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The existence, uniqueness and asymptotic estimates of solutions for third-order full nonlinear singularly perturbed vector boundary value problems

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Abstract

In this paper, we discuss third-order full nonlinear singularly perturbed vector boundary value problems. We first present the existence of solutions for the nonlinear vector boundary value problems without perturbation by using the upper and lower solutions method and topological degree theory. Then the existence, uniqueness and asymptotic estimates of solutions for the singularly perturbed vector boundary value problems are established by constructing appropriate a lower solution–upper solution pair, as well as analysis technique. Some known results are extended.

MSC: 34D15; 34E10; 34B15

Keywords: Vector boundary value problems; Singular perturbation; Existence and uniqueness; Upper and lower solutions; Asymptotic estimates

1 Introduction

In the past few decades, nonlinear boundary value problems (BVPs) and singularly perturbed boundary value problems (SPBVPs) have been studied widely [1–11]. For example, Zhao [5] discussed the existence and asymptotic estimates of the solutions for a third-order boundary value problem with perturbation. Du et al. [9] were concerned with a more generalized third-order singularly perturbed differential equations with multi-point boundary conditions and obtained the existence and uniqueness as well as the asymptotic estimates of solutions. Lodhi and Mishra [12] discussed second order singularly perturbed nonlinear boundary value problems by using the quintic B-spline method. Recently, the geometric singular perturbation theory has also received a great deal of interests in studying the Burgers–KdV equation [13], the vector-disease model [14], the perturbed BBM equation [15], the perturbed Camassa–Holm equation [16] and the perturbed shallow water wave model [17] etc.

However, the boundary value problems in the above-mentioned references are all scalar and little work has been published for vector systems [18–20]. Motivated by the above work, in this article, we discuss the singular perturbations of third-order nonlinear differential system

$$\varepsilon \mathbf{x}'''(t) + \mathbf{F}\big(t, \mathbf{x}(t), \mathbf{x}'(t), \mathbf{x}''(t), \varepsilon\big) = \mathbf{0}, \quad 0 \le t \le 1, 0 < \varepsilon \ll 1, \tag{1.1}$$

with full nonlinear multi-point boundary value conditions

$$\begin{cases} \mathbf{x}(0,\varepsilon) = \mathbf{0}, \\ \mathbf{G}(\mathbf{x}'(0,\varepsilon), \mathbf{x}''(0,\varepsilon), \mathbf{x}(\xi_1,\varepsilon), \mathbf{x}(\xi_2,\varepsilon), \ldots, \mathbf{x}(\xi_{m-2},\varepsilon)) = \mathbf{A}, \\ \mathbf{H}(\mathbf{x}'(1,\varepsilon), \mathbf{x}''(1,\varepsilon), \mathbf{x}(\eta_1,\varepsilon), \mathbf{x}(\eta_2,\varepsilon), \ldots, \mathbf{x}(\eta_{n-2},\varepsilon)) = \mathbf{B}, \end{cases} \tag{1.2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_N)^T$, $\mathbf{F}(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'', \varepsilon) = (f_1, f_2, \ldots, f_N)^T \in R^N$, $f_i = f_i(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'', \varepsilon) \in R$, $\mathbf{G}(\mathbf{x}'(0,\varepsilon), \mathbf{x}''(0,\varepsilon), \mathbf{x}(\xi_1,\varepsilon), \ldots, \mathbf{x}(\xi_{m-2},\varepsilon)) = (g_1, g_2, \ldots, g_N)^T \in R^N$, $g_i = g_i(\mathbf{x}'(0,\varepsilon), \mathbf{x}''(0,\varepsilon), \mathbf{x}(\xi_1,\varepsilon), \ldots, \mathbf{x}(\xi_{m-2},\varepsilon)) \in R$, $\mathbf{H}(\mathbf{x}'(1,\varepsilon), \mathbf{x}''(1,\varepsilon), \mathbf{x}(\eta_1,\varepsilon), \ldots, \mathbf{x}(\eta_{n-2},\varepsilon)) = (h_1, h_2, \ldots, h_N)^T \in R^N$, $h_i = h_i(\mathbf{x}'(1,\varepsilon), \mathbf{x}''(1,\varepsilon), \mathbf{x}(\eta_1,\varepsilon), \ldots, \mathbf{x}(\eta_{n-2},\varepsilon)) \in R$, $i = 1, 2, \ldots, N$, $\mathbf{A} = (A_1, A_2, \ldots, A_N)^T$, $\mathbf{B} = (B_1, B_2, \ldots, B_N)^T \in R^N$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < 1$, ε is a small positive parameter.

In order to study SPBVP (1.1), (1.2), we need to study the following nonlinear unperturbed vector multi-point boundary value problem:

$$\mathbf{x}'''(t) + \mathbf{F}\big(t, \mathbf{x}(t), \mathbf{x}'(t), \mathbf{x}''(t)\big) = \mathbf{0}, \quad 0 \le t \le 1, \tag{1.3}$$

$$\begin{cases} \mathbf{x}(0) = \mathbf{0}, \\ \mathbf{G}(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})) = \mathbf{A}, \\ \mathbf{H}(\mathbf{x}'(1), \mathbf{x}''(1), \mathbf{x}(\eta_1), \mathbf{x}(\eta_2), \ldots, \mathbf{x}(\eta_{n-2})) = \mathbf{B}. \end{cases} \tag{1.4}$$

The remaining part of this paper is organized as follows. In Sect. 2, we present some definitions and lemmas. In Sect. 3, we obtain the existence of solutions for BVP (1.3), (1.4) by using the differential inequality technique and topological degree theory. Furthermore, we give the existence and asymptotic estimates of solutions of SPBVP (1.1), (1.2). In Sect. 4, we establish the uniqueness result of SPBVP (1.1), (1.2).

2 Preliminaries

For the simplicity, for $\forall \mathbf{x} = (x_1, \ldots, x_N)^T, \mathbf{y} = (y_1, \ldots, y_N)^T \in R^N$, we denote $\mathbf{x} \preceq \mathbf{y}$ ($\mathbf{x} \prec \mathbf{y}$), if and only if $x_i \le y_i$ ($x_i < y_i$), $i = 1, 2, \ldots, N$. Similarly, we can define $\mathbf{x} \succeq \mathbf{y}$ ($\mathbf{x} \succ \mathbf{y}$). We use the norm $\|\mathbf{x}\| = (\sum_{i=1}^N x_i^2)^{\frac{1}{2}}$, for $\forall \mathbf{x} = (x_1, \ldots, x_N) \in R^N$.

Definition 1 The vector function $\mathbf{F}(t, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in R^N$ is increasing in $\mathbf{x}_1$, if for $\forall \mathbf{y}_1 \succeq \mathbf{x}_1$, such that

$$\mathbf{F}(t, \mathbf{y}_1, \mathbf{x}_2, \mathbf{x}_3) \succeq \mathbf{F}(t, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).$$

The vector function $\mathbf{G}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) \in R^N$ is increasing in $\mathbf{x}_k$, $k = 1, 2, \ldots, m$, if, for $\forall \mathbf{y}_k \succeq \mathbf{x}_k$,

$$\mathbf{G}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}, \mathbf{y}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_m) \succeq \mathbf{G}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_m).$$

The vector function $\mathbf{H}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \in R^N$ is decreasing in $\mathbf{x}_j$, $j = 1, 2, \ldots, n$, if, for $\forall \mathbf{y}_j \succeq \mathbf{x}_j$,

$$\mathbf{H}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{j-1}, \mathbf{y}_j, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_n) \preceq \mathbf{H}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_n).$$

Similarly, we define the case that $\mathbf{G}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m)$ is decreasing in $\mathbf{x}_k$, $k = 1, 2, \dots, m$. $\mathbf{H}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ is increasing in $\mathbf{x}_j$, $j = 1, 2, \dots, n$.

Definition 2 We define a function δ as follows:

$$\delta(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = \begin{cases} \mathbf{z}_1, & \mathbf{z}_2 \prec \mathbf{z}_1, \\ \mathbf{z}_2, & \mathbf{z}_1 \preceq \mathbf{z}_2 \preceq \mathbf{z}_3, \\ \mathbf{z}_3, & \mathbf{z}_2 \succ \mathbf{z}_3, \end{cases} \tag{2.1}$$

where $\mathbf{z}_\nu = (z_{\nu 1}, z_{\nu 2}, \dots, z_{\nu N})^T \in R^N$, $\nu = 1, 2, 3$, $\mathbf{z}_1 \preceq \mathbf{z}_3$.

