Wei et al. *Boundary Value Problems* (2020) 2020:13
https://doi.org/10.1186/s13661-020-01325-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Liouville-type theorem for Kirchhoff equations involving Grushin operators

Yunfeng Wei[1,2]*, Caisheng Chen[2] and Hongwei Yang[3]

*Correspondence:
weiyunfeng@nau.edu.cn
[1]School of Statistics and Mathematics, Nanjing Audit University, Nanjing, P.R. China
[2]College of Science, Hohai University, Nanjing, P.R. China
Full list of author information is available at the end of the article

Abstract

The aim of this paper is to prove the Liouville-type theorem of the following weighted Kirchhoff equations:

$$-M\Big(\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\,dz\Big)\,\mathrm{div}_G(\omega(z)\nabla_G u)=f(z)e^u,\\ z=(x,y)\in R^N=R^{N_1}\times R^{N_2} \tag{0.1}$$

and

$$M\Big(\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\,dz\Big)\,\mathrm{div}_G(\omega(z)\nabla_G u)=f(z)u^{-q},\\ z=(x,y)\in \mathbb{R}^N=\mathbb{R}^{N_1}\times \mathbb{R}^{N_2}, \tag{0.2}$$

where $M(t)=a+bt^k$, $t\geq 0$, with $a>0$, $b,k\geq 0$, $k=0$ if and only if $b=0$. $q>0$ and $\omega(z),f(z)\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative functions satisfying $\omega(z)\leq C_1\|z\|_G^{\theta}$ and $f(z)\geq C_2\|z\|_G^{d}$ as $\|z\|_G\geq R_0$ with $d>\theta-2$, R_0, C_i $(i=1,2)$ are some positive constants, here $\alpha\geq 0$ and $\|z\|_G=(|x|^{2(1+\alpha)}+|y|^2)^{\frac{1}{2(1+\alpha)}}$ is the norm corresponding to the Grushin distance. $N_\alpha=N_1+(1+\alpha)N_2$ is the homogeneous dimension of $\mathbb{R}^N$. div_G (resp., ∇_G) is Grushin divergence (resp., Grushin gradient). Under suitable assumptions on k, θ, d, and N_α, the nonexistence of stable weak solutions to equations (0.1) and (0.2) is investigated.

MSC: 35J61; 35B53; 35A01

Keywords: Kirchhoff equations; Grushin operator; Stable weak solutions; Liouville-type theorem

1 Introduction and main result

In this paper, we study the nonexistence of stable weak solutions for the weighted Kirchhoff equations

$$-M\Big(\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\,dz\Big)\,\mathrm{div}_G\big(\omega(z)\nabla_G u\big)=f(z)e^u,\\ z=(x,y)\in \mathbb{R}^N=\mathbb{R}^{N_1}\times \mathbb{R}^{N_2} \tag{1.1}$$

and

$$M\left(\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\,dz\right)\operatorname{div}_G\big(\omega(z)\nabla_G u\big)=f(z)u^{-q},$$
$$z=(x,y)\in\mathbb{R}^N=\mathbb{R}^{N_1}\times\mathbb{R}^{N_2},\tag{1.2}$$

where $M(t)=a+bt^k$, $t\ge 0$, with $a>0$, $b,k\ge 0$, $k=0$ if and only if $b=0$. $q>0$ and $\omega(z),f(z)\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative functions verifying $\omega(z)\le C_1\|z\|_G^\theta$ and $f(z)\ge C_2\|z\|_G^d$ as $\|z\|_G\ge R_0$ with $d>\theta-2$, where R_0, C_i $(i=1,2)$ are some positive constants. Here $\alpha\ge 0$ and

$$\|z\|_G=\big(|x|^{2(1+\alpha)}+|y|^2\big)^{\frac{1}{2(1+\alpha)}},\quad z=(x,y)\in\mathbb{R}^N=\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}$$

is the norm corresponding to the Grushin distance, where $|x|$ and $|y|$ are the usual Euclidean norms in $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$, respectively.

Set ∇_x and ∇_y as Euclidean gradients with respect to the variables $x\in\mathbb{R}^{N_1}$ and $y\in\mathbb{R}^{N_2}$, respectively. The Grushin gradient is defined by

$$\nabla_G=\big(\nabla_x,(1+\alpha)|x|^\alpha\nabla_y\big).$$

Moreover, we define

$$\begin{aligned}\operatorname{div}_G(g,h)&=\sum_{i=1}^{N_1}\frac{\partial g_i}{\partial x_i}+(1+\alpha)|x|^\alpha\sum_{j=1}^{N_2}\frac{\partial h_j}{\partial y_j}\\&=\operatorname{div}_x g+(1+\alpha)|x|^\alpha\operatorname{div}_y h,\quad (g,h)\in C^1\big(\mathbb{R}^N,\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\big)\end{aligned}$$

as the Grushin divergence. Then the Grushin operator is given by

$$\triangle_G u=\operatorname{div}_G(\nabla_G u)=\triangle_x u+(1+\alpha)^2|x|^{2\alpha}\triangle_y u,$$

where Δ_x and Δ_y represent the usual Laplacians on $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$ respectively. This operator is uniformly elliptic for $x\neq 0$ and degenerate when $x=(x_1,x_2,\dots,x_{N_1})$ goes to 0.

The anisotropic dilation attached to $\triangle_G$ is defined by

$$\delta_\lambda(z)=\big(\lambda x,\lambda^{1+\alpha}y\big),\quad \lambda>0, z=(x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}.$$

It is not hard to see that

$$d\delta_\lambda(z)=\lambda^{N_\alpha}\,dx\,dy=\lambda^{N_\alpha}\,dz,$$

where

$$N_\alpha=N_1+(1+\alpha)N_2$$

is usually called the homogeneous dimension of $\mathbb{R}^N$, $dx\,dy$ denotes the Lebesgue measure in $\mathbb{R}^N$. For more details about Grushin operators and their basic properties, we refer the reader for instance to [34].

In the case $\alpha = 0$ and $\omega(z) \equiv 1$, problems (1.1) and (1.2) are related to the stationary analogue of the following Kirchhoff model:

$$\frac{\partial^2 u}{\partial t^2} - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dz\right)\Delta u = h(z,u),$$

which was proposed by Kirchhoff in 1883 as a generalization of the well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial z}\right|^2 dz\right)\frac{\partial^2 u}{\partial z^2} = h(z,u)$$

for free vibrations of elastic strings, see [26], where ρ, p_0, h, E, L are constants which represent some physical meanings respectively. Indeed, Kirchhoff's model considers the changes in length of the string produced by transverse vibrations. Up to now, a great attention has been paid to the study of the Kirchhoff-type problems involving nonlocal operators, because nonlinear equations with nonlocal operators have a broad application background and play an important role in physics, probability, biology, finance, etc. With the help of variational calculus, some important and interesting results for this direction, especially those concerning the existence and multiplicity of solutions, have been established, we refer the interested reader to [17–19, 24, 31, 37] and the references therein.

On the other hand, the nonexistence and stability of solutions to nonlinear elliptic equations have drawn much attention in the last decades. For some physical motivation and recent development on the topic of stable solutions, we refer to [13]. Also, see [2, 33] for related problems.

The motivation of writing this article is to prove a Liouville-type theorem for stable solutions of equations (1.1) and (1.2). We recall that Liouville-type theorem focuses on the nonexistence of nontrivial solution in the entire space $\mathbb{R}^N$. In 1981, in their pioneering article [22], Gidas and Spruck established the optimal Liouville-type result for positive solutions to the equation

$$-\Delta u = |u|^{q-1}u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

They proved that (1.3) has no positive solution if and only if $1 < q < q_s = \frac{N+2}{N-2}$ ($= \infty$) if $N = 2$. Farina [14, 16] also considered problem (1.3). He proved that there is no nontrivial stable solution if $1 < q < q_c(N)$, where $q_c(N)$ is explicitly given and is always greater than the classical critical exponent $\frac{N+2}{N-2}$. It is worth pointing out that his proof makes a delicate application of the classical Moser iterative method. Later, these results were extended to the quasilinear case $-\Delta_p u = |u|^{q-1}u$ in [7] and the weighted quasilinear case $-\Delta_p u = f(z)|u|^{q-1}u$ in [3].

