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Null controllability of a semilinear degenerate parabolic equation with a gradient term

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Abstract

This paper concerns the null controllability of a semilinear control system governed by degenerate parabolic equation with a gradient term, where the nonlinearity of the problem is involved with the first derivative. We first establish the well-posedness and prove the approximate null controllability of the linearized system, then we can get the approximate null controllability of the semilinear control system by a fixed point argument. Finally, the semilinear control system with a gradient term is shown to be null controllable.

Keywords: Null controllability; The semilinear problem; Degenerate parabolic equation

1 Introduction

In this paper, we investigate the null controllability of the following semilinear degenerate system:

$$u_t - \left(x^\alpha u_x\right)_x + g(x,t,u,u_x) = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0 \quad \text{if } 0 < \alpha < 1, t \in (0,T), \tag{1.2}$$

$$x^\alpha u_x(0,t) = u(1,t) = 0 \quad \text{if } 1 \le \alpha < 2, t \in (0,T), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{1.4}$$

where $Q_T = (0,1) \times (0,T)$, ω is a nonempty open subset of $(0,1)$, $u_0 \in L^2(0,1)$, $h(x,t) \in L^2(Q_T)$ is a control function, $g(x,t,s,p)$ is Lebesgue measurable in $Q_T \times \mathbb{R} \times \mathbb{R}$ and C^1 continuous with respect to s, p uniformly for $(x,t) \in Q_T$. Furthermore, we assume that g satisfies $g(\cdot,\cdot,0,0) = 0$ and

$$\left|g_s(x,t,s,p)\right| + x^{-\alpha/2}\left|g_p(x,t,s,p)\right| \le K, \quad \forall(x,t,s,p) \in Q_T \times \mathbb{R} \times \mathbb{R}, \tag{1.5}$$

where $K > 0$ is a constant. Equation (1.1) is degenerate at the boundary $x = 0$, and it can be used to describe some physical models, for example, in [6, 8], we can find a motivating

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example of a Crocco-type equation coming from the study on the velocity field of a laminar flow on a flat plate.

In the last forty years, many authors have been devoted to studying control systems, the interested readers can refer to [1–26] and the references therein. For instance, Wang in [27–29] studied the approximate controllability of a class of systems governed by degenerate parabolic equations. In 2013, Du and Wang in [11] investigated the null controllability of a class of coupled degenerate systems. Later, Du and Xu in [13] studied the boundary controllability of a semilinear degenerate system with convection term. Recently, Xu, Wang and Nie in [30] considered the Carleman estimate and null controllability of a cascade control system with convection terms. For degenerate equations, one must overcome some technical difficulties to get some necessary estimates for controllability theory. In particular, the following system governed by a single degenerate parabolic equation has been widely studied:

$$w_t - \left(x^\alpha w_x\right)_x + k(x,t)w = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{1.6}$$

$$w(0,t) = 0 \quad \text{if } 0 < \alpha < 1, \qquad \left(x^\alpha w_x\right)(0,t) = 0 \quad \text{if } \alpha \geq 1, t \in (0,T), \tag{1.7}$$

$$w(1,t) = 0, \quad t \in (0,T), \tag{1.8}$$

$$w(x,0) = w_0(x), \quad x \in (0,1), \tag{1.9}$$

where $k \in L^\infty(Q_T)$. The system is null controllable if $0 < \alpha < 2$ [8, 9, 26], while not if $\alpha \geq 2$ [7]. It is noted that the degeneracy of (1.6) is weak if $0 < \alpha < 1$ and strong if $\alpha \geq 1$. The null controllability of system (1.6)–(1.9) for $0 < \alpha < 2$ is based on the Carleman estimate for solutions to its conjugate problem

$$-W_t - \left(x^\alpha W_x\right)_x + k(x,t)W = F(x,t), \quad (x,t) \in Q_T, \tag{1.10}$$

