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The existence of solutions for perturbed fractional differential equations with impulses via Morse theory

Peiluan Li[1*], Liang Xu[1], Hui Wang[2] and Yumeng Wang[3]

*Correspondence:
lpllpl_lpl@163.com
[1]School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, P.R. China
Full list of author information is available at the end of the article

Abstract

In the present paper, the existence of nontrivial solutions of impulsive fractional differential equations with Dirichlet boundary conditions is studied. We apply Morse theory coupled with local linking arguments to solve the topic, and we prove the existence of at least one nontrivial solution for the impulsive fractional differential equations.

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Keywords: Morse theory; Perturbed fractional differential equations; Impulses; Nontrivial solutions

1 Introduction

Fractional calculus is a powerful tool for describing the genetic properties and memory processes of various materials [1–3]. Fractional differential equations (FDEs) have been widely used in the field of medical, physical, economic and technological sciences in recent times. Though fractional differential equations containing Riemann–Liouville fractional derivatives or Caputo fractional derivatives have got more and more attentions, the fixed point theorems, coincidence degree theory and monotone iteration methods are still the main approaches. For the critical point theory, we refer to [4–6] and the references therein.

In [7], the fractional boundary-value problems considered by Jiao and Zhou is listed as follows:

$$\begin{cases} -\frac{1}{2}\frac{d}{dt}({}_0D_t^{-\beta} + {}_tD_T^{-\beta})u'(t) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\beta \in [0,1)$, ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann–Liouville fractional derivatives respectively. $F : [0,T] \times R^N \to R$ (with $N \geq 1$) is a suitable given function and $\nabla F(t,x)$ is the gradient of F with respect to x. In this paper, the sufficient conditions for the existence of solutions are obtained by using the least action principle and the mountain path theorem. Since then, the variational methods have been applied to study fractional differential equations; see [7–10].

The problem (1.1) arose from the phenomenon of advection dispersion and was first scrutinized by Ervin and Loop in [11]. From then on, the existence and multiplicity of solutions for the above problem (1.1) or related problems were further studied by the authors in [12–15] with the critical point theory.

The impulsive differential equations originated from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. Due to their significance, many researchers established the solvability of impulsive differential equations. If you are interested in the general theory and applications of such equations, please refer to [16–18] and the references therein.

Up to now, there are few papers that use variational methods and critical point theory to study the fractional boundary-value problems with impulses [19–24].

In [22], the authors use variational methods and critical point theory to study the following fractional differential systems with impulsive effects:

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t){}_0^cD_t^{\alpha_i}u_i(t)) = \lambda F_{u_i}(t,u) + h_i(u_i(t)), \quad 0 < t < T, t \neq t_j, \\ \Delta({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i)(t_j) = I_{ij}(u_i(t_j)), \quad j = 1,2,\ldots,m, \\ u_i(0) = u_i(T) = 0, \quad 1 \leq i \leq N. \end{cases}$$

In [23], the authors have considered the following boundary-value problems of impulsive fractional differential equations:

$$\begin{cases} -\frac{1}{2}\frac{d}{dt}({}_0D_t^{-\beta_i} + {}_tD_T^{-\beta_i})u_i'(t) \\ \quad = a_i(t)u_i(t) + F_{u_i}(t,u(t)), \quad 1 \leq i \leq N, t \neq t_j, \text{a.e. } t \in [0,T], \\ \Delta(D_t^{\alpha_i}u_i)(t_j) = I_{ij}(u_i(t_j)), \quad t_j \in (0,T), j = 1,2,\ldots,l, \\ u_i(0) = u_i(T) = 0, \quad 1 \leq i \leq N, \end{cases} \tag{1.2}$$

where $u = (u_1,\ldots,u_N)$, $|u| = \sqrt{\sum_{i=1}^N u_i^2}$, $\beta_i \in [0,1)$, $\alpha_i = 1 - \frac{\beta_i}{2} \in (\frac{1}{2},1]$ for $1 \leq i \leq N$, ${}_0D_t^{-\beta_i}$, ${}_tD_T^{-\beta_i}$ are the left and right Riemann–Liouville fractional integrals of order β_i, ${}_0^cD_t^{\alpha_i}$ and ${}_t^cD_T^{\alpha_i}$ are the left and right Caputo fractional derivative of order a_i, $a_i \in L^\infty[0,T]$, $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$, $I_{ij} \in C([0,T],R)$, $F : [0,T] \times R^N \to R$ is measurable, continuously differentiable, F_{u_i} denotes the partial derivative of F with respect to u_i for $1 \leq i \leq N$, and

$$\begin{aligned} \left(D_t^{\alpha_i}u_i\right)(t_j) &= \frac{1}{2}\left\{{}_0D_t^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right) - {}_tD_T^{\alpha_i-1}\left({}_t^cD_T^{\alpha_i}u_i\right)\right\}(t_j), \\ \Delta\left(D_t^{\alpha_i}u_i\right)(t_j) &= \frac{1}{2}\left\{{}_0D_t^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right) - {}_tD_T^{\alpha_i-1}\left({}_t^cD_T^{\alpha_i}u_i\right)\right\}\left(t_j^+\right) \\ &\quad - \frac{1}{2}\left\{{}_0D_t^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right) - {}_tD_T^{\alpha_i-1}\left({}_t^cD_T^{\alpha_i}u_i\right)\right\}\left(t_j^-\right), \\ \left\{{}_0D_t^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right) - {}_tD_T^{\alpha_i-1}\left({}_t^cD_T^{\alpha_i}u_i\right)\right\}\left(t_j^+\right) &= \lim_{t\to t_j^+}\left\{{}_0D_t^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right) - {}_tD_T^{\alpha_i-1}\left({}_t^cD_T^{\alpha_i}u_i\right)\right\}(t), \\ \left\{{}_0D_t^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right) - {}_tD_T^{\alpha_i-1}\left({}_t^cD_T^{\alpha_i}u_i\right)\right\}\left(t_j^-\right) &= \lim_{t\to t_j^-}\left\{{}_0D_t^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right) - {}_tD_T^{\alpha_i-1}\left({}_t^cD_T^{\alpha_i}u_i\right)\right\}(t), \end{aligned}$$

for $j = 1,\ldots,l$, $1 \leq i \leq N$.

