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Ground states for planar axially Schrödinger–Newton system with an exponential critical growth

Wenbo Wang[1], Quanqing Li[2] and Yongkun Li[1*]

*Correspondence: yklie@ynu.edu.cn
[1]School of Mathematics and Statistics, Yunnan University, Kunming, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, we study the following planar Schrödinger–Newton system:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u) \quad \text{in } \mathbb{R}^2, \\ \Delta\phi = u^2 \quad \text{in } \mathbb{R}^2, \end{cases}$$

where V, f are axially symmetric about x, V is positive, and f is super-linear at zero and exponential critical at infinity. Using a weaker condition

$$\left[\frac{f(x,u)}{u^3} - \frac{f(x,tu)}{(tu)^3}\right]\operatorname{sign}(1-t) + \theta V(x)\frac{|1-t^2|}{(tu)^2} \geq 0, \quad \forall x \in \mathbb{R}^2, t > 0, u \neq 0$$

with $\theta \in [0,1)$ instead of the Nehari type monotonic condition on $\frac{f(x,u)}{|u|^3}$, we obtain a ground state solution of the above problem via variational methods.

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Keywords: Schrödinger–Newton system; Axial symmetry; The exponential critical growth; Ground states

1 Introduction and main results

In the present paper, we are concerned with the wave solutions of the Schrödinger–Newton system

$$\begin{cases} -i\psi_t - \Delta\psi + W(x)\psi + \lambda\phi u = g(x,\psi) \quad \text{in } \mathbb{R}^d, \\ \Delta\phi = |\psi|^2 \quad \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ is the wave function, $W(x)$ is a real external potential, $\lambda > 0$ is a parameter. Problems of the type (1.1) arise in many problems from physics. We refer the readers to [15], therein (1.1) appears in a quantum mechanical context in the case $d \leq 3$.

A standing wave solution of (1.1) is a solution of the form $\psi(x,t) = e^{-iEt}u(x)$ and its existence reduces (1.1) to the system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u) & \text{in } \mathbb{R}^d, \\ \Delta\phi = u^2 & \text{in } \mathbb{R}^d, \end{cases} \tag{1.2}$$

where $V(x) = W(x) - E$, $g(x, e^{-iEt}u) = f(x,u)e^{-iEt}$. For the case $d = 3$, it is called the Schrödinger–Poisson system and it has been well studied. For the existence, multiplicity, and concentration, we refer the readers to [2, 3, 9, 10, 13, 20] and the references therein. For Kirchhoff type equations involving subcritical and critical growth in three dimensions, please see [19] and the references therein. We also quote the paper [12] for Hardy–Schrödinger–Kirchhoff systems.

However, much less is known about the case $d = 2$. For $\Delta\phi = u^2$, in $\mathbb{R}^2$, one has

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln\bigl(|x-y|\bigr)\bigl|u(y)\bigr|^2\,dy. \tag{1.3}$$

Substituting it into (1.2), we obtain the integro-differential equation

$$-\Delta u + V(x)u + \frac{\lambda}{2\pi}\bigl(\ln\bigl(|\cdot|\bigr) * u^2\bigr)u = f(x,u) \quad \text{in } \mathbb{R}^2. \tag{1.4}$$

For simplicity, throughout this paper, let $\lambda = 2\pi$. The approach for $d = 3$ cannot be easily adapted to $d = 2$ since

$$\frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\bigl(|x-y|\bigr)\bigl|u(y)\bigr|^2\bigl|u(x)\bigr|^2\,dy\,dx, \tag{1.5}$$

which is the functional associated with the third term in (1.4), is sign-changing, and is neither bounded from above nor from below on $H^1(\mathbb{R}^2)$. This difficulty has been overcome recently in [7] or [16]. For

$$-\Delta u + \bigl(\ln\bigl(|\cdot|\bigr) * u^2\bigr)u = \mu u \quad \text{in } \mathbb{R}^2, \tag{1.6}$$

by introducing the following subspace of $H^1(\mathbb{R}^2)$

$$X := \left\{ u \in H^1\bigl(\mathbb{R}^2\bigr) : \int_{\mathbb{R}^2} \ln\bigl(1 + |x|\bigr)u^2\,dx < \infty \right\}$$

endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^2} \bigl(|\nabla u|^2 + u^2 + \ln\bigl(1+|x|\bigr)u^2\bigr)\,dx,$$

Stubbe considered the L^2-constraint minimization problem and proved that (1.6) admits a ground state.