Obviously, equation (1.1) becomes the following Laplace equation with exponential nonlinearity:

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^N \tag{1.4}$$

for the case $M(t) \equiv 1$, $\alpha = 0$, and $\omega(z) \equiv 1 \equiv f(z)$. Problem (1.4) has been studied by several experts; for example, Farina [15] proved that all stable C^2 solutions of (1.4) must be zero if $2 \le N \le 9$; Dancer and Farina [9] proved that equation (1.4) admits classical entire solutions which are stable outside a compact set of $\mathbb{R}^N$ if and only if $N \ge 10$. Later, Wang and Ye [38] proved the following theorem.

Theorem 1.1 *Let $\alpha = 0$, $M(t) \equiv 1 \equiv \omega(z)$, and $f(z) = |z|^d$ with $d > -2$. If $2 \le N < 10 + 4d$, then equation (1.1) admits no stable weak solution.*

For the case of negative exponent nonlinearity, the authors [32] obtained the following.

Theorem 1.2 *Let $q > 0$, $M(t) \equiv 1$, $\alpha = 0$, and $\omega(z) \equiv 1 \equiv f(z)$ in (1.2). If*

$$2 \le N < 2 + \frac{4}{1+q}\left(q + \sqrt{q(q+1)}\right),$$

then there are no positive stable solutions to (1.2) in $\mathbb{R}^N$.

Remark 1.3 It is clear that if $2 < N < 10$, then from the above inequality we have

$$q > p_0 := \frac{N^2 - 8N + 4 + 8\sqrt{N-1}}{(N-2)(10-N)}. \tag{1.5}$$

Recently, Du and Guo [10] studied equation (1.2) with $\alpha = 0$, $M(t) \equiv 1 \equiv \omega(z)$ and $f(z) = |z|^d$ with $d > -2$. It was proved that there is no positive stable solution provided $2 \le N < 10 + 4d$ and $q > p_c(d)$ hold, where $p_c(d)$ is a critical exponent depending on d and N.

Very recently, by Farina's approach, Cowan and Fazly [6] established the nonexistence of nontrivial stable solution of the weighted elliptic equation

$$-\operatorname{div}\big(\omega_1(z)\nabla u\big) = \omega_2(z)g(u) \quad \text{in } \mathbb{R}^N \tag{1.6}$$

with positive smooth weights $\omega_i(z)$, $i = 1, 2$, where the nonlinearity $g(u) = e^u$, $|u|^{p-1}u$ with $p > 1$ and $-u^{-p}$ with $p > 0$. After that, these results were extended to the quasilinear case $-\triangle_p u = f(z)g(u)$ in [4, 27] and the weighted quasilinear case $-\operatorname{div}(\omega(z)|\nabla u|^{p-2}\nabla u) = f(z)g(u)$ in [28, 29], where $g(u) = e^u$ or $g(u) = -u^{-q}$, $q > 0$. Similar works can be found in [5, 21, 23, 25, 41].

We now turn to the case where $\alpha > 0$, equations (1.1) and (1.2) become nonlinear elliptic equations involving Grushin operator. It is well known that the Grushin operator belongs to the wide class of subelliptic operators studied by Franchi et al. [20] (also see [1]). The Liouville-type theorem has been recently proved by Monticelli [35] for nonnegative classical solutions and by Yu [40] for nonnegative weak solutions of the problem

$$-\triangle_G u = u^\tau \quad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

The optimal exponent is $\tau < \frac{N_\alpha+2}{N_\alpha-2}$, where $N_\alpha = N_1 + (1+\alpha)N_2$ is the homogeneous dimension. The main tool they used [35, 40] is the Kelvin transform combined with the moving planes technique. On the other hand, Monti and Morbidelli [34] considered the classification results for equation

$$-\triangle_G u = u^{\frac{N_\alpha+2}{N_\alpha-2}} \quad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

the main tool they used is the moving spheres technique, which is a variant of the moving plane technique and was widely used in elliptic equations such as [30]. For other results of Liouville-type theorem related to Grushin operators, we refer the reader to [8, 11, 12, 36] and the references therein.

However, as far as we know, there are few results on the Liouville-type theorem for problem (1.1) or (1.2) with $\alpha \neq 0$ and $M(t), \omega(z), f(z) \not\equiv 1$. Motivated by the above works, in the present paper, we try to establish the Liouville property for the class of stable weak solutions of (1.1) and (1.2).

Since solutions to elliptic equations with Hardy potentials may possess singularities, it is natural to study weak solutions of (1.1) and (1.2) in a suitable weighted Sobolev space. Based on this reality, we define

$$\|\varphi\|_{\omega} = \left(\int_{\mathbb{R}^N} \omega(z)|\nabla_G\varphi|^2\,dz\right)^{1/2}$$

for $\varphi \in C_c^\infty(\mathbb{R}^N)$ and denote by $H_0^1(\mathbb{R}^N,\omega)$ the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the $\|\cdot\|_{\omega}$-norm. Note that, for $\omega(z) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we have $C_c^1(\mathbb{R}^N) \subset H_0^1(\mathbb{R}^N,\omega)$. Denote also by $H^1_{\mathrm{loc}}(\mathbb{R}^N,\omega)$ the space of all functions u such that $u\varphi \in H_0^1(\mathbb{R}^N,\omega)$ for all $\varphi \in C_c^1(\mathbb{R}^N)$. Here and in the following $C_c^k(\mathbb{R}^N)$ denotes the set of C^k functions with compact support in $\mathbb{R}^N$.

To facilitate the writing, we unify equations (1.1) and (1.2) into the following equation:

$$\begin{aligned} &{}^{-}M\left(\int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2\,dz\right)\operatorname{div}_G\bigl(\omega(z)\nabla_G u\bigr) = f(z)g(u), \\ &z=(x,y)\in\mathbb{R}^N=\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}, \end{aligned} \tag{1.7}$$

where $g(u) = e^u$ or $g(u) = -u^{-q}$.

Definition 1.4 Let $X = H_0^1(\mathbb{R}^N,\omega)\cap H^1_{\mathrm{loc}}(\mathbb{R}^N,\omega)$, we say that $u \in X$ is a weak solution of (1.7) if $f(z)g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\bigl(a + b\|u\|_{\omega}^{2k}\bigr)\int_{\mathbb{R}^N}\omega(z)\nabla_G u\cdot\nabla_G\varphi\,dz = \int_{\mathbb{R}^N} f(z)g(u)\varphi\,dz, \quad \forall\varphi\in C_c^1\bigl(\mathbb{R}^N\bigr), \tag{1.8}$$

where $g(u) = e^u$ or $g(u) = -u^{-q}$.

The energy functional $\mathcal{J}: X \to \mathbb{R}$ corresponding to (1.7) is

$$\mathcal{J}(u) = \frac{a}{2}\|u\|_{\omega}^2 + \frac{b}{2(k+1)}\|u\|_{\omega}^{2(k+1)} - \int_{\mathbb{R}^N} f(z)G(u)\,dz,$$

where $G(u) = \int_0^u g(s)\,ds$.

Obviously, if $u \in X$ is a weak solution of (1.7), then for any $\varphi \in C_c^1(\mathbb{R}^N)$, the function $E(t) := \mathcal{J}(u+t\varphi)$ satisfies $E'(0) = 0$. As in [7], we say that the solution u of (1.7) is stable if $E''(0) \geq 0$. Now, we compute $E''(0)$. Observe that

$$\begin{aligned} &\frac{E'(t)-E'(0)}{t} \\ &\quad = \frac{\mathcal{J}'(u+t\varphi)\varphi - \mathcal{J}'(u)\varphi}{t} \end{aligned}$$

$$= \frac{b(\|u+t\varphi\|_\omega^{2k} - \|u\|_\omega^{2k})}{t} \int_{\mathbb{R}^N} \omega(z)\nabla_G u \cdot \nabla_G \varphi \, dz + \left(a + b\|u+t\varphi\|_\omega^{2k}\right) \int_{\mathbb{R}^N} \omega(z)|\nabla_G \varphi|^2 \, dz - \frac{1}{t}\int_{\mathbb{R}^N} f(z)\bigl(g(u+t\varphi) - g(u)\bigr)\varphi \, dz,$$

we obtain

$$E''(0) = \lim_{t\to 0} \frac{E'(t) - E'(0)}{t} = 2bk\|u\|_\omega^{2(k-1)} \left(\int_{\mathbb{R}^N} \omega(z)\nabla_G u \cdot \nabla_G \varphi \, dz\right)^2 + \left(a + b\|u\|_\omega^{2k}\right) \int_{\mathbb{R}^N} \omega(z)|\nabla_G \varphi|^2 \, dz - \int_{\mathbb{R}^N} f(z) g'(u)\varphi^2 \, dz.$$

We are ready to state the stability as follows.

