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Solvability of some boundary value problems involving p-Laplacian and non-autonomous differential operators

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Abstract

The paper deals with the existence and non-existence of solutions of the following nonlinear non-autonomous boundary value problem governed by the p-Laplacian operator:

$$(P)\qquad \begin{cases} (h(t,x(t))|x'(t)|^{p-2}x'(t))' = g(t,x(t),x'(t)) & \text{a.e. } t \in \mathbb{R}, \\ x(-\infty) = a, \qquad x(+\infty) = b \end{cases}$$

with $a < b$, where a is a positive, continuous function and g is a Carathéodory nonlinear function.

We prove an existence result, underlying the relationship between the behavior of $g(t,x,\cdot)$ as $y \to 0$ related to that of $g(\cdot,x,y)$ and $h(\cdot,x)$ as $|t| \to +\infty$.

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1 Introduction

Differential equations involving the p-Laplacian operator and its generalization, the so-called Φ-Laplacian, have been widely studied due to several applications in various sciences. Indeed, many models in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity, and theory of capillary surfaces can be expressed in terms of such differential operators.

The simplest form of a differential equation involving the Φ-Laplacian operator is

$$\left(\Phi\left(x'\right)\right)' = g\left(t,x,x'\right), \tag{1.1}$$

and it has been studied in many papers [1, 2, 7, 8, 12, 14–16, 23]. We also quote [19] for systems of differential equations, [13] for differential inclusions, [17, 18, 24] for systems of differential inclusions.

More recently, other types of differential operators, governed by an increasing function Φ, possibly singular and not necessarily surjective, have been considered. The theory on this subject can be found in [2] for operators having a bounded domain and in [1] for non-surjective operators.

A different type of generalization consists in dealing with mixed-type operators, that is, equations of type

$$\big(h(t,x)\Phi\big(x'\big)\big)' = g\big(t,x,x'\big) \quad \text{or} \quad h(t,x)\big(\Phi\big(x'\big)\big)' = g\big(t,x,x'\big)$$

(see, e.g., [3–6, 9–11, 22]).

In recent papers (see [20, 21]), some existence and non-existence results were proved for the boundary value problem

$$\begin{cases} (h(t,x(t))\Phi(x'(t)))' = g(t,x(t),x'(t)) & \text{a.e. } t \in \mathbb{R}, \\ x(-\infty) = a, \qquad x(+\infty) = b, \end{cases}$$

where Φ is a general increasing homeomorphism in $\mathbb{R}$. The generality of the differential operator Φ required a rather strong growth assumption on the right-hand side $g(t,x,x')$ with respect to x' (see conditions 3.3 and 3.4 in [20, Theorem 3.1]). For instance, when $g(t,x,y) = a(t)b(x)c(y)$ and Φ has a superlinear growth at infinity, then the condition $c(x')/|\Phi(x')| \to 0$ as $|x'| \to \infty$ is needed (among others) in order to obtain the existence of solutions (see [20, Corollary 4.13]). So, in the special case of the p-Laplacian operator, the function $g(t,x,x')$ has to grow less than $p-1$ as $|x'| \to +\infty$.

The aim of the present paper is to show that when dealing with the p-Laplacian operator, that is, when one has the following equation:

$$(P) \qquad \begin{cases} (h(t,x(t))|x'(t)|^{p-2}x'(t))' = g(t,x(t),x'(t)) & \text{a.e. } t \in \mathbb{R}, \\ x(-\infty) = a, \qquad x(+\infty) = b \end{cases}$$

the growth assumption on the right-hand side g considered in [20] can be improved and the class of solvable problems can be widened. More in detail, when $g(t,x,y) = a(t)b(x)c(y)$, the assumption $c(x')/|\Phi(x')| \to 0$ as $|x'| \to \infty$ can be replaced by $c(y) = O(|y|^p)$ (see condition (3.26)). To our knowledge, the existence result here presented is new also for the classical case $p = 2$. For instance, as an application of our results, we have that the differential equation

$$\big(|t|^n\beta(x)x'(t)\big)' = -t^m g(x)\big(x'(t)\big)^2,$$

where $\beta(x)$, $g(x)$ are generic positive continuous functions, admits solutions satisfying $x(-\infty) = a$, $x(+\infty) = b$ for every a, b with $a < b$, provided that m is odd and $m > 2n+1$ (see Example 3.7). We underline that the previous equation can not be treated by means of the results in [20], since in this case $p = 2$ and the growth of f with respect to x' is greater than $p-1$.

2 Existence and non-existence theorem

In the whole paper we will consider a positive continuous function $h : \mathbb{R} \times [a,b] \to \mathbb{R}$ and a Carathéodory function $g : \mathbb{R}^3 \to \mathbb{R}$.