$$W(0,t) = 0 \quad \text{if } 0 < \alpha < 1, \qquad \left(x^\alpha W_x\right)(0,t) = 0 \quad \text{if } 1 \leq \alpha < 2, t \in (0,T), \tag{1.11}$$

$$W(1,t) = 0, \quad t \in (0,T), \tag{1.12}$$

$$W(x,T) = W_T(x), \quad x \in (0,1). \tag{1.13}$$

Since the problem may be not null controllable, the authors introduced some new concepts on controllability, the regional null controllability and the persistent regional null controllability, which is weaker than the null controllability [7]. They proved that the problem is regional null controllable and persistent regional null controllable for all $\alpha > 0$. For semilinear problem (1.1)–(1.4), the authors also showed the regional and persistent regional null controllability in [3, 5]. Moreover, the approximate controllability of degenerate equation (1.1) with suitable boundary and initial conditions has been proved in [12, 27–29] for all $\alpha > 0$. In [1, 19], the authors proved the null controllability of problem (1.1)–(1.4) with

$$g(x,t,u,u_x) = f(x,t,u)$$

and

$$g(x,t,u,u_x) = x^{\alpha/2}b(x,t)u_x + c(x,t)u, \tag{1.14}$$

respectively.

In this paper, we investigate the null controllability of semilinear problem (1.1)–(1.4). First, we prove the approximate null controllability of linear problem (1.1)–(1.4) with (1.14). Next, we prove the approximate null controllability of semilinear problem (1.1)–(1.4) by using the Schauder fixed point theorem. At last, we state the null controllability of semilinear problem (1.1)–(1.4) with the method inspired by [3]. The paper is organized as follows: In Sect. 2, we introduce function spaces that are needed for the well-posedness and prove the well-posedness of system (1.1)–(1.4). In Sect. 3, we prove that the semilinear system is null controllable.

2 Well-posedness

In this section, we first consider the linear problem

$$u_t - \left(x^\alpha u_x\right)_x + x^{\alpha/2} b(x,t) u_x + c(x,t) u = f(x,t), \quad (x,t) \in Q_T, \tag{2.1}$$

$$u(0,t) = u(1,t) = 0 \quad \text{if } 0 < \alpha < 1, t \in (0,T), \tag{2.2}$$

$$x^\alpha u_x(0,t) = u(1,t) = 0 \quad \text{if } 1 \le \alpha < 2, t \in (0,T), \tag{2.3}$$

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{2.4}$$

where $b, c \in L^\infty(Q_T), f \in L^2(Q_T)$, and $u_0 \in L^2(0,1)$.

Define that $H_\alpha^1(0,1)$ and $H_\alpha^2(0,1)$ are the closure of $C_0^\infty(0,1)$ with respect to the following norm:

$$\|u\|_{H_\alpha^1(0,1)} = \left(\int_0^1 \left(u^2 + x^\alpha u_x^2\right) dx\right)^{1/2}, \quad u \in H_\alpha^1(0,1)$$

and

$$\|u\|_{H_\alpha^2(0,1)} = \left(\int_0^1 \left(u^2 + x^\alpha u_x^2 + \left(x^\alpha u_x\right)_x^2\right) dx\right)^{1/2}, \quad u \in H_\alpha^2(0,1),$$

respectively.

For readers' convenience, we denote

$$\mathcal{M} = C\left(0,T;L^2(0,1)\right) \cap L^2\left(0,T;H_\alpha^1(0,1)\right)$$

and

$$\mathcal{N} = H^1\left(0,T;L^2(0,1)\right) \cap L^2\left(0,T;H_\alpha^2(0,1)\right).$$

Lemma 2.1 $\mathcal{N}$ *is compactly imbedded in* $\mathcal{M}$.

Proof Using Aubin's theorem ([3], Theorem 4.3) with $r_0 = r_1 = 2$, $X_0 = H_\alpha^2(0,1)$, $X_1 = H_\alpha^1(0,1)$, $X_2 = L^2(0,1)$, and $a = 0$, $b = T$, one can get that $\mathcal{N}$ is compactly imbedded in $L^2(0,T;H_\alpha^1(0,1))$.

