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On the Boussinesq system: local well-posedness of the strong solution and inviscid limits

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Abstract

In this paper, we consider the solvability, regularity and vanishing viscosity limit of the 3D viscous Boussinesq equations with a Navier-slip boundary condition. We also obtain the rate of convergence of the solution of viscous Boussinesq equations to the corresponding ideal Boussinesq equations.

Keywords: Boussinesq equations; Vanishing viscosity limit; Navier-slip boundary conditions

1 Introduction

Let $\Omega \subset R^3$ be a class of bounded smooth domains. We investigate the 3D Boussinesq equations, which are governed by the following equations:

$$\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = \theta e_3, \quad \text{in } \Omega, \tag{1.1}$$

$$\partial_t \theta - \kappa \Delta \theta + u \cdot \nabla \theta = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{1.3}$$

$$u(0,x) = u_0(x), \qquad \theta(0,x) = \theta_0(x), \quad \text{in } \Omega, \tag{1.4}$$

with the Navier-slip boundary condition

$$u \cdot n = 0, \quad n \times (\nabla \times u) = [\beta u]_\tau, \qquad \partial_n \theta = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

where $\nabla \cdot$ and $\nabla \times$ denote the div and curl operators, n is the unit outward normal vector. β is a given smooth symmetric tensor on the boundary, τ is the unit tangential vector of Ω. $[\cdot]_\tau$ represents the tangential component. $p = p(x,t)$ the pressure, $e_3 = (0,0,1)$ the unit vector in the vertical direction, $\nu \geq 0$ and $k \geq 0$ are parameters representing the fluid viscosity and the thermal diffusivity, respectively.

The corresponding ideal Boussinesq system is the following:

$$\partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p = \theta^0 e_3, \quad \text{in } \Omega,$$

$$\partial_t \theta^0 + u^0 \cdot \nabla \theta^0 = 0, \quad \text{in } \Omega,$$

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$$\nabla \cdot u^0 = 0, \quad \text{in } \Omega.$$

With slip boundary condition,

$$u^0 \cdot n = 0, \quad \text{on } \partial\Omega.$$

In physics, the Boussinesq system (1.1)–(1.4) is commonly used to model large scale atmospheric and oceanic flows, for example, tornadoes, cyclones, and hurricanes. It describes the dynamics of fluid influenced by gravitational force, which plays a very important role in the study of Raleigh–Bernard convection; see [1, 2]. If we set $\theta = 0$, then the Boussinesq system (1.1)–(1.4) reduces to the incompressible Navier–Stokes equations. The vanishing viscosity limit problem for the Navier–Stokes equations has been a well studied domain [3–8]. The well-posedness of the Boussinesq system has been studied extensively in recent years. The global well-posedness of weak solutions, or strong solutions in the case of small data for the Boussinesq equations has been considered by many authors. See, e.g. [9–12]. While, the research about inviscid limit of viscous Boussinesq equations is less than the Navier–Stokes equations. Concerning the mathematical analysis early and more recent results for the Boussinesq system are those in [13–17]. More interesting results are in [18–25].

In this paper, we follow the approach of [6, 7] and formulate the boundary value problem in a suitable functional setting so that the Stokes operator is well behaved. In the functional setting, the nonlinear terms naturally fall into desired functional spaces. These facts allow us to establish the existence and regularity of solutions through the Galerkin approximation and appropriate a priori bounds. Thus, to obtain a unform convergence about the solution of the Boussinesq system (1.1)–(1.5), one needs to obtain some uniform estimates on vorticity. Our approach here is motivated by the idea introduced in [6, 7] to study the same problem for the Boussinesq equations.

The rest of the paper is organized as follows: in Sect. 2, we introduce some notations of function spaces and some basic results. In Sect. 3 we present the basic a priori estimates for the existence theory to the strong solutions. The last section is devoted to providing the detailed proof for the rate of convergence on system (1.1)–(1.5) converge to the idea Boussinesq equations.