On the other hand, in recent years, Morse theory has been used to discuss the existence of solutions of differential equations [25, 26]. However, to the best of our knowledge, Morse theory is rarely applied to the impulsive fractional boundary-value problems.

In [27], based on Morse theory coupled with local linking arguments, the authors studied the following impulsive fractional differential equation:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + k(t)u(t) = f(t,u), \quad 0 < t < T, t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha_i}u))(t_j) = I_j(u(t_j)), \quad j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \end{cases}$$

Motivated by the work above, we will investigate the existence of at least one nontrivial weak solution of problem (1.2) by Morse theory. Compared with the research in [23], the method of this paper is different.

To investigate problem (1.2), we make the following assumptions.

$(I0)$ $I_{ij} \in C([0,T],R)$, $I_{ij}(0) = 0$, $I_{ij}(u_i)u_i \geq 0$ and there exist constants $a_j, b_j > 0$ and $c_j, \gamma_j \in [0,1)$, such that $|I_{ij}(u)| \leq a_j|u_i|^{\gamma_j}$, $\lim_{|u_i|\to 0} \frac{|I_j(u_i)|}{|u_i|^{c_j}} = b_j$, $i = 1,\ldots,N$, $j = 1,\ldots,l$;

$(I1)$ there exists $\theta_1 \geq 1$ such that $\theta_1 I_{ij}^*(u) \geq I_{ij}^*(\zeta_1 u_i)$, $\forall u_i \in R$ and $\zeta_1 \in [0,1]$, where

$$I_{ij}^*(u) := 2\int_0^{u_i} I_{ij}(s)\,ds - I_{ij}(u_i)u_i.$$

We introduce the following conditions on the nonlinearity function $F_{u_i}(t,u)$:

$(F0)$ $F_{u_i}(t,0) = 0$, $\lim_{|u|\to 0}\sup \frac{|F(t,u)|}{|u|^2} < \sum_{i=1}^N \frac{\Gamma^2(\alpha_i+1)}{2T^{2\alpha_i}}|\cos(\pi\alpha_i)|$ uniformly for $t \in (0,T)$, and there are constants $C > 0$, r, r_0, γ with $\gamma \in (1, \max_{j\in\{1,2,\ldots,l\}}\{\gamma_j + 1\})$ such that

$$F(t,u) \geq C|u_i|^{\gamma}, \quad r \leq |u_i| \leq r_0 \text{ a.e. } t \in [0,T];$$

$(F1)$ there exists $\theta_2 \geq 1$ such that $\theta_2 F_*(t,u) \geq F_*(t,\zeta_2 u)$, $\forall (t,u) \in [0,T] \times R$, $\zeta_2 \in [0,1]$, where $F_*(t,u) := F_{u_i}(t,u)u_i - 2F(t,u)$;

$(F2)$ $F_{u_i}(t,u)u_i \geq 0$, $\forall (t,u) \in [0,T] \times R$; $\lim_{|u|\to\infty} \frac{F_{u_i}(t,u)}{u} = +\infty$ uniformly for $t \in (0,T)$;

$(F3)$ $\lim_{|u|\to\infty} \frac{|F(t,u)|}{|u|^2} = +\infty$ uniformly for $t \in (0,T)$.

Theorem 1.1 *Assume that* $(I0)$, $(I1)$, $(F0)$, $(F1)$, $(F2)$ *hold. Then problem* (1.2) *has at least one nontrivial weak solution.*

Theorem 1.2 *Assume that* $(I0)$, $(I1)$, $(F0)$, $(F1)$, $(F3)$ *are satisfied. Then problem* (1.2) *has at least one nontrivial weak solution.*

2 Preliminaries

As discussed in [23], we can transfer problem (1.2) to the following problem:

$$\begin{cases} \frac{d}{dt}\{\frac{1}{2}{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i(t)) - \frac{1}{2}{}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i(t))\} + a_i(t)u_i(t) + F_{u_i}(t,u(t)) = 0, \\ \quad 1 \leq i \leq N, t \neq t_j, \text{a.e.}, t \in [0,T], \\ \Delta(D_t^{\alpha_i}u_i)(t_j) = I_{ij}(u_i(t_j)), \quad t_j \in (0,T), j = 1,2,\ldots,l, \\ u_i(0) = u_i(T) = 0, \quad 1 \leq i \leq N, \end{cases} \tag{2.1}$$

The problem (1.2) is equivalent to problem (2.1). Therefore, a solution of problem (2.1) corresponds to a solution of the BVP (1.2).

A variational structure is established to transform the existence of solutions to problem (2.1) into the existence of corresponding functional critical points. We construct the following appropriate function spaces.

Let us recall that, for any fixed $t \in [0,T]$ and $1 \le p \le \infty$,

$$\|u\|_\infty = \max_{t\in[0,T]} \bigl|u(t)\bigr|, \qquad \|u\|_{L^p} = \left(\int_0^T \bigl|u(s)\bigr|^p \,ds\right)^{\frac{1}{p}}.$$

For $\alpha_i \in [0,1)$, $1 \le i \le N$, we define the fractional derivative spaces $E_0^{\alpha_i}$ by the closure of $C_0^\infty([0,T],R^N)$ with $u_i(0) = u_i(T)$ under the norm

$$\|u_i\|_{\alpha_i} = \left(\int_0^T \bigl|{}_0^cD_t^{\alpha_i}u_i(t)\bigr|^2\,dt + \int_0^T \bigl|u_i(t)\bigr|^2\,dt\right)^{\frac{1}{2}}, \quad \forall u_i \in E_0^{\alpha_i}.$$

Obviously, the fractional derivative space $E_0^{\alpha_i}$ is the space of functions $u_i \in L^2(0,T)$ having α_i-order Caputo left and right fractional derivatives and Riemann–Liouville left and right fractional derivatives, ${}_0^cD_t^{\alpha_i}u_i$, ${}_t^cD_T^{\alpha_i}u_i$, ${}_0D_t^{\alpha_i}u_i$, ${}_tD_T^{\alpha_i}u_i \in L^2(0,T)$ and $u_i(0) = u_i(T) = 0$.