Definition 1.5 We say that a weak solution u of (1.7) is stable if $f(z)g'(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$2bk\|u\|_\omega^{2(k-1)} \left(\int_{\mathbb{R}^N} \omega(z)\nabla_G u \cdot \nabla_G \varphi \, dz\right)^2 + \left(a + b\|u\|_\omega^{2k}\right) \int_{\mathbb{R}^N} \omega(z)|\nabla_G \varphi|^2 \, dz \geq \int_{\mathbb{R}^N} f(z) g'(u)\varphi^2 \, dz \tag{1.9}$$

for every $\varphi \in C^1_c(\mathbb{R}^N)$, where $g(u) = e^u$ or $g(u) = -u^{-q}$.

Remark 1.6 If u is a stable weak solution of (1.1), in view of (1.9) with $g(u) = e^u$, it can be deduced that

$$\int_{\mathbb{R}^N} f(z) e^u \varphi^2 \, dz \leq A \int_{\mathbb{R}^N} \omega(z)|\nabla_G \varphi|^2 \, dz, \quad \forall \varphi \in C^1_c\left(\mathbb{R}^N\right) \tag{1.10}$$

with

$$A = a + b(1+2k)\|u\|_\omega^{2k}. \tag{1.11}$$

Similarly, if u is a positive stable weak solution of (1.2), by virtue of (1.9) with $g(u) = -u^{-q}$, it follows that

$$q\int_{\mathbb{R}^N} f(z) u^{-(1+q)} \varphi^2 \, dz \leq A \int_{\mathbb{R}^N} \omega(z)|\nabla_G \varphi|^2 \, dz, \quad \forall \varphi \in C^1_c\left(\mathbb{R}^N\right), \tag{1.12}$$

where A is given in (1.11). Note that (1.8)–(1.10) and (1.12) hold for all $\varphi \in H^1_0(\mathbb{R}^N, \omega)$ by density arguments.

Throughout this paper, the functions $\omega(z), f(z)$ satisfy the following assumptions:

(H_1) $\omega(z), f(z) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative functions. In addition, there exist $d > \theta - 2$, $C_i > 0$ $(i = 1, 2)$ and $R_0 > 0$ such that

$$\omega(z) \leq C_1\|z\|_G^\theta, \qquad f(z) \geq C_2\|z\|_G^d, \quad \forall \|z\|_G \geq R_0.$$

To simplify the notations, we denote

$$\mu_0(k,\theta,d) = 2 - \theta + \frac{4(2-\theta+d)}{1+2k}, \qquad \mu_1(k,\theta,d) = 2 - \theta + \frac{1+\sqrt{1+2k}}{2k}(2-\theta+d).$$

Our results can be stated as follows.

Theorem 1.7 *Let $M(t) = a + bt^k$, $t \geq 0$, $a > 0$, $b, k \geq 0$, $k = 0$ if and only if $b = 0$. Suppose that* (H_1) *and $N_\alpha < \mu_0(k,\theta,d)$ hold. Then there is no stable weak solution $u \in X$ to* (1.1).

Remark 1.8 If $\alpha = \theta = 0$ and $k = 1$, then the result in Theorem 1.7 coincides with that in [39]. If $\alpha = k = 0$, we get a result similar to that in [6, 23]. If $\alpha = k = \theta = 0$ we derive the result in [38]. Finally, if $\alpha = k = \theta = d = 0$, we have the Liouville theorem in the pioneering article [15].

Theorem 1.9 *Let $M(t) = a + bt^k$, $t \geq 0$, $a > 0$, $b, k \geq 0$, $k = 0$ if and only if $b = 0$. Assume that* (H_1) *holds. Then* (1.2) *has no positive stable weak solution $u \in X$ provided that one of the following conditions is satisfied*:

(H_2) $k \geq 0$ *and* $N_\alpha \leq 2 - \theta$, $q > 0$;
(H_3) $0 \leq k \leq \frac{3}{2}$ *and* $2 - \theta < N_\alpha < \mu_0(k,\theta,d)$, $q > q_c$;
(H_4) $k > \frac{3}{2}$ *and* $2 - \theta < N_\alpha < \mu_0(k,\theta,d)$, $q > \widetilde{q}_c$;
(H_5) $k > \frac{3}{2}$ *and* $N_\alpha = \mu_0(k,\theta,d)$, $q > \frac{4}{2k-3}$;
(H_6) $k > \frac{3}{2}$ *and* $\mu_0(k,\theta,d) < N_\alpha < \mu_1(k,\theta,d)$, $q_1 < q < q_2$,

where

$$q_c = -1 - \frac{2(2-\theta+d)[N_\alpha - 4 + 2\theta - d + \sqrt{(N_\alpha+d)^2 - (N_\alpha - 2 + \theta)^2(1+2k)}]}{(N_\alpha - 2 + \theta)(1+2k)(N_\alpha - \mu_0(k,\theta,d))}; \tag{1.13}$$

$$\widetilde{q}_c = -1 - \frac{2(2-\theta+d)[N_\alpha - 4 + 2\theta - d + \sqrt{(N_\alpha+d)^2 - (N_\alpha - 2 + \theta)^2(1+2k)}]}{(N_\alpha - 2 + \theta)(1+2k)(N_\alpha - \mu_0(k,\theta,d))}; \tag{1.14}$$

$$q_{1,2} = -1 - \frac{2(2-\theta+d)[N_\alpha - 4 + 2\theta - d \pm \sqrt{(N_\alpha+d)^2 - (N_\alpha - 2 + \theta)^2(1+2k)}]}{(N_\alpha - 2 + \theta)(1+2k)(N_\alpha - \mu_0(k,\theta,d))}. \tag{1.15}$$

Remark 1.10 If $\alpha = k = \theta = 0$, we obtain

$$q_c = \frac{2(2+d)^2 + 2(N-2)(2+d) - (N-2)^2 - 2(2+d)\sqrt{(2+d)(d+2N-2)}}{(N-2)(N-10-4d)},$$

which equals the critical exponent p_c in [10]. If $\alpha = k = \theta = d = 0$, we find

$$q_c = \frac{N^2 - 8N + 4 + 8\sqrt{N-1}}{(N-2)(10-N)},$$

which is the critical exponent p_0 in (1.5) and equals the exponent in [32]. Obviously, equations (1.1) and (1.2) are an extension of problems in [5, 6, 10, 15, 23, 24, 28, 29, 32, 38, 39], respectively. Therefore, our conclusions in Theorem 1.7 and Theorem 1.9 extend some results in the above references.

This paper is organized as follows. In Sect. 2, we give the proof of Theorem 1.7. The proof of Theorem 1.9 is finally finished in Sect. 3. In the sequel, we denote by C some constant, which may vary from line to line. If this constant depends on an arbitrary small number ε, then we denote it by C_ε.

2 Proof of Theorem 1.7

We first give the following proposition which plays a crucial role in the proof of Theorem 1.7.

Proposition 2.1 *Assume that u is a stable weak solution of* (1.1). *Then, for any $s \in (0, \frac{4}{1+2k})$, there exists a constant $C = C(k,s) > 0$ such that*

$$\int_{\mathbb{R}^N} f(z)e^{(1+s)u}\psi^{2(1+s)}\,dz \le CA^{1+s}\int_{\mathbb{R}^N}\omega(z)^{1+s}f(z)^{-s}|\nabla_G\psi|^{2(1+s)}\,dz \tag{2.1}$$

holds for all functions $\psi \in C_c^1(\mathbb{R}^N)$ satisfying $0 \le \psi \le 1$ and $\nabla_G\psi = 0$ in a neighborhood of $\{z \in \mathbb{R}^N \mid f(z) = 0\}$.