Soon afterwards, in [8], Cingolani and Weth processed successfully the two dimensional Schrödinger–Newton equations with nonlinear term $|u|^{p-2}u$, $p \geq 4$. Du and Weth [11] provided some results about $p > 2$ and $p \geq 3$. The key tool is Pohozaev type identity (see [11, Lemma 2.4]). Chen, Shi, and Tang [4] used the same idea to obtain a ground state but they could deal with the general nonlinearity $f(u)$. Simultaneously, Chen and Tang [5] investigated the existence of an axially symmetric Nehari type ground state and nontrivial solution for

$$-\Delta u + V(x)u + \big(\ln\big(|\cdot|\big) * u^2\big)u = f(x,u) \quad \text{in } \mathbb{R}^2, \tag{1.7}$$

where V, f is axially symmetric about x. Please see [6, 17] for further results about two dimensional Schrödinger–Newton equations with the axially symmetric assumptions. Recently, when $V(x) = 1$, Alves and Figueiredo [1] proved that (1.4) admits a positive ground state, where f is a continuous function with the exponential critical growth.

In this paper, motivated by the papers [1] and [5], we shall study the existence of ground state solutions of planar problem (1.1) with an exponential critical growth. In order to state our main result, we assume that

(V_1) $V \in C(\mathbb{R}^2, \mathbb{R})$, $\inf_{x\in\mathbb{R}^2} V(x) > 0$, $V(x) := V(x_1, x_2) = V(|x_1|, |x_2|)$ for all $x \in \mathbb{R}^2$.

(V_2) There exists a sequence $\{t_n\} \subset (0, \infty)$ such that $t_n \to \infty$ and

$$\sup_{x\in\mathbb{R}^2, n\in\mathbb{N}} \frac{V(t_n^{-1}x)}{V(x)} < \infty.$$

(f_1) $f \in C^1(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}), f(x,u) := f((x_1,x_2),u) = f((|x_1|,|x_2|),u)$.

(f_2) $f(x,u) = o(|u|)$ as $u \to 0$, uniformly in $x \in \mathbb{R}^2$.

(f_3) There exists $\alpha_0 > 0$ such that

$$\lim_{|u|\to\infty} \frac{f(x,u)}{\exp(\alpha u^2)} = 0 \quad \text{for } \alpha > \alpha_0, \qquad \lim_{|u|\to\infty} \frac{f(x,u)}{\exp(\alpha u^2)} = +\infty \quad \text{for } \alpha < \alpha_0.$$

(f_4) There exists $\theta \in [0, 1)$ such that

$$\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right] \operatorname{sign}(1-t) + \theta V(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0, \quad \forall x \in \mathbb{R}^2, t > 0, \tau \neq 0;$$

(f_5) $\inf_{x\in\mathbb{R}^2, u\neq 0} \frac{F(x,u)}{u^2} > -\infty$, where $F(u) = \int_0^u f(t)\,dt$.

Remark 1.1 A simple example of satisfying the hypotheses of (V_1)–(V_2) is the function $V(x) = 1 + |x_2|[1 + \sin(\pi|x_1|)]$ with $t_n = n$. Here we also give an example which satisfies (f_1)–(f_5):

$$f(x,u) = \big(K(x)|u|^3 u - V(x)|u|^{\frac{3}{2}} u + V(x)|u|u\big) \exp\left(\frac{\frac{1}{2} - \frac{\theta}{2}}{m}\pi u^2\right),$$

where $K \in (\mathbb{R}^2, \mathbb{R})$ is axially symmetric and $\inf_{x\in\mathbb{R}^2} K(x) > 0$, V satisfies (V_1) and (V_2). But it does not satisfy the Nehari type monotonic condition

$$\frac{f(x,u)}{|u|^3} \text{ is a strictly increasing function of } u \in \mathbb{R} \setminus \{0\}.$$

Now we state our main result as follows.

Theorem 1 *For $d = 2$, suppose that (V_1), (V_2) and (f_1)–(f_5) are satisfied. Then, for any $\alpha \in (0, \frac{\pi(1-\theta)}{m})$, where m is the least energy (it will be defined in (2.22)), θ is from (f_3), (1.7) possesses a ground state solution.*

Remark 1.2 The condition $\alpha \in (0, \frac{\pi(1-\theta)}{m})$ is used to prove the minimizing sequence of m is bounded, and please see Lemma 3.3. Up to now, we have not been able to remove it.