2 Preliminaries

Throughout the rest of this paper, $\Omega \subset R^3$ denotes a simply connected smooth domain. We will use the classical Lebesgue spaces $(L^2(\Omega), \|\cdot\|_{L^2} = \|\cdot\|)$ and the Sobolev spaces $(H^s(\Omega), \|\cdot\|_{H^m} = \|\cdot\|_m)$, for $s \geq 0$, and $H^{-s}(\Omega)$ with $s \geq 0$ denotes the dual of $H_0^s(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$). For convenience, Ω may be omitted when we write the spaces without confusion. Let

$$X = \{u \in L^2(\Omega); \nabla \cdot u = 0, u \cdot n = 0\},$$

be the Hilbert space with the L^2 inner product, and let

$$V = \{H^1 \cap X \subset X\}, \quad V^* \text{ is the dual of } V,$$
$$W = \{u \in V \cap H^2; n \times (\nabla \times u) = [\beta u]_\tau, \text{ on } \partial\Omega\}.$$

The following lemma (see [6, 7]) allows us to control the H^s-norm of a vector-valued function $u \in V$ by its H^{s-1}-norms of $\nabla \times u$ and $\nabla \cdot u$, together with the $H^{s-\frac{1}{2}}(\partial\Omega)$-norm of $u \cdot n$.

Lemma 2.1 *Let $s \geq 0$ be an integer, $u \in H^s$ be a vector-valued function. Then*

$$\|u\|_s \leq C\big(\|\nabla \times u\|_{s-1} + \|\nabla \cdot u\|_{s-1} + |n \cdot u|_{s-\frac{1}{2}} + \|u\|_{s-1}\big). \tag{2.1}$$

As a special consequence of (2.1), for any $u \in V$,

$$\|u\|_1 \leq C\big(\|\nabla \times u\|\big).$$

It is easy to check that, for any $u \in W$ and $v \in V$,

$$(-\Delta u, v) = (\nabla \times u, \nabla \times v).$$

Therefore, $-\Delta$ can be extended to the closure of W in V. The extended operator is denoted by A and its domain by $D(A)$. Obviously,

$$W \subseteq D(A) \subset V.$$

The following lemma states that A is well-behaved in these functional settings.

Lemma 2.2 *The Stokes operator $A = -\Delta$ with $D(A) = W \subset V$ is the self-adjoint extension of the positive closed bilinear form*

$$(Au, v) = a(u, v) = \int_\Omega (\nabla \times u) \cdot (\nabla \times v)\, dx + \int_{\partial\Omega} \beta u \cdot v\, dx \tag{2.2}$$

with its inverse being compact, and there is a countable eigenvalues λ_j such that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty,$$

the corresponding eigenvector $\{e_j\} \subset W \cap C^\infty(\Omega)$ makes an orthogonal complete basis of X.

For notational convenience, we still write $-\Delta$ for A. Now, we consider the nonlinear terms in these functional settings. We define

$$B_1(u, \theta) = u \cdot \nabla u - \theta e_3, \qquad B_2(u, \theta) = u \cdot \nabla \theta,$$

where p satisfies

$$\Delta p = \nabla \cdot (\theta e_3 - u \cdot \nabla u), \qquad \nabla p \cdot n = (\theta e_3 - u \cdot \nabla u) \cdot n;$$

thus, we can obtain $B_1(u, \theta) \in X$.

3 The weak solutions

This section establishes the global existence of weak solutions to the Boussinesq system (1.1)–(1.5). The approach is the Galerkin approximation following the argument of Constantin and Foias [26]. Here as in the next section, we consider a general smooth bounded simply connected domain in R^3.

Definition 3.1 The pair (u,θ) is called a weak solution of (1.1)–(1.5) with the initial data $(u_0,\theta_0) \in X$ on the time interval $[0,T)$ if $(u,\theta) \in L^2(0,T;V) \cap C_w([0,T);X)$ satisfies $(u',\theta') \in L^1(0,T;V^*)$ and

$$\begin{aligned} &\bigl(u',\phi\bigr) + \nu(\omega, \nabla \times \phi) + \nu \int_{\partial\Omega} \beta u \cdot \phi \, ds + (u \cdot \nabla u - \theta e_3, \phi) = 0, \\ &\bigl(\theta',\varphi\bigr) + k(\nabla\theta, \nabla\varphi) + (u \cdot \nabla\theta, \varphi) = 0, \end{aligned}$$

for all $\phi \in V, \varphi \in H_0^1$ and for a.e. $t \in [0,T)$, and $u(0) = u_0, \theta(0) = \theta_0$, where $\omega = \nabla \times u$.