We deal with the following nonlinear differential equation:

$$\left(h\left(t,x(t)\right)\left|x'(t)\right|^{p-2}x'(t)\right)' = g\left(t,x(t),x'(t)\right) \quad \text{a.e. } t \tag{2.1}$$

and we use the following notations:

$$\begin{aligned} m(t) &:= \min_{x\in[a,b]} h(t,x), \qquad M(t) := \max_{x\in[a,b]} h(t,x), \\ m^*(t) &:= \min_{(s,x)\in[-t,t]\times[a,b]} h(s,x), \qquad M^*(t) := \max_{(s,x)\in[-t,t]\times[a,b]} h(s,x). \end{aligned} \tag{2.2}$$

Of course, $M^*(t) \geq M(t) \geq m(t) \geq m^*(t) > 0$ for every $t \in \mathbb{R}$, with $\inf_{t\in\mathbb{R}} m(t)$ possibly zero.

Our main results are the following general existence and non-existence theorems.

Theorem 2.1 *Suppose that*

$$g(t,a,0) \leq 0 \leq g(t,b,0) \quad \text{for a.e. } t \in \mathbb{R} \tag{2.3}$$

and that there exist constants $C_1, C_2 > 0$, *a continuous function* $\mu : \mathbb{R}^+ \to \mathbb{R}^+$, *and a function* $\lambda \in L^q([-C_1, C_1])$ *with* $1 \leq q \leq \infty$ *such that*

$$\left|g(t,x,y)\right| \leq \lambda(t)\mu\left(h(t,x)|y|^{p-1}\right) \quad \text{for a.e. } |t| \leq C_1, \text{ every } x \in [a,b], |y| \geq C_2, \tag{2.4}$$

$$\int^{+\infty} \frac{\tau^{\frac{q-1}{q(p-1)}}}{\mu(\tau)}\, \mathrm{d}\tau = +\infty \tag{2.5}$$

(*with* $\frac{q-1}{q(p-1)} = \frac{1}{p-1}$ *if* $q = +\infty$).

At last, suppose that there exists a constant $\gamma > 1$ *such that, for every* $C > 0$, *there exists a function* $\Lambda_C \in L^1_{\mathrm{loc}}([0,+\infty))$, *null in* $[0, C_1]$ *and positive in* $(C_1, +\infty)$, *such that*:

$$m(t)^{1-\gamma} \cdot \left(\int_0^t \frac{\Lambda_C(|s|)}{M(s)^\gamma}\, \mathrm{d}s\right)^{-1} \in L^{\frac{1}{(p-1)(\gamma-1)}}(\mathbb{R}) \tag{2.6}$$

and there exists a function $\eta_C \in L^1(\mathbb{R})$ *such that putting*

$$N_C(t) := m(t)^{-\frac{1}{p-1}} \left\{\left(M^*(C_1)C^{p-1}\right)^{1-\gamma} + (\gamma-1)\int_0^t \frac{\Lambda_C(|s|)}{M(s)^\gamma}\, \mathrm{d}s\right\}^{-\frac{1}{\gamma-1}\frac{1}{p-1}} \tag{2.7}$$

for every $x \in [a,b]$ *and every* $|y| \leq N_C(t)$, *we have*

$$\begin{cases} g(t,x,y) \leq -\Lambda_C(t)|y|^{\gamma(p-1)} \\ g(-t,x,y) \geq \Lambda_C(t)|y|^{\gamma(p-1)} \end{cases} \quad \text{for a.e. } t \geq C_1, \tag{2.8}$$

$$\left|g(t,x,y)\right| \leq \eta_C(t) \quad \text{for a.e. } t \in \mathbb{R}. \tag{2.9}$$

Then there exists a function $x \in C^1(\mathbb{R})$ such that $t \mapsto h(t,x(t))|x'(t)|^{p-1}x'(t))$ belongs to $W^{1,1}(\mathbb{R})$ and

$$\begin{cases} (h(t,x(t))|x'(t)|^{p-1}x'(t))' = g(t,x(t),x'(t)), & \text{for a.e. } t \in \mathbb{R}, \\ a \le x(t) \le b, & \text{for every } t \in \mathbb{R}, \\ x(-\infty) = a, \qquad x(+\infty) = b. \end{cases} \tag{2.10}$$

Proof With no restriction we may assume $C_2 > \frac{1}{2C_1}(b-a)$. By (2.5) there exists a constant C such that

$$C > \left(\frac{M^*(C_1)}{m^*(C_1)}\right)^{\frac{1}{p-1}} C_2 \ge C_2 \tag{2.11}$$

and

$$\int_{M^*(C_1)C_2^{p-1}}^{m^*(C_1)C^{p-1}} \frac{\tau^{\frac{q-1}{q(p-1)}}}{\mu(\tau)}\,d\tau > \|\lambda\|_q \big[M^*(C_1)^{\frac{1}{p-1}}(b-a)\big]^{1-\frac{1}{q}}. \tag{2.12}$$

Fix $n \in \mathbb{N}$, $n > C_1$, and put $I_n := [-n,n]$. Consider the truncation operator $U : W^{1,1}(I_n) \to W^{1,1}(I_n)$ defined by

$$U(x) := U_x, \quad \text{where } U_x(t) := \max\{a, \min\{b, x(t)\}\},$$

and for every $x \in W^{1,1}_{\text{loc}}(\mathbb{R})$, put

$$V_x(t) := \max\{-N_C(t), \min\{U_x'(t), N_C(t)\}\}.$$

Moreover, for every $x \in \mathbb{R}$, put $w(x) := \max\{x-b,0\} + \min\{x-a,0\}$.