Proof We will use some of the ideas in [23, 28] to complete the proof. For each $i \in \mathbb{N}$, we define

$$\beta_i(t) = \begin{cases} e^{\frac{st}{2}}, & t < i,\\ [\frac{s}{2}(t-i)+1]e^{\frac{si}{2}}, & t \ge i, \end{cases}$$

and

$$\gamma_i(t) = \begin{cases} e^{st}, & t < i,\\ [s(t-i)+1]e^{si}, & t \ge i. \end{cases}$$

It is not difficult to verify that β_i, γ_i are increasing positive $C^1(\mathbb{R})$ functions and

$$\beta_i^2(t) \ge \gamma_i(t), \qquad \beta_i'(t)^2 = \frac{s}{4}\gamma_i'(t) \quad \text{and} \quad \beta_i^2(t) + \gamma_i^2(t)\gamma_i'(t)^{-1} \le Ce^{st} \tag{2.2}$$

for all $t \in \mathbb{R}$, where $C > 0$ depends only on s. Owing to $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N,\omega)$, we conclude $\beta_i(u), \gamma_i(u) \in H^1_{\mathrm{loc}}(\mathbb{R}^N,\omega)$ for any $i \in \mathbb{N}$.

Let $\varepsilon \in (0,1)$ and $\psi \in C_c^1(\mathbb{R}^N)$ be a nonnegative function.

Set $\varphi = \gamma_i(u)\psi^2$ as a test function in (1.8) with $g(u) = e^u$, we have

$$\begin{aligned} &B\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\gamma_i'(u)\psi^2\,dz + 2B\int_{\mathbb{R}^N}\omega(z)\gamma_i(u)\psi\nabla_G u\cdot\nabla_G\psi\,dz\\ &\quad= \int_{\mathbb{R}^N} f(z)e^u\gamma_i(u)\psi^2\,dz, \end{aligned}$$

where $B = a + b\|u\|_\omega^{2k}$. By Young's inequality, it yields

$$\begin{aligned} &B\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\gamma_i'(u)\psi^2\,dz\\ &\quad\le 2B\int_{\mathbb{R}^N}\omega(z)\gamma_i(u)\psi|\nabla_G u||\nabla_G\psi|\,dz + \int_{\mathbb{R}^N} f(z)e^u\gamma_i(u)\psi^2\,dz\\ &\quad\le \varepsilon B\int_{\mathbb{R}^N}\left(\omega(z)^{1/2}|\nabla_G u|\gamma_i'(u)^{1/2}\psi\right)^2 dz\\ &\qquad + C_\varepsilon B\int_{\mathbb{R}^N}\left(\omega(z)^{1/2}\gamma_i(u)\gamma_i'(u)^{-1/2}|\nabla_G\psi|\right)^2 dz + \int_{\mathbb{R}^N} f(z)e^u\gamma_i(u)\psi^2\,dz \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon B \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 \gamma_i'(u)\psi^2\, dz \\
&\quad + C_\varepsilon B \int_{\mathbb{R}^N} \omega(z)\gamma_i^2(u)\gamma_i'(u)^{-1}|\nabla_G\psi|^2\, dz + \int_{\mathbb{R}^N} f(z)e^u\gamma_i(u)\psi^2\, dz,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1-\varepsilon)B \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 \gamma_i'(u)\psi^2\, dz \\
&\quad \le C_\varepsilon B \int_{\mathbb{R}^N} \omega(z)\gamma_i^2(u)\gamma_i'(u)^{-1}|\nabla_G\psi|^2\, dz + \int_{\mathbb{R}^N} f(z)e^u\gamma_i(u)\psi^2\, dz.
\end{aligned} \tag{2.3}
$$

On the other hand, according to the stability assumption, we take $\varphi = \beta_i(u)\psi$ in (1.10) and get

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(z)e^u\beta_i^2(u)\psi^2\, dz \\
&\quad \le A \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2\beta_i'(u)^2\psi^2\, dz + 2A \int_{\mathbb{R}^N} \omega(z)\beta_i(u)\beta_i'(u)\psi|\nabla_G u||\nabla_G\psi|\, dz \\
&\qquad + A \int_{\mathbb{R}^N} \omega(z)\beta_i^2(u)|\nabla_G\psi|^2\, dz.
\end{aligned} \tag{2.4}
$$

We use Young's inequality to estimate the middle term of the right-hand side of the above inequality:

$$
\begin{aligned}
&2A \int_{\mathbb{R}^N} \omega(z)\beta_i(u)\beta_i'(u)\psi|\nabla_G u||\nabla_G\psi|\, dz \\
&\quad \le \varepsilon A \int_{\mathbb{R}^N} \bigl(\omega(z)^{1/2}|\nabla_G u|\beta_i'(u)\psi\bigr)^2\, dz + C_\varepsilon A \int_{\mathbb{R}^N} \bigl(\omega(z)^{1/2}\beta_i(u)|\nabla_G\psi|\bigr)^2\, dz \\
&\quad = \varepsilon A \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2\beta_i'(u)^2\psi^2\, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z)\beta_i^2(u)|\nabla_G\psi|^2\, dz.
\end{aligned}
$$

Substituting this estimate into (2.4), there holds

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(z)e^u\beta_i^2(u)\psi^2\, dz \\
&\quad \le (1+\varepsilon)A \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2\beta_i'(u)^2\psi^2\, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z)\beta_i^2(u)|\nabla_G\psi|^2\, dz.
\end{aligned} \tag{2.5}
$$

Together with (2.2), (2.3), (2.5), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(z)e^u\beta_i^2(u)\psi^2\, dz \\
&\quad \le \frac{(1+\varepsilon)sA}{4} \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2\gamma_i'(u)\psi^2\, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z)\beta_i^2(u)|\nabla_G\psi|^2\, dz \\
&\quad \le \frac{(1+\varepsilon)sA}{4(1-\varepsilon)B} \int_{\mathbb{R}^N} f(z)e^u\gamma_i(u)\psi^2\, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z)\bigl[\beta_i^2(u) + \gamma_i^2(u)\gamma_i'(u)^{-1}\bigr]|\nabla_G\psi|^2\, dz \\
&\quad \le \frac{(1+\varepsilon)sA}{4(1-\varepsilon)B} \int_{\mathbb{R}^N} f(z)e^u\beta_i^2(u)\psi^2\, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z)e^{su}|\nabla_G\psi|^2\, dz.
\end{aligned}
$$

This combined with the expressions of A and B gives

$$\begin{aligned}&\int_{\mathbb{R}^N} f(z)e^u\beta_i^2(u)\psi^2\,dz\\&\quad\le \frac{(1+\varepsilon)(1+2k)s}{4(1-\varepsilon)}\int_{\mathbb{R}^N} f(z)e^u\beta_i^2(u)\psi^2\,dz + C_\varepsilon A\int_{\mathbb{R}^N}\omega(z)e^{su}|\nabla_G\psi|^2\,dz,\end{aligned}$$

that is,

$$\lambda_\varepsilon\int_{\mathbb{R}^N} f(z)e^u\beta_i^2(u)\psi^2\,dz \le C_\varepsilon A\int_{\mathbb{R}^N}\omega(z)e^{su}|\nabla_G\psi|^2\,dz, \tag{2.6}$$

where $\lambda_\varepsilon = 1-\frac{(1+\varepsilon)(1+2k)s}{4(1-\varepsilon)}$. Since $\lim_{\varepsilon\to0^+}\lambda_\varepsilon = 1-\frac{(1+2k)s}{4}$. Thanks to $s\in(0,\frac{4}{1+2k})$, we let $\varepsilon>0$ be so small that $\lambda_\varepsilon>0$.

