint_{Q_{2R}} \left(|\tilde{g}|^{pq_0} + |\tilde{\tilde{g}}|^p\right)dz \\
&\leq c\left(\|\tilde{g}\|_{L^{pq_0}}^{pq_0} + \|\tilde{\tilde{g}}\|_{L^p}^p\right).
\end{aligned}$$

The proof is completed. □

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