Let us consider the following auxiliary boundary value problem on the compact interval I_n:

$$(P_n^*) \quad \begin{cases} (h(t,U_x(t))|x'(t)|^{p-2}x'(t))' = g(t,U_x(t),V_x(t)) + \frac{w(x(t))}{|w(x(t))|+1}, & \text{a.e. in } I_n, \\ x(-n) = a, \qquad x(n) = b. \end{cases} \tag{2.13}$$

By the same argument developed in the proof of [20, Theorem 3.1], it is possible to prove that problem (2.13) admits a solution u_n for every $n > C_1$ such that $a \le u_n(t) \le b$ for all $t \in I_n$, so that $U_{u_n}(t) = u_n(t)$ for every $t \in I_n$. Moreover, u_n is increasing in $[-n,-C_1]$ and in $[C_1,n]$ and if $u_n'(t_0) = 0$ for some $C_1 < |t_0| < n$, then $u_n'(t) = 0$ whenever $|t| > |t_0|$ (see Steps 1–2 in the proof of [20, Theorem 3.1]).

Now our goal is to prove that $|u_n'(t)| \le N_C(t)$ for every $t \in I_n$, so that also $V_{u_n}(t) = u_n'(t)$ in I_n and consequently u_n is a solution of equation (2.1) too.

To this aim, put

$$f(t) := h(t,u_n(t))u_n'(t)|u_n'(t)|^{p-2} \quad \text{in } I_n,$$

we claim that

$$\left|f(t)\right| \le m^*(C_1)C^{p-1} \quad \text{for every } t \in [-C_1, C_1] \tag{2.14}$$

implying that $|u_n'(t)| \le C$ for every $t \in [-C_1, C_1]$.

Indeed, notice that by the Lagrange theorem there exists a point $\tau_0 \in I_n$ such that

$$\left|u_n'(\tau_0)\right| = \frac{1}{2C_1}\left|u_n(C_1) - u_n(-C_1)\right| \le \frac{b-a}{2C_1} < C_2 < C,$$

so, by (2.11), we have

$$\left|f(\tau_0)\right| \le M^*(C_1)C_2^{p-1} < m^*(C_1)C^{p-1}.$$

Assume, by contradiction, the existence of an interval $(\tau_1, \tau_2) \subset (-C_1, C_1)$ such that $|f(t)| < m^*(C_1)C^{p-1}$ in (τ_1, τ_2) and $|f(\tau_1)| = M^*(C_1)C_2^{p-1}$, $|f(\tau_2)| = m^*(C_1)C^{p-1}$ or vice versa.

Then we have $C_2 \le u_n'(t) \le C$ in (τ_1, τ_2) and since $N_C(t) = C(\frac{M^*(C_1)}{m(t)})^{\frac{1}{p-1}} \ge C$ for every $t \in (-C_1, C_1)$, we have $|u_n'(t)| < N_C(t)$ for every $t \in (\tau_1, \tau_2)$. Then, by Step 1, the definition of (P_n^*), and assumption (2.4), for a.e. $t \in (\tau_1, \tau_2)$, we have

$$\left|f'(t)\right| = \left|g\big(t, U_{u_n}(t), V_{u_n}(t)\big)\right| \le \lambda(t)\mu\big(\left|f(t)\right|\big).$$

Therefore, by the Hölder inequality we get

$$\begin{aligned}
\int_{M^*(C_1)C_2^{p-1}}^{m^*(C_1)C^{p-1}} \frac{\tau^{\frac{q-1}{q(p-1)}}}{\mu(\tau)}\,d\tau &\le \int_{\tau_1}^{\tau_2} \frac{|f(t)|^{\frac{q-1}{q(p-1)}}}{\mu(|f(t)|)}\left|f'(t)\right| dt \\
&\le \int_{\tau_1}^{\tau_2} \lambda(t)h\big(t, u_n(t)\big)^{\frac{q-1}{q(p-1)}}\left|u_n(t)\right|^{1-\frac{1}{q}} dt \\
&\le \|\lambda\|_q M^*(C_1)^{\frac{q-1}{q(p-1)}}\left(\int_{\tau_1}^{\tau_2}\left|u_n'(t)\right| dt\right)^{1-\frac{1}{q}} \\
&\le \|\lambda\|_q\left(M^*(C_1)^{\frac{1}{p-1}}\int_{\tau_1}^{\tau_2}\left|u_n'(t)\right| dt\right)^{1-\frac{1}{q}} \\
&\le \|\lambda\|_q\big[M^*(C_1)^{\frac{1}{p-1}}(b-a)\big]^{1-\frac{1}{q}}
\end{aligned}$$

in contradiction with (2.12). Thus, claim (2.14) is proved, and consequently we have $|u_n'(t)| < C \le N_C(t)$ for every $t \in [-C_1, C_1]$.

We now prove that $u_n'(t) \le N_C(t)$ for every $t \in I_n \setminus [-C_1, C_1]$.