Letting $i\to\infty$ in (2.6), by Fatou's lemma, we have

$$\int_{\mathbb{R}^N} f(z)e^{(1+s)u}\psi^2\,dz \le CA\int_{\mathbb{R}^N}\omega(z)e^{su}|\nabla_G\psi|^2\,dz \tag{2.7}$$

for some constant $C>0$ depending only on k and s.

On the other hand, replacing ψ by ψ^{1+s} in (2.7) and using Hölder's inequality, we derive

$$\begin{aligned}&\int_{\mathbb{R}^N} f(z)e^{(1+s)u}\psi^{2(1+s)}\,dz\\&\quad\le CA\int_{\mathbb{R}^N}\omega(z)e^{su}\psi^{2s}|\nabla_G\psi|^2\,dz\\&\quad\le CA\left(\int_{\mathbb{R}^N}\left(f(z)^{\frac{s}{1+s}}e^{su}\psi^{2s}\right)^{\frac{1+s}{s}}dz\right)^{\frac{s}{1+s}}\left(\int_{\mathbb{R}^N}\left(\omega(z)f(z)^{-\frac{s}{1+s}}|\nabla_G\psi|^2\right)^{1+s}dz\right)^{\frac{1}{1+s}}\\&\quad= CA\left(\int_{\mathbb{R}^N} f(z)e^{(1+s)u}\psi^{2(1+s)}\,dz\right)^{\frac{s}{1+s}}\left(\int_{\mathbb{R}^N}\omega(z)^{1+s}f(z)^{-s}|\nabla_G\psi|^{2(1+s)}\,dz\right)^{\frac{1}{1+s}}.\end{aligned}$$

Hence, (2.1) is obtained immediately. □

Set $R>0$, $\Omega_{2R}=B_1(0,2R)\times B_2(0,2R^{1+\alpha})$, where $B_i\subset\mathbb{R}^{N_i}$, $i=1,2$, are open balls centered at 0, the radii are $2R$ and $2R^{1+\alpha}$, respectively. Let $\chi(t)\in C_c^\infty([0,\infty);[0,1])$ be cut-off functions such that

$$\chi(t)=\begin{cases}1, & 0\le t\le 1,\\ 0, & t\ge 2.\end{cases}$$

Define

$$\psi_{1,R}(x)=\chi\left(\frac{|x|}{R}\right),\quad x\in\mathbb{R}^{N_1},\qquad \psi_{2,R}(y)=\chi\left(\frac{|y|}{R^{1+\alpha}}\right),\quad y\in\mathbb{R}^{N_2}$$

and

$$\psi_R(x,y)=\psi_{1,R}(x)\psi_{2,R}(y),\quad (x,y)\in\mathbb{R}^N=\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}. \tag{2.8}$$

By a series of calculations, one can verify that there exists a constant $C > 0$ independent of R such that

$$
\begin{aligned}
&|\nabla_x \psi_{1,R}| \le CR^{-1}, \qquad |\nabla_y \psi_{2,R}| \le CR^{-(1+\alpha)},\\
&|\triangle_x \psi_{1,R}| \le CR^{-2}, \qquad |\triangle_y \psi_{2,R}| \le CR^{-2(1+\alpha)},\\
&|\nabla_G \psi_R|^2 + |\triangle_G \psi_R| \le CR^{-2},\\
&R \le \|z\|_G \le CR, \quad \forall z = (x,y) \in \Omega_{2R} \backslash \Omega_R.
\end{aligned}
\tag{2.9}
$$

Proof of Theorem 1.7 Arguing by contradiction, we assume that u is a stable weak solution of (1.1). Let $\psi = \psi_R(x,y) = \psi_{1,R}(x)\psi_{2,R}(y)$ with $R \ge R_0$ in (2.1), then there exists a positive constant C independent of R such that

$$
\begin{aligned}
\int_{\Omega_R} f(z)e^{(1+s)u}\,dz &\le CA^{1+s}R^{-2(1+s)} \int_{\Omega_{2R}\backslash\Omega_R} \|z\|_G^{\theta(1+s)-ds}\,dz\\
&\le CA^{1+s}R^{N_\alpha+\theta(1+s)-ds-2(1+s)},
\end{aligned}
\tag{2.10}
$$

where assumption (H_1) and (2.9) have been used.

Since $N_\alpha < \mu_0(k,\theta,d)$ and

$$
\lim_{s\to \frac{4}{1+2k}^-} \bigl[N_\alpha + \theta(1+s) - ds - 2(1+s)\bigr] = N_\alpha - \mu_0(k,\theta,d) < 0,
$$

we may choose some $s \in (0, \frac{4}{1+2k})$ suitably near $\frac{4}{1+2k}$ such that $N_\alpha + \theta(1+s) - ds - 2(1+s) < 0$. Letting $R \to +\infty$ in (2.10), we have

$$
\int_{\mathbb{R}^N} f(z)e^{(1+s)u}\,dz = 0.
$$

A contradiction! The proof is completed. □

3 Proof of Theorem 1.9

In this section, we give the proof of Theorem 1.9, which mainly relies on the following a priori estimate.

Proposition 3.1 *Suppose that u is a positive stable weak solution of* (1.2) *with $q > 0$. Then, for every $s \in (h(q), -1)$, where*

$$
h(t) = -\frac{1+2k+2t+2\sqrt{t(t+1+2k)}}{1+2k}, \quad t > 0,
\tag{3.1}
$$

and for any constant $\tau \ge \frac{q-s}{q+1}$, there exists a constant $C > 0$ depending on q, s, τ, a, and k such that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \bigl(f(z)u^{s-q} + \omega(z)|\nabla_G u|^2 u^{s-1}\bigr)\psi^{2\tau}\,dz\\
&\quad \le CA^{\frac{q-s}{q+1}} \int_{\mathbb{R}^N} \omega(z)^{\frac{q-s}{q+1}} f(z)^{\frac{s+1}{q+1}} |\nabla_G \psi|^{\frac{2(q-s)}{q+1}}\,dz
\end{aligned}
\tag{3.2}
$$

holds for all functions $\psi \in C_c^1(\mathbb{R}^N)$ *verifying* $0 \le \psi \le 1$ *and* $\nabla_G \psi = 0$ *in a neighborhood of* $\{z \in \mathbb{R}^N \mid f(z) = 0\}$.

Proof Some ideas in this proof are inspired by [29]. For each $i \in \mathbb{N}\backslash\{0\}$, we define

$$\delta_i(t) = \begin{cases} \frac{1+s}{2i^{\frac{s-1}{2}}}(t + \frac{1-s}{i(1+s)}), & 0 \le t < \frac{1}{i}, \\ t^{\frac{1+s}{2}}, & t \ge \frac{1}{i}, \end{cases}$$

and

$$\eta_i(t) = \begin{cases} \frac{s}{i^{s-1}}(t + \frac{1-s}{is}), & 0 \le t < \frac{1}{i}, \\ t^s, & t \ge \frac{1}{i}. \end{cases}$$

A direct calculation yields that δ_i, η_i are decreasing positive $C^1([0,+\infty))$ functions and

$$\delta_i^2(t) \ge t\eta_i(t), \qquad \delta_i'(t)^2 = \frac{(1+s)^2}{4|s|}\left|\eta_i'(t)\right|, \qquad \delta_i^2(t) + \eta_i^2(t)\left|\eta_i'(t)\right|^{-1} \le Ct^{1+s} \tag{3.3}$$

for all $t \ge 0$, where $C > 0$ is a constant depending only on s. Owning to $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N, \omega)$, we have $\delta_i(u), \eta_i(u) \in H^1_{\mathrm{loc}}(\mathbb{R}^N, \omega)$ for any $i \in \mathbb{N}\backslash\{0\}$.

Let $\varepsilon \in (0,1)$ and $\psi \in C_c^1(\mathbb{R}^N)$ be a nonnegative function.