The paper is organized as follows. Section 2 is to establish the variational setting and to give some preliminaries. Section 3 is to prove the existence of ground states. Throughout the paper, we always assume that (V_1), (V_2) and (f_1)–(f_5) hold and make use of the following notations:

- C, C_i $(i = 0, 1, 2, \dots)$ for positive constants (possibly different from line to line).
- $L^s(\mathbb{R}^2) := \{u : \mathbb{R}^2 \to \mathbb{R} : \int_{\mathbb{R}^2} |u|^s \, dx < \infty\}$ and $\|\cdot\|_s$ denotes the usual L^s-norm in $L^s(\mathbb{R}^2)$.

2 Variational setting and preliminaries

In this section, we begin our study by establishing the variational setting for (1.7). Let $H^1(\mathbb{R}^2)$ be the usual fractional Sobolev space with the usual norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2 \right) dx \right)^{\frac{1}{2}}$$

and

$$H^1_{as}(\mathbb{R}^2) := \left\{ u \in H^1(\mathbb{R}^2) : u(x) := u(x_1, x_2) = u(|x_1|, |x_2|), \forall x \in \mathbb{R}^2 \right\}.$$

By (V_1) and (f_1), similar to [5], let E be defined as

$$E := \left\{ u \in H^1_{as}(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x) u^2 \, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x) u^2 + \ln(1 + |x|) u^2 \right) dx \right)^{\frac{1}{2}}.$$

Denote

$$\|u\| := \left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x) u^2 \right) dx \right)^{\frac{1}{2}}, \qquad \|u\|_* := \left(\int_{\mathbb{R}^2} \ln(1 + |x|) u^2 \, dx \right)^{\frac{1}{2}}.$$

According to [1, Lemma 2.1], we have the following.

Proposition 2.1 *$E \hookrightarrow L^t(\mathbb{R}^2)$ is compact for all $t \in [2, \infty)$.*

We formally formulate problem (1.7) in a variational way as follows:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x) u^2 \right) dx + \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) \left| u(y) \right|^2 \left| u(x) \right|^2 dy \, dx$$
$$- \int_{\mathbb{R}^2} F(x, u) \, dx, \quad u \in E. \tag{2.1}$$

For simplicity of notations, denote

$$I_0(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\bigl(|x-y|\bigr)\bigl|u(y)\bigr|^2\bigl|u(x)\bigr|^2\,dy\,dx.$$

Similar to [8], using $\ln(r) = \ln(1+r) - \ln(1+\frac{1}{r})$, $\forall r > 0$, it holds that

$$\begin{aligned} I_0(u) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\bigl(1+|x-y|\bigr)\bigl|u(y)\bigr|^2\bigl|u(x)\bigr|^2\,dy\,dx \\ &\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\biggl(1+\frac{1}{|x-y|}\biggr)\bigl|u(y)\bigr|^2\bigl|u(x)\bigr|^2\,dy\,dx \\ &:= I_1(u) - I_2(u). \end{aligned}$$

We give the following proposition which is used to estimate the nonlinearity.

Proposition 2.2 ([1, Lemma 2.5]) *For every $\alpha > 0$ and for all $u \in H^1(\mathbb{R}^2)$, we have*

$$\exp\bigl(\alpha u^2\bigr) - 1 \in L^1\bigl(\mathbb{R}^2\bigr). \tag{2.2}$$

Moreover, if $\|\nabla u\|_2 \le 1$, $\|u\|_2 \le M$, and $\alpha < 4\pi$, then there exists $C > 0$ independent of u such that

$$\int_{\mathbb{R}^2} \bigl[\exp\bigl(\alpha u^2\bigr) - 1\bigr]\,dx \le C. \tag{2.3}$$

Lemma 2.3 $I \in C^1(E,\mathbb{R})$.

Proof Noting that $\ln(1+|x-y|) \le \ln(1+|x|) - \ln(1+|y|)$, $\forall x,y \in \mathbb{R}^2$, we get

$$\bigl|I_1(u)\bigr| \le 2\|u\|_2^2\|u\|_*^2. \tag{2.4}$$

In view of $\ln(1+r) \le r$, $\forall r > 0$, jointly with the Hardy–Littlewood–Sobolev inequality [14], we obtain

$$\bigl|I_2(u)\bigr| \le C\|u\|_{\frac{8}{3}}^4. \tag{2.5}$$

So I_0 is well defined in E.