To this aim, let $\hat{t} := \sup\{t > C_1 : u_n'(\tau) < N_C(\tau) \text{ in } [C_1, t]\}$. Assume, by contradiction, $\hat{t} < n$. By the definition of V_{u_n}, we have

$$\big(h\big(t, u_n(t)\big)u_n'(t)^{p-1}\big)' = g\big(t, U_{u_n}(t), V_{u_n}(t)\big) = g\big(t, u_n(t), u_n'(t)\big) \quad \text{a.e. in } [C_1, \hat{t}].$$

Recalling that $u_n'(t) \ge 0$ in $[C_1, n)$, by (2.8) we have

$$\big(h\big(t, u_n(t)\big)u_n'(t)^{p-1}\big)' \le -\Lambda_C(t)u_n(t)^{\gamma(p-1)} \le -\frac{\Lambda_C(t)}{M(t)^\gamma}\big[h\big(t, u_n(t)\big)u_n'(t)^{p-1}\big]^\gamma$$

for a.e. $t \in [C_1, \hat{t}]$. Then

$$\frac{1}{1-\gamma}\{[h(t,u_n(t))u_n'(t)^{p-1}]^{1-\gamma} - [h(C_1,u_n(C_1))u_n'(C_1)^{p-1}]^{1-\gamma}\}$$
$$= \int_{C_1}^t \frac{(h(u_n(s))u_n'(s)^{p-1})'}{(h(u_n(s))u_n'(s)^{p-1})^\gamma}\,ds \le -\int_{C_1}^t \frac{\Lambda_C(s)}{M(s)^\gamma}\,ds = -\int_0^t \frac{\Lambda_C(s)}{M(s)^\gamma}\,ds$$

for every $t \in [C_1, \bar{t}]$. Therefore,

$$\begin{aligned}(h(t,u_n(t))u_n'(t)^{p-1})^{1-\gamma} &\ge (h(C_1,u_n(C_1))u_n'(C_1)^{p-1})^{1-\gamma} + (\gamma-1)\int_0^t \frac{\Lambda_C(s)}{M(s)^\gamma}\,ds \\ &> (M^*(C_1)C^{p-1})^{1-\gamma} + (\gamma-1)\int_0^t \frac{\Lambda_C(s)}{M(s)^\gamma}\,ds\end{aligned}$$

implying that

$$u_n'(t) < \left(\frac{1}{m(t)}\left\{(M^*(C_1)C^{p-1})^{1-\gamma} + (\gamma-1)\int_0^t \frac{\Lambda_C(s)}{M(s)^\gamma}\,ds\right\}^{\frac{1}{1-\gamma}}\right)^{\frac{1}{p-1}} = N_C(t)$$

for every $t \in [C_1, \hat{t}]$, a contradiction when $\hat{t} < n$. So, $\hat{t} = n$ and the claim is proved. The same argument works in the interval $[-n, -C_1]$ too.

Therefore, we have $|u_n'(t)| \le N_C(t)$ for every $t \in [-n, n]$ implying that

$$h(t,u_n(t))|u_n'(t)|^{p-2}u_n'(t) = g(t,u_n(t),u_n'(t)) \quad \text{a.e. in } I_n.$$

Now, following the same argument as in [20, Theorem 3.1], one can show that the sequence $(\tilde{u}_n)_n$ of the functions u_n continued in a constant way in the whole $\mathbb{R}$ converges to a solution x of problem (2.10), satisfying all the properties stated in the assertion. □

The main tool in the previous existence theorem is the summability of function $N_C(t)$ (condition (2.6)) combined with assumption (2.8). Such conditions are not improvable in the sense that if (2.8) is satisfied with the reversed inequality and N_C is not summable, then problem (P) does not admit solutions, as stated in the following result.

Theorem 2.2 *Suppose that there exist three constants $C_1 \ge 0$, $\rho > 0$, $\gamma > 1$ and a positive function $\Lambda \in L^1_{\mathrm{loc}}([C_1,+\infty))$ such that one of the following pairs of conditions holds:*

$$g(t,x,y) \ge -\Lambda(t)y^{\gamma(p-1)} \quad \text{for a.e. } t \ge C_1, \text{ every } x \in [a,b], y \in (0,\rho) \tag{2.15}$$

or

$$g(t,x,y) \le \Lambda(-t)y^{\gamma(p-1)} \quad \text{for a.e. } t \le -C_1, \text{ every } x \in [a,b], y \in (0,\rho) \tag{2.16}$$

and for every constant C, the function

$$N_C(t) := \left(\frac{1}{M(t)}\left\{C + (\gamma-1)\int_{C_1}^t \frac{\Lambda(s)}{m(s)^\gamma}\,ds\right\}^{\frac{1}{1-\gamma}}\right)^{\frac{1}{p-1}} \tag{2.17}$$

does not belong to $L^1([1,+\infty))$.