Since $H^1(0,T;L^2(0,1))$ is compactly imbedded in $C(0,T;L^2(0,1))$ and $\mathcal{N}$ is continuously imbedded in $H^1(0,T;L^2(0,1))$, one has that $\mathcal{N}$ is compactly imbedded in $C(0,T;L^2(0,1))$.

Moreover, since $\mathcal{N}$ is compactly imbedded in $L^2(0, T; H_\alpha^1(0, 1))$ and $C(0, T; L^2(0, 1))$, respectively, it is obvious that $\mathcal{N}$ is compactly imbedded in $\mathcal{M} = C(0, T; L^2(0, 1)) \cap L^2(0, T; H_\alpha^1(0, 1))$, the proof is complete. □

Due to the degeneracy of the coefficient x^α, problem (2.1)–(2.4) may not have classical solutions, so we need to give the definition of weak solutions.

Definition 2.1 If $u \in \mathcal{M}$, for any function $\varphi \in \mathcal{M}$ with $\varphi_t \in L^2(Q_T)$ and $\varphi(\cdot, T)|_{(0,1)} = 0$, it holds that

$$\iint_{Q_T} \left(-u\varphi_t + x^\alpha u_x \varphi_x + x^{\alpha/2} b u_x \varphi + c u \varphi \right) dx\, dt = \iint_{Q_T} f \varphi \, dx\, dt + \int_0^1 u_0(x) \varphi(x, 0)\, dx,$$

then the function u is called the weak solution of the system (2.1)–(2.4).

On the basis of Theorem 2.4 [3] and Lemma 2.1 [12], the problem (2.1)–(2.4) is well posed. Furthermore, we can get the following proposition.

Proposition 2.1 *If* $\|b\|_{L^\infty(Q_T)} \leq C_1$, $\|c\|_{L^\infty(Q_T)} \leq C_2$, $f \in L^2(Q_T)$, *and* $u_0 \in L^2(0, 1)$, *then the problem* (2.1)–(2.4) *uniquely admits a weak solution* $u \in \mathcal{M}$. *Furthermore*, u *satisfies that*
(i)

$$\sup_{t \in (0,T]} \int_0^1 u^2(x, t)\, dx + \iint_{Q_T} x^\alpha u_x^2 \, dx\, dt \leq C \left(\iint_{Q_T} f^2(x, t)\, dx\, dt + \int_0^1 u_0^2(x)\, dx \right),$$

where $C > 0$ *is a constant depending on* T, C_1, *and* C_2.
(ii) *If* $u_0 \in H_\alpha^1(0, 1)$, *then* $u \in \mathcal{N}$ *and it holds that*

$$\iint_{Q_T} \left(u_t^2 + \left(x^\alpha u_x \right)_x^2 \right) dx\, dt$$

$$\leq C \left(\iint_{Q_T} f^2(x, t)\, dx\, dt + \int_0^1 \left(u_0^2(x) + x^\alpha \left(u_0' \right)^2(x) \right) dx \right),$$

where $C > 0$ *is a constant depending on* T, C_1, *and* C_2.

Similar to the linear problem (2.1)–(2.4), one can give the definition of weak solution to the following semilinear problem:

$$u_t - \left(x^\alpha u_x \right)_x + g(x, t, u, u_x) = f(x, t), \quad (x, t) \in Q_T, \tag{2.5}$$

$$u(0, t) = u(1, t) = 0 \quad \text{if } 0 < \alpha < 1, t \in (0, T), \tag{2.6}$$

$$x^\alpha u_x(0, t) = u(1, t) = 0 \quad \text{if } 1 \leq \alpha < 2, t \in (0, T), \tag{2.7}$$

$$u(x, 0) = u_0(x), \quad x \in (0, 1). \tag{2.8}$$

Definition 2.2 A function u is called the weak solution of the problem (2.5)–(2.8) if $u \in \mathcal{M}$, and for any function $\varphi \in \mathcal{M}$ with $\varphi_t \in L^2(Q_T)$ and $\varphi(\cdot, T)|_{(0,1)} = 0$, the following integral equality holds:

$$\iint_{Q_T} \left(-u\varphi_t + x^\alpha u_x \varphi_x + g(x, t, u, u_x)\varphi \right) dx\, dt = \iint_{Q_T} f \varphi \, dx\, dt + \int_0^1 u_0(x) \varphi(x, 0)\, dx.$$