Definition 2.1 ([23]) We denote $u = (u_1,\dots,u_N)$, $u_i \in E_0^{\alpha_i}$, $(i = 1,\dots,N)$ this being a weak solution of the problem (2.1) if the following identity:

$$\begin{aligned}&\sum_{i=1}^N \int_0^T \left\{-\frac{1}{2}\bigl[{}_0^cD_t^{\alpha_i}u_i(t){}_t^cD_T^{\alpha_i}v_i(t) + {}_t^cD_T^{\alpha_i}u_i(t){}_0^cD_t^{\alpha_i}v_i(t)\bigr] - a_i(t)u_i(t)v_i(t)\right\}dt\\&\quad + \sum_{i=1}^N\sum_{j=1}^l I_{ij}\bigl(u_i(t_j)\bigr)v_i(t_j) - \sum_{i=1}^N \int_0^T F_{u_i}\bigl(t,u(t)\bigr)v_i(t)\,dt = 0\end{aligned}$$

holds for all $\forall v_i \in E_0^{\alpha_i}$.

Consider the functional $\Phi: E_0^{\alpha_1} \times \cdots \times E_0^{\alpha_N} \to R$ defined by

$$\begin{aligned}\Phi(u) &= \sum_{i=1}^N \int_0^T \left\{-\frac{1}{2}{}_0^cD_t^{\alpha_i}u_i(t){}_t^cD_T^{\alpha_i}u_i(t) - \frac{1}{2}a_i(t)u_i^2(t)\right\}dt - \int_0^T F\bigl(t,u(t)\bigr)\,dt\\&\quad + \sum_{i=1}^N\sum_{j=1}^l \int_0^{u_i(t_j)} I_{ij}(s)\,ds.\end{aligned}$$

From ($I0$) and ($F0$), we can infer that Φ is continuous, differentiable and for all $u = (u_1,\dots,u_N)$, $v = (v_1,\dots,v_N)$, $u_i, v_i \in E_0^{\alpha_i}$ $(i = 1,\dots,N)$, and we have

$$\begin{aligned}\bigl\langle\Phi'(u),v\bigr\rangle &= \sum_{i=1}^N \int_0^T \left\{-\frac{1}{2}\bigl[{}_0^cD_t^{\alpha_i}u_i(t){}_t^cD_T^{\alpha_i}v_i(t) + {}_t^cD_T^{\alpha_i}u_i(t){}_0^cD_t^{\alpha_i}v_i(t)\bigr] - a_i(t)u_i(t)v_i(t)\right\}dt\\&\quad + \sum_{i=1}^N\sum_{j=1}^l I_{ij}\bigl(u_i(t_j)\bigr)v_i(t_j) - \sum_{i=1}^N \int_0^T F_{u_i}\bigl(t,u(t)\bigr)v_i(t)\,dt. \qquad (2.2)\end{aligned}$$

Then, the critical point of Φ is the weak solution of (2.1).

Lemma 2.2 ([7]) *Let* $\frac{1}{2} < \alpha \le 1$ *and* $1 < p < \infty$, *for all* $u \in E_0^\alpha$, *one has*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \left\| {}_0^cD_t^\alpha u \right\|_{L^p}. \tag{2.3}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \le \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \left\| {}_0^cD_t^\alpha u \right\|_{L^p}. \tag{2.4}$$

It is easy to verify that the norm $\|u_i\|_{\alpha_i} = (\int_0^T |{}_0^cD_t^{\alpha_i} u_i(t)|^2\,dt + \int_0^T |u_i(t)|^2\,dt)^{\frac{1}{2}}$ is equivalent to $\|u_i\|_{\alpha_i} = (\int_0^T |{}_0^cD_t^{\alpha_i} u_i(t)|^2\,dt)^{\frac{1}{2}}$, $\forall u_i \in E_0^{\alpha_i}$. In the following, we will consider the fractional derivative spaces $E_0^{\alpha_i}$ with respect to the norm $\|u_i\|_{\alpha_i} = (\int_0^T |{}_0^cD_t^{\alpha_i} u_i(t)|^2\,dt)^{\frac{1}{2}}$.

Lemma 2.3 ([23]) *For* $\alpha_i \in [\frac{1}{2}, 1)$, $1 \le i \le N$, *one has*

$$\sum_{i=1}^N \|u_i\|_{L^p}^p \le A_p \sum_{i=1}^N \|u_i\|_{\alpha_i}^p, \qquad \sum_{i=1}^N \|u_i\|_\infty^2 \le B \sum_{i=1}^N \|u_i\|_{\alpha_i}^2,$$

where $A_p = \max\{\frac{T^{p\alpha_i}}{\Gamma^p(\alpha_i+1)}, 1 \le i \le N\}$, $B = \max\{\frac{T^{2\alpha_i-1}}{\Gamma^2(\alpha_i)(2\alpha_i-1)}, 1 \le i \le N\}$.

Lemma 2.4 ([23]) *Let* $\frac{1}{2} < \alpha_i \le 1$ *for* $1 \le i \le N$. *Assume the sequence* $\{x_n\}$ *converges weakly to* x *in* $E_0^{\alpha_i}$. *Then* $x_n \to x$ *strongly in* $C([0,T],R)$, *i.e.*, $\|x_n - x\|_\infty \to 0$, *as* $n \to \infty$.