Moreover, assume that

$$tg(t,x,y) \le 0 \quad \text{for a.e. } |t| \ge C_1, \text{ every } (x,y) \in [a,b] \times \mathbb{R}, \tag{2.18}$$

and there exist two constants $k, C_2 > 0$ *such that*

$$h(t,x_1) \le C_2 h(t+\delta,x_2) \quad \text{for every } t > C_1, x_1,x_2 \in [a,b] \text{ and } \delta < k, \tag{2.19}$$

$$h(t+\delta,x_1) \le C_2 h(t,x_2) \quad \text{for every } t < -C_1, x_1,x_2 \in [a,b] \text{ and } \delta < k. \tag{2.20}$$

Then each possible solution x of problem (P) is constant in $[C_1,+\infty)$ *(when (2.15) holds) or constant in* $(-\infty,-C_1]$ *(when (2.16) holds).*

Therefore, if both (2.15) and (2.16) hold and $C_1 = 0$, *then problem (P) does not admit solutions, that is, there exists no function* $x \in C^1(\mathbb{R})$ *such that* $t \mapsto h(t,x(t))|x'(t)|^{p-2}x'(t)$ *is almost everywhere differentiable, satisfying the conditions of problem (P).*

Remark 2.1 Of course, (2.19), (2.20) are satisfied (for $C_1 = 0$) when $h(t,x) = h_1(t)h_2(x)$, provided that $h_1(t)$ is decreasing in $(-\infty,0)$ and increasing in $(0,+\infty)$, or h_1 is uniformly continuous in $\mathbb{R}$ and $\inf_{t\in\mathbb{R}} h_1(t) > 0$, or $h_1(t) \sim |t|^{-k}$ as $|t| \to +\infty$ for some $k > 0$, since $h_2(x) > 0$ on $[a,b]$.

3 Asymptotic criteria

We now provide some applications for operators and right-hand side having the product structure

$$h(t,x) = h_1(t)h_2(x) \quad \text{and} \quad g(t,x,y) = g_1(t,x)g_2(x,y).$$

We emphasize the link between the local behavior of $g_2(x,\cdot)$ at $y = 0$ and of $g_1(\cdot,x)$, $h_1(\cdot)$ at infinity, which is crucial for the existence or non-existence of solutions.

In what follows we assume that h_1, h_2 are continuous positive functions, g_1 is a Carathéodory function, and g_2 is a continuous function satisfying

$$g_2(x,y) > 0 \quad \text{for every } y \ne 0 \text{ and } x \in [a,b]; \qquad g_2(a,0) = g_2(b,0) = 0.$$

Put $\tilde{m} := \min_{x\in[a,b]} h_2(x)$ and $\tilde{M} := \max_{x\in[a,b]} h_2(x)$, we have

$$m(t) = \tilde{m}h_1(t) \quad \text{and} \quad M(t) = \tilde{M}h_1(t) \quad \text{for every } t \in \mathbb{R},$$

where recall that $m(t) := \min_{x\in[a,b]} h(t,x)$ and $M(t) := \max_{x\in[a,b]} h(t,x)$.

Moreover, from now on we put

$$m_\infty := \inf_{t\in\mathbb{R}} h_1(t) \ge 0. \tag{3.1}$$

The following existence theorems are application of Theorem 2.1.

Proposition 3.1 *Suppose that, for some* $C_1 > 0$, *we have*

$$t \cdot g_1(t,x) < 0 \quad \text{for a.e. } t \text{ such that } |t| \ge C_1, \text{ every } x \in [a,b], \tag{3.2}$$

and there exists a function $\lambda \in L^q_{\mathrm{loc}}(\mathbb{R})$, $1 \le q \le +\infty$, *such that*

$$|g_1(t,x)| \le \lambda(t) \quad \textit{for a.e. } t \in \mathbb{R}, \textit{ every } x \in [a,b]. \tag{3.3}$$

Moreover, assume that there exist real constants σ, δ, *and* $\gamma > 1$ *satisfying one of the following pairs of conditions*:

$$\delta + 1 > \sigma\gamma, \quad (p-1)(\gamma-1) < \delta + 1 - \sigma, \tag{3.4}$$

$$\delta + 1 < \sigma\gamma, \quad \sigma > p-1, \tag{3.5}$$

such that, for every $x \in [a,b]$, *we have*

$$h_1|t|^\sigma \le h_1(t) \le h_2|t|^\sigma, \quad \textit{a.e. } |t| > C_1, \tag{3.6}$$

$$h_1|t|^\delta \le |g_1(t,x)| \le h_2|t|^\delta, \quad \textit{a.e. } |t| > C_1, \tag{3.7}$$

$$g_2(x,y) \le k_2|y|^{\gamma(p-1)} \quad \textit{whenever } |y| < \rho, \tag{3.8}$$

$$g_2(x,y) \le k_2|y|^{p-\frac{1}{q}} \quad \textit{whenever } |y| > C_2, \tag{3.9}$$

$$g_2(x,y) \ge k_1|y|^{\gamma(p-1)} \quad \textit{for every } y \in \mathbb{R}. \tag{3.10}$$

for certain positive constants h_1, h_2, k_1, k_2, ρ, C_2.

Then problem (*P*) *admits solutions.*

Proof It is not restrictive to assume $C_2 > \max\{C_1, \frac{b-a}{2C_1}\}$. Put $\mu(r) := k_2(\frac{r}{m^*(C_1)})^{\frac{qp-1}{q(p-1)}}$ for $r > 0$ (see (2.2)), from (3.3) and (3.9) the validity of conditions (2.4) and (2.5) follows.