Definition 3 ([20]) $\mathbf{F}(t, \mathbf{x}, \mathbf{y}, \mathbf{z})$ is said to satisfy Nagumo condition with respect to $\mathbf{z}$, for $(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) \in [0, 1] \times R^{3N}$, if $\mathbf{F}(t, \mathbf{x}, \mathbf{y}, \mathbf{z})$ satisfies one of the following conditions:

(i) There exist nondecreasing functions $\Phi_i \in C([0, +\infty), (0, +\infty))$, $i = 1, 2, \dots, N$, such that

$$\bigl|f_i(t, \mathbf{x}, \mathbf{y}, \mathbf{z})\bigr| \le \Phi_i\bigl(|z_i|\bigr) \quad \text{and} \quad \int_0^{+\infty} \frac{s\,ds}{\Phi_i(s)} = +\infty.$$

(ii) There exist nondecreasing functions $\Phi \in C([0, +\infty), (0, +\infty))$, such that

$$\bigl\|\mathbf{F}(t, \mathbf{x}, \mathbf{y}, \mathbf{z})\bigr\| \le \Phi\bigl(\|\mathbf{z}\|\bigr) \quad \text{and} \quad \frac{s^2}{\Phi(s)} = +\infty, \quad s \to +\infty.$$

Definition 4 ([10, 20]) A vector function $\boldsymbol{\alpha}(t) = (\alpha_1(t), \dots, \alpha_N(t))^T \in C^3([0, 1], R^N)$ is called a lower solution of BVP (1.3), (1.4), if for $i = 1, 2, \dots, N$,

$$\alpha_i'''(t) + f_i\bigl(t, \mathbf{x}_{\alpha_i}(t), \mathbf{x}'_{\alpha_i}(t), \mathbf{x}''_{\alpha_i}(t)\bigr) \ge 0, \quad 0 \le t \le 1,$$

and

$$\begin{aligned} &\alpha_i(0) \le 0, \\ &g_i\bigl(\boldsymbol{\alpha}'(0), \boldsymbol{\alpha}''(0), \boldsymbol{\alpha}(\xi_1), \dots, \boldsymbol{\alpha}(\xi_{m-2})\bigr) \le A_i, \\ &h_i\bigl(\boldsymbol{\alpha}'(1), \boldsymbol{\alpha}''(1), \boldsymbol{\alpha}(\eta_1), \dots, \boldsymbol{\alpha}(\eta_{n-2})\bigr) \le B_i. \end{aligned}$$

Similarly, a vector function $\boldsymbol{\beta}(t) = (\beta_1(t), \dots, \beta_N(t))^T \in C^3([0, 1], R^N)$ is called an upper solution of BVP (1.3), (1.4), if for $i = 1, 2, \dots, N$,

$$\beta_i'''(t) + f_i\bigl(t, \mathbf{x}_{\beta_i}(t), \mathbf{x}'_{\beta_i}(t), \mathbf{x}''_{\beta_i}(t)\bigr) \le 0, \quad 0 \le t \le 1,$$

and

$$\begin{aligned} &\beta_i(0) \ge 0, \\ &g_i\bigl(\boldsymbol{\beta}'(0), \boldsymbol{\beta}''(0), \boldsymbol{\beta}(\xi_1), \dots, \boldsymbol{\beta}(\xi_{m-2})\bigr) \ge A_i, \\ &h_i\bigl(\boldsymbol{\beta}'(1), \boldsymbol{\beta}''(1), \boldsymbol{\beta}(\eta_1), \dots, \boldsymbol{\beta}(\eta_{n-2})\bigr) \ge B_i, \end{aligned}$$

where

$$\mathbf{x}_{\alpha_i} = (x_1, \dots, x_{i-1}, \alpha_i, x_{i+1}, \dots, x_N),$$
$$\mathbf{x}'_{\alpha_i} = (x'_1, \dots, x'_{i-1}, \alpha'_i, x'_{i+1}, \dots, x'_N),$$
$$\mathbf{x}''_{\alpha_i} = (x''_1, \dots, x''_{i-1}, \alpha''_i, x''_{i+1}, \dots, x''_N),$$

$\mathbf{x}_{\beta_i}$, $\mathbf{x}'_{\beta_i}$, $\mathbf{x}''_{\beta_i}$ are defined analogously.

Similar to [10, 20], we have Lemma 2.1 and we omit the proof.

Lemma 2.1 *Assume that* $\rho_s(t,\varepsilon) = \operatorname{diag}(\rho_{s1}(t,\varepsilon), \dots, \rho_{sN}(t,\varepsilon)) \in C([0,1]\times[0,\varepsilon_0], R^{N\times N})$, $s = 1,2,3$, $\rho_{3i}(t,\varepsilon) \geq 0$, $(t,\varepsilon) \in [0,1]\times[0,\varepsilon_0]$ *and there exists* $\boldsymbol{\beta}(t,\varepsilon) = (\beta_1(t,\varepsilon), \dots, \beta_N(t,\varepsilon))^T \in C^3([0,1]\times[0,\varepsilon_0], R^N)$, *such that* $\boldsymbol{\beta}'(t,\varepsilon) \succ \mathbf{0}$ *and*

$$\varepsilon\boldsymbol{\beta}'''(t,\varepsilon) + \rho_1(t,\varepsilon)\boldsymbol{\beta}''(t,\varepsilon) + \rho_2(t,\varepsilon)\boldsymbol{\beta}'(t,\varepsilon) + \rho_3(t,\varepsilon)\boldsymbol{\beta}(t,\varepsilon) \prec \mathbf{0}, \quad 0 \leq t \leq 1, \tag{2.2}$$

$$\begin{cases} \boldsymbol{\beta}(0,\varepsilon) \succeq \mathbf{0}, \\ P_1\boldsymbol{\beta}'(0,\varepsilon) + Q_1\boldsymbol{\beta}''(0,\varepsilon) + \sum_{k=1}^{m-2}\mu_k\boldsymbol{\beta}(\xi_k,\varepsilon) \succ \mathbf{0}, \\ P_2\boldsymbol{\beta}'(1,\varepsilon) + Q_2\boldsymbol{\beta}''(1,\varepsilon) + \sum_{j=1}^{n-2}\nu_j\boldsymbol{\beta}(\eta_j,\varepsilon) \succ \mathbf{0}, \end{cases} \tag{2.3}$$

where $P_l = \operatorname{diag}(p_{l1}, p_{l2}, \dots, p_{lN})$, $Q_l = \operatorname{diag}(q_{l1}, q_{l2}, \dots, q_{lN})$, $l = 1,2$, $\mu_k = \operatorname{diag}(\mu_{k1}, \dots, \mu_{kN})$, $\nu_j = \operatorname{diag}(\nu_{j1}, \dots, \nu_{jN})$ *satisfy* $q_{1i} \leq 0$, $q_{2i} \geq 0$, $\mu_{ki} \leq 0$, $\nu_{ji} \leq 0$, $i = 1,2,\dots,N$, $k = 1,2,\dots,m-2$, $j = 1,2,\dots,n-2$.

Then the singularly perturbed boundary value problem

$$\varepsilon\mathbf{x}'''(t,\varepsilon) + \rho_1(t,\varepsilon)\mathbf{x}''(t,\varepsilon) + \rho_2(t,\varepsilon)\mathbf{x}'(t,\varepsilon) + \rho_3(t,\varepsilon)\mathbf{x}(t,\varepsilon) = \mathbf{0}, \quad 0 \leq t \leq 1, \tag{2.4}$$

$$\begin{cases} \mathbf{x}(0,\varepsilon) = \mathbf{0}, \\ P_1\mathbf{x}'(0,\varepsilon) + Q_1\mathbf{x}''(0,\varepsilon) + \sum_{k=1}^{m-2}\mu_k\mathbf{x}(\xi_k,\varepsilon) = \mathbf{0}, \\ P_2\mathbf{x}'(1,\varepsilon) + Q_2\mathbf{x}''(1,\varepsilon) + \sum_{j=1}^{n-2}\nu_j\mathbf{x}(\eta_j,\varepsilon) = \mathbf{0}, \end{cases} \tag{2.5}$$

has only a zero solution.

3 Existence results

3.1 Existence result of the modified problem

Assume that $\boldsymbol{\alpha}(t) = (\alpha_1(t), \dots, \alpha_N(t))^T$, $\boldsymbol{\beta}(t) = (\beta_1(t), \dots, \beta_N(t))^T \in C^3([0,1], R^N)$, $\boldsymbol{\alpha}(t) \preceq \boldsymbol{\beta}(t)$, $\boldsymbol{\alpha}'(t) \preceq \boldsymbol{\beta}'(t)$, $0 \leq t \leq 1$. We define the modified function as

$$\bar{\mathbf{F}}(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'') = \mathbf{F}(t, \bar{\mathbf{x}}, \bar{\mathbf{x}}', \bar{\mathbf{x}}'') - \boldsymbol{\omega}(\mathbf{x}'), \tag{3.1}$$

where

$$\bar{\mathbf{x}}(t) = \delta(\mathbf{x}_{\alpha_i}(t), \mathbf{x}(t), \mathbf{x}_{\beta_i}(t)), \tag{3.2}$$
$$\bar{\mathbf{x}}'(t) = \delta(\mathbf{x}'_{\alpha_i}(t), \mathbf{x}'(t), \mathbf{x}'_{\beta_i}(t)), \tag{3.3}$$
$$\bar{\mathbf{x}}''(t) = \delta(-\mathbf{D}, \mathbf{x}''(t), \mathbf{D}), \tag{3.4}$$

$\mathbf{D} = (D_1, \dots, D_N)^T \in R^N$ is a positive constant vector, such that

$$D_i > \max_{t\in I}\{2M_i, |\alpha_i''(t)|, |\beta_i''(t)|\} \quad \text{and} \quad \int_{2M_i}^{D_i} \frac{s\,ds}{\Phi_i(s)} > 2M_i, \tag{3.5}$$

$$M_i > \max_{t\in I}\{|\alpha_i'(t)|, |\beta_i'(t)|\}, \quad i = 1, 2, \dots, N. \tag{3.6}$$

$\boldsymbol{\omega}(\mathbf{x}')$ is continuous and bounded, satisfying

$$\boldsymbol{\omega}(\mathbf{x}') \begin{cases} \prec \mathbf{0}, & \mathbf{x}' \prec \boldsymbol{\alpha}', \\ = \mathbf{0}, & \boldsymbol{\alpha}' \preceq \mathbf{x}' \preceq \boldsymbol{\beta}', \\ \succ \mathbf{0}, & \mathbf{x}' \succ \boldsymbol{\beta}', \end{cases} \tag{3.7}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_N)^T$, and such a function $\boldsymbol{\omega}(\cdot)$ can be easily obtained. For example, similar to [21], let $\boldsymbol{\omega}(\mathbf{x}') = \mathbf{x}' - \bar{\mathbf{x}}'$.