The semilinear problem (2.5)–(2.8) is well posed, which is proved in Theorem 3.1 [12] and Theorem 3.7 [3].

For any $w \in L^2(0, T; H_\alpha^1(0, 1))$, define the functions

$$c(x, t, w) = \int_0^1 g_s(x, t, \lambda w, \lambda w_x) \, d\lambda,$$

$$b(x, t, w) = x^{-\alpha/2} \int_0^1 g_p(x, t, \lambda w, \lambda w_x) \, d\lambda.$$

Then (1.5) yields that

$$\left\| c(x, t, w) \right\|_{L^\infty(Q_T \times \mathbb{R})} \leq K, \left\| b(x, t, w) \right\|_{L^\infty(Q_T \times \mathbb{R})} \leq K. \tag{2.9}$$

Moreover, we can obtain that

$$
\begin{aligned}
g(x, t, u, u_x) - g(x, t, 0, 0) &= \int_0^1 \frac{\partial g(x, t, \lambda u, \lambda u_x)}{\partial \lambda} \, d\lambda \\
&= \int_0^1 \frac{\partial g(x, t, \lambda u, \lambda u_x)}{\partial s} u \, d\lambda + \int_0^1 \frac{\partial g(x, t, \lambda u, \lambda u_x)}{\partial p} u_x \, d\lambda \\
&= c(x, t, u)u + x^{\alpha/2} b(x, t, u) u_x.
\end{aligned}
$$

Furthermore, $c(x, t, w)$ and $b(x, t, w)$ satisfy the following property.

Lemma 2.2 *Assume that $\{w_k\}_{k=1}^\infty$ converges to w in $L^2(0, T; H_\alpha^1(0, 1))$, then*

$$c(x, t, w_k(x, t)) \rightharpoonup c(x, t, w(x, t)) \text{ weakly} * \text{ in } L^\infty(Q_T), \quad as \ k \to \infty, \tag{2.10}$$

$$b(x, t, w_k(x, t)) \rightharpoonup b(x, t, w(x, t)) \text{ weakly} * \text{ in } L^\infty(Q_T), \quad as \ k \to \infty \tag{2.11}$$

and

$$(c(x, t, w_k(x, t) - c(x, t, w(x, t)))^2 \rightharpoonup 0 \text{ weakly} * \text{ in } L^\infty(Q_T), \quad as \ k \to \infty, \tag{2.12}$$

$$(b(x, t, w_k(x, t) - b(x, t, w(x, t)))^2 \rightharpoonup 0 \text{ weakly} * \text{ in } L^\infty(Q_T), \quad as \ k \to \infty. \tag{2.13}$$

Proof For convenience, we denote

$$c[w](x, t) = c(x, t, w(x, t)), \qquad b[w](x, t) = b(x, t, w(x, t)), \quad w \in L^2(0, T; H_\alpha^1(0, 1)).$$

First, we will prove

$$\lim_{k \to \infty} \iint_{Q_T} \left| c[w_k](x, t) - c[w](x, t) \right| dx \, dt = 0. \tag{2.14}$$

For each $\delta > 0$, let

$$E_\delta = \{(x, t) \in Q_T : x^\alpha \leq \delta\}, \qquad F_\delta = \{(x, t) \in Q_T : x^\alpha > \delta\}.$$