Using (f_1)–(f_3), for each $\varepsilon > 0$, we have

$$\bigl|F(x,u)\bigr| \le \varepsilon|u|^2 + C(\varepsilon)|u|^p\bigl[\exp\bigl(\alpha|u|^2\bigr) - 1\bigr], \tag{2.6}$$

where $p > 2$. Thus, using Hölder's inequality with $s > 1$, $\frac{1}{s} + \frac{1}{s'} = 1$, we get

$$\begin{aligned} \int_{\mathbb{R}^2} F(x,u)\,dx &\le \varepsilon\int_{\mathbb{R}^2}|u|^2\,dx + C(\varepsilon)\int_{\mathbb{R}^2}|u|^p\bigl[\exp\bigl(\alpha|u|^2\bigr) - 1\bigr]\,dx \\ &\le \varepsilon\int_{\mathbb{R}^2}|u|^2\,dx + C(\varepsilon)\biggl(\int_{\mathbb{R}^2}|u|^{ps}\,dx\biggr)^{\frac{1}{s}}\biggl(\int_{\mathbb{R}^2}\bigl[\exp\bigl(s'\alpha|u|^2\bigr) - 1\bigr]\,dx\biggr)^{\frac{1}{s'}}. \end{aligned}$$

By Propositions 2.1 and 2.2, I is well defined in E. By [8, Lemma 2.2], $I_0 \in C^1(E,\mathbb{R})$. It is easy to check that $\int_{\mathbb{R}^2} F(x,u)\,dx$ belongs to $C^1(E,\mathbb{R})$. Thus, $I \in C^1(E,\mathbb{R})$. □

Based on Lemma 2.3, we have

$$\begin{aligned}\langle I'(u),v\rangle &= \int_{\mathbb{R}^2}\big(\nabla u\nabla v + V(x)uv\big)\,dx + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\big(|x-y|\big)\big|u(y)\big|^2 u(x)v(x)\,dy\,dx \\ &\quad - \int_{\mathbb{R}^2} f(x,u)v\,dx. \end{aligned} \tag{2.7}$$

Lemma 2.4 *For every $u \in E$, we have*

$$I(u) \geq I(tu) + \frac{1-t^4}{4}\langle I'(u),u\rangle + \frac{(1-\theta)(1-t^2)^2}{4}\|u\|^2, \quad \forall t \geq 0. \tag{2.8}$$

Proof Since the proof is similar to [5, Lemma 2.3], we omit it here. □

Now, we define the Nehari manifold

$$\mathcal{N} := \big\{u \in E\setminus\{0\} : \langle I'(u),u\rangle = 0\big\}.$$

Since the Nehari type monotonic condition on $\frac{f(x,u)}{|u|^3}$ and super-cubic condition are not satisfied, we need to prove that $\mathcal{N} \neq \emptyset$. To the end, we introduce the following new set:

$$\mathcal{E} := \left\{u \in E\setminus\{0\} : \int_{\mathbb{R}^2} V(x)u^2\,dx + I_0(u) < \int_{\mathbb{R}^2} f(x,u)u\,dx\right\}.$$

Lemma 2.5 $\mathcal{E} \neq \emptyset$.

Proof Let $u \in E$ with $u \neq 0$. $u_t(x) := u(tx)$. By (V_2), there exists $C_1 > 0$ such that

$$V\big(t_n^{-1}x\big) \leq C_1 V(x), \quad \forall x \in \mathbb{R}^2, n \in \mathbb{N}. \tag{2.9}$$

It follows that

$$\begin{aligned}&\int_{\mathbb{R}^2} V(x)(t_n u_{t_n})^2\,dx + I_0(t_n u_{t_n}) - \int_{\mathbb{R}^2} f(x,t_n u_{t_n})t_n u_{t_n}\,dx \\ &\quad \leq C_1\|u\|^2 + I_0(u) - \ln t_n\|u\|_2^4 - \int_{\mathbb{R}^2}\frac{f(t_n^{-1}x,t_n u)t_n u}{t_n^2}\,dx.\end{aligned}$$

In view of (f_4), $t \geq 0$, $\tau \neq 0$, it holds that

$$\begin{aligned}&\frac{1-t^4}{4}\tau f(x,\tau) + F(x,t\tau) - F(x,\tau) + \frac{\theta V(x)}{4}\big(1-t^2\big)^2\tau^2 \\ &\quad = \int_t^1\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,s\tau)}{(s\tau)^3}\right]\operatorname{sign}(1-t) + \theta V(x)\frac{|1-t^2|}{(s\tau)^2}\tau^3\tau^4\,ds \geq 0. \end{aligned}\tag{2.10}$$