The major result of this section is the global existence of a weak solution.

Theorem 3.1 *Let $(u_0,\theta_0) \in X$. Let $T > 0$. Then there exists at least one weak solution (u,θ) of (1.1)–(1.5) on $[0,T)$ which satisfies the energy inequality*

$$\frac{d}{dt}\bigl(\|u\|^2 + \|\theta\|^2\bigr) + 2\bigl(\nu\|\nabla \times u\|^2 + k\|\nabla\theta\|^2\bigr) + 2\nu\beta\|u\|_{L^2_{\partial\Omega}} \le \|\theta\|^2 + \|u\|^2, \tag{3.1}$$

in the sense of distribution.

Proof We start with a sequence of approximate functions $(u^{(m)},\theta^{(m)})$,

$$u^{(m)}(t) = \sum_{j=1}^{m} u_j(t)e_j, \qquad \theta^{(m)}(t) = \sum_{j=1}^{m} \theta_j(t)e_j,$$

where ν_j, θ_j for $j = 1,\dots,m$, solve the following ordinary differential equations:

$$u_j'(t) + \nu\lambda_j u_j(t) + h_j^1(U) = 0, \tag{3.2}$$

$$\theta_j'(t) + k\lambda_j\theta_j(t) + h_j^2(U) = 0, \tag{3.3}$$

$$u_j(0) = (u_0, e_j), \qquad \theta_j(0) = (\theta_0, e_j),$$

where $U = (u_1, u_2, \dots, u_m, \theta_1, \theta_2, \dots, \theta_m)$, and

$$\begin{aligned} &h_j^1(U) = \bigl(B_1\bigl(u^{(m)},\theta^{(m)}\bigr), e_j\bigr), \\ &h_j^2(U) = \bigl(B_2\bigl(u^{(m)},\theta^{(m)}\bigr), e_j\bigr). \end{aligned}$$

Since $(h_j^k(u))$ are Lipshitz in U, (3.2)–(3.3) is locally well posed, say on $[0,T)$. Consequently, for any $t \in [0,T)$, $(u^{(m)},\theta^{(m)})$ solves the following system of equations:

$$\bigl(u^{(m)}\bigr)' - \nu\Delta u^{(m)} + P_m B_1\bigl(u^{(m)},\theta^{(m)}\bigr) = 0, \tag{3.4}$$

$$\big(\theta^{(m)}\big)' - k\Delta\theta^{(m)} + P_m B_2\big(u^{(m)},\theta^{(m)}\big) = 0, \tag{3.5}$$
$$u^{(m)}(0) = P_m u_0, \qquad \theta^{(m)}(0) = P_m\theta_0,$$

where P_m denotes the projection of X onto the space spanned by $\{e_j\}_1^m$.

Taking the inner products ((3.4), $u^{(m)}(t)$), ((3.5), $\theta^{(m)}(t)$), adding them up, and noting that

$$\big(P_m B_1\big(u^{(m)},\theta^{(m)}\big),u^{(m)}\big) = \big(u^{(m)}\cdot\nabla u^{(m)} - \theta^{(m)}e_3, u^{(m)}\big),$$
$$\big(P_m B_2\big(u^{(m)},\theta^{(m)}\big),\theta^{(m)}\big) = \big(u^{(m)}\cdot\nabla\theta^{(m)},\theta^{(m)}\big),$$

we obtain by simple algebraic identities

$$\begin{aligned}
&\big\|u^{(m)}(t)\big\|^2 + \big\|\theta^{(m)}(t)\big\|^2 + 2\int_0^t \big(\nu\big\|\nabla\times u^{(m)}(t)\big\|^2 + k\big\|\nabla\theta^{(m)}(t)\big\|^2\big)\,d\tau \\
&\quad + 2\nu\int_{\partial\Omega}\big\|u^{(m)}(t)\big\|^2\,d\tau \\
&\quad \le \|u_0\|^2 + \|\theta_0\|^2 + \int_0^t \big(\big\|\theta^{(m)}(t)\big\|^2 + \big\|u^{(m)}(t)\big\|^2\big)\,dt.
\end{aligned} \tag{3.6}$$