Put $\Lambda(t) := 0$ for $0 \le t \le C_1$, and

$$\Lambda(t) := k_1 \min\Big\{\min_{x\in[a,b]} g_1(-t,x), \min_{x\in[a,b]} -g_1(t,x)\Big\} \quad \text{for } t \ge C_1.$$

By condition (3.3) we have $\Lambda \in L^1_{\mathrm{loc}}([0,+\infty))$, and by (3.2) we have that Λ is positive. Observe that by (3.10) it follows that

$$g(t,x,y) = g_1(t,x)g_2(x,y) \le k_1 g_1(t,x)|y|^{\gamma(p-1)} \le -\Lambda(t)|y|^{\gamma(p-1)}$$

and

$$g(-t,x,y) = g_1(-t,x)g_2(x,y) \ge k_1 g_1(-t,x)|y|^{\gamma(p-1)} \ge \Lambda(t)|y|^{\gamma(p-1)}$$

for a.e. $t \ge C_1$, every $x \in [a,b]$, and every $y \in \mathbb{R}$. Then condition (2.8) of Theorem 2.1 holds, with $\Lambda_C(\cdot) := \Lambda(\cdot)$ for every $C > 0$.

Now, from (3.7) it follows that $h_1k_1t^\delta \le \Lambda(t)$ for a.e. $t \ge C_1$ and by (3.6) we deduce that, for some positive constant c_1, we have

$$\int_0^t \frac{\Lambda(|s|)}{h_1(s)^\gamma}\,\mathrm{d}s = \int_{C_1}^t \frac{\Lambda(|s|)}{h_1(s)^\gamma}\,\mathrm{d}s \ge c_1|t|^{\delta-\gamma\sigma+1} \quad \text{for every } t \text{ large enough.} \tag{3.11}$$

Hence, by condition (3.6) we obtain

$$m(t)^{1-\gamma}\left(\int_0^t \frac{\Lambda(|s|)}{h_1(s)^\gamma}\,\mathrm{d}s\right)^{-1} \le c_2|t|^{-(\delta+1-\sigma)}$$

for some positive constant c_2, implying by the second inequality in (3.4) the validity of assumption (2.6). Moreover, the function $N_C(t)$ defined in (2.7) satisfies

$$N_C(t)^{p-1} \le c_3|t|^{\frac{\delta+1-\sigma\gamma}{1-\gamma}-\sigma} = c_3|t|^{\frac{\delta+1-\sigma}{1-\gamma}} \quad \text{for } t \text{ large enough} \tag{3.12}$$

for some constant c_3. So, by the first inequality in (3.4) we have $\lim_{|t|\to+\infty} N_C(t) = 0$, and then a constant $L_C^* > C_1$ exists such that $N_C(t) \le \rho$ for every $|t| \ge L_C^*$. Let us define $\hat{C} := \max_{|t|\le L_C^*} N_C(t)$ and

$$\eta_C(t) := \begin{cases} \max_{x\in[a,b]} |g_1(t,x)| \cdot \max_{(x,y)\in[a,b]\times[-\hat{C},\hat{C}]} g_2(x,y) & \text{if } |t| \le L_C^*, \\ h_2k_2|t|^\delta N_C(t)^{\gamma(p-1)} & \text{if } |t| > L_C^*. \end{cases}$$

By (3.7) and (3.8), for a.e. $t\in\mathbb{R}$, for every $x\in[a,b]$ and every $y\in\mathbb{R}$ such that $|y| \le N_C(t)$, we have

$$\bigl|g(t,x,y)\bigr| = \bigl|g_1(t,x)\bigr|g_2(x,y) \le \eta_C(t),$$

so it remains to prove that $\eta_C \in L^1(\mathbb{R})$.

By (3.3) and the continuity of the function $g_2(\cdot,\cdot)$, we have $\eta_C \in L^1([-L_C^*, L_C^*])$. Moreover, when $|t| > L_C^*$, by (3.12) we have

$$\eta_C(t) \le \text{Const.}\ |t|^{\delta+\gamma\frac{\delta+1-\sigma}{1-\gamma}} = \text{Const.}\ |t|^{\frac{\delta+\gamma-\sigma\gamma}{1-\gamma}}$$

implying that $\eta_C(t) \in L^1(\mathbb{R}\setminus[-L_C^*, L_C^*])$ by the first condition in (3.4).

Therefore, we can apply Theorem 2.1 and obtain the assertion of the present result.

The case of (3.5) is similar.