Lemma 2.5 ([23]) *Let* $\frac{1}{2} < \alpha_i \le 1$ *for* $1 \le i \le N$. *For any* $u_i \in E_0^{\alpha_i}$, *one has*

$$\sum_{i=1}^N \left|\cos(\pi\alpha_i)\right| \|u_i\|_{\alpha_i}^2 \le \sum_{i=1}^N -\int_0^T {}_0^cD_t^{\alpha_i} u_i(t) {}_t^cD_T^{\alpha_i} u_i(t)\,dt \le \sum_{i=1}^N \frac{1}{|\cos(\pi\alpha_i)|} \|u_i\|_{\alpha_i}^2.$$

In the sequel, we denote $X = E_0^{\alpha_1} \times \cdots \times E_0^{\alpha_N}$, then X is a reflexive and separable Banach space with the norm

$$\|u\|_X = \left\|(u_1, \ldots, u_N)\right\|_X = \sum_{i=1}^N \|u_i\|_{\alpha_i} = \sum_{i=1}^N \left(\int_0^T \left| {}_0^cD_t^{\alpha_i} u_i(t) \right|^2 dt \right)^{\frac{1}{2}}.$$

Definition 2.6 The sequence $\{u^{(k)}\} \subset X$ is said to be a C sequence of the functional Φ if for $u^{(k)} = (u_1^{(k)}, \ldots, u_i^{(k)}, \ldots, u_N^{(k)})$, $1 \le k < \infty$, $c \in R$, one has $\Phi(u^{(k)}) \to c$, $\|u^{(k)}\|_X \to +\infty$ and $\langle \Phi'(u^{(k)}), u^{(k)} \rangle \to 0$, as $k \to \infty$. The functional Φ satisfies the C-condition if ever the C sequence of Φ has a convergent subsequence.

Let E be a real Banach space and $\Phi \in C^1[E,R)$, $Q = \{u \in E : \Phi'(u) = 0\}$.

Definition 2.7 ([28]) For $c \in R$, we define u as an isolated critical point of Φ with $\Phi(u) = c$, and define U as a neighborhood of u such that Φ has the only u as a critical point in U.

We call

$$C_q(\Phi,u) := H_q\big(\Phi^c \cap U, \Phi^c \cap U\backslash\{u\}\big), \quad \big(q \in N := \{0,1,2,\dots\}\big)$$

the qth critical group of Φ at u, where $\Phi^c := \{u \in E : \Phi(u) \le c\}$ is the c-sublevel set, and H_q is the singular relative homology group with coefficients in an Abelian group G.

Lemma 2.8 ([28]) *If $Q = \{0\}$, then $C_q(\Phi,\infty) = C_q(\Phi,0)$, $\forall q \in N$. It follows that if $C_q(\Phi,\infty) \neq C_q(\Phi,0)$ for some $q \in N$, then Φ must have a nontrivial critical point.*

Lemma 2.9 ([29]) *Let 0 be a critical point of Φ with $\Phi(0) = 0$. Suppose that Φ has a local linking at 0 with respect to $E = V \oplus W$, $k = \dim V < \infty$, that is, there exists $\rho > 0$ small such that*

$$\Phi(u) \le 0, \quad \forall u \in V, \|u\| \le \rho \quad \textit{and} \quad \Phi(u) > 0, \quad \forall u \in W, 0 < \|u\| \le \rho.$$

Then $C_k(\Phi,0) \ncong 0$, hence 0 is a homological nontrivial point of Φ.

3 Proofs of main results

Lemma 3.1 *Assume that $(I0)$, $(F0)$, $(F2)$ hold, then Φ satisfies the C-condition.*

Proof Assume $\{u^{(k)}\}$ is a C sequence in X, where $u^{(k)} = (u_1^{(k)},\dots,u_i^{(k)},\dots,u_N^{(k)})$.

First, we address the boundedness of C sequence $\{u^{(k)}\}$.

Assume that C sequence $\{u^{(k)}\}$ is unbounded. Up to a subsequence we have

$$\Phi\big(u^{(k)}\big) \to c, \qquad \big\|u^{(k)}\big\|_X \to +\infty, \qquad \big\langle\Phi'\big(u^{(k)}\big),u^{(k)}\big\rangle \to 0, \quad k \to \infty, c \in R. \tag{3.1}$$

Set $v_i^{(k)} := \|u_i^{(k)}\|_{\alpha_i}^{-1} u_i^{(k)}(t) \in E_0^{\alpha_i}\backslash\{0\}$, then $\|v_i^{(k)}\|_{\alpha_i} = 1$ for all $n \in N$, where $v_0 = (v_{01},\dots,v_{0i},\dots,v_{0N})$, $v^{(k)} = (v_1^{(k)},\dots,v_i^{(k)},\dots,v_N^{(k)})$, $1 \le i \le n$, $1 \le k < \infty$. By Lemma 2.4, we have

$$v_i^{(k)} \to v_{0i} \quad \text{in } L^p\big([0,T]\big), \qquad v_i^{(k)} \to v_{0i} \quad \text{a.e. } t \in [0,T].$$

Obvious $v_{0i} \neq 0$, set $\Sigma_1 := \{t \in [0,T] : v(t) \neq 0\}$ and $\Sigma_2 := [0,T]\backslash\Sigma_1$. Then $[0,T] = \Sigma_1 \cup \Sigma_2$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$. So $\operatorname{meas}(\Sigma_1) > 0$. By (3.1), we obtain

$$\begin{aligned} &\sum_{i=1}^{N}\int_0^T \big\{-{}_0^cD_t^{\alpha_i}u_i^{(k)}(t){}_t^cD_T^{\alpha_i}u_i^{(k)}(t) - a_i(t)\big(u_i^{(k)}\big)^2(t)\big\}\,dt + \sum_{i=1}^{N}\sum_{j=1}^{l} I_{ij}\big(u_i^{(k)}(t_j)\big)u_i^{(k)}(t_j) \\ &\quad - \sum_{i=1}^{N}\int_0^T F_{u_i^{(k)}}\big(t,u(t)\big)u_i^{(k)}(t)\,dt = \big\langle\Phi'(u),u\big\rangle = o(1). \end{aligned} \tag{3.2}$$

Combining $(I0)$ and $\|u^{(k)}\|_X \to +\infty$, as $k \to \infty$, we can derive

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{N}\sum_{j=1}^{l} I_{ij}(u_i^{(k)}(t_j))u_i^{(k)}(t_j)}{\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}\|u_i^{(k)}\|_{\alpha_i}^2} \to 0. \tag{3.3}$$