Therefore,

$$\big(u^{(m)},\theta^{(m)}\big) \quad \text{is bounded in } L^\infty(0,T;X) \text{ uniformly for } m,$$
$$\big(\nabla\times u^{(m)},\nabla\theta^{(m)}\big) \quad \text{is bounded in } L^2(0,T;V) \text{ uniformly for } m.$$

Note that, for $\phi\in V$, we have

$$\big|\big(-\Delta u^{(m)},\phi\big)\big| = \left|\big(\nabla\times u^{(m)},\nabla\phi\big) + \int_{\partial\Omega}\beta u^{(m)}\cdot\phi\right|.$$

Therefore,

$$\big\{-\Delta u^{(m)}\big\} \quad \text{is bounded in } L^2\big(0,T;V^*\big).$$

Similarly,

$$\big\{-\Delta\theta^{(m)}\big\} \quad \text{is bounded in } L^2\big(0,T;V^*\big).$$

For the nonlinear terms, we have, for any $\phi\in V$,

$$\begin{aligned}
&\big|\big(P_m B_1\big(u^{(m)},\theta^{(m)}\big),\phi\big)\big| \\
&\quad = \big|\big(B_1\big(u^{(m)},\theta^{(m)}\big),P_m\phi\big)\big| = \big|\big(B_1\big(u^{(m)},\theta^{(m)}\big),\phi^{(m)}\big)\big| \\
&\quad \le \big\|u^{(m)}\big\|_{L^3}\big\|\nabla u^{(m)}\big\|\big\|\phi^{(m)}\big\|_{L^6} + \big\|\theta^{(m)}\big\|_{L^3}\big\|\nabla\theta^{(m)}\big\|\big\|\phi^{(m)}\big\|_{L^6} \\
&\quad \le C\big(\big\|u^{(m)}\big\|^{\frac{1}{2}}\big\|u^{(m)}\big\|_1^{\frac{3}{2}} + \big\|\theta^{(m)}\big\|^{\frac{1}{2}}\big\|u^{(m)}\big\|_1^{\frac{3}{2}}\big)\big\|\theta^{(m)}\big\|_1,
\end{aligned} \tag{3.7}$$

where $P_m\phi^{(m)} = \phi^{(m)}$. Because of the uniform bound for $\|u^{(m)}\|$ and the bound for $\|u^{(m)}\|_1$, we obtain

$$\{B_1(u^{(m)},\theta^{(m)})\} \quad \text{is bounded in } L^{\frac{4}{3}}(0,T;V^*).$$

Similarly,

$$\{B_2(u^{(m)},\theta^{(m)})\} \quad \text{is bounded in } L^{\frac{4}{3}}(0,T;V^*).$$

Therefore,

$$\{(u^{(m)})',(\theta^{(m)})'\} \quad \text{is bounded in } L^{\frac{4}{3}}(0,T;V^*).$$

The rest of the proof of is similar to the arguments in Constantin and Foias [26] and thus further details are omitted. This completes the proof Theorem 3.1. □

4 The strong solutions

This section studies the local well-posedness of the strong solution of (1.1)–(1.5) corresponding to an initial data $(u_0,\theta_0) \in V$, and its higher regularities. Let $(u_0,\theta_0 \in V$ and let $(u^{(m)},\theta^{(m)})$ be the Galerkin approximation constructed in the previous section. To obtain regularity estimates for $(u^{(m)},\theta^{(m)})$, we set $\omega^m = \nabla \times u^{(m)}$.

Theorem 4.1 *Let $(u_0,\theta_0) \in V$, then we have $T^* > 0$, depending on ν, k, and the H^1-norm of (u_0,θ_0) only such that (1.1)–(1.5) has a unique strong solution (u,θ) on $[0,T^*)$ satisfying*

$$\begin{aligned}
&(u,\theta) \in L^2(0,T;W) \cap C(0,T^*;V)\\
&(u',\theta') \in L^2(0,T;V),
\end{aligned}$$

for any $T \in (0,T^*)$. In addition, the energy equation

$$\begin{aligned}
&\frac{d}{dt}(\|\omega\|^2 + \|\nabla\theta\|^2) + \int_{\partial\Omega} \beta u \cdot u\,ds + 2(\nu\|\Delta u\|^2 + k\|\nabla^2\theta\|^2)\\
&\quad + 2((\nabla \times B_1(u,\theta),\omega) + (\nabla B_2(u,\theta),\nabla\theta)) = 0
\end{aligned}$$

holds where $\omega = \nabla \times u$.