If Λ is the function defined above, by the first inequality in (3.5) we get that the integral function $t \mapsto \int_0^t \frac{\Lambda(|s|)}{M(s)^\gamma}\,\mathrm{d}s$ is bounded. So, the second condition in (3.5) implies the validity of assumption (2.6). Moreover, if η_C is defined as above, then

$$\eta_C(t) \sim \text{Const.}\ |t|^\delta \frac{1}{h_1(t)^\gamma} \le \text{Const.}\ t^{\delta-\sigma\gamma} \quad \text{as } t \to +\infty$$

so η_C is summable by condition (3.5) and the proof proceeds as in the first part. □

Remark 3.1 We underline that conditions (3.9) and (3.10) are compatible with each other for large $|y|$ only if $q > 1$ and $\gamma \le \frac{p-\frac{1}{q}}{p-1}$. However, if $m_\infty > 0$ (see (3.1)), condition (3.10) can be improved requiring that it holds only for $|y|$ small enough, as the following result states.

Proposition 3.2 *Let all the assumption of Proposition 3.1 be satisfied, with the exception of (3.10), replaced by*

$$g_2(x,y) \ge k_1|y|^{\gamma(p-1)} \quad \textit{for every } x\in[a,b], |y| < \rho. \tag{3.13}$$

Moreover, assume that $m_\infty > 0$. Then problem (P) admits solutions.

Proof For every fixed $C > 0$, let

$$\Gamma_C := \max\left\{\rho, C\left(\frac{M^*(C_1)}{m_\infty}\right)^{\frac{1}{p-1}}\right\}, \qquad \hat{m}_C := \min_{(x,y)\in[a,b]\times[\rho,\Gamma_C]} g_2(x,y),$$

and finally

$$h_C := \min\left\{k_1, \frac{\hat{m}_C}{\Gamma_C^{\gamma(p-1)}}\right\}.$$

Let us define the function $\Lambda(t)$ as in the proof of Proposition (3.1), with k_1 replaced by h_C.

Notice that $N_C(t) \le \Gamma_C$ for every $t \ge C_1$; moreover $g_2(x,y) \ge h_C|y|^{\gamma(p-1)}$ whenever $|y| \le \Gamma_C$. So, condition (2.8) holds whenever $|y| < \Gamma_C$ and the proof proceeds as that of Proposition 3.1, replacing everywhere k_1 with h_C. □

We state now two non-existence results, obtained applying Theorem 2.2.

Proposition 3.3 *Suppose that*

$$t \cdot g_1(t,x) \le 0 \quad \text{for a.e. } t \in \mathbb{R} \text{ and every } x \in [a,b], \tag{3.14}$$

and let there exist real constants $\delta, \gamma > 1$, $\Lambda > 0$ *and a positive function* $\ell(t) \in L^1([0,\Lambda])$ *such that*

$$\left|g_1(t,x)\right| \le \lambda_1|t|^\delta \quad \text{for every } x \in [a,b], \text{ a.e. } |t| > \Lambda, \tag{3.15}$$

$$\left|g_1(t,x)\right| \le \ell\left(|t|\right) \quad \text{for a.e. } |t| \le \Lambda, x \in [a,b], \tag{3.16}$$

$$g_2(x,y) \le \lambda_2 y^{\gamma(p-1)} \quad \text{for every } x \in [a,b], 0 < y < \rho \tag{3.17}$$

for some positive constants $\lambda_1, \lambda_2, \rho$. *Moreover, assume that* (3.6) *holds for some constants* h_1, h_2, σ *such that one of the following pairs of conditions is satisfied*:

$$\delta + 1 > \sigma\gamma, \quad (p-1)(\gamma-1) \ge \delta + 1 - \sigma, \tag{3.18}$$

$$\delta + 1 \le \sigma\gamma, \quad \sigma \le p - 1. \tag{3.19}$$

At last, suppose that there exist two constants $\epsilon, C_2 > 0$ *such that*

$$h_1(t) \le C_2 h_1(t+r) \quad \text{for every } t > 0 \text{ and } r < \epsilon \tag{3.20}$$

$$h_1(t+r) \le C_2 h_1(t) \quad \text{for every } t < 0 \text{ and } r < \epsilon. \tag{3.21}$$

Then problem (*P*) *does not admit solutions*.

Proof Put

$$\Lambda(t) := \begin{cases} \lambda_2\ell(t) & \text{for } t \in [0,\Lambda], \\ \lambda_1\lambda_2 t^\delta & \text{for } t > \Lambda \end{cases}$$

we have that Λ is a positive function belonging to $L^1_{\mathrm{loc}}([0,+\infty))$ and one can easily verify that conditions (3.15), (3.16), and (3.17) ensure the validity of (2.15) and (2.16) with $C_1 = 0$. Moreover, by (3.6) and the very definition of Λ, one has

$$\begin{aligned}\rho(t) &:= \int_0^t \frac{\Lambda(\tau)}{m(\tau)^\gamma}\,\mathrm{d}\tau \geq \int_\Lambda^t \frac{\Lambda(\tau)}{m(\tau)^\gamma}\,\mathrm{d}\tau = \frac{\lambda_1\lambda_2}{\tilde{m}^\gamma}\int_\Lambda^t \frac{\tau^\delta}{h_1(\tau)^\gamma}\,\mathrm{d}\tau \\ &\geq \frac{\lambda_1\lambda_2}{\tilde{m}^\gamma h_1^\gamma}\int_\Lambda^t \tau^{\delta-\gamma\delta}\,\mathrm{d}\tau = \text{Const.}\, t^{\delta-\sigma\gamma+1} \quad \text{(provided } t \text{ is sufficiently large).}\end{aligned}$$