Combined $\lim_{\delta \to 0} \text{meas} \, E_\delta = 0$ with (2.9), we only need to prove

$$\lim_{k \to \infty} \iint_{F_\delta} \left| c[w_k](x,t) - c[w](x,t) \right| dx \, dt = 0, \quad \delta > 0. \tag{2.15}$$

Fix $\delta > 0$, since $\{w_k\}_{k=1}^\infty$ converges to w in $L^2(0,T;H_\alpha^1(0,1))$, then

$$\lim_{k \to \infty} \iint_{F_\delta} (w_k - w)^2 \, dx \, dt = 0, \qquad \lim_{k \to \infty} \iint_{F_\delta} (w_{kx} - w_x)^2 \, dx \, dt = 0. \tag{2.16}$$

For any integers $m, j > 0$, denote

$$F_{\delta,m} = \left\{ (x,t) \in F_\delta, \left| w(x,t) \right| + \left| w_x(x,t) \right| > m \right\}$$

and

$$F_{\delta,m,j} = \left\{ (x,t) \in F_\delta : \text{there exsits } k \geq j, \text{such that } \left| w_k(x,t) \right| + \left| w_{kx}(x,t) \right| > m + 1 \right\},$$

then

$$\lim_{m \to \infty} \text{meas} \, F_{\delta,m} = 0. \tag{2.17}$$

Furthermore, (2.16) shows that

$$\lim_{j \to \infty} \text{meas}(F_{\delta,m,j} \setminus F_{\delta,m}) = 0, \quad m = 1, 2, \cdots. \tag{2.18}$$

It follows from the definition of $c[w](x,t)$ and (2.9) that

$$\iint_{F_\delta} \left| c[w_k](x,t) - c[w](x,t) \right| dx \, dt$$

$$= \iint_{F_\delta \setminus (F_{\delta,m} \cup F_{\delta,m,j})} \left| c[w_k](x,t) - c[w](x,t) \right| dx \, dt + \iint_{F_{\delta,m}} \left| c[w_k](x,t) - c[w](x,t) \right| dx \, dt$$

$$+ \iint_{F_{\delta,m,j} \setminus F_{\delta,m}} \left| c[w_k](x,t) - c[w](x,t) \right| dx \, dt$$

$$\leq \iint_{F_\delta \setminus (F_{\delta,m} \cup F_{\delta,m,j})} \left| c[w_k](x,t) - c[w](x,t) \right| dx \, dt$$

$$+ 2K \, \text{meas} \, F_{\delta,m} + 2K \, \text{meas}(F_{\delta,m,j} \setminus F_{\delta,m}). \tag{2.19}$$

Since $g(x,t,s,p)$ is C^1 continuous with respect to s, p uniformly for $(x,t) \in Q_T$, then

$$\lim_{k \to \infty} \iint_{F_\delta \setminus (F_{\delta,m} \cup F_{\delta,m,j})} \left| \frac{\partial g}{\partial s}(x,t,\lambda w_k, \lambda w_{kx}) - \frac{\partial g}{\partial s}(x,t,\lambda w, \lambda w_x) \, d\lambda \right| dx \, dt = 0,$$

$$m, j = 1, 2, \cdots. \tag{2.20}$$

Let $k \to \infty$, $j \to \infty$, $m \to \infty$ in turn in (2.19), one can deduce (2.15) from (2.17), (2.18), (2.20), and thus (2.14) holds.

Fix $\varphi(x,t) \in L^1(Q_T)$, for any integer $n > 0$, we can deduce from (2.9) that

$$
\iint_{Q_T} \big(c[w_k](x,t) - c[w](x,t) \big) \varphi(x,t)\, dx\, dt
$$

$$
= \iint_{\{(x,t)\in Q_T : |\varphi(x,t)|>n\}} \big(c[w_k] - c[w] \big) \varphi(x,t)\, dx\, dt
$$

$$
+ \iint_{\{(x,t)\in Q_T : |\varphi(x,t)|\leq n\}} \big(c[w_k] - c[w] \big) \varphi(x,t)\, dx\, dt
$$