Furthermore, we define

$$\begin{aligned} &\bar{\mathbf{G}}(\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}(\xi_1), \dots, \mathbf{x}(\xi_{m-2})) \\ &\quad = \delta(\boldsymbol{\alpha}'(t), \mathbf{x}'(t) + \mathbf{A} - \mathbf{G}(\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}(\xi_1), \dots, \mathbf{x}(\xi_{m-2})), \boldsymbol{\beta}'(t)), \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\bar{\mathbf{H}}(\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}(\eta_1), \dots, \mathbf{x}(\eta_{n-2})) \\ &\quad = \delta(\boldsymbol{\alpha}'(t), \mathbf{x}'(t) + \mathbf{B} - \mathbf{H}(\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}(\eta_1), \dots, \mathbf{x}(\eta_{n-2})), \boldsymbol{\beta}'(t)). \end{aligned} \tag{3.9}$$

Then we consider the following modified problem:

$$\begin{cases} \mathbf{x}'''(t) + \bar{\mathbf{F}}(t, \mathbf{x}(t), \mathbf{x}'(t), \mathbf{x}''(t)) = \mathbf{0}, \\ \mathbf{x}(0) = \mathbf{0}, \\ \mathbf{x}'(0) = \bar{\mathbf{G}}(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \dots, \mathbf{x}(\xi_{m-2})), \\ \mathbf{x}'(1) = \bar{\mathbf{H}}(\mathbf{x}'(1), \mathbf{x}''(1), \mathbf{x}(\eta_1), \dots, \mathbf{x}(\eta_{n-2})). \end{cases} \tag{3.10}$$

Lemma 3.1 *Assume that*

(i) $(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ *is a lower solution-upper solution pair of BVP* (1.3), (1.4), *such that*

$$\boldsymbol{\alpha}_i'(t) \le \boldsymbol{\beta}_i'(t), \quad 0 \le t \le 1, i = 1, 2, \dots, N.$$

(ii) *For* $(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) \in [0,1] \times R^{3N}$, $\mathbf{F}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) \in C([0,1] \times R^{3N}, R^N)$ *is continuous and increasing with respect to* $\mathbf{x}$, *and* $\mathbf{F}(t, \mathbf{x}, \mathbf{y}, \mathbf{z})$ *satisfies Nagumo condition with respect to* $\mathbf{z}$.

Then BVP (3.10) *has a solution* $\mathbf{x}(t) = (x_1(t), \dots, x_N(t))^T \in C^3([0,1], R^N)$, *such that*

$$\alpha_i(t) \le x_i(t) \le \beta_i(t), \qquad \alpha_i'(t) \le x_i'(t) \le \beta_i'(t), \quad 0 \le t \le 1; \tag{3.11}$$

$$|x_i''(t)| \le D_i, \quad i = 1, 2, \dots, N, \tag{3.12}$$

where $\mathbf{D} = (D_1, \dots, D_N)^T \in R^N$ *is concerned by* (3.5), (3.6).

Proof First, we prove that (3.10) has a solution $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))^T \in C^3([0,1], R^N)$. We consider the following differential systems:

$$\begin{cases} \mathbf{x}'''(t) = -\lambda \bar{\mathbf{F}}(t, \mathbf{x}(t), \mathbf{x}'(t), \mathbf{x}''(t)) =: \boldsymbol{\Psi}(t), \\ \mathbf{x}(0) = \mathbf{0}, \\ \mathbf{x}'(0) = \lambda \bar{\mathbf{G}}(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \ldots, \mathbf{x}(\xi_{m-2})) =: \boldsymbol{\Psi}_*(0), \\ \mathbf{x}'(1) = \lambda \bar{\mathbf{H}}(\mathbf{x}'(1), \mathbf{x}''(1), \mathbf{x}(\eta_1), \ldots, \mathbf{x}(\eta_{n-2})) =: \boldsymbol{\Psi}_*(1), \end{cases} \tag{3.13}$$

where $\lambda \in [0,1]$. From the representations of $\bar{\mathbf{F}}$, $\bar{\mathbf{G}}$, $\bar{\mathbf{H}}$, we see that $\mathbf{x}'''(t)$, $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ in (3.13) are bounded. Thus $\mathbf{x}''(t)$, $\mathbf{x}'(t)$, $\mathbf{x}(t)$, $0 \le t \le 1$ are bounded. Consider the set

$$\begin{aligned} \Omega = \bigl\{ &\mathbf{x}(t) \in R^N : \bigl\| \mathbf{x}^{(s)}(t) \bigr\| < K, s = 0, 1, 2, K \text{ is some sufficiently} \\ &\text{large positive constant}, t \in [0,1] \bigr\}. \end{aligned}$$

Then Ω is a bounded open set. BVP (3.13) can be equal to the following integral equation:

$$\mathbf{x}(t) = \mathbf{c}_1 + \mathbf{c}_2 t + \mathbf{c}_3 t^2 + \int_0^t \int_0^{t_2} \int_1^{t_1} \boldsymbol{\Psi}(s)\, ds\, dt_1\, dt_2 =: T_\lambda \mathbf{x}, \tag{3.14}$$

where T_λ is an integral operator with a parameter λ, and $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3) \in R^N \times R^N \times R^N$ is determined as

$$\begin{cases} \mathbf{c}_1 = \mathbf{0}, \\ \mathbf{c}_2 = \boldsymbol{\Psi}_*(0), \\ \mathbf{c}_2 + 2\mathbf{c}_3 = \boldsymbol{\Psi}_*(1) - \int_0^1 \int_1^{t_1} \boldsymbol{\Psi}(s)\, ds\, dt_1. \end{cases}$$

Let $\mathbf{W}(\lambda, \mathbf{x}) = (I - T_\lambda)(\mathbf{x})$, thus $\mathbf{W} : [0,1] \times \bar{\Omega} \to R^N$ is continuous, where I is identical mapping. Let $\mathbf{w}_\lambda(\mathbf{x}) = \mathbf{W}(\lambda, \mathbf{x})$, $\forall \mathbf{x} \in \partial\Omega$, due to K is sufficiently large, we have

$$\bigl\| \mathbf{w}_\lambda(\mathbf{x}) \bigr\| = \| \mathbf{x} - T_\lambda \mathbf{x} \| \ge \| \mathbf{x} \| - \| T_\lambda \mathbf{x} \| = K - \| T_\lambda \mathbf{x} \| > 0, \quad \forall \lambda \in [0,1].$$

Thus, $\mathbf{0} \notin \mathbf{w}_\lambda(\partial\Omega)$. According to the homotopy invariance theorem of topological degree, $\deg(\mathbf{w}_\lambda, \Omega, \mathbf{0})$ keeps constant, in particular, $\deg(\mathbf{w}_1, \Omega, \mathbf{0}) = \deg(\mathbf{w}_0, \Omega, \mathbf{0})$. Noticing that $\mathbf{0} \in \Omega$, by the normality of topological degree, we have

$$\deg\bigl(\mathbf{w}_0(\mathbf{x}), \Omega, \mathbf{0}\bigr) = \deg(\mathbf{x} - T_0 \mathbf{x}, \Omega, \mathbf{0}) = \deg(\mathbf{x}, \Omega, \mathbf{0}) = 1$$

and

$$\deg\bigl(\mathbf{w}_1(\mathbf{x}), \Omega, \mathbf{0}\bigr) = \deg(\mathbf{x} - T_1 \mathbf{x}, \Omega, \mathbf{0}) = \deg(\mathbf{x} - T_0 \mathbf{x}, \Omega, \mathbf{0}) = 1.$$

Hence, by the solvability theorem of topological degree, $\mathbf{w}_1(\mathbf{x}) = \mathbf{0}$ has at least one solution. That is to say, $\mathbf{x}(t) = T_1 \mathbf{x}$ has solutions $\mathbf{x}(t)$, it is clear that there exists some $\mathbf{x}(t) \in C^3([0,1], R^N)$ satisfying (3.10).

Next, we prove that every solution $\mathbf{x}(t)$ of BVP (3.10) satisfies (3.11). First of all, we prove

$$\alpha_i'(t) \le x_i'(t) \le \beta_i'(t), \quad 0 \le t \le 1, i = 1, 2, \ldots, N, \tag{3.15}$$

if $\alpha_i'(t) \le x_i'(t)$, $i = 1, 2, \ldots, N$, is not true, then there exist some $i \in \{1, 2, \ldots, N\}$ and $\zeta \in [0, 1]$, such that

$$\max_{0 \le t \le 1} \bigl(\alpha_i'(t) - x_i'(t)\bigr) = \alpha_i'(\zeta) - x_i'(\zeta) > 0.$$

Obviously, from the boundary conditions of BVP (3.10), we know $\zeta \neq 0, 1$. Thus

$$\alpha_i''(\zeta) - x_i''(\zeta) = 0, \tag{3.16}$$

$$\alpha_i'''(\zeta) - x_i'''(\zeta) \le 0. \tag{3.17}$$

From conditions (i), (ii) and (2.1), (3.1)–(3.5), (3.7), (3.16), Definition 2 and the fact that $\mathbf{x}(t)$ is a solution of (3.10), we have

$$\begin{aligned} \alpha_i'''(\zeta) - x_i'''(\zeta) &\ge -f_i\bigl(\zeta, \mathbf{x}_{\alpha_i}(\zeta), \mathbf{x}_{\alpha_i}'(\zeta), \mathbf{x}_{\alpha_i}''(\zeta)\bigr) + \bar{f}_i\bigl(\zeta, \mathbf{x}(\zeta), \mathbf{x}'(\zeta), \mathbf{x}''(\zeta)\bigr) \\ &= -f_i\bigl(\zeta, \mathbf{x}_{\alpha_i}(\zeta), \mathbf{x}_{\alpha_i}'(\zeta), \mathbf{x}_{\alpha_i}''(\zeta)\bigr) + f_i\bigl(\zeta, \bar{\mathbf{x}}(\zeta), \bar{\mathbf{x}}'(\zeta), \bar{\mathbf{x}}''(\zeta)\bigr) - \omega_i\bigl(\mathbf{x}'(\zeta)\bigr) \\ &= -f_i\bigl(\zeta, \mathbf{x}_{\alpha_i}(\zeta), \mathbf{x}_{\alpha_i}'(\zeta), \mathbf{x}''(\zeta)\bigr) + f_i\bigl(\zeta, \bar{\mathbf{x}}(\zeta), \mathbf{x}_{\alpha_i}'(\zeta), \mathbf{x}''(\zeta)\bigr) - \omega_i\bigl(\mathbf{x}'(\zeta)\bigr) \\ &\ge 0 - \omega_i\bigl(\mathbf{x}'(\zeta)\bigr) > 0, \end{aligned}$$

it is contradictory to (3.17), hence we obtain $\alpha_i'(t) \le x_i'(t)$, $0 \le t \le 1$.