From $(F2)$, (3.2), (3.3) and Lemma 2.5, we have

$$
\begin{aligned}
1 &= \lim_{k\to\infty} \frac{\sum_{i=1}^{N}\int_0^T\{-{}_0^cD_t^{\alpha_i}u_i^{(k)}(t){}_t^cD_T^{\alpha_i}u_i^{(k)}(t)-a_i(t)(u_i^{(k)})^2(t)\}\,dt-o(1)}{\sum_{i=1}^{N}\int_0^T\{-{}_0^cD_t^{\alpha_i}u_i^{(k)}(t){}_t^cD_T^{\alpha_i}u_i^{(k)}(t)-a_i(t)(u_i^{(k)})^2(t)\}\,dt} \\
&\geq \lim_{k\to\infty} \frac{\sum_{i=1}^{N}\int_0^T\{-{}_0^cD_t^{\alpha_i}u_i^{(k)}(t){}_t^cD_T^{\alpha_i}u_i^{(k)}(t)-a_i(t)(u_i^{(k)})^2(t)\}\,dt-o(1)}{\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}\|u_i^{(k)}\|_{\alpha_i}^2} \\
&\geq \lim_{k\to\infty}\int_{\sum_2}\frac{\sum_{i=1}^{N}F_{u_i^{(k)}}(t,u(t))u_i^{(k)}(t)\,dt}{\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}|u_i^{(k)}|^2}\bigl|v_i^{(k)}(t)\bigr|^2\,dt \\
&\quad+\lim_{k\to\infty}\int_{\sum_1}\frac{\sum_{i=1}^{N}F_{u_i^{(k)}}(t,u(t))u_i^{(k)}(t)\,dt}{\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}|u_i^{(k)}|^2}\bigl|v_i^{(k)}(t)\bigr|^2\,dt \\
&\quad-\lim_{k\to\infty}\frac{\sum_{i=1}^{N}\sum_{j=1}^{l}I_{ij}(u_i^{(k)}(t_j))u_i^{(k)}(t_j)}{\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}\|u_i^{(k)}\|_{\alpha_i}^2} \\
&\geq \lim_{k\to\infty}\int_{\sum_1}\frac{\sum_{i=1}^{N}F_{u_i^{(k)}}(t,u(t))u_i^{(k)}(t)\,dt}{\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}|u_i^{(k)}|^2}\bigl|v_i^{(k)}(t)\bigr|^2\,dt.
\end{aligned}
\tag{3.4}
$$

By $(F2)$, we have, for any $t\in[0,T]$,

$$
\lim_{k\to\infty}\frac{\sum_{i=1}^{N}\int_0^T F_{u_i^{(k)}}(t,u(t))u_i^{(k)}(t)\,dt}{\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}|u_i^{(k)}|^2}\bigl|v_i^{(k)}(t)\bigr|^2=+\infty.
$$

From Fatou's lemma, we can get

$$
\lim_{k\to\infty}\int_{\sum_1}\frac{\sum_{i=1}^{N}F_{u_i^{(k)}}(t,u(t))u_i^{(k)}(t)\,dt}{\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}|u_i^{(k)}|^2}\bigl|v_i^{(k)}(t)\bigr|^2\,dt=+\infty,
$$

which is contradictory with (3.4), thus $\{u^{(k)}\}$ is bounded in X.

Second, we verify C sequence $\{u^{(k)}\}$ have convergent subsequence in X.

Since X is reflexive, we know that $\{u^{(k)}\}$ have a weakly convergent subsequence in X. Hence, we have

$$
\begin{aligned}
&u_i^{(k)}\rightharpoonup u_{0i}\in E_0^{\alpha_i},\quad i=1,\dots,N,\ \text{as } k\to\infty, u_0=(u_{01},\dots,u_{0i},\dots,u_{0N}),\\
&u_i^{(k)}\to u_{0i}\quad\text{in } C\bigl([0,T]\bigr),\quad i=1,\dots,N,\ \text{a.e. } t\in[0,T].
\end{aligned}
$$

Thus $\|u_i^{(k)}-u_{01}\|_\infty\to 0$, as $k\to\infty$. According to (2.2), it is easy to prove

$$
\bigl\langle \Phi'\bigl(u^{(k)}\bigr)-\Phi'(u_0),u^{(k)}-u_0\bigr\rangle\to 0,\quad k\to\infty.
$$

From Lemma 2.5, we can get

$$
\begin{aligned}
0&\leftarrow\bigl\langle \Phi'\bigl(u^{(k)}\bigr)-\Phi'(u_0),u^{(k)}-u_0\bigr\rangle\\
&=\sum_{i=1}^{N}\int_0^T\bigl\{-\bigl[{}_0^cD_t^{\alpha_i}\bigl(u_i^{(k)}-u_{0i}\bigr){}_t^cD_T^{\alpha_i}\bigl(u_i^{(k)}-u_{0i}\bigr)\bigr]-a_i(t)\bigl(u_i^{(k)}-u_{0i}\bigr)\bigl(u_i^{(k)}-u_{0i}\bigr)\bigr\}\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\sum_{j=1}^{l}\big[I_{ij}\big(u_i^{(k)}(t_j)-u_{0i}(t_j)\big)\big]\big(u_i^{(k)}(t_j)-u_{0i}(t_j)\big)\\
&-\sum_{i=1}^{N}\int_0^T\big[F_{u_i^{(k)}}\big(t,u(t)\big)-F_{u_{0i}}\big(t,u_0(t)\big)\big]\big(u_i^{(k)}-u_{0i}\big)\,dt\\
\geq{}&\sum_{i=1}^{N}\big|\cos(\pi\alpha_i)\big|\big\|u_i^{(k)}-u_{0i}\big\|_{\alpha_i}^2+\left|\sum_{i=1}^{N}\sum_{j=1}^{l}\big[I_{ij}\big(u_i^{(k)}(t_j)-u_{0i}(t_j)\big)\big]\right|\big\|u_i^{(k)}-u_{0i}\big\|_\infty\\
&-\left|\sum_{i=1}^{N}\int_0^T\big[F_{u_i^{(k)}}\big(t,u(t)\big)-F_{u_{0i}}\big(t,u_0(t)\big)\big]\right|\big\|u_i^{(k)}-u_{0i}\big\|_\infty.
\end{aligned}
$$

Combining $(I0)$, $(F0)$ and $\|u_i^{(k)}-u_{0i}\|_\infty\to 0$, we know $\sum_{i=1}^N\|u_i^{(k)}-u_{0i}\|_{\alpha_i}\to 0$ as $k\to\infty$ and $u_i^{(k)}\to u_{0i}$ in $E_0^{\alpha_i}$, $i=1,\dots,N$. Thus, $\{u^{(k)}\}$ admits a convergent subsequence, which implies that Φ satisfies the C-condition. □

Corollary 3.2 *Assume* $(I0)$, $(F0)$, $(F3)$ *hold, then* Φ *satisfies the* C*-condition.*

Lemma 3.3 *Assume* $(I0)$, $(I1)$, $(F1)$, $(F3)$ *hold, then* $C_q(\Phi,\infty)=0$ *for every* $q\in N$.