Choosing $\varphi = \eta_i(u)\psi^2$ as a test function in (1.8) with $g(u) = -u^{-q}$, $q > 0$, we obtain

$$\begin{aligned} &B\int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 \eta_i'(u)\psi^2\,dz + 2B\int_{\mathbb{R}^N} \omega(z)\eta_i(u)\psi\nabla_G u\cdot\nabla_G\psi\,dz \\ &\quad = -\int_{\mathbb{R}^N} f(z)u^{-q}\eta_i(u)\psi^2\,dz. \end{aligned}$$

By use of Young's inequality, we deduce that

$$\begin{aligned} &B\int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2\left|\eta_i'(u)\right|\psi^2\,dz \\ &\quad \le 2B\int_{\mathbb{R}^N} \omega(z)\eta_i(u)\psi|\nabla_G u||\nabla_G\psi|\,dz + \int_{\mathbb{R}^N} f(z)u^{-q}\eta_i(u)\psi^2\,dz \\ &\quad \le \varepsilon B\int_{\mathbb{R}^N}\left(\omega(z)^{1/2}|\nabla_G u|\left|\eta_i'(u)\right|^{1/2}\psi\right)^2 dz \\ &\qquad + C_\varepsilon B\int_{\mathbb{R}^N}\left(\omega(z)^{1/2}\eta_i(u)\left|\eta_i'(u)\right|^{-1/2}|\nabla_G\psi|\right)^2 dz + \int_{\mathbb{R}^N} f(z)u^{-q}\eta_i(u)\psi^2\,dz \\ &\quad = \varepsilon B\int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2\left|\eta_i'(u)\right|\psi^2\,dz + C_\varepsilon B\int_{\mathbb{R}^N} \omega(z)\eta_i^2(u)\left|\eta_i'(u)\right|^{-1}|\nabla_G\psi|^2\,dz \\ &\qquad + \int_{\mathbb{R}^N} f(z)u^{-q}\eta_i(u)\psi^p\,dz, \end{aligned}$$

that is,

$$\begin{aligned} &(1-\varepsilon)B\int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2\left|\eta_i'(u)\right|\psi^2\,dz \\ &\quad \le C_\varepsilon B\int_{\mathbb{R}^N} \omega(z)\eta_i^2(u)\left|\eta_i'(u)\right|^{-1}|\nabla_G\psi|^2\,dz + \int_{\mathbb{R}^N} f(z)u^{-q}\eta_i(u)\psi^2\,dz. \end{aligned} \tag{3.4}$$

On the other hand, by virtue of the stability assumption, we take $\varphi = \delta_i(u)\psi$ in (1.12) and get

$$\begin{aligned} q\int_{\mathbb{R}^N} f(z)u^{-(1+q)}\delta_i^2(u)\psi^2\,dz &\le A\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\delta_i'(u)^2\psi^2\,dz + 2A\int_{\mathbb{R}^N}\omega(z)\delta_i(u)\psi|\nabla_G u|\left|\delta_i'(u)\right||\nabla_G\psi|\,dz \\ &\quad + A\int_{\mathbb{R}^N}\omega(z)\delta_i^2(u)|\nabla_G\psi|^2\,dz. \end{aligned} \tag{3.5}$$

Moreover, by Young's inequality, we conclude

$$\begin{aligned} &2\int_{\mathbb{R}^N}\omega(z)\delta_i(u)\psi|\nabla_G u|\left|\delta_i'(u)\right||\nabla_G\psi|\,dz \\ &\quad\le \varepsilon\int_{\mathbb{R}^N}\left(\omega(z)^{1/2}|\nabla_G u|\left|\delta_i'(u)\right|\psi\right)^2 dz + C_\varepsilon\int_{\mathbb{R}^N}\left(\omega(z)^{1/2}\delta_i(u)|\nabla_G\psi|\right)^2 dz \\ &\quad= \varepsilon\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\delta_i'(u)^2\psi^2\,dz + C_\varepsilon\int_{\mathbb{R}^N}\omega(z)\delta_i^2(u)|\nabla_G\psi|^2\,dz. \end{aligned}$$

Substituting this inequality into (3.5), it holds that

$$\begin{aligned} q\int_{\mathbb{R}^N} f(z)u^{-(1+q)}\delta_i^2(u)\psi^2\,dz &\le (1+\varepsilon)A\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\delta_i'(u)^2\psi^2\,dz \\ &\quad + C_\varepsilon A\int_{\mathbb{R}^N}\omega(z)\delta_i^2(u)|\nabla_G\psi|^2\,dz. \end{aligned} \tag{3.6}$$

Combining (3.6) with (3.3) and (3.4), we can find

$$\begin{aligned} q\int_{\mathbb{R}^N} f(z)u^{-(1+q)}\delta_i^2(u)\psi^2\,dz &\le \frac{(1+\varepsilon)(1+s)^2}{4|s|}A\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\left|\eta_i'(u)\right|\psi^2\,dz \\ &\quad + C_\varepsilon A\int_{\mathbb{R}^N}\omega(z)\delta_i^2(u)|\nabla_G\psi|^2\,dz. \\ &\le \frac{(1+\varepsilon)(1+s)^2A}{4|s|(1-\varepsilon)B}\int_{\mathbb{R}^N} f(z)u^{-q}\eta_i(u)\psi^2\,dz \\ &\quad + C_\varepsilon A\int_{\mathbb{R}^N}\omega(z)\left[\delta_i^2(u)+\eta_i^2(u)\left|\eta_i'(u)\right|^{-1}\right]|\nabla_G\psi|^2\,dz \\ &\le \frac{(1+\varepsilon)(1+s)^2(1+2k)}{4|s|(1-\varepsilon)}\int_{\mathbb{R}^N} f(z)u^{-(1+q)}\delta_i^2(u)\psi^2\,dz \\ &\quad + C_\varepsilon A\int_{\mathbb{R}^N}\omega(z)u^{1+s}|\nabla_G\psi|^2\,dz, \end{aligned}$$

that is,

$$\mu_\varepsilon\int_{\mathbb{R}^N} f(z)u^{-(1+q)}\delta_i^2(u)\psi^2\,dz \le C_\varepsilon A\int_{\mathbb{R}^N}\omega(z)u^{1+s}|\nabla_G\psi|^2\,dz, \tag{3.7}$$

where

$$\mu_\varepsilon = q - \frac{(1+\varepsilon)(1+s)^2(1+2k)}{4|s|(1-\varepsilon)}.$$

Clearly,

$$\lim_{\varepsilon\to 0^+}\mu_\varepsilon=\mu_0:=q-\frac{(1+s)^2(1+2k)}{4|s|}.$$

Thanks to $s\in(h(q),-1)$, we get $\mu_0>0$. Thus, we can fix some sufficiently small $\varepsilon>0$ such that $\mu_\varepsilon>0$. Furthermore, the monotone convergence theorem implies

$$\int_{\mathbb{R}^N}f(z)u^{s-q}\psi^2\,dz\le CA\int_{\mathbb{R}^N}\omega(z)u^{1+s}|\nabla_G\psi|^2\,dz \tag{3.8}$$

as $i\to+\infty$ in (3.7), where $C>0$ depending on q, k, and s.

Taking advantage of (3.4) with $\varepsilon=1/2$, (3.3), and (3.7), it follows that

$$\begin{aligned}
&B\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2\big|\eta_i'(u)\big|\psi^2\,dz\\
&\quad\le CB\int_{\mathbb{R}^N}\omega(z)\eta_i^2(u)\big|\eta_i'(u)\big|^{-1}|\nabla_G\psi|^2\,dz+2\int_{\mathbb{R}^N}f(z)u^{-q}\eta_i(u)\psi^2\,dz\\
&\quad\le CB\int_{\mathbb{R}^N}\omega(z)u^{1+s}|\nabla_G\psi|^2\,dz+2\int_{\mathbb{R}^N}f(z)u^{-q}\eta_i(u)\psi^2\,dz\\
&\quad\le C(A+B)\int_{\mathbb{R}^N}\omega(z)u^{1+s}|\nabla_G\psi|^2\,dz.
\end{aligned}\tag{3.9}$$

Setting $i\to+\infty$ in (3.9), the monotone convergence theorem leads to

$$B\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2u^{s-1}\psi^2\,dz\le C(A+B)\int_{\mathbb{R}^N}\omega(z)u^{1+s}|\nabla_G\psi|^2\,dz. \tag{3.10}$$