Proof Taking the curl of (3.4), taking the grad of (3.5), we can obtain the following system:

$$(\omega^{(m)})' - \nu\omega^{(m)} + \Sigma g_j^1 \nabla \times e_j = 0, \tag{4.1}$$

$$(\nabla\theta^{(m)})' - k\Delta\nabla\theta^{(m)} + \Sigma g_j^2 \nabla e_j = 0, \tag{4.2}$$

$$\omega^{(m)}(0) = \nabla \times u_0^{(m)},$$

where we recall that g_j^1 satisfies $\sum_{j=1}^m g_j^1 e_j = P_m B_1(u^{(m)},\theta^{(m)})$. Taking the inner product $((4.1),\omega^{(m)}) + ((4.2),\nabla\theta^{(m)})$ and noting that

$$(\nabla \times e_i, \nabla \times e_j) = \lambda_j(e_i,e_j),$$

we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\omega^{(m)}\right\|^2+\left\|\nabla\theta^{(m)}\right\|^2\right)+\int_{\partial\Omega}\beta u^{(m)}\cdot u^{(m)}\,ds\\&\quad+2\left(\nu\left\|\nabla\times\omega^{(m)}\right\|^2+k\left\|\nabla^2\theta^{(m)}\right\|^2\right)\\&\quad+2\left(\left(\nabla\times B_1\left(u^{(m)},\theta^{(m)}\right),\omega^{(m)}\right)+\left(\nabla B_2\left(u^{(m)},\theta^{(m)}\right),\nabla\theta^{(m)}\right)\right)=0.\end{aligned}\tag{4.3}$$

Applying the Agmon inequality

$$\|\phi\|_{L^\infty}\le\|\phi\|_1^{\frac{1}{2}}\|\phi\|_2^{\frac{1}{2}},\quad\forall\phi\in H^2,$$

we find

$$\begin{aligned}&\left(\nabla\times B_1\left(u^{(m)},\theta^{(m)}\right),\omega^{(m)}\right)\le C\left\|\omega^{(m)}\right\|^{\frac{3}{2}}\left\|\omega^{(m)}\right\|_1^{\frac{3}{2}}+\left\|\nabla\theta^{(m)}\right\|\left\|\omega^{(m)}\right\|,\\&\left(\nabla B_2\left(u^{(m)},\theta^{(m)}\right),\nabla\theta^{(m)}\right)\le C\left\|\omega^{(m)}\right\|\left\|\nabla\theta^{(m)}\right\|^{\frac{1}{2}}\left\|\nabla^2\theta^{(m)}\right\|_1^{\frac{3}{2}},\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\omega^{(m)}\right\|^2+\left\|\nabla\theta^{(m)}\right\|^2+\int_{\partial\Omega}\beta u^{(m)}\cdot u^{(m)}\,ds\right)+2\left(\nu\left\|\Delta u^{(m)}\right\|^2+k\left\|\nabla^2\theta^{(m)}\right\|^2\right)\\&\quad\le C\left(\left\|\omega^{(m)}\right\|+\left\|\nabla\theta^{(m)}\right\|\right)^6+\left\|\omega^{(m)}\right\|^2+\left\|\nabla\theta^{(m)}\right\|^2,\end{aligned}$$

where C depends on ν,k. Comparing with the ordinary equation

$$\frac{d}{dt}y=Cy^3,\tag{4.4}$$

we find that there exists a time $T_0>0$, such that, for any fixed $T\in(0,T_0)$,

$$\begin{aligned}&\left(u^{(m)},\theta^{(m)}\right)\quad\text{is bounded in }L^\infty\left(0,T;H^1\right),\\&\left(u^{(m)},\theta^{(m)}\right)\quad\text{is bounded in }L^2\left(0,T;H^2\right).\end{aligned}$$

Note that

$$\left\|P_m(u\times v)\right\|\le C\|u\|_{L^\infty}\|v\|,$$

it follows that

$$\left\{\left(u^{(m)}\right)'\right\},\left\{\left(\theta^{(m)}\right)'\right\}\quad\text{is bounded in }L^2\left(0,T;L^2\right).$$