Proof Let $\Omega=\{u_i\in E_0^{\alpha_i}:\|u_i\|_{\alpha_i}=1\}$. For $u_i\in\Omega$, by $(I0)$ we have

$$
\left|\int_0^{\tau u_i(t_j)}I_{ij}(s)\,ds\right|\leq\frac{a_j}{\gamma_j+1}\big|\tau u_i(t_j)\big|^{\gamma_j+1}\leq\frac{A_0^{\gamma_j+1}a_j}{\gamma_j+1}|\tau|^{\gamma_j+1}\|u_i\|_{\alpha_i}^{\gamma_j+1}\leq\frac{A_0^{\gamma_j+1}a_j}{\gamma_j+1}|\tau|^{\gamma_j+1},
$$

where $A_0=\frac{T^{\alpha_i-\frac{1}{2}}}{\Gamma(\alpha_i)\sqrt{2\alpha_i-1}}$. According to Fatou's lemma and $(F3)$, we can get

$$
\lim_{\tau\to\infty}\int_0^T\frac{F(t,\tau u)}{|\tau|^2}\,dt\geq\int_0^T\lim_{\tau\to\infty}\frac{F(t,\tau u)}{|\tau u|^2}|u|^2\,dt=+\infty. \tag{3.5}
$$

Hence $\forall u_i\in\Omega$, by (3.5) and Lemma 2.5, we obtain

$$
\begin{aligned}
\Phi(\tau u)={}&\sum_{i=1}^{N}\int_0^T-\frac{1}{2}{}_0^cD_t^{\alpha_i}(\tau u_i)(t){}_t^cD_T^{\alpha_i}(\tau u_i)(t)\,dt-\sum_{i=1}^{N}\frac{1}{2}a_i(t)\big(\tau^2u_i^2\big)(t)\\
&-\int_0^TF\big(t,\tau u(t)\big)\,dt+\sum_{i=1}^{N}\sum_{j=1}^{l}\int_0^{\tau u(t_j)}I_{ij}(s)\,ds\\
\leq{}&\frac{\tau^2}{2}\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}\|u_i\|_{\alpha_i}^2-\int_0^T\frac{F(t,\tau u)}{|\tau|^2}|\tau|^2\,dt+\sum_{i=1}^{N}\sum_{j=1}^{l}\frac{A_0^{\gamma_j+1}a_j}{\gamma_j+1}|\tau|^{\gamma_j+1}\\
={}&\tau^2\left(\frac{1}{2}\sum_{i=1}^{N}\frac{1}{|\cos(\pi\alpha_i)|}\|u_i\|_{\alpha_i}^2-\int_0^T\frac{F(t,\tau u)}{|\tau|^2}\,dt+\sum_{i=1}^{N}\sum_{j=1}^{l}\frac{A_0^{\gamma_j+1}a_j}{(\gamma_j+1)|\tau|^{1-\gamma_j}}\right)\\
\to{}&-\infty,
\end{aligned}
$$

as $\tau\to\infty$.

Let $a < \min\{\inf \|u_i\|_{\alpha_i \le 1} \Phi(u), 0\}$, for any $u_i \in \Omega$, then there exists $\tau_0 > 1$ such that $\Phi(\tau u) \le a$ for $\tau > \tau_0$. We can derive

$$\begin{aligned}
&\sum_{i=1}^{N} \int_0^T -{}_0^cD_t^{\alpha_i}(\tau u_i){}_t^cD_T^{\alpha_i}(\tau u_i)\,dt \\
&\quad\le 2a + \sum_{i=1}^{N} a_i(t)\tau^2 u_i^2 + 2\int_0^T F(t,\tau u)\,dt - 2\sum_{i=1}^{N}\sum_{j=1}^{l}\int_0^{\tau u_i(t_j)} I_{ij}(s)\,ds.
\end{aligned} \tag{3.6}$$

Combining $(I1)$, $(F1)$ with (3.6), we obtain

$$F_*(t,x) \ge 0, \quad \text{for } \forall (t,x) \in [0,T] \times R, \quad \text{and} \quad I_j^*(x) \ge 0, \quad \text{for } \forall x \in R, j = 1, 2, \ldots, l.$$

It follows (3.6) that

$$\begin{aligned}
&\frac{d}{d\tau}\Phi(\tau u) \\
&\quad = \frac{1}{\tau}\Biggl(\sum_{i=1}^{N}\int_0^T -{}_0^cD_t^{\alpha_i}(\tau u_i){}_t^cD_T^{\alpha_i}(\tau u_i)\,dt - \sum_{i=1}^{N} a_i(t)\bigl(\tau^2 u_i^2(t)\bigr)\,dt \\
&\qquad + \sum_{i=1}^{N}\sum_{j=1}^{l} I_{ij}\bigl(\tau u_i(t_j)\bigr)\tau u_i(t_j) \\
&\qquad - \sum_{i=1}^{N}\int_0^T F_{u_i}(t,\tau u)\tau u_i\,dt\Biggr) \\
&\quad \le \frac{1}{\tau}\Biggl(2a + 2\int_0^T F(t,\tau u)\,dt - \sum_{i=1}^{N}\int_0^T F_{u_i}(t,\tau u)\tau u_i\,dt + \sum_{i=1}^{N}\sum_{j=1}^{l} I_{ij}\bigl(\tau u_i(t_j)\bigr)\tau u_i(t_j) \\
&\qquad - 2\sum_{i=1}^{N}\sum_{j=1}^{l}\int_0^{\tau u_i(t_j)} I_{ij}(s)\,ds\Biggr) \\
&\quad \le \frac{1}{\tau}\Biggl(2a - \int_0^T F_*(t,\tau u)\,dt - \sum_{i=1}^{N}\sum_{j=1}^{l} I_{ij}^*\bigl(\tau u_i(t_j)\bigr)\Biggr) < 0.
\end{aligned}$$