Similarly, we could prove that $x_i'(t) \le \beta_i'(t)$, $0 \le t \le 1$.

Thus, (3.15) is true. According to condition (i) and Definition 4, we have $\alpha_i(0) \le x_i(0) \le \beta_i(0)$, by integrating the inequalities (3.15) on $[0, t]$, we obtain

$$\alpha_i(t) \le x_i(t) \le \beta_i(t), \quad 0 \le t \le 1.$$

Finally, we prove (3.12) holds. We suppose that $|x_i''(t)| \le D_i$ is not true. Then there exists $\sigma \in [0, 1]$, such that $x_i''(\sigma) > D_i$, or $x_i''(\sigma) < -D_i$. Suppose that the first case holds. From (3.5), (3.6) and $\mathbf{F}(t)$ is continuous, there exists $\varsigma \in [0, 1]$ such that

$$x_i''(\varsigma) = \frac{x_i'(1) - x_i'(0)}{1 - 0} \le \beta_i'(1) - \alpha_i'(0) \le 2M_i < D_i.$$

Because $\mathbf{x}''(t)$ is continuous and $x_i''(\sigma) > D_i$, there exists some subinterval $[a, b]$ (or $[b, a]$) $\subset [0, 1]$ such that

$$\begin{aligned} &x_i''(a) = 2M_i, \qquad x_i''(b) = D_i, \\ &2M_i < x_i''(t) < D_i, \quad \forall t \in [a, b] \bigl(\text{or } [b, a]\bigr). \end{aligned}$$

From condition (ii) and Definition 3, one has

$$\left| \int_a^b \frac{x_i''(s) x_i'''(s)}{\Phi_i(x_i''(s))} \, ds \right| \le \left| \int_a^b x_i''(s) \, ds \right| = \bigl| x_i'(b) - x_i'(a) \bigr| \le 2M_i.$$

On the other hand, from (3.5) and (3.6), we know that

$$\left| \int_a^b \frac{x_i''(s) x_i'''(s)}{\Phi_i(x_i''(s))} \, ds \right| = \left| \int_{2M_i}^{D_i} \frac{s \, ds}{\Phi_i(s)} \right| = \int_{2M_i}^{D_i} \frac{s \, ds}{\Phi_i(s)} > 2M_i.$$

This inequality is contradictory to the above one. So we show that $x_i''(\sigma) > D_i$ is not true. Similarly, we can prove that $x_i''(\sigma) < -D_i$ is not true too. Therefore, (3.12) holds. □

3.2 Existence result of BVP (1.3), (1.4)

Theorem 3.1 *Assume that conditions* (i), (ii) *in Lemma* 3.1 *hold and*

(iii) $\mathbf{G}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ *is continuous and decreasing with respect to* $\mathbf{x}_2, \ldots, \mathbf{x}_m$; $\mathbf{H}(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ *is continuous and increasing in* $\mathbf{y}_2$ *and decreasing with respect to* $\mathbf{y}_3, \ldots, \mathbf{y}_n$.

Then BVP (1.3), (1.4) *has a solution* $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))^T \in C^3([0,1], R^N)$ *satisfying inequalities* (3.11) *and* (3.12).

Proof From (2.1), (3.1)–(3.4), (3.7) and Lemma 3.1, there exists a solution $\mathbf{x}(t)$ of the modified BVP (3.10) satisfying (1.3), (3.11) and (3.12).

Now we show the solution $\mathbf{x}(t)$ satisfying the boundary conditions (1.4). From the boundary conditions of (3.10), it is easy to get $\mathbf{x}(0) = \mathbf{0}$.

First, we prove

$$\mathbf{G}\big(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\big) = \mathbf{A}. \tag{3.18}$$

Case 1. Suppose that $\boldsymbol{\alpha}'(0) \preceq \mathbf{x}'(0) + \mathbf{A} - \mathbf{G}(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})) \preceq \boldsymbol{\beta}'(0)$. By (2.1), (3.8) and (3.10), we obtain

$$\begin{aligned}\mathbf{x}'(0) &= \bar{\mathbf{G}}\big(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\big) \\ &= \mathbf{x}'(0) + \mathbf{A} - \mathbf{G}\big(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\big).\end{aligned}$$

Thus (3.18) holds.

Case 2. Suppose that $\boldsymbol{\alpha}'(0) \succ \mathbf{x}'(0) + \mathbf{A} - \mathbf{G}(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2}))$. By (2.1), (3.8) and (3.10), we obtain

$$\mathbf{x}'(0) = \bar{\mathbf{G}}\big(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\big) = \boldsymbol{\alpha}'(0). \tag{3.19}$$

Then

$$\mathbf{G}\big(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\big) \succ \mathbf{A}. \tag{3.20}$$

According to (3.11), (3.19) and condition (iii), we know

$$\mathbf{G}\big(\boldsymbol{\alpha}'(0), \boldsymbol{\alpha}''(0), \boldsymbol{\alpha}(\xi_1), \ldots, \boldsymbol{\alpha}(\xi_{m-2})\big) \succeq \mathbf{G}\big(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\big).$$

Therefore,

$$\mathbf{G}\big(\boldsymbol{\alpha}'(0), \boldsymbol{\alpha}''(0), \boldsymbol{\alpha}(\xi_1), \ldots, \boldsymbol{\alpha}(\xi_{m-2})\big) \succ \mathbf{A}. \tag{3.21}$$

From condition (i), it is easy to see that (3.21) is contradictory to Definition 4. Therefore, (3.20) is not true.

Case 3. Suppose that $\mathbf{x}'(0) + \mathbf{A} - \mathbf{G}(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})) \succ \boldsymbol{\beta}'(0)$. By (2.1), (3.8) and (3.10), we obtain

$$\begin{aligned} \mathbf{x}'(0) &= \bar{\mathbf{G}}\bigl(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\bigr) \\ &= \boldsymbol{\beta}'(0). \end{aligned} \tag{3.22}$$

So

$$\mathbf{G}\bigl(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\bigr) \prec \mathbf{A}. \tag{3.23}$$

In view of (3.11), (3.22) and condition (iii), we know

$$\mathbf{G}\bigl(\boldsymbol{\beta}'(0), \boldsymbol{\beta}''(0), \boldsymbol{\beta}(\xi_1), \ldots, \boldsymbol{\beta}(\xi_{m-2})\bigr) \preceq \mathbf{G}\bigl(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})\bigr),$$

thus,

$$\mathbf{G}\bigl(\boldsymbol{\beta}'(0), \boldsymbol{\beta}''(0), \boldsymbol{\beta}(\xi_1), \ldots, \boldsymbol{\beta}(\xi_{m-2})\bigr) \prec \mathbf{A}. \tag{3.24}$$

By condition (i), it is easy to see that (3.24) is also contradictory to Definition 4. Therefore, (3.23) is not true too. Thus, we show that (3.18) holds.

Similar to the above argument, we could prove that

$$\mathbf{H}\bigl(\mathbf{x}'(1), \mathbf{x}''(1), \mathbf{x}(\eta_1), \mathbf{x}(\eta_2), \ldots, \mathbf{x}(\eta_{n-2})\bigr) = \mathbf{B}.$$

Thus $\mathbf{x}(t)$ is a solution of BVP (1.3), (1.4) and satisfies (3.11), (3.12). □

3.3 Existence result of SPBVP (1.1), (1.2)

Theorem 3.2 *Assume that*

(i) *The reduced problem of SPBVP* (1.1), (1.2)

$$\begin{cases} \mathbf{F}(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'', 0) = \mathbf{0}, \\ \mathbf{x}(0) = \mathbf{0}, \qquad \mathbf{G}(\mathbf{x}'(0), \mathbf{x}''(0), \mathbf{x}(\xi_1), \mathbf{x}(\xi_2), \ldots, \mathbf{x}(\xi_{m-2})) = \mathbf{A}, \end{cases} \tag{3.25}$$

has a reduced solution $\mathbf{v}(t) = (v_1(t), \ldots, v_N(t))^T \in C^3([0,1], R^N)$. *For* $i = 1, 2, \ldots, N$, $v_i(t)$ *satisfies*

$$\begin{aligned} &f_i\bigl(t, \mathbf{x}_{v_i}(t,\varepsilon), \mathbf{x}'_{v_i}(t,\varepsilon), \mathbf{x}''_{v_i}(t,\varepsilon), 0\bigr) = f_i\bigl(t, \mathbf{x}_{v_i}(t,0), \mathbf{x}'_{v_i}(t,0), \mathbf{x}''_{v_i}(t,0), 0\bigr) = 0, \\ &v_i(0) = 0, \qquad g_i\bigl(\mathbf{v}'(0), \mathbf{v}''(0), \mathbf{v}(\xi_1), \mathbf{v}(\xi_2), \ldots, \mathbf{v}(\xi_{m-2})\bigr) = A_i; \end{aligned}$$