Taking $t = 0$, we obtain

$$\frac{1}{4}\tau f(x,\tau) - F(x,\tau) + \frac{\theta V(x)}{4}\tau^2 \geq 0, \quad \forall x \in \mathbb{R}^2, \tau \in \mathbb{R}. \tag{2.11}$$

By (f_5), one has

$$F(x,\tau) \geq -C_2\tau^2, \quad \forall x \in \mathbb{R}^2, \tau \in \mathbb{R}. \tag{2.12}$$

Thus, we get

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{f(t_n^{-1}x, t_n u)t_n u}{t_n^2}\,dx &\geq \int_{\mathbb{R}^2} \left[\frac{4F(t_n^{-1}x, t_n u)}{t_n^2} - \theta V\left(t_n^{-1}x\right)u^2\right] dx \\ &\geq -4C_2 \int_{\mathbb{R}^2} u^2\,dx - \theta C_1 \int_{\mathbb{R}^2} V(x)u^2\,dx. \end{aligned} \tag{2.13}$$

So

$$\begin{aligned} &\int_{\mathbb{R}^2} V(x)(t_n u_{t_n})^2\,dx + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(|x-y|\right)\left|t_n u_{t_n}(y)\right|^2\left|t_n u_{t_n}(x)\right|^2 dx\,dy \\ &\quad - \int_{\mathbb{R}^2} f(x, t_n u_{t_n})t_n u_{t_n}\,dx \to -\infty, \end{aligned}$$

which implies that $\mathcal{E} \neq \emptyset$. □

The following lemma shows that $\mathcal{N} \neq \emptyset$.

Lemma 2.6 *For any $u \in \mathcal{E}$, there exists unique $t > 0$ such that $tu \in \mathcal{N}$.*

Proof Given $u \in \mathcal{E}$, let $\gamma_u(t) := \langle I'(tu), tu\rangle$ for $t > 0$. Then $tu \in \mathcal{N}$ if and only if $\gamma_u(t) = 0$. Taking $\varepsilon > 0$ sufficiently small, jointly with Sobolev embedding, we obtain

$$\begin{aligned} \gamma_u(t) &\geq t^2\|u\|^2 - t^4 I_2(u) - \int_{\mathbb{R}^2} f(x,tu)tu\,dx \\ &\geq t^2\|u\|^2 - t^4 C_1\|u\|^{\frac{3}{2}} - t^2\varepsilon C_2\|u\|^2 - t^p C(\varepsilon)\int_{\mathbb{R}^2} |u|^p\left[\exp\left(\alpha|tu|^2\right) - 1\right] dx \\ &\geq t^2(1-\varepsilon C_2)\|u\|^2 - t^4 C_1\|u\|^{\frac{3}{2}} \\ &\quad - t^p C(\varepsilon)\left(\int_{\mathbb{R}^2} |u|^{sp}\,dx\right)^{\frac{1}{s}}\left(\int_{\mathbb{R}^2}\left[\exp\left(\alpha s'\|tu\|^2\left(\frac{u}{\|u\|}\right)^2\right) - 1\right] dx\right)^{\frac{1}{s'}}. \end{aligned}$$

Choosing $t > 0$ small such that $\alpha s'\|tu\|^2 < 4\pi$, it follows from Proposition 2.2 that there exists $\bar{t} > 0$ small enough such that

$$\gamma_u(t) > 0 \quad \text{for all } 0 < t < \bar{t}. \tag{2.14}$$

Now, by (f_4), one has

$$f(x,t\tau)t\tau \geq f(x,\tau)\tau t^4 - \theta V(x)\left(t^2-1\right)(t\tau)^2, \quad \forall x \in \mathbb{R}^2, t \geq 1, \tau \in \mathbb{R}, \tag{2.15}$$

which implies that

$$\int_{\mathbb{R}^2}\left[\theta V(x)(tu)^2 - f(x,tu)tu\right] dx \leq t^4 \int_{\mathbb{R}^2}\left[\theta V(x)u^2 - f(x,u)u\right] dx, \quad \forall t \geq 1. \tag{2.16}$$

Therefore,

$$\begin{aligned}\gamma_u(t) &= t^2\|u\|^2 + t^4 I_0(u) - \int_{\mathbb{R}^2} f(x,tu)tu\,dx \\ &\le t^2\|u\|^2 + t^4\left[\int_{\mathbb{R}^2}\left[V(x)u^2 - f(x,u)u\right]dx + I_0(u)\right] \\ &\quad - \theta t^2 \int_{\mathbb{R}^2} V(x)u^2\,dx, \quad \forall t \ge 1. \end{aligned} \tag{2.17}$$