Consequently,

$$\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2u^{s-1}\psi^2\,dz\le C\int_{\mathbb{R}^N}\omega(z)u^{1+s}|\nabla_G\psi|^2\,dz. \tag{3.11}$$

Now we claim that (3.2) holds true. Indeed, we can choose some positive constant $\tau\gg 1$ such that

$$\frac{(\tau-1)(s-q)}{s+1}\ge\tau,\quad\text{or}\quad \tau\ge\frac{q-s}{q+1}.$$

By virtue of $0\le\psi(z)\le 1$ in $\mathbb{R}^N$, it follows

$$\big(\psi(z)\big)^{\frac{2(\tau-1)(s-q)}{s+1}}\le\big(\psi(z)\big)^{2\tau},\quad\forall z\in\mathbb{R}^N.$$

Replacing ψ by ψ^τ in (3.8) and applying Hölder's inequality, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^N}f(z)u^{s-q}\psi^{2\tau}\,dz\\
&\quad\le CA\int_{\mathbb{R}^N}\omega(z)u^{1+s}\psi^{2(\tau-1)}|\nabla_G\psi|^2\,dz
\end{aligned}$$

$$
\begin{aligned}
&\le CA\left(\int_{\mathbb{R}^N}\left(f(z)^{\frac{s+1}{s-q}}u^{s+1}\psi^{2(\tau-1)}\right)^{\frac{s-q}{s+1}}dz\right)^{\frac{s+1}{s-q}}\left(\int_{\mathbb{R}^N}\left(\omega(z)f(z)^{\frac{s+1}{q-s}}|\nabla_G\psi|^2\right)^{\frac{q-s}{q+1}}dz\right)^{\frac{q+1}{q-s}}\\
&= CA\left(\int_{\mathbb{R}^N}f(z)u^{s-q}\psi^{\frac{2(\tau-1)(s-q)}{s+1}}dz\right)^{\frac{s+1}{s-q}}\left(\int_{\mathbb{R}^N}\omega(z)^{\frac{q-s}{q+1}}f(z)^{\frac{s+1}{q+1}}|\nabla_G\psi|^{\frac{2(q-s)}{q+1}}dz\right)^{\frac{q+1}{q-s}}\\
&\le CA\left(\int_{\mathbb{R}^N}f(z)u^{s-q}\psi^{2\tau}dz\right)^{\frac{s+1}{s-q}}\left(\int_{\mathbb{R}^N}\omega(z)^{\frac{q-s}{q+1}}f(z)^{\frac{s+1}{q+1}}|\nabla_G\psi|^{\frac{2(q-s)}{q+1}}dz\right)^{\frac{q+1}{q-s}},
\end{aligned}
\tag{3.12}
$$

which implies that

$$
\int_{\mathbb{R}^N}f(z)u^{s-q}\psi^{2\tau}dz\le CA^{\frac{q-s}{q+1}}\int_{\mathbb{R}^N}\omega(z)^{\frac{q-s}{q+1}}f(z)^{\frac{s+1}{q+1}}|\nabla_G\psi|^{\frac{2(q-s)}{q+1}}dz.
\tag{3.13}
$$

Analogously, we replace the function ψ by ψ^τ in (3.11); it follows from (3.11)–(3.13) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\omega(z)|\nabla_G u|^2u^{s-1}\psi^{2\tau}dz\\
&\quad\le C\int_{\mathbb{R}^N}\omega(z)u^{1+s}\psi^{2(\tau-1)}|\nabla_G\psi|^2dz\\
&\quad\le CA^{\frac{q-s}{q+1}}\int_{\mathbb{R}^N}\omega(z)^{\frac{q-s}{q+1}}f(z)^{\frac{s+1}{q+1}}|\nabla_G\psi|^{\frac{2(q-s)}{q+1}}dz.
\end{aligned}
$$

This together with (3.13) derives (3.2), and the proof is finished. □

Proof of Theorem 1.9 On the contrary, u is a positive stable weak solution of (1.2). We apply (3.2) for a test function $\psi_R(x,y)$ defined in (2.8). Similar to the proof of (2.10), we can prove that, for all $R\ge R_0$, there exists a constant $C>0$ independent of R satisfying

$$
\int_{\Omega_R}\left(f(z)u^{s-q}+\omega(z)|\nabla_G u|^2u^{s-1}\right)dz\le CA^{\frac{q-s}{q+1}}R^m
\tag{3.14}
$$

with

$$
m=N_\alpha-\frac{(2-\theta)(q-s)-d(s+1)}{q+1}.
$$

Evidently, if $m<0$, the desired result is obtained immediately by tending $R\to+\infty$ in (3.14). Define the function

$$
l(t)=\frac{(2-\theta)(t-h(t))-d(h(t)+1)}{t+1},\quad t>0,
$$

where $h(t)$ is given in (3.1). The elementary calculus shows that

$$
\lim_{t\to0^+}h(t)=-1,\qquad\lim_{t\to+\infty}h(t)=-\infty,\quad h'(t)<0,\forall t>0
$$

and

$$
\lim_{t\to0^+}l(t)=2-\theta,\qquad\lim_{t\to+\infty}l(t)=\mu_0(k,\theta,d),
$$

$$
l'(t)=\frac{(2-\theta+d)(2\sqrt{t(t+1+2k)}+1+2k+t(1-2k))}{(1+2k)(t+1)^2\sqrt{t(t+1+2k)}},\quad t>0.
$$

A straightforward calculation yields that if $0 \le k \le \frac{3}{2}$, then $l(t)$ is increasing on $(0,\infty)$. If $k > \frac{3}{2}$, then $l(t)$ is increasing on $(0, \frac{1+2k+2\sqrt{1+2k}}{2k-3})$ and decreasing on $(\frac{1+2k+2\sqrt{1+2k}}{2k-3}, \infty)$. Moreover, $l(\frac{1+2k+2\sqrt{1+2k}}{2k-3}) = \mu_1(k,\theta,d)$, $l(\frac{4}{2k-3}) = \mu_0(k,\theta,d)$.

Thus, if $N_\alpha \le 2-\theta$ and $k \ge 0$, then $N_\alpha < l(t)$, $\forall t > 0$. Hence if we fix $s \in (h(q), -1)$ sufficiently near to $h(q)$, we have

$$N_\alpha < \frac{(2-\theta)(q-s) - d(s+1)}{q+1}, \quad q > 0,$$

which means that $m < 0$. Then we reach a contradiction by letting $R \to +\infty$ in (3.14).

If $2-\theta < N_\alpha < \mu_0(k,\theta,d)$ and $0 \le k \le \frac{3}{2}$. Make use of the monotonicity of $l(t)$, there exists unique $q_c > 0$ such that $N_\alpha < l(t)$ for $t > q_c$. So if we fix $s \in (h(q), -1)$ sufficiently close to $h(q)$, we obtain

$$N_\alpha < \frac{(2-\theta)(q-s) - d(s+1)}{q+1}, \quad q > q_c,$$

which means that $m < 0$. Letting $R \to +\infty$ in (3.14), we get a contradiction and the desired result is obtained. Obviously, q_c can be derived from the equation $N_\alpha = l(q)$, which is given in (1.13).

If $2-\theta < N_\alpha < \mu_0(k,\theta,d)$ and $k > \frac{3}{2}$. According to the monotonicity of $l(t)$, there exists unique $\widetilde{q}_c > 0$ such that $N_\alpha < l(t)$ for $t > \widetilde{q}_c$. Taking $R \to +\infty$ in (3.14), we reach a contradiction, and q_c may be deduced from the equation $N_\alpha = l(q)$, which is given in (1.14).

If $N_\alpha = \mu_0(k,\theta,d)$ and $k > \frac{3}{2}$. Combining $l(\frac{4}{2k-3}) = \mu_0(k,\theta,d)$ and the monotonicity of $h(t)$, we have $l(t) > N_\alpha$ for $t > \frac{4}{2k-3}$. We get a contradiction by tending $R \to +\infty$ in (3.14).