(ii) *Let* ε_0 *be a sufficiently small constant,* $f_i(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'', \varepsilon)$, $i = 1, 2, \ldots, N$, *is continuously differentiable and satisfies Nagumo condition on* $[0,1] \times R^{3N} \times [0, \varepsilon_0]$ *and there exist some positive constants* l_i, r_i, c_i, $i = 1, 2, \ldots, N$, *such that*

$$\begin{aligned} &0 < f_{ix_i}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \varepsilon) \le l_i, \qquad f_{iy_i}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \varepsilon) \le -r_i < 0, \\ &f_{iz_i}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \varepsilon) \le 0, \qquad \bigl|f_{i\varepsilon}(t, \mathbf{x}, \mathbf{y}, \mathbf{z}, \varepsilon)\bigr| \le c_i, \end{aligned}$$

where $f_{ix_i} = \dfrac{\partial f_i(t, \mathbf{x}, \mathbf{y}, \mathbf{z})}{\partial x_i}$, *the others are defined analogously.*

(iii) $\mathbf{G}(\mathbf{x}_1,\dots,\mathbf{x}_m)$ *is continuous and increasing in* $\mathbf{x}_1$ *and decreasing with respect to* $\mathbf{x}_2,\dots,\mathbf{x}_m$; $\mathbf{H}(\mathbf{y}_1,\dots,\mathbf{y}_n)$ *is continuous and increasing with respect to* $\mathbf{y}_1$, $\mathbf{y}_2$ *and decreasing with respect to* $\mathbf{y}_3,\dots,\mathbf{y}_n$. *And there exist some vectors* $\mathbf{M}_s=(M_{s1},M_{s2},\dots,M_{sN})^T\succ\mathbf{0}$, $s=1,2,\dots,6$, *such that* $\mathbf{v}''(0)\prec-\mathbf{M}_1$, $\mathbf{v}''(1)\succ\mathbf{M}_2$, *and*

$$g_i\big(\mathbf{v}'(0),\mathbf{M}_1,\mathbf{M}_5,\dots,\mathbf{M}_5\big)\le A_i\le g_i\big(\mathbf{v}'(0),-\mathbf{M}_1,\mathbf{M}_3,\dots,\mathbf{M}_3\big),\tag{3.26}$$

$$h_i\big(\mathbf{v}'(1),-\mathbf{M}_2,\mathbf{M}_6,\dots,\mathbf{M}_6\big)\le B_i\le h_i\big(\mathbf{v}'(1),\mathbf{M}_2,\mathbf{M}_4,\dots,\mathbf{M}_4\big).\tag{3.27}$$

Then SPBVP (1.1), (1.2) *has a solution* $\mathbf{x}(t,\varepsilon)=(x_1(t,\varepsilon),\dots,x_N(t,\varepsilon))^T$ *such that*

$$\big|x_i(t,\varepsilon)-v_i(t)\big|\le T_{1i}e^{\lambda_{1i}t}+T_{2i}e^{\lambda_{2i}(t-1)}+T_{3i}\varepsilon,\quad i=1,2,\dots,N,\tag{3.28}$$

where $T_\kappa=\operatorname{diag}(T_{\kappa1},T_{\kappa2},\dots,T_{\kappa N})$, $T_{\kappa i}$ ($\kappa=1,2,3$, $i=1,2,\dots,N$) *are positive numbers.* ε *is sufficiently small,* λ_{1i}, λ_{2i} *are two roots of equation* $\varepsilon\lambda^3-r_i\lambda+l_i=0$, *such that*

$$-2\sqrt{\frac{r_i}{\varepsilon}}<\lambda_{1i}<-\sqrt{\frac{r_i}{\varepsilon}},\qquad \frac{1}{2}\sqrt{\frac{r_i}{\varepsilon}}<\lambda_{2i}<\sqrt{\frac{r_i}{\varepsilon}}.\tag{3.29}$$

Proof From condition (i), there exists a positive constant vector $\mathbf{M}^*=(M_1^*,M_2^*,\dots,M_N^*)^T$, such that $|v_i'''(t)|\le M_i^*$, $i=1,2,\dots,N$, since $\mathbf{v}(t)\in C^3([0,1],R^N)$. Then the equation $\varepsilon\lambda^3-r_i\lambda+l_i=0$ has three different real roots λ_{1i}, λ_{2i}, and λ_{3i}, since

$$\frac{1}{4}\left(\frac{l_i}{\varepsilon}\right)^2+\frac{1}{27}\left(-\frac{r_i}{\varepsilon}\right)^3=\frac{1}{\varepsilon^2}\left(\frac{l_i^2}{4}-\frac{r_i^3}{27\varepsilon}\right)<0.$$

Furthermore, for $i=1,2,\dots,N$, the estimates of λ_{1i}, λ_{2i} are given in (3.29) and have the estimate of λ_{3i} satisfies

$$\frac{l_i}{r_i}<\lambda_{3i}<\frac{l_i+r_i}{r_i}.\tag{3.30}$$

To construct the upper and lower solutions, we define

$$\gamma_i(t,\varepsilon)=\varepsilon^{\frac{1}{2}}\left[\frac{d_{1i}}{\lambda_{1i}}e^{\lambda_{1i}t}+\frac{d_{2i}}{\lambda_{2i}}e^{\lambda_{2i}(t-1)}\right]+\frac{d_{3i}}{\lambda_{3i}}\big[2e^{\lambda_{3i}t}-1\big],\tag{3.31}$$

where

$$d_{1i}=-\frac{M_{1i}+|v_i''(0)|+1}{\lambda_{1i}\varepsilon^{\frac{1}{4}}},\qquad d_{2i}=\frac{M_{2i}+|v_i''(1)|+1}{\lambda_{2i}\varepsilon^{\frac{1}{2}}},\qquad d_{3i}=\frac{\lambda_{3i}(c_i+M_i^*+1)}{l_i}\varepsilon^{\frac{1}{5}}.$$

Then we have

$$\gamma_i'(t,\varepsilon)=\varepsilon^{\frac{1}{2}}\big[d_{1i}e^{\lambda_{1i}t}+d_{2i}e^{\lambda_{2i}(t-1)}\big]+2d_{3i}e^{\lambda_{3i}t},$$

$$\gamma_i''(t,\varepsilon)=\varepsilon^{\frac{1}{2}}\big[d_{1i}\lambda_{1i}e^{\lambda_{1i}t}+d_{2i}\lambda_{2i}e^{\lambda_{2i}(t-1)}\big]+2d_{3i}\lambda_{3i}e^{\lambda_{3i}t},$$

$$\gamma_i'''(t,\varepsilon)=\varepsilon^{\frac{1}{2}}\big[d_{1i}\lambda_{1i}^2e^{\lambda_{1i}t}+d_{2i}\lambda_{2i}^2e^{\lambda_{2i}(t-1)}\big]+2d_{3i}\lambda_{3i}^2e^{\lambda_{3i}t}.$$

In view of $d_{1i} > 0$, $d_{2i} > 0$, $d_{3i} > 0$, we obtain

$$\gamma_i'(t,\varepsilon) > 0, \qquad \gamma_i'''(t,\varepsilon) > 0, \quad 0 \le t \le 1, \varepsilon > 0.$$

For sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
\gamma_i(0,\varepsilon) &= \varepsilon^{\frac{1}{2}}\left(\frac{d_{1i}}{\lambda_{1i}} + \frac{d_{2i}}{\lambda_{2i}}e^{-\lambda_{2i}}\right) + \frac{d_{3i}}{\lambda_{3i}} \\
&= -\frac{M_{1i} + |v_i''(0)| + 1}{\lambda_{1i}^2}\varepsilon^{\frac{1}{4}} + \frac{M_{2i} + |v_i''(1)| + 1}{\lambda_{2i}^2}e^{-\lambda_{2i}} + \frac{c_i + M_i^* + 1}{l_i}\varepsilon^{\frac{1}{5}} \\
&> -\frac{M_{1i} + |v_i''(0)| + 1}{r_i}\varepsilon^{\frac{5}{4}} + \frac{M_{2i} + |v_i''(1)| + 1}{r_i}\varepsilon e^{-\sqrt{\frac{r_i}{\varepsilon}}} + \frac{c_i + M_i^* + 1}{l_i}\varepsilon^{\frac{1}{5}} \\
&> 0
\end{aligned}$$

since $\gamma_i'(s,\varepsilon) > 0$, we have $\gamma_i(t,\varepsilon) = \gamma_i(0,\varepsilon) + \int_0^t \gamma_i'(s,\varepsilon)\,ds > 0$, for $0 \le t \le 1$.

Similarly, we obtain

$$\begin{aligned}
\gamma_i''(0,\varepsilon) &= \varepsilon^{\frac{1}{2}}\left(d_{1i}\lambda_{1i} + d_{2i}\lambda_{2i}e^{-\lambda_{2i}}\right) + 2d_{3i}\lambda_{3i} \\
&> -\left(M_{1i} + \left|v_i''(0)\right| + 1\right)\varepsilon^{\frac{1}{4}} + \left(M_{2i} + \left|v_i''(1)\right| + 1\right)\varepsilon^{-\sqrt{\frac{r_i}{\varepsilon}}} + \frac{2l_i(c_i + M_i^* + 1)}{r_i^2}\varepsilon^{\frac{1}{5}} \\
&> 0.
\end{aligned}$$

Thus, $\gamma_i''(t,\varepsilon) = \gamma_i''(0,\varepsilon) + \int_0^t \gamma_i'''(s,\varepsilon)\,ds > 0$, for $0 \le t \le 1$, since $\gamma_i'''(s,\varepsilon) > 0$.