Thus, we have $\gamma_u(t) \to -\infty$, as $t \to \infty$. So there exists $t_0 > 0$ such that $\gamma_u(t_0) = 0$. Next, we shall prove that t_0 is unique. Suppose to the contrary that there are $t_1, t_2 > 0$ with $t_1 \neq t_2$ such that $\gamma_u(t_1) = \gamma_u(t_2) = 0$. For $t_1 u \in E$, using Lemma 2.4, for all $t > 0$, we have

$$I(t_1 u) \ge I(tt_1 u) + \frac{(1-\theta)(1-t^2)^2 t_1^2\|u\|^2}{4}. \tag{2.18}$$

Taking $t = \frac{t_2}{t_1}$, it yields that

$$I(t_1 u) \ge I(t_2 u) + \frac{(1-\theta)(1-(\frac{t_2}{t_1})^2)^2 t_1^2\|u\|^2}{4}. \tag{2.19}$$

Similarly, one has

$$I(t_2 u) \ge I(t_1 u) + \frac{(1-\theta)(1-(\frac{t_1}{t_2})^2)^2 t_2^2\|u\|^2}{4}. \tag{2.20}$$

We obtain $t_1 = t_2$, so it is absurd. □

Since $u \in \mathcal{N}$, by Lemma 2.4, one has

$$I(u) = I(u) - \frac{1}{4}\langle I'(u), u\rangle \ge \frac{1-\theta}{4}\|u\|^2. \tag{2.21}$$

So we can define

$$m := \inf_{u\in\mathcal{N}} I(u). \tag{2.22}$$

Up to this stage, preparations have been made. We point out that we can define m without using the condition $\alpha \in (0, \frac{\pi(1-\theta)}{m})$. In the next section, taking full advantage of the condition $\alpha \in (0, \frac{\pi(1-\theta)}{m})$, we shall prove the existence of ground state solutions of (1.7).

3 Existence of ground states

In this section, with the additional condition $\alpha \in (0, \frac{\pi(1-\theta)}{m})$, we are devoted to showing that m is achieved and the minimizer is a ground state solution of equation (1.7).

Lemma 3.1 *There exists $C > 0$ such that $\|u\| \ge C$ for all $u \in \mathcal{N}$; furthermore, $m > 0$.*

Proof Assume by contradiction that there is $\{u_n\} \subset \mathcal{N}$ such that $\|u_n\| \to 0$. Obviously,

$$\|u_n\|^2 + 4\langle I_1'(u_n), u_n\rangle = 4\langle I_2'(u_n), u_n\rangle + \int_{\mathbb{R}^2} f(x,u_n)u_n\,dx.$$

In view of (f_1) – (f_3), combining Hölder's inequality, it follows that

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} f(x,u_n)u_n\,dx\right|\\&\quad\le \varepsilon\int_{\mathbb{R}^2}|u_n|^2\,dx + C(\varepsilon)\int_{\mathbb{R}^2}|u_n|^p\bigl[\exp\bigl(\alpha|u_n|^2\bigr)-1\bigr]\,dx\\&\quad\le \varepsilon\int_{\mathbb{R}^2}|u_n|^2\,dx\\&\qquad + C(\varepsilon)\biggl(\int_{\mathbb{R}^2}|u_n|^{sp}\,dx\biggr)^{\frac{1}{s}}\biggl(\int_{\mathbb{R}^2}\biggl[\exp\biggl(\alpha s'\|u_n\|^2\biggl(\frac{u_n}{\|u_n\|}\biggr)^2\biggr)-1\biggr]\,dx\biggr)^{\frac{1}{s'}}.\end{aligned}$$

With Proposition 2.2 in hand, using the Sobolev embedding, it leads to

$$\int_{\mathbb{R}^2} f(x,u_n)u_n\,dx = o_n(1).$$

By direct calculation, it holds that

$$4\bigl\langle I_2'(u_n),u_n\bigr\rangle \le C\|u_n\|_{\frac{8}{3}}^4 = o_n(1).$$