As a consequence, for every constant C, we have

$$\begin{aligned}N_C(t) &= \left(\frac{1}{M(t)[C+(\gamma-1)\rho(t)]^{\frac{1}{\gamma-1}}}\right)^{\frac{1}{p-1}} \\ &\geq \text{Const.}\left(\frac{1}{t^{\sigma+\frac{\delta-\sigma\gamma+1}{\gamma-1}}}\right)^{\frac{1}{p-1}} = \text{Const.}\, t^{-\frac{\delta+1-\sigma}{(p-1)(\gamma-1)}},\end{aligned}$$

provided $t \in (0,+\infty)$ is sufficiently large. At last, the second assumption in (3.18) implies that $N_C(t)$ is not summable in $[1,+\infty)$ and the assertion follows as an application of Theorem 2.2 in the case $C_1 = 0$.

The case of (3.19) is similar.

By using the same notations, notice that under the first condition in (3.19) the integral function $\rho(t) = \int_0^t \frac{\Lambda(\tau)}{m(\tau)^\gamma}\,\mathrm{d}\tau$ is bounded, hence $N_C(t)^{p-1} \geq$ Const. $t^{-\sigma}$, implying that $N_C(t) \geq$ Const. $t^{-\sigma/(p-1)}$. Therefore N_C is not summable at infinity owing to the second condition in assumption (3.19), and the assertion follows from Theorem 2.2, applied for $C_1 = 0$. □

As an immediate application of the previous theorems, the following criteria hold.

Corollary 3.4 *Let $g(t,x,y) = g_1(t)g_2(x)g_3(y)$, where $g_1 \in L^q_{\mathrm{loc}}(\mathbb{R})$ for some $1 \leq q \leq +\infty$, g_3 is continuous in $\mathbb{R}$, and g_2 is continuous and positive in $[a,b]$.*

Assume that $g_3(y) > 0$ for $y \neq 0$; $t \cdot g_1(t) \leq 0$ for every t and suppose that there exist constants $c_1,\ldots,c_3 > 0$ such that

$$h_1(t) \sim c_1|t|^\sigma \text{ as } |t| \to +\infty \quad \text{for some } \sigma \in \mathbb{R}, \tag{3.22}$$

$$|g_1(t)| \sim c_2|t|^\delta \text{ as } |t| \to +\infty \quad \text{for some } \delta \in \mathbb{R}, \tag{3.23}$$

$$g_3(y) \sim c_3|y|^\beta \text{ as } y \to 0 \quad \text{for some } \beta > 0, \tag{3.24}$$

with

$$\delta + 1 > \frac{\sigma\beta}{p-1} \quad \text{and} \quad \beta > p-1. \tag{3.25}$$

Then, if conditions (3.20), (3.21) hold and $p \leq \beta + \sigma - \delta$, (P) has no solution.

Vice versa, if $p > \beta + \sigma - \delta$ and

$$\limsup_{|y|\to+\infty} g_3(y)/y^{p-\frac{1}{q}} \in [0,+\infty) \tag{3.26}$$

$$g_3(y) \geq k_1 |y|^\beta \quad \forall y \in \mathbb{R}, \tag{3.27}$$

then (*P*) *admits solutions.*

Proof The assertion is an immediate consequence of Propositions 3.1 and 3.3 taking $\gamma = \beta/(p-1)$. □

Taking into account what we have observed in Remark 3.1, the following result holds in the particular case $m_\infty > 0$.

Corollary 3.5 *Let all the assumption of Corollary 3.4 hold, with the exception of* (3.27). *Then, if* $m_\infty > 0$, *problem* (*P*) *admits solutions.*

When assumption (3.25) is not satisfied, we can use the following result, the consequence of Propositions 3.1 and 3.3.

Corollary 3.6 *Let all the assumptions of Corollary 3.4 be satisfied, with the exception of* (3.25), *that is, assume that*

$$\delta + 1 < \frac{\sigma\beta}{p-1}. \tag{3.28}$$

Then, if conditions (3.20), (3.21) *hold and* $\sigma + 1 \leq p < \beta + 1$, (*P*) *has no solution.*

Vice versa, if $p < \sigma + 1$, $p < \beta + 1$, *and we further assume* (3.26) *and* (3.27), *then* (*P*) *admits solutions.*

At last, a result analogous to Corollary 3.6 holds when condition (3.27) is removed, provided that $m_\infty > 0$, as in Corollary 3.5.

Example 3.7 Let us consider the following differential equation:

$$\left(\left(1+|t|^n\right)\beta(x)\left|x'(t)\right|^{p-2}x'(t)\right)' = -t^m g(x)\left(x'(t)\right)^2,$$

where β, g are generic positive continuous functions. By virtue of Corollary 3.4 taking $q = \infty$, $\sigma = n$, $\delta = m$, $s = 2$, with $m > 2n - 1$, we deduce that the differential equations admit solutions satisfying $x(-\infty) = a$, $x(+\infty) = b$, for any pair of ordered data a, b.

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