Define functions $\boldsymbol{\beta}(t,\varepsilon)$, $\boldsymbol{\alpha}(t,\varepsilon)$ as

$$\boldsymbol{\beta}(t,\varepsilon) = \mathbf{v}(t) + \boldsymbol{\gamma}(t,\varepsilon), \qquad \boldsymbol{\alpha}(t,\varepsilon) = \mathbf{v}(t) - \boldsymbol{\gamma}(t,\varepsilon),$$

where

$$\boldsymbol{\gamma}(t,\varepsilon) = \left(\gamma_1(t,\varepsilon), \gamma_2(t,\varepsilon), \dots, \gamma_N(t,\varepsilon)\right)^T.$$

Hence

$$\beta_i(t,\varepsilon) = v_i(t) + \gamma_i(t,\varepsilon), \qquad \alpha_i(t,\varepsilon) = v_i(t) - \gamma_i(t,\varepsilon), \quad i = 1, 2, \dots, N.$$

For $(t,\varepsilon) \in [0,1] \times [0,\varepsilon_0]$, we have

$$\begin{aligned}
&\alpha_i(t,\varepsilon) \le \beta_i(t,\varepsilon), \qquad \alpha_i'(t,\varepsilon) \le \beta_i'(t,\varepsilon), \\
&\alpha_i''(t,\varepsilon) \le \beta_i''(t,\varepsilon), \qquad \alpha_i(0,\varepsilon) \le 0 \le \beta_i(0,\varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon\beta_i'''(t,\varepsilon) + f_i\left(t, \mathbf{x}_{\beta_i}(t,\varepsilon), \mathbf{x}_{\beta_i}'(t,\varepsilon), \mathbf{x}_{\beta_i}''(t,\varepsilon), \varepsilon\right) \\
&\quad = \varepsilon\beta_i'''(t,\varepsilon) + f_i\left(t, \mathbf{x}_{\beta_i}(t,\varepsilon), \mathbf{x}_{\beta_i}'(t,\varepsilon), \mathbf{x}_{\beta_i}''(t,\varepsilon), \varepsilon\right) - f_i\left(t, \mathbf{x}_{\beta_i}(t,\varepsilon), \mathbf{x}_{\beta_i}'(t,\varepsilon), \mathbf{x}_{v_i}''(t,\varepsilon), \varepsilon\right) \\
&\qquad + f_i\left(t, \mathbf{x}_{\beta_i}(t,\varepsilon), \mathbf{x}_{\beta_i}'(t,\varepsilon), \mathbf{x}_{v_i}''(t,\varepsilon), \varepsilon\right) - f_i\left(t, \mathbf{x}_{\beta_i}(t,\varepsilon), \mathbf{x}_{v_i}'(t,\varepsilon), \mathbf{x}_{v_i}''(t,\varepsilon), \varepsilon\right)
\end{aligned}$$

$$
\begin{aligned}
&+f_i\big(t,\mathbf{x}_{\beta_i}(t,\varepsilon),\mathbf{x}'_{\nu_i}(t,\varepsilon),\mathbf{x}''_{\nu_i}(t,\varepsilon),\varepsilon\big)-f_i\big(t,\mathbf{x}_{\nu_i}(t,\varepsilon),\mathbf{x}'_{\nu_i}(t,\varepsilon),\mathbf{x}''_{\nu_i}(t,\varepsilon),\varepsilon\big)\\
&+f_i\big(t,\mathbf{x}_{\nu_i}(t,\varepsilon),\mathbf{x}'_{\nu_i}(t,\varepsilon),\mathbf{x}''_{\nu_i}(t,\varepsilon),\varepsilon\big)-f_i\big(t,\mathbf{x}_{\nu_i}(t,\varepsilon),\mathbf{x}'_{\nu_i}(t,\varepsilon),\mathbf{x}''_{\nu_i}(t,\varepsilon),0\big)\\
&+f_i\big(t,\mathbf{x}_{\nu_i}(t,\varepsilon),\mathbf{x}'_{\nu_i}(t,\varepsilon),\mathbf{x}''_{\nu_i}(t,\varepsilon),0\big)\\
={}&\varepsilon\beta_i'''(t,\varepsilon)+\int_0^1 f_{iz_i}\big(t,\mathbf{x}_{\beta_i}(t,\varepsilon),\mathbf{x}'_{\beta_i}(t,\varepsilon),\mathbf{x}''_{\nu_i+\theta(\beta_i-\nu_i)}(t,\varepsilon),\varepsilon\big)\,d\theta\cdot\gamma_i''(t,\varepsilon)\\
&+\int_0^1 f_{iy_i}\big(t,\mathbf{x}_{\beta_i}(t,\varepsilon),\mathbf{x}'_{\nu_i+\theta(\beta_i-\nu_i)}(t,\varepsilon),\mathbf{x}''_{\nu_i}(t,\varepsilon),\varepsilon\big)\,d\theta\cdot\gamma_i'(t,\varepsilon)\\
&+\int_0^1 f_{ix_i}\big(t,\mathbf{x}_{\nu_i+\theta(\beta_i-\nu_i)}(t,\varepsilon),\mathbf{x}'_{\nu_i}(t,\varepsilon),\mathbf{x}''_{\nu_i}(t,\varepsilon),\varepsilon\big)\,d\theta\cdot\gamma_i(t,\varepsilon)\\
&+\int_0^1 f_{i\varepsilon}\big(t,\mathbf{x}_{\nu_i}(t,\varepsilon),\mathbf{x}'_{\nu_i}(t,\varepsilon),\mathbf{x}''_{\nu_i}(t,\varepsilon),\theta\varepsilon\big)\,d\theta\cdot\varepsilon\\
\le{}&\varepsilon\big(\nu_i'''(t)+\gamma_i'''(t,\varepsilon)\big)-r_i\gamma_i'(t,\varepsilon)+l_i\gamma_i(t,\varepsilon)+c_i\varepsilon\\
\le{}&\varepsilon\big(c_i+M_i^*\big)+\frac{\varepsilon^{\frac12}d_{1i}}{\lambda_{1i}}e^{\lambda_{1i}t}\big(\varepsilon\lambda_{1i}^3-r_i\lambda_{1i}+l_i\big)+\frac{\varepsilon^{\frac12}d_{2i}}{\lambda_{2i}}e^{\lambda_{2i}(t-1)}\big(\varepsilon\lambda_{2i}^3-r_i\lambda_{2i}+l_i\big)\\
&+\frac{2d_{3i}}{\lambda_{3i}}e^{\lambda_{3i}t}\big(\varepsilon\lambda_{3i}^3-r_i\lambda_{3i}+l_i\big)-\frac{l_id_{3i}}{\lambda_{3i}}\\
={}&\varepsilon\big(c_i+M_i^*\big)-\frac{l_id_{3i}}{\lambda_{3i}}\\
={}&-\varepsilon^{\frac15}\big[\big(1+c_i+M_i^*\big)-\big(c_i+M_i^*\big)\varepsilon^{\frac45}\big]<0,
\end{aligned}
$$

i.e.

$$\varepsilon\beta_i'''(t,\varepsilon)+f_i\big(t,\mathbf{x}_{\beta_i}(t,\varepsilon),\mathbf{x}'_{\beta_i}(t,\varepsilon),\mathbf{x}''_{\beta_i}(t,\varepsilon),\varepsilon\big)\le 0.$$

Similarly, from the expression of $\beta_i'(t,\varepsilon)$, we obtain $\beta_i'(0,\varepsilon)=\nu_i'(0)+\gamma_i'(0,\varepsilon)\ge\nu_i'(0)$, and $\beta_i'(1,\varepsilon)\ge\nu_i'(1)$. From condition (iii), there exists $\varepsilon_{i1}>0$, for $0<\varepsilon\le\varepsilon_{i1}$, one has $\beta_i''(0,\varepsilon)<-M_{1i}$, since $\gamma_i''(0,\varepsilon)>0$ is sufficient small. Furthermore, there exists $\varepsilon_{i2}>0$, for $0<\varepsilon\le\varepsilon_{i2}$, we have $\beta_i''(1,\varepsilon)\ge M_{2i}$. Then there exists $\widetilde{\varepsilon}_{ik}>0$, for $0<\varepsilon\le\widetilde{\varepsilon}_{ik}$ ($k=1,2,\dots,m-2$), we have

$$
\begin{aligned}
\beta_i(\xi_k,\varepsilon)&=\nu_i(\xi_k)+\varepsilon^{\frac12}\left[\frac{d_{1i}}{\lambda_{1i}}e^{\lambda_{1i}\xi_k}+\frac{d_{2i}}{\lambda_{2i}}e^{\lambda_{2i}(\xi_k-1)}\right]+\frac{d_{3i}}{\lambda_{3i}}\big[2e^{\lambda_{3i}\xi_k}-1\big]\\
&\le\nu_i(\xi_k)-\frac{\varepsilon^{\frac54}}{4r_i}\big(M_{1i}+\big|\nu_i''(0)\big|+1\big)e^{-2\sqrt{\frac{r_i}{\varepsilon}}\xi_k}+\frac{\varepsilon^{\frac12}}{4r_i}\big(M_{2i}+\big|\nu_i''(1)\big|+1\big)e^{\frac12\sqrt{\frac{r_i}{\varepsilon}}(\xi_k-1)}\\
&\quad+\frac{c_i+M_i^*+1}{l_i}\big(2e^{\frac{l_i+r_i}{r_i}\xi_i}-1\big)\varepsilon^{\frac15}\\
&\le\nu_i(\xi_k)+1\le\big|\nu_i(\xi_k)\big|+1:=\widetilde{m}_{ik},\quad k=1,2,\dots,m-2.
\end{aligned}
$$