The standard compactness results allow us to find a subsequence of $(u^{(m)},\theta^{(m)})$ and (u,θ) such that

$$\begin{aligned}&\left(u^{(m)},\theta^{(m)}\right)\to(u,\theta)\in L^\infty\left(0,T;H^1\right)\quad\text{weak-star},\\&\left(u^{(m)},\theta^{(m)}\right)\to(u,\theta)\in L^2\left(0,T;H^2\right)\quad\text{weakly},\end{aligned}$$

$$\big(u^{(m)},\theta^{(m)}\big) \to (u,\theta) \in L^2\big(0,T;H^1\big) \quad \text{strongly.}$$

Passing to the limit, we find the weak solution obtained in the previous section may be chosen such that $(u,\theta) \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$. We find $(u',\theta') \in L^2(0,T;L^2)$ and thus $(u,\theta) \in C([0,T];H^1)$. We call such a solution a strong solution.

To show that strong solutions are unique, we consider two strong solutions (u_1,θ_1) and (u_2,θ_2). Then their difference $\widetilde{u} = u_1 - u_2$, $\widetilde{\theta} = \theta_1 - \theta_2$ satisfies

$$\partial_t\widetilde{u} - \nu\Delta\widetilde{u} + B_1(u_1,\theta_1) - B_1(u_2,\theta_2) = 0, \tag{4.5}$$

$$\partial_t\widetilde{\theta} - k\Delta\widetilde{\theta} + B_2(u_1,\theta_1) - B_2(u_2,\theta_2) = 0. \tag{4.6}$$

Taking the inner products $((4.5),\widetilde{u}) + ((4.6),\widetilde{\theta})$, we find

$$\frac{d}{dt}\big(\|\widetilde{u}\|^2 + \|\widetilde{\theta}\|^2\big) \le g(t)\big(\|\widetilde{u}\|^2 + \|\widetilde{\theta}\|^2\big) \tag{4.7}$$

on $[0,T]$ for some positive integrable function $g(t)$. Then, $u_1 = u_2, \theta_1 = \theta_2$ follows from $\widetilde{u}(0) = 0, \widetilde{\theta}(0) = 0$ and the Gronwall inequality. From the standard extension method of time evolution, we complete the proof of Theorem 4.1. □

5 The vanishing viscosity limit

This section focuses on the vanishing viscosity limit of the Boussinesq equations. We start with the following uniform bound estimate.

Proposition 5.1 *Let $u_0,\theta_0 \in H^2(\Omega)$. Then there is a T_0 depending on $\|(u_0,\theta_0)\|_{H^2}$ such that the strong solution $u = u(\nu,k), \theta = \theta(\nu,k)$ of the system (1.1)–(1.5) with the initial data u_0,θ_0 obeys the following bound*:

$$\big\|u(\cdot,t)\big\|_1 + \big\|\theta(\cdot,t)\big\|_1 \le C, \quad \text{for } t \in [0,T_0],$$

where C is a constant independent of ν and k.

Proof Using the Hölder inequality and the Young inequality, we have

$$\begin{aligned} &\big(\nabla\times B_1(u,\theta),\omega\big) + \big(\nabla B_2(u,\theta),\nabla\theta\big) \\ &\quad\le \|\nabla\times\omega\|^{\frac{3}{2}}\|\omega\|^{\frac{3}{2}} + \|\omega\|^2 + \|\nabla\theta\|^2 + \|\nabla u\|\big\|\nabla^2\theta\big\|^{\frac{3}{2}}\|\nabla\theta\|^{\frac{1}{2}} \\ &\quad\le \frac{\nu}{2}\|\nabla\times\omega\|^2 + C\|\omega\|^2 + C\|\nabla\theta\|^2 + \frac{\kappa}{2}\big\|\nabla^2\theta\big\|^2. \end{aligned} \tag{5.1}$$

According to Theorems 4.1, thus, we can obtain $T(\nu,\kappa) \ge T_0$ for all $\nu,\kappa > 0$. This completes the proof of Proposition 5.1. □