According to the implicit function theorem, there exists a unique $S \in C(\Omega, R)$, such that $\Phi(S(u)u) = a$. Similarly to discussing in [29], there exists a strong deformation retract from $E_0^{\alpha_i} \backslash \{0\}$ to Φ^{a_i}. Thus

$$C_q(\Phi, \infty) = H_q\bigl(E_0^{\alpha_i}, E_0^{\alpha_i} \backslash \{0\}\bigr) = 0, \quad \forall q \in N.$$

So we completed the conclusion. □

Corollary 3.4 *Assume* $(I0)$, $(I1)$, $(F1)$, $(F2)$ *hold, then* $C_q(\Phi, \infty) = 0$ *for every* $q \in N$.

Since $E_0^{\alpha_i}$ $(i = 1,\dots,N)$ is a reflexive and separable Banach space, there exists an orthogonal basis $\{e_{ik}\}$ of $E_0^{\alpha_i}$ such that $E_0^{\alpha_i} = \overline{\operatorname{span}\{e_{ik} : k = 1,2,\dots\}}$. For $m = 1,2,\dots$, denote

$$
\begin{aligned}
&Y_{ik} := \operatorname{span}\{e_{ik}\}, \qquad V_{im} = \bigoplus_{k=1}^{m} Y_{ik}, \qquad Z_{im} = \overline{\bigoplus_{k=m}^{\infty} Y_{ik}},\\
&V = V_{1m} \times \cdots \times V_{Nm}, \qquad W = Z_{1m} \times \cdots \times Z_{Nm}.
\end{aligned}
$$

Then $E_0^{\alpha_i} = V_{im} \oplus Z_{im}$, $i = 1,\dots,N$, $X = V \oplus W$.

Lemma 3.5 *Assume* $(I0)$, $(F0)$ *hold, then there exists* $k_0 \in N$ *such that* $C_{k_0}(\Phi, 0) \neq 0$.

Proof From $(I0)$, $(F0)$ we know, $F_{u_i}(t,0) = 0$, $I_{ij}(0) = 0$ $(i = 1,\dots,N; j = 1,\dots,l)$. Then we found that the functional Φ has a trivial critical point at zero. So it has a local linking at zero in X.

Owing to the fact that all norms of a finite dimensional normed space are equivalent, there exist positive constants M_1, M_2, M_1', M_2', such that

$$
M_1\|u_i\|_{\alpha_i} \le \|u_i\|_\infty \le M_2\|u_i\|_{\alpha_i}, \qquad M_1'\|u_i\|_{\alpha_i} \le \|u_i\|_{L^\gamma} \le M_2'\|u_i\|_{\alpha_i}, \quad u_i \in V_{ik}.
$$

First, we verify there exists $0 < \rho_1 < 1$, such that

$$
\Phi(u) < 0 \quad \text{for } \|u\|_X < \rho_1, \forall u = (u_1,\dots,u_N), u_i \in V_{im}, i = 1,\dots,N.
$$

Because V_{im} is finite dimensional, then, for given r_0, such that

$$
\begin{aligned}
&\bigl|u_i(t)\bigr| \le \frac{r_0}{N}, \quad \text{for } \|u_i\|_{\alpha_i} \le \frac{\rho_1}{N}, \forall 1 \le i \le N,\\
&|u| \le r_0, \quad \text{for } \|u\|_X < \rho_1
\end{aligned}
\tag{3.7}
$$

For any $r \in (0, r_0)$, we set

$$
\begin{aligned}
&\Omega_1 = \bigl\{t \in [0,T] : |u| \le r\bigr\},\\
&\Omega_2 = \bigl\{t \in [0,T] : r \le |u| \le r_0\bigr\},\\
&\Omega_3 = \bigl\{t \in [0,T] : r_0 \le |u|\bigr\}.
\end{aligned}
$$

Then $[0,T] = \bigcup_{i=1}^{3} \Omega_i$ and Ω_i $(i = 1,2,3)$ are pairwise disjoint.
Set $F^*(t,u) = F(t,u) - C|u_i|^\gamma$, it follows from $(I0)$, $(F0)$ and Lemma 2.5 that

$$
\begin{aligned}
\Phi(u) ={}& \sum_{i=1}^{N}\int_0^T \biggl\{-\frac{1}{2}{}_0^cD_t^{\alpha_i}u_i(t){}_t^cD_T^{\alpha_i}u_i(t) - \frac{1}{2}a_i(t)u_i^2(t)\biggr\}\,dt + \sum_{i=1}^{N}\sum_{j=1}^{l}\int_0^{u_i(t_j)} I_{ij}(s)\,ds\\
&- \int_0^T C|u_i|^\gamma\,dt - \int_{\Omega_1} F^*\bigl(t,u(t)\bigr)\,dt - \int_{\Omega_2} F^*\bigl(t,u(t)\bigr)\,dt - \int_{\Omega_3} F^*\bigl(t,u(t)\bigr)\,dt\\
\le{}& \sum_{i=1}^{N}\int_0^T \biggl\{-\frac{1}{2}{}_0^cD_t^{\alpha_i}u_i(t){}_t^cD_T^{\alpha_i}u_i(t) - \frac{1}{2}a_i(t)u_i^2(t)\biggr\}\,d + \sum_{i=1}^{N}\sum_{j=1}^{l}\frac{a_j}{\gamma_j+1}\bigl|u_i(t_j)\bigr|^{\gamma_j+1}
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^T C|u_i|^\gamma \, dt - \int_{\Omega_1} F^*\bigl(t, u(t)\bigr) \, dt - \int_{\Omega_2} F^*\bigl(t, u(t)\bigr) \, dt \\
&- \int_{\Omega_3} F^*\bigl(t, u(t)\bigr) \, dt. \qquad (3.8)
\end{aligned}
$$