Similarly there exists $\widehat{\varepsilon}_{ij}>0$, for $0<\varepsilon\le\widehat{\varepsilon}_{ij}$ ($j=1,2,\dots,n-2$), we have

$$\beta_i(\eta_j,\varepsilon)\le\big|\nu_i(\eta_j)\big|+1:=\widehat{m}_{ij},\quad j=1,2,\dots,n-2.$$

Let

$$M_{3i} = \max_{k=1,2,\ldots,m-2}\{\widetilde{m}_{ik}\}, \qquad M_{4i} = \max_{j=1,2,\ldots,n-2}\{\widehat{m}_{ij}\},$$

$$\varepsilon_0 = \min_{i=1,2,\ldots,N}\Big\{\varepsilon_{i1}, \varepsilon_{i2}, \min_{k=1,2,\ldots,m-2}\{\widetilde{\varepsilon}_{ik}\}, \min_{j=1,2,\ldots,n-2}\{\widehat{\varepsilon}_{ij}\}\Big\}.$$

For $0 < \varepsilon \le \varepsilon_0$, we have $\boldsymbol{\beta}'(0,\varepsilon) \succeq \mathbf{v}'(0)$, $\boldsymbol{\beta}'(1,\varepsilon) \succeq \mathbf{v}'(1)$, $\boldsymbol{\beta}''(0,\varepsilon) \prec -\mathbf{M}_1$, $\boldsymbol{\beta}''(1,\varepsilon) \succeq \mathbf{M}_2$, $\boldsymbol{\beta}(\xi_k,\varepsilon) \preceq \mathbf{M}_3$, $\boldsymbol{\beta}(\eta_j,\varepsilon) \preceq \mathbf{M}_4$, $k = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$. Here $\mathbf{M}_s = (M_{s1}, M_{s2}, \ldots, M_{sN})^T$, $s = 1,2,\ldots,6$. From condition (iii), we have

$$\begin{aligned} g_i\big(\boldsymbol{\beta}'(0,\varepsilon), \boldsymbol{\beta}''(0,\varepsilon), \boldsymbol{\beta}(\xi_1,\varepsilon), \ldots, \boldsymbol{\beta}(\xi_{m-2},\varepsilon)\big) &> g_i\big(\mathbf{v}'(0), -\mathbf{M}_1, \mathbf{M}_3, \ldots, \mathbf{M}_3\big) \\ &\ge A_i, \\ h_i\big(\boldsymbol{\beta}'(1,\varepsilon), \boldsymbol{\beta}''(1,\varepsilon), \boldsymbol{\beta}(\eta_1,\varepsilon), \ldots, \boldsymbol{\beta}(\eta_{m-2},\varepsilon)\big) &\ge h_i\big(\mathbf{v}'(1), \mathbf{M}_2, \mathbf{M}_4, \ldots, \mathbf{M}_4\big) \\ &\ge B_i. \end{aligned}$$

Thus $\boldsymbol{\beta}(t,\varepsilon) = (\beta_1(t,\varepsilon), \ldots, \beta_N(t,\varepsilon))^T$ is an upper solution of SPBVP (1.1), (1.2). Similarly, we could show $\boldsymbol{\alpha}(t,\varepsilon) = (\alpha_1(t,\varepsilon), \ldots, \alpha_N(t,\varepsilon))^T$ is a lower solution of SPBVP (1.1), (1.2). From Theorem 3.1, SPBVP (1.1), (1.2) has a solution $\mathbf{x}(t,\varepsilon) = (x_1(t,\varepsilon), \ldots, x_N(t,\varepsilon))^T$ satisfying

$$\boldsymbol{\alpha}(t,\varepsilon) \preceq \mathbf{x}(t,\varepsilon) \preceq \boldsymbol{\beta}(t,\varepsilon), \quad 0 \le t \le 1,$$

and the inequality (3.28) holds on $[0,1] \times [0,\varepsilon_0]$. □

4 Uniqueness result of SPBVP (1.1), (1.2)

Theorem 4.1 *Assume that all conditions of Theorem* 3.2 *hold, and for* $i = 1,2,\ldots,N$, *the following inequalities hold*:

$$\bar{p}_{1i} + \left(\sum_{k=1}^{m-2} \bar{\mu}_{ki}\right)\frac{r_i}{l_i}\left(2e^{\frac{l_i+r_i}{r_i}} - 1\right) > 0, \tag{4.1}$$

$$2\left(\bar{p}_{2i} + \frac{\bar{q}_{2i} l_i}{r_i}\right)e^{\frac{r_i}{l_i}} + \left(\sum_{j=1}^{n-2} \bar{\nu}_{ji}\right)\frac{r_i}{l_i}\left(2e^{\frac{l_i+r_i}{r_i}} - 1\right) > 0, \tag{4.2}$$

where

$$\bar{p}_{1i} = \int_0^1 g_{iz_{1i}}\big(\mathbf{x}_1'(0,\varepsilon) + \theta\mathbf{x}_0'(0,\varepsilon), \mathbf{x}_1''(0,\varepsilon), \tau\mathbf{x}_1(t,\varepsilon)\big)\,d\theta,$$

$$\bar{p}_{2i} = \int_0^1 h_{iz_{1i}}\big(\mathbf{x}_1'(1,\varepsilon) + \theta\mathbf{x}_0'(1,\varepsilon), \mathbf{x}_1''(1,\varepsilon), \rho\mathbf{x}_1(t,\varepsilon)\big)\,d\theta,$$

$$\bar{q}_{2i} = \int_0^1 h_{iz_{2i}}\big(\mathbf{x}_1'(1,\varepsilon), \mathbf{x}_1''(1,\varepsilon) + \theta\mathbf{x}_0''(1,\varepsilon), \rho\mathbf{x}_1(t,\varepsilon)\big)\,d\theta,$$

$$\bar{\mu}_{ki} = \int_0^1 g_{iz_{(k+2)i}}\big(\mathbf{x}_1'(0,\varepsilon), \mathbf{x}_1''(0,\varepsilon), \tau\mathbf{x}_1(t,\varepsilon) + \theta\mathbf{x}_0(\xi_k,\varepsilon)\big)\,d\theta, \quad k = 1,2,\ldots,m-2,$$

$$\bar{\nu}_{ji} = \int_0^1 h_{iz_{(j+2)i}}\big(\mathbf{x}_1'(1,\varepsilon), \mathbf{x}_1''(1,\varepsilon), \rho\mathbf{x}_1(t,\varepsilon) + \theta\mathbf{x}_0(\eta_j,\varepsilon)\big)\,d\theta, \quad j = 1,2,\ldots,n-2,$$

$$\tau \mathbf{x}_1(t,\varepsilon) := \big(\mathbf{x}_1(\xi_1,\varepsilon), \mathbf{x}_1(\xi_2,\varepsilon), \ldots, \mathbf{x}_1(\xi_{m-2},\varepsilon)\big),$$
$$\rho \mathbf{x}_1(t,\varepsilon) := \big(\mathbf{x}_1(\eta_1,\varepsilon), \mathbf{x}_1(\eta_2,\varepsilon), \ldots, \mathbf{x}_1(\eta_{n-2},\varepsilon)\big),$$
$$\tau \mathbf{x}_1(t,\varepsilon) + \theta \mathbf{x}_0(\xi_k,\varepsilon) := \big(\mathbf{x}_1(\xi_1,\varepsilon), \ldots, \mathbf{x}_1(\xi_k,\varepsilon) + \theta \mathbf{x}_0(\xi_k,\varepsilon), \ldots, \mathbf{x}_1(\xi_{m-2},\varepsilon)\big),$$
$$\rho \mathbf{x}_1(t,\varepsilon) + \theta \mathbf{x}_0(\eta_j,\varepsilon) := \big(\mathbf{x}_1(\eta_1,\varepsilon), \ldots, \mathbf{x}_1(\eta_j,\varepsilon) + \theta \mathbf{x}_0(\eta_j,\varepsilon), \ldots, \mathbf{x}_1(\eta_{n-2},\varepsilon)\big),$$
$$g_{iz_{ki}} = \frac{\partial g_i(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m)}{\partial z_{ki}}, \quad k = 1,2,\ldots,m,$$
$$h_{iz_{ji}} = \frac{\partial h_i(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n)}{\partial z_{ji}}, \quad j = 1,2,\ldots,n,$$

and l_i, r_i, $i = 1,2,\ldots,N$ are given in Theorem 3.2. Then SPBVP (1.1), (1.2) has a unique solution.

Proof From Theorem 3.2, for SPBVP (1.1), (1.2) there exist solutions. In order to show the uniqueness of the solutions, we only need to show (1.1), (1.2) has at most one solution. If the assertion is not true, then SPBVP (1.1), (1.2) has two different solutions $\mathbf{x}_1(t,\varepsilon)$, $\mathbf{x}_2(t,\varepsilon)$. Let

$$\mathbf{y}(t,\varepsilon) = \mathbf{x}_2(t,\varepsilon) - \mathbf{x}_1(t,\varepsilon),$$

then $\mathbf{y}(t,\varepsilon)$ is a solution of the boundary value problem

$$\varepsilon \mathbf{x}'''(t,\varepsilon) + \bar{\rho}_1(t,\varepsilon)\mathbf{x}''(t,\varepsilon) + \bar{\rho}_2(t,\varepsilon)\mathbf{x}'(t,\varepsilon) + \bar{\rho}_3(t,\varepsilon)\mathbf{x}(t,\varepsilon) = \mathbf{0}, \quad 0 \le t \le 1, \tag{4.3}$$

$$\begin{cases} \mathbf{x}(0,\varepsilon) = 0, \\ \bar{P}_1\mathbf{x}'(0,\varepsilon) + \bar{Q}_1\mathbf{x}''(0,\varepsilon) + \sum_{k=1}^{m-2} \bar{\mu}_k \mathbf{x}(\xi_k,\varepsilon) = \mathbf{0}, \\ \bar{P}_2\mathbf{x}'(1,\varepsilon) + \bar{Q}_2\mathbf{x}''(1,\varepsilon) + \sum_{j=1}^{n-2} \bar{\nu}_j \mathbf{x}(\eta_i,\varepsilon) = \mathbf{0}, \end{cases} \tag{4.4}$$