Thus, one has

$$\bigl\langle I_1'(u_n),u_n\bigr\rangle = o_n(1).$$

Therefore, we obtain

$$\begin{aligned}\|u_n\|^2 &\le 4\bigl\langle I_1'(u_n),u_n\bigr\rangle + \int_{\mathbb{R}^2} f(x,u_n)u_n\,dx\\&\le o_n(1) + \varepsilon\int_{\mathbb{R}^2}|u_n|^2\,dx + C(\varepsilon)\int_{\mathbb{R}^2}|u_n|^p\bigl[\exp\bigl(\alpha|u_n|^2\bigr)-1\bigr]\,dx.\end{aligned}\tag{3.1}$$

That is,

$$\begin{aligned}&(1-\varepsilon C)\|u_n\|^2\\&\quad\le o_n(1) + C(\varepsilon)\biggl(\int_{\mathbb{R}^2}|u_n|^{sp}\,dx\biggr)^{\frac{1}{s}}\biggl(\int_{\mathbb{R}^2}\biggl[\exp\biggl(\alpha s'\|u_n\|^2\biggl(\frac{u_n}{\|u_n\|}\biggr)^2\biggr)-1\biggr]\,dx\biggr)^{\frac{1}{s'}}.\end{aligned}\tag{3.2}$$

Noting that $\|u_n\|\to 0$, using Proposition 2.2 again, we get

$$(1-\varepsilon C)\|u_n\|^2 \le C(\varepsilon)\|u_n\|^p,\tag{3.3}$$

which is ridiculous. Combining with (2.21), we have $m>0$. □

Next, we give the following lemma which shall be used later.

Lemma 3.2 *For every $u\in E$, it holds that $I_1(u)\ge \frac{1}{16}\|u\|_2^2\|u\|_*^2$.*

Proof The proof is similar to [5, Lemma 2.2]. Let

$$\Lambda_1 := \{(x_1,x_2) \in \mathbb{R}^2, x_1 > 0, x_2 \geq 0\}, \qquad \Lambda_3 := \{(x_1,x_2) \in \mathbb{R}^2, x_1 < 0, x_2 \leq 0\}.$$

For any $(x,y) \in \Lambda_1 \times \Lambda_3$, it holds that

$$|x-y| = \sqrt{|x|^2 + |y|^2 - 2x \cdot y} \geq \sqrt{|x|^2 + |y|^2} \geq |x|.$$

Thus,

$$\begin{aligned} I_1(u) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\bigl(1+|x-y|\bigr)\bigl|u(y)\bigr|^2 \bigl|u(x)\bigr|^2 \,dy\,dx \\ &\geq \int_{\Lambda_3} \int_{\Lambda_1} \ln\bigl(1+|x-y|\bigr)\bigl|u(y)\bigr|^2 \bigl|u(x)\bigr|^2 \,dy\,dx \\ &\geq \int_{\Lambda_3} \bigl|u(y)\bigr|^2 \,dy \int_{\Lambda_1} \ln\bigl(1+|x|\bigr)\bigl|u(x)\bigr|^2 \,dx \\ &= \frac{1}{16}\|u\|_2^2 \|u\|_*^2. \end{aligned}$$

□

Let $\{u_n\} \subset \mathcal{N}$ be a minimizing sequence of m. On the additional condition $\alpha \in (0, \frac{\pi(1-\theta)}{m})$, we want to prove that $\{u_n\}$ is bounded in E.

Lemma 3.3 *If* $\alpha \in (0, \frac{\pi(1-\theta)}{m})$, *we have* $\{u_n\}$ *is bounded in* E.

Proof Similar to (2.21), $\{\|u_n\|\}$ is bounded. Similar to (2.5), $\{I_2(u_n)\}$ is bounded. Next, we want to estimate the $\{I_1(u_n)\}$. Note that

$$\left|\int_{\mathbb{R}^2} f(x,u_n)u_n \,dx\right| \leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 \,dx + C(\varepsilon) \int_{\mathbb{R}^2} |u_n|^p \bigl[\exp\bigl(\alpha |u_n|^2\bigr) - 1\bigr] \,dx. \tag{3.4}$$

For the second term on the right, using Hölder's inequality with $s' > 1$ and $s' \approx 1$, it holds that

$$\begin{aligned} &\int_{\mathbb{R}^2} |u_n|^p \bigl[\exp\bigl(\alpha |u_n|^2\bigr) - 1\bigr] \,dx \\ &\quad \leq \left(\int_{\mathbb{R}^2} |u_n|^{sp} \,dx\right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^2} \left[\exp\left(\alpha s' \|u_n\|^2 \left(\frac{u_n}{\|u_n\|}\right)^2\right) - 1\right] dx\right)^{\frac{1}{s'}}. \end{aligned}$$