$$
\leq 2K \iint_{\{(x,t)\in Q_T : |\varphi(x,t)|>n\}} \big| \varphi(x,t) \big|\, dx\, dt
$$

$$
+ n \iint_{\{(x,t)\in Q_T : |\varphi(x,t)|\leq n\}} \big| c[w_k](x,t) - c[w](x,t) \big|\, dx\, dt \tag{2.21}
$$

and

$$
\iint_{Q_T} \big(c[w_k](x,t) - c[w](x,t) \big)^2 \varphi(x,t)\, dx\, dt
$$

$$
= \iint_{\{(x,t)\in Q_T : |\varphi(x,t)|>n\}} \big(c[w_k] - c[w] \big)^2 \varphi(x,t)\, dx\, dt
$$

$$
+ \iint_{\{(x,t)\in Q_T : |\varphi(x,t)|\leq n\}} \big(c[w_k] - c[w] \big)^2 \varphi(x,t)\, dx\, dt
$$

$$
\leq 4K^2 \iint_{\{(x,t)\in Q_T : |\varphi(x,t)|>n\}} \big| \varphi(x,t) \big|\, dx\, dt
$$

$$
+ 2nK \iint_{\{(x,t)\in Q_T : |\varphi(x,t)|\leq n\}} \big| c[w_k](x,t) - c[w](x,t) \big|\, dx\, dt. \tag{2.22}
$$

Let $k \to \infty$ and then $n \to \infty$ in (2.21) and (2.22), it follows from $\varphi \in L^1(Q_T)$ and (2.14) that

$$
\lim_{k\to\infty} \iint_{Q_T} \big(c[w_k](x,t) - c[w](x,t) \big) \varphi(x,t)\, dx\, dt = 0
$$

and

$$
\lim_{k\to\infty} \iint_{Q_T} \big(c[w_k](x,t) - c[w](x,t) \big)^2 \varphi(x,t)\, dx\, dt = 0.
$$

The convergence for $b[w](x,t)$ can be proved similarly, the proof is complete.   □

Theorem 2.1 *For any $f \in L^2(Q_T)$ and $u_0 \in L^2(0,1)$, the problem* (2.5)–(2.8) *has a unique weak solution.*

Proof We divide the proof into two steps.

Step 1. Let us prove the existence of the weak solution to the problem by using the Schauder fixed point theorem. It follows from Proposition 2.1 that the problem

$$
u_t - \big(x^\alpha u_x \big)_x + x^{\alpha/2} b(x,t,w) u_x + c(x,t,w) u = f(x,t) - g(x,t,0,0), \quad (x,t) \in Q_T, \tag{2.23}
$$

$$
u(0,t) = u(1,t) = 0 \quad \text{if } 0 < \alpha < 1, t \in (0,T), \tag{2.24}
$$

$$x^{\alpha/2}u_x(0,t) = u(1,t) = 0 \quad \text{if } 1 \le \alpha < 2, t \in (0,T), \tag{2.25}$$

$$u(x,0) = 0, \quad x \in (0,1) \tag{2.26}$$

admits a unique weak solution. Define an operator Λ:

$$\Lambda(w) = u, \quad w \in L^2\big(0,T;H^1_\alpha(0,1)\big),$$

where u is the weak solution to problem (2.23)–(2.26). For any $\{w_k\}_{k=1}^\infty \subset L^2(0,T;H^1_\alpha(0,1))$, it follows from (1.5) that $\{c(x,t,w_k)\}_{k=1}^\infty$ and $\{b(x,t,w_k)\}_{k=1}^\infty$ are uniformly bounded in $L^\infty(Q_T)$, respectively. Therefore, there exists a subsequence of the integer set k, denoted by itself for convenience, such that $\{c(x,t,w_k)\}_{k=1}^\infty$ and $\{b(x,t,w_k)\}_{k=1}^\infty$ converge weakly $*