where $\bar{\rho}_s(t,\varepsilon) = \operatorname{diag}(\bar{\rho}_{s1}(t,\varepsilon), \ldots, \bar{\rho}_{sN}(t,\varepsilon))$, $s = 1,2,3$, $\bar{P}_1 = \operatorname{diag}(\bar{p}_{11}, \bar{p}_{12}, \ldots, \bar{p}_{1N})$, $\bar{P}_2 = \operatorname{diag}(\bar{p}_{21}, \bar{p}_{22}, \ldots, \bar{p}_{2N})$, $\bar{Q}_1 = \operatorname{diag}(\bar{q}_{11}, \bar{q}_{12}, \ldots, \bar{q}_{1N})$, $\bar{Q}_2 = \operatorname{diag}(\bar{q}_{21}, \bar{q}_{22}, \ldots, \bar{q}_{2N})$, $\bar{\mu}_k = \operatorname{diag}(\bar{\mu}_{k1}, \ldots, \bar{\mu}_{kN})$, $\bar{\nu}_j = \operatorname{diag}(\bar{\nu}_{j1}, \ldots, \bar{\nu}_{jN})$, $k = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$,

$$\bar{\rho}_{1i}(t,\varepsilon) = \int_0^1 f_{ix_i''}\big(t, \mathbf{x}_1(t,\varepsilon), \mathbf{x}_1'(t,\varepsilon), \mathbf{x}_1''(t,\varepsilon) + \theta \mathbf{y}''(t,\varepsilon), \varepsilon\big)\, d\theta,$$
$$\bar{\rho}_{2i}(t,\varepsilon) = \int_0^1 f_{ix_i'}\big(t, \mathbf{x}_1(t,\varepsilon), \mathbf{x}_1'(t,\varepsilon) + \theta \mathbf{y}'(t,\varepsilon), \mathbf{x}_1''(t,\varepsilon), \varepsilon\big)\, d\theta,$$
$$\bar{\rho}_{3i}(t,\varepsilon) = \int_0^1 f_{ix_i}\big(t, \mathbf{x}_1(t,\varepsilon) + \theta \mathbf{y}(t,\varepsilon), \mathbf{x}_1'(t,\varepsilon), \mathbf{x}_1''(t,\varepsilon), \varepsilon\big)\, d\theta,$$
$$\bar{q}_{1i} = \int_0^1 g_{iz_{2i}}\big(\mathbf{x}_1'(0,\varepsilon), \mathbf{x}_1''(0,\varepsilon) + \theta \mathbf{x}_0''(0,\varepsilon), \tau \mathbf{x}_1(t,\varepsilon)\big)\, d\theta.$$

From conditions (ii), (iii) in Theorem 3.2, we obtain $\bar{\rho}_{si} \in C([0,1] \times [0,\varepsilon_0], R)$, $s = 1,2,3$ and $\bar{\rho}_{1i}(t,\varepsilon) \le 0$, $\bar{\rho}_{2i}(t,\varepsilon) \le -r_i < 0$, $0 \le \bar{\rho}_{3i}(t,\varepsilon) \le l_i$, $(t,\varepsilon) \in [0,1] \times [0,\varepsilon_0]$, and $\bar{q}_{1i} \le 0$, $\bar{q}_{2i} \ge 0$, $\bar{\mu}_{ki} \le 0$, $\bar{\nu}_{ji} \le 0$, $i = 1,2,\ldots,N$, $k = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$. That is, $\bar{\rho}_s(t,\varepsilon)$, $s = 1,2,3$, $\bar{Q}_1$, $\bar{Q}_2$, $\bar{\mu}_k$, $\bar{\nu}_j$, satisfy Eq. (2.4) and boundary conditions (2.5).

Define

$$\phi_i(t,\varepsilon)=\frac{2e^{\lambda_{3i}t}-1}{\lambda_{3i}}-\frac{2\lambda_{3i}e^{\lambda_{1i}t}}{\lambda_{1i}^2}.$$

It is obvious that $\phi_i(t,\varepsilon)>0$, $\phi_i'(t,\varepsilon)>0$, $\phi_i''(t,\varepsilon)\geq 0$, and

$$\begin{aligned}
&\varepsilon\phi_i'''(t,\varepsilon)+\bar{\rho}_{1i}(t,\varepsilon)\phi_i''(t,\varepsilon)+\bar{\rho}_{2i}(t,\varepsilon)\phi_i'(t,\varepsilon)+\bar{\rho}_{3i}(t,\varepsilon)\phi_i(t,\varepsilon)\\
&\quad\leq \varepsilon\phi_i'''(t,\varepsilon)-r_i\phi_i'(t,\varepsilon)+l_i\phi_i(t,\varepsilon)\\
&\quad=\frac{2}{\lambda_{3i}}e^{\lambda_{3i}t}\bigl(\varepsilon\lambda_{3i}^3-r_i\lambda_{3i}+l_i\bigr)-\frac{2\lambda_{3i}}{\lambda_{1i}^2}e^{\lambda_{1i}t}\bigl(\varepsilon\lambda_{1i}^3-r_i\lambda_{1i}+l_i\bigr)-\frac{l_i}{\lambda_{3i}}\\
&\quad=-\frac{l_i}{\lambda_{3i}}<0.
\end{aligned}$$

For $0<\varepsilon\leq\varepsilon_0$, from (4.1), (4.2), we have

$$\begin{aligned}
&\bar{p}_{1i}\phi_i'(0,\varepsilon)+\bar{q}_{1i}\phi_i''(0,\varepsilon)+\sum_{k=1}^{m-2}\bar{\mu}_{ki}\phi_i(\xi_{ki},\varepsilon)\\
&\quad=2\bar{p}_{1i}\left(1-\frac{\lambda_{3i}}{\lambda_{1i}}\right)+\sum_{k=1}^{m-2}\bar{\mu}_{ki}\left(\frac{2e^{\lambda_{3i}\xi_{ki}}-1}{\lambda_{3i}}-\frac{2\lambda_{3i}e^{\lambda_{1i}\xi_{ki}}}{\lambda_{1i}^2}\right)\\
&\quad\geq\bar{p}_{1i}+\sum_{k=1}^{m-2}\bar{\mu}_{ki}\frac{r_i}{l_i}\bigl(2e^{\frac{l_i+r_i}{r_i}\xi_{ki}}-1\bigr)\\
&\quad\geq\bar{p}_{1i}+\left(\sum_{k=1}^{m-2}\bar{\mu}_{ki}\right)\frac{r_i}{l_i}\bigl(2e^{\frac{l_i+r_i}{r_i}}-1\bigr)>0,\\
&\bar{p}_{2i}\phi_i'(1,\varepsilon)+\bar{q}_{2i}\phi_i''(1,\varepsilon)+\sum_{j=1}^{n-2}\bar{\nu}_{ji}\phi_i(\eta_j,\varepsilon)\\
&\quad\geq 2\left(\bar{p}_{2i}+\frac{\bar{q}_{2i}l_i}{r_i}\right)e^{\frac{r_i}{l_i}}+\sum_{j=1}^{n-2}\bar{\nu}_{ji}\frac{r_i}{l_i}\bigl(2e^{\frac{l_i+r_i}{r_i}\eta_{ji}}-1\bigr)\\
&\quad\geq 2\left(\bar{p}_{2i}+\frac{\bar{q}_{2i}l_i}{r_i}\right)e^{\frac{r_i}{l_i}}+\left(\sum_{j=1}^{n-2}\bar{\nu}_{ji}\right)\frac{r_i}{l_i}\bigl(2e^{\frac{l_i+r_i}{r_i}}-1\bigr)>0.
\end{aligned}$$

Then $\boldsymbol{\Phi}(t,\varepsilon)=(\phi_1(t,\varepsilon),\ldots,\phi_N(t,\varepsilon))^T$ satisfies the conditions in Lemma 2.1. Hence SPBVP (4.3), (4.4) has only a zero solution, which contradicts $\mathbf{x}_1(t,\varepsilon)\neq\mathbf{x}_2(t,\varepsilon)$. Therefore, SPBVP (1.1), (1.2) has a unique solution. □

Remark 4.1 If we take $N=1$, we find that SPBVP (1.1), (1.2) becomes the singularly perturbed boundary value problem (3), (4) in [10]. It is notable that our results agree well with the corresponding ones in [10].

Remark 4.2 If we choose $N = 1$, $m = n$, and take the nonlinear boundary functions g, h to occur in the following linear functions:

$$g(x_1, x_2, \ldots, x_n) = ax_1 - bx_2 + \sum_{i=3}^{n} \alpha_i x_i,$$

$$h(y_1, y_2, \ldots, y_n) = cy_1 + dy_2 + \sum_{j=3}^{n} \beta_j y_j,$$

then SPBVP (1.1), (1.2) becomes the singularly perturbed boundary value problem (1.1), (1.2) in [9].

Remark 4.3 If we choose the nonlinear boundary functions $\mathbf{G}$, $\mathbf{H}$ to be the following linear functions:

$$\mathbf{G} = P_1 \mathbf{x}'(0, \varepsilon) - P_2 \mathbf{x}''(0, \varepsilon), \qquad \mathbf{H} = Q_1 \mathbf{x}'(1, \varepsilon) - Q_2 \mathbf{x}''(1, \varepsilon),$$

then SPBVP (1.1), (1.2) becomes the singularly perturbed boundary value problem (1), (2) in [20]. In this paper, we get the existence and uniqueness of solutions. We also discuss the asymptotic estimates of solutions.

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