Theorem 5.2 *Let $(u_0,\theta_0) \in W$. Let there exist a positive $T_0 > 0$ and $u = u(\nu,k), \theta = \theta(\nu,k)$, the corresponding strong solution of the Boussinesq equations (1.1)–(1.5). Then, as $\nu,k \to 0$, (u,θ) converges to the unique solution (u^0,θ^0) of the ideal Boussinesq equations with the same initial data in the sense that*

$$\big(u(\nu,k),\theta(\nu,k)\big) \to \big(u^0,\theta^0\big) \quad \text{in } L^2(0,T;W),$$

$$\big(u(\nu,k),\theta(\nu,k)\big)\to\big(u^0,\theta^0\big)\quad in\ C(0,T;X).$$

Proof It follows from Proposition 5.1 that

$$\begin{aligned}&u(\nu,k),\theta(\nu,k)\quad\text{is uniformly bounded in } L^2(0,T_0;W)\cap C\big(0,T^*;V\big),\\&u'(\nu,k),\theta'(\nu,k)\quad\text{is uniformly bounded in } L^2(0,T;X),\end{aligned}$$

for all $\nu,k>0$. By the standard compactness result, there is a subsequence ν_n,k_n of ν,k and vector functions u^0,θ^0 such that

$$\begin{aligned}&\big(u(\nu_n,k_n),\theta(\nu_n,k_n)\big)\to\big(u^0,\theta^0\big)\quad\text{in } L^2(0,T;V),\\&\big(u(\nu_n,k_n),\theta(\nu_n,k_n)\big)\to\big(u^0,\theta^0\big)\quad\text{in } C(0,T;X).\end{aligned}$$

As $\nu_n,k_n\to 0$, passing to the limit, we find the limit (u^0,θ^0) solves the following limit equations:

$$\begin{aligned}&\partial_t u^0+u^0\cdot\nabla u^0+\nabla p=\theta^0 e_3,\quad\text{in }\Omega,\\&\partial_t\theta^0+u^0\cdot\nabla\theta^0=0,\quad\text{in }\Omega,\\&\nabla\cdot u^0=0,\quad\text{in }\Omega,\end{aligned}$$

with the slip boundary condition,

$$\nabla\cdot u^0=0,\qquad u^0\cdot n=0,\quad\text{on }\partial\Omega,$$

and p satisfies

$$\Delta p=\nabla\cdot\big(\theta^0 e_3-u^0\cdot\nabla u^0\big),\qquad\nabla p\cdot n=\big(\theta^0 e_3-u^0\cdot\nabla u^0\big)\cdot n.$$

As in the proof of the uniqueness of the strong solutions of the Boussinesq equations in the previous section, we can show that (u^0,θ^0) is unique. Then we show the convergence of whole sequence. □

Finally, we present the convergence rate.

Theorem 5.3 *Let $(u_0,\theta_0)\in V$ satisfy the assumptions state in Theorems 4.1, and (u^0,θ^0) be the solution of the ideal Boussinesq equations on $(0,T_0)$ with $(u^0(0)=u_0,\theta^0(0)=\theta_0)$, and $(u,\theta)=(u(\nu,\kappa),\theta(\nu,\kappa))$ be the solution of the viscous Boussinesq equations (1.1)–(1.5). Then*

$$\begin{aligned}&\big|u(\nu,\kappa)-u^0\big\|^2+\big\|\theta(\nu,\kappa)-\theta^0\big\|^2+\int_0^{T_0}\nu\big\|u(\nu,\kappa)-u^0\big\|_1^2+\kappa\big\|\theta(\nu,\kappa)-\theta^0\big\|_1^2\\&\quad\le C(T_0)\big(\nu^{2-s}+\kappa^{2-s}\big),\end{aligned}$$

where $s\in(0,1), t\in[0,T_0]$.

Proof Denote $\widetilde{u} = u(\nu,\kappa) - u^0, \widetilde{\theta} = \theta(\nu,\kappa) - \theta^0$. We find that

$$\partial_t \widetilde{u} - \nu\Delta\widetilde{u} + B_1(u,\theta) - B_1(u^0,\theta^0) = \nu\Delta u^0, \tag{5.2}$$

$$\partial_t \widetilde{\theta} - \kappa\Delta\widetilde{\theta} + B_2(u,\theta) - B_2(u^0,\theta^0) = \kappa\Delta \theta^0, \tag{5.3}$$