Taking into account $\alpha \in (0, \frac{\pi(1-\theta)}{m})$, jointly with

$$\frac{1-\theta}{4}\|u_n\|^2 \leq I(u_n) \to m, \tag{3.5}$$

for n large enough, we obtain $\alpha s' \|u_n\|^2 < 4\pi$. So, by Proposition 2.2, we get

$$\left|\int_{\mathbb{R}^2} f(x,u_n)u_n \,dx\right| \leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 \,dx + C(\varepsilon) C \left(\int_{\mathbb{R}^2} |u_n|^{sp} \,dx\right)^{\frac{1}{s}}. \tag{3.6}$$

Since

$$\|u_n\|^2 + I_1(u_n) = I_2(u_n) + \int_{\mathbb{R}^2} f(x,u_n)u_n\,dx, \tag{3.7}$$

which yields that $\{I_1(u_n)\}$ is bounded. And it follows from Lemma 3.2 that $\{u_n\}$ is bounded in E. □

Next, we claim that there are $R, \eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2\,dx \geq \eta. \tag{3.8}$$

If it is false, using Lions' lemma (see [18, Lemma 1.21]), we get $u_n \to 0$ in $L^t(\mathbb{R}^2)$ for all $t \in [2,\infty)$. Noting that

$$\left|I_1(u_n)\right| \leq 2\|u_u\|_2^2\|u_n\|_*^2 = o_n(1), \qquad \left|I_2(u_n)\right| \leq C\|u_n\|_{\frac{8}{3}}^4 = o_n(1), \tag{3.9}$$

similar to (3.5), it holds that

$$\begin{aligned}\|u_n\|^2 &= o_n(1) + \int_{\mathbb{R}^2} f(x,u_n)u_n\,dx \\ &\leq o_n(1) + \varepsilon \int_{\mathbb{R}^2} |u_n|^2\,dx + C(\varepsilon)C\left(\int_{\mathbb{R}^2} |u_n|^{sp}\,dx\right)^{\frac{1}{s}} \\ &= o_n(1),\end{aligned} \tag{3.10}$$

which contradicts Lemma 3.1.

Lemma 3.4 *m is achieved and the minimizer is a weak solution of* (1.7).

Proof Now, we can assume that $u_n \rightharpoonup u_0 \neq 0$ in E, $u_n \to u_0$ in $L^t(\mathbb{R}^2)$ for all $t \in [2,\infty)$ and $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^2$. By a standard argument, one can deduce that $I'(u_0) = 0$. Obviously, we have

$$\int_{\mathbb{R}^2} F(x,u_n)\,dx = \int_{\mathbb{R}^2} F(x,u_0)\,dx + o_n(1), \tag{3.11}$$

$$\int_{\mathbb{R}^2} f(x,u_n)u_n\,dx = \int_{\mathbb{R}^2} f(x,u_0)u_0\,dx + o_n(1). \tag{3.12}$$

Here, we only check 3.12 since (3.11) is similar. We have already known that

$$\left|f(x,u_n)u_n\right| \leq \varepsilon|u_n|^2 + C(\varepsilon)|u_n|^p\left[\exp\left(\alpha\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2\right) - 1\right]. \tag{3.13}$$

Noting that $\alpha \in (0, \frac{\pi(1-\theta)}{m})$ and (3.5), we obtain that $\alpha\|u_n\|^2 < 4\pi$ for n large enough. By Proposition 2.2, there exists $C > 0$ independent of n such that

$$\int_{\mathbb{R}^2}\left[\exp\left(\alpha\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2\right) - 1\right]dx \leq C.$$

It follows from [18, Lemma A.1] and Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f(x,u_n)u_n\,dx = \int_{\mathbb{R}^2} f(x,u_0)u_0\,dx. \tag{3.14}$$

Thus, we have

$$\begin{aligned} m &= \lim_{n\to\infty}\left[I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle\right] \\ &\geq \frac{1}{4}\|u_0\|^2 + \int_{\mathbb{R}^2}\left[\frac{1}{4}f(x,u_0) - F(x,u_0)\right]dx \\ &= I(u_0) - \frac{1}{4}\langle I'(u_0), u_0\rangle \\ &\geq m. \end{aligned}$$

□

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Author details
[1]School of Mathematics and Statistics, Yunnan University, Kunming, P.R. China. [2]Department of Mathematics, Honghe University, Mengzi, P.R. China.

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