According to (3.7) and the definition of Ω_3, we have $\int_{\Omega_3} F^*(t, u(t)) \, dt = 0$, $\forall u_i \in V_{im}$. From $(F0)$, we have $F_* > 0$ on Ω_2, $|u| < r$ on Ω_1. From $(F0)$, one has

$$
F^*(t, u) \leq \sum_{i=1}^N \biggl[\frac{\Gamma^2(\alpha_i + 1)}{2T^{2\alpha_i}} \bigl|\cos(\pi\alpha_i)\bigr| |u|^2 - C|u_i|^\gamma \biggr] \to 0, \quad \text{as } r \to 0.
$$

Hence, we can obtain $\int_{\Omega_1} F^*(t, u(t)) \, dt \to 0$. Then, $\forall u \in X$, $\|u\|_X \leq \rho_1 \leq 1$, $1 < \gamma < \max\{\gamma_j + 1\} < 2$, $r \in (0, r_0)$, from (3.8), we can get

$$
\begin{aligned}
\Phi(u) &\leq \frac{1}{2} \sum_{i=1}^N \frac{1}{|\cos(\pi\alpha_i)|} \|u_i\|_{\alpha_i}^2 + \sum_{i=1}^N \sum_{j=1}^l \frac{a_j A_o^{\gamma_j+1}}{\gamma_j + 1} \|u_i\|_{\alpha_i}^{\gamma_j+1} - M_1^\gamma C \|u_i\|_{\alpha_i}^\gamma \\
&\leq \max_{1 \leq i \leq N} \|u_i\|_{\alpha_i}^\gamma \Biggl(\frac{1}{2} \sum_{i=1}^N \frac{1}{|\cos(\pi\alpha_i)|} \|u_i\|_{\alpha_i}^{2-\gamma} + \sum_{i=1}^N \sum_{j=1}^l \frac{a_j A_o^{\gamma_j+1}}{\gamma_j + 1} \|u_i\|_{\alpha_i}^{\gamma_j+1-\gamma} - M_1^\gamma C \Biggr) \\
&\leq 0.
\end{aligned}
$$

Next, we will prove that there exists $0 < \rho_2 < 1$, $\forall u \in X$, such that $\|u\|_X < \rho_2$, and we have $\Phi(u) \geq 0$.

Because the continuous embedding $X \to C_0^\infty([0, T], R^N)$ is compact. $\forall u \in X$, then, for given $\varepsilon > 0$, there exists $0 < \rho_2 < 1$ such that $|u| < \|u\|_\infty < \varepsilon$, for $\|u\|_X \leq \rho_2$, $t \in [0, T]$.

From $(F0)$, $\forall |u| < \varepsilon$, for $\|u\|_X \leq \rho_2$, $t \in [0, T]$, there exists $\zeta \in (0, 1)$ we know that

$$
F(t, u) \leq \sum_{i=1}^N (1 - \zeta) \frac{\Gamma^2(\alpha_i + 1)}{2T^{2\alpha_i}} \bigl|\cos(\pi\alpha_i)\bigr| |u|^2. \qquad (3.9)
$$

Let $c = \min_{1 \leq j \leq l} b_j$, $c = \max_{1 \leq j \leq l} c_j$. By $(I0)$, we know $e \in [0, 1)$. $\forall u \in X$, $|u| < \varepsilon$ combining $(I0)$, Lemma 2.3 and (3.9), we obtain

$$
\begin{aligned}
\Phi(u) &\geq \frac{1}{2} \sum_{i=1}^N \bigl|\cos(\pi\alpha_i)\bigr| \|u_i\|_{\alpha_i}^2 - \frac{1}{2} \sum_{i=1}^N (1 - \zeta) \bigl|\cos(\pi\alpha_i)\bigr| \|u_i\|_{\alpha_i}^2 \\
&\quad - \frac{1}{2} AT \bigl|u_i(\xi)\bigr|^2 + \frac{1}{2} \frac{bl}{c + 1} |u_i|^{c+1} \\
&\geq \frac{1}{2} \Biggl(\sum_{i=1}^N \bigl|\cos(\pi\alpha_i)\bigr| \zeta \|u_i\|_{\alpha_i}^2 + |u_i|^{c+1} \biggl(\frac{bl}{e + 1} - AT |u_i|^{1-c} \biggr) \Biggr) > 0.
\end{aligned}
$$

Then,

$$
\Phi(u) > 0, \quad \forall u \in X, \qquad \|u\|_X \leq \rho_2 \leq 1, \quad t \in [0, T]. \qquad (3.10)
$$

Let $\rho = \min\{\rho_1, \rho_2\}$, according to (3.8), (3.10), we can get

$$
\Phi(u) \leq 0, \quad \forall u \in V, \|u\|_X \leq \rho; \qquad \Phi(u) > 0, \quad \forall u \in W, 0 < \|u\|_X \leq \rho,
$$

From Lemma 2.9, we have, $C_k(\Phi,0) \ncong 0$. □

Proof of Theorem 1.1 It follows from Lemma 3.1 that Φ satisfies the C-condition. By Corollary 3.4 and Lemma 3.5, we have $C_{k_0}(\Phi,\infty) \ncong C_{k_0}(\Phi,0)$ for some $k_0 \in N$. Then we can conclude Φ has a nontrivial critical point from Lemma 2.8. Hence, problem (1.2) has at least one nontrivial weak solution. □

Proof of Theorem 1.1 It follows from Corollary 3.2 that Φ satisfies the C-condition. By Lemma 3.3 and Lemma 3.5, we have $C_{k_0}(\Phi,\infty) \ncong C_{k_0}(\Phi,0)$ for some $k_0 \in N$. Then we can conclude Φ have a nontrivial critical point from Lemma 2.8. Hence, problem (1.2) has at least one nontrivial weak solution. □

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Author details
[1]School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, P.R. China. [2]College of Information Engineering, Henan University of Science and Technology, Luoyang, P.R. China. [3]Statistics Bureau of Shihe District, Xinyang, P.R. China.

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