Assume now $\mu_0(k,\theta,d) < N_\alpha < \mu_1(k,\theta,d)$ and $k > \frac{3}{2}$, by the monotonicity of $l(t)$, there exists $q_{1,2} > \frac{4}{2k-3}$ such that $l(t) > N_\alpha$ for $q_1 < t < q_2$. We get a contradiction by tending $R \to +\infty$ in (3.14). Evidently, $q_{1,2}$ can be derived from the equation $N_\alpha = l(q)$, which is given in (1.15). So we are done. □

Acknowledgements
The authors would like to express their sincere gratitude to the anonymous reviewer for their valuable comments and suggestions which improved the presentation of this manuscript.

Funding
This work was supported by the National Natural Science Foundation of China (Grant No. 11571092), the Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (Grant No. 19KJD100002), the Natural Science Foundation of Shandong Province (Grant No. ZR2018MA017), and the China Postdoctoral Science Foundation (Grant No. 2017M610436).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally, all authors read and approved the final manuscript.

Author details
[1]School of Statistics and Mathematics, Nanjing Audit University, Nanjing, P.R. China. [2]College of Science, Hohai University, Nanjing, P.R. China. [3]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, P.R. China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 16 September 2019 Accepted: 9 January 2020 Published online: 23 January 2020

References

1. Birindelli, I., Capuzzo Dolcetta, I., Cutrì, A.: Liouville theorems for semilinear equations on the Heisenberg group. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **14**(3), 295–308 (1997)
2. Castorina, D., Sanchón, M.: Regularity of stable solutions to semilinear elliptic equations on Riemannian models. Adv. Nonlinear Anal. **4**, 295–309 (2015)
3. Chen, C.S.: Liouville type theorem for stable solutions of p-Laplace equation in $\mathbb{R}^N$. Appl. Math. Lett. **68**, 62–67 (2017)
4. Chen, C.S., Song, H.X., Yang, H.W.: Liouville type theorems for stable solutions of p-Laplace equation in $\mathbb{R}^N$. Nonlinear Anal. **160**, 44–52 (2017)
5. Chen, C.S., Song, H.X., Yang, H.W.: Liouville-type theorems for stable solutions of singular quasilinear elliptic equations $\mathbb{R}^N$. Electron. J. Differ. Equ. **2018**, 81 (2018)
6. Cowan, C., Fazly, M.: On stable entire solutions of semi-linear elliptic equations with weights. Proc. Am. Math. Soc. **140**(6), 2003–2012 (2012)
7. Damascelli, L., Farina, A., Sciunzi, B., Valdinoci, E.: Liouville results for m-Laplace equations of Lane–Emden–Fowler type. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **26**(4), 1099–1119 (2009)
8. D'Ambrosio, L., Lucente, S.: Nonlinear Liouville theorems for Grushin and Tricomi operators. J. Differ. Equ. **193**, 511–541 (2003)
9. Dancer, E.N., Farina, A.: On the classification of solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$: stability outside a compact set and applications. Proc. Am. Math. Soc. **137**(4), 1333–1338 (2009)
10. Du, Y.H., Guo, Z.M.: Positive solutions of an elliptic equation with negative exponent: stability and critical power. J. Differ. Equ. **246**(6), 2387–2414 (2009)
11. Duong, A.T., Nguyen, N.T.: Liouville type theorems for elliptic equations involving Grushin operator and advection. Electron. J. Differ. Equ. **2017**, 108 (2017)
12. Duong, A.T., Phan, Q.H.: Liouville type theorem for nonlinear elliptic system involving Grushin operator. J. Math. Anal. Appl. **454**(2), 785–801 (2017)
13. Dupaigne, L.: Stable Solutions of Elliptic Partial Differential Equations. Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 143 (2011)
14. Farina, A.: Liouville-type results for solutions of $-\Delta u = |u|^{p-1}u$ on unbounded domains of $\mathbb{R}^N$. C. R. Acad. Sci. Paris, Ser. I **341**, 415–418 (2005)
15. Farina, A.: Stable solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$. C. R. Acad. Sci. Paris, Ser. I **345**, 63–66 (2007)
16. Farina, A.: On the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^N$. J. Math. Pures Appl. **87**, 537–561 (2007)
17. Figueiredo, G.M.: Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument. J. Math. Anal. Appl. **401**, 706–713 (2013)
18. Figueiredo, G.M., Nascimento, R.G.: Existence of a nodal solution with minimal energy for a Kirchhoff equation. Math. Nachr. **288**(1), 48–60 (2015)
19. Fiscella, A., Valdinoci, E.: A critical Kirchhoff type problem involving a nonlocal operator. Nonlinear Anal. **94**, 156–170 (2014)
20. Franchi, B., Gutiérrez, C.E., Wheeden, R.L.: Weighted Sobolev–Poincaré inequalities for Grushin type operators. Commun. Partial Differ. Equ. **19**, 523–604 (1994)
21. Ghergu, M., Rădulescu, V.: Singular elliptic problems with lack of compactness. Ann. Mat. Pura Appl. (4) **185**, 63–79 (2006)
22. Gidas, B., Spruck, J.: Global and local behavior of positive solutions of nonlinear elliptic equations. Commun. Pure Appl. Math. **34**(4), 525–598 (1981)
23. Huang, X.: Stable weak solutions of weighted nonlinear elliptic equations. Commun. Pure Appl. Anal. **13**, 293–305 (2014)
24. Huynh, N.V., Le, P., Nguyen, D.P.: Liouville theorems for Kirchhoff equations in $\mathbb{R}^N$. J. Math. Phys. **60**(6), 061506 (2019)
25. Jeong, W., Lee, Y.: Stable solutions and finite Morse index solutions of nonlinear elliptic equations with Hardy potential. Nonlinear Anal. **87**, 126–145 (2013)
26. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)
27. Le, P.: Nonexistence of stable solutions to p-Laplace equations with exponential nonlinearities. Electron. J. Differ. Equ. **2016**, 326 (2016)
28. Le, P., Le, D.H.T., Le, K.A.T.: On stable solutions to weighted quasilinear problems of Gelfand type. Mediterr. J. Math. **15**, 94 (2018)
29. Le, P., Nguyen, H.T., Nguyen, T.Y.: On positive stable solutions to weighted quasilinear problems with negative exponent. Complex Var. Elliptic Equ. **63**, 1739–1751 (2018)
30. Li, Y., Zhang, L.: Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations. J. Anal. Math. **90**, 27–87 (2003)
31. Li, Y.H., Li, F.Y., Shi, J.P.: Existence of a positive solution to Kirchhoff type problems without compactness conditions. J. Differ. Equ. **253**, 2285–2294 (2012)
32. Ma, L., Wei, J.C.: Properties of positive solutions to an elliptic equation with negative exponent. J. Funct. Anal. **254**, 1058–1087 (2008)
33. Mironescu, P., Rădulescu, V.D.: The study of a bifurcation problem associated to an asymptotically linear function. Nonlinear Anal. **26**(4), 857–875 (1996)
34. Monti, R., Morbidelli, D.: Kelvin transform for Grushin operators and critical semilinear equations. Duke Math. J. **131**, 167–202 (2006)
35. Monticelli, D.D.: Maximum principles and the method of moving planes for a class of degenerate elliptic linear operators. J. Eur. Math. Soc. **12**, 611–654 (2010)

36. Morbidelli, D.: Liouville theorem, conformally invariant cones and umbilical surfaces for Grushin-type metrics. Isr. J. Math. **173**, 379–402 (2009)
37. Pucci, P., Saldi, S.: Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators. Rev. Mat. Iberoam. **32**(1), 1–22 (2016)
38. Wang, C., Ye, D.: Some Liouville theorems for Hénon type elliptic equations. J. Funct. Anal. **262**, 1705–1727 (2012)
39. Wei, Y.F., Chen, C.S., Song, H.X., Yang, H.W.: Liouville-type theorems for stable solutions of Kirchhoff equations with exponential and superlinear nonlinearities. Complex Var. Elliptic Equ. **64**, 1297–1309 (2019)
40. Yu, X.H.: Liouville type theorem for nonlinear elliptic equation involving Grushin operators. Commun. Contemp. Math. **17**(5), 1450050 (2015)
41. Yu, X.H.: Liouville type theorems for singular integral equations and integral systems. Commun. Pure Appl. Anal. **15**, 1825–1840 (2016)