$$\nabla\cdot\widetilde{u} = 0, \qquad \widetilde{u}\cdot n = 0.$$

Taking the L^2 inner product of (5.2) with $\widetilde{u}$, and (5.3) with $\widetilde{\theta}$, integrating by parts, one can obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\|\widetilde{u}\|^2 + \|\widetilde{\theta}\|^2\big) + 2\big(\nu\|\nabla\times\widetilde{u}\|^2 + \kappa\|\nabla\widetilde{\theta}\|^2\big)\\
&\quad + 2\big(B_1(u,\theta) - B_1(u^0,\theta^0) - \nu\Delta u^0, \widetilde{u}\big) + 2\big(B_2(u,\theta) - B_2(u^0,\theta^0) - \kappa\theta^0, \widetilde{\theta}\big)\\
&\quad + 2\nu\int_{\partial\Omega} n\times(\nabla\times\widetilde{u})\widetilde{u}\,ds + 2\kappa\int_{\partial\Omega} n\cdot(\nabla\widetilde{\theta})\widetilde{\theta}\,ds\\
&\le \nu^2 + \|\widetilde{u}\|^2 + \kappa^2 + \|\widetilde{\theta}\|^2.
\end{aligned}\tag{5.4}$$

By using the trace theorem

$$\begin{aligned}
\nu\int_{\partial\Omega} n\times(\nabla\widetilde{u})\widetilde{u}\,ds &= \nu\int_{\partial\Omega} n\times\nabla\times\big(u-u^0\big)\widetilde{u}\,ds\\
&= \nu\int_{\partial\Omega}\big(\beta\widetilde{u} + \beta u^0 - n\times\nabla\times u^0\big)\widetilde{u}\,ds \le \nu\int_{\partial\Omega}|\widetilde{u}|^2 + |\widetilde{u}|^2\,ds\\
&\le \nu\|\nabla u\|\|u\| + \nu\big(\|\nabla\times\widetilde{u}\|^2 + \|\widetilde{u}\|^2\big) + \nu^{2-s}.
\end{aligned}\tag{5.5}$$

Similarly,

$$\begin{aligned}
\left|\kappa\int_{\partial\Omega} n\cdot(\nabla\widetilde{\theta})\widetilde{\theta}\,ds\right| &= \left|\kappa\int_{\partial\Omega} n\cdot\nabla\big(\theta-\theta^0\big)\widetilde{\theta}\,ds\right|\\
&= \left|\kappa\int_{\partial\Omega} n\cdot\big(\nabla\theta^0\big)\widetilde{\theta}\,ds\right|\\
&\le \kappa\int_{\partial\Omega}|\widetilde{\theta}|\,ds \le \kappa\big(\|\nabla\widetilde{\theta}\|^2 + \|\widetilde{\theta}\|\big) + \kappa^{2-s},
\end{aligned}\tag{5.6}$$

for any $s\in(0,1)$. Also note that

$$\big|\big(\nu\Delta u^0,\widetilde{u}\big)\big| \le c\nu^2 + \|\widetilde{u}\|^2, \qquad \big|\big(\kappa\Delta\theta^0,\widetilde{\theta}\big)\big| \le c\kappa^2 + \|\widetilde{\theta}\|^2. \tag{5.7}$$

Because $H^1\hookrightarrow L^6$ and $\|u\|_{L^3} \le \|u\|^{\frac{1}{2}}\|u\|_2^{\frac{1}{2}}$, by calculation we have

$$\left|\int_0^{T_0}\big(B_1(u,\theta) - B_1\big(u^0,\theta^0\big),\widetilde{u}\big)\,dt\right| \le C\left(\|\widetilde{\theta}\|^2 + \big\||\widetilde{u}|^2\big\| + \int_0^{T_0}\|\widetilde{u}\|^2 + \|\widetilde{\theta}\|^2\,dt\right), \tag{5.8}$$

$$\left|\int_0^{T_0}\big(B_2(u,\theta) - B_2\big(u^0,\theta^0\big),\widetilde{\theta}\big)\,dt\right| \le C\left(\|\widetilde{\theta}\|^2 + \big\||\widetilde{u}|^2\big\| + \int_0^{T_0}\|\widetilde{u}\|^2 + \|\widetilde{\theta}\|^2\,dt\right). \tag{5.9}$$

Collecting these estimates (5.4)–(5.9), by the Gronwall inequality, we can complete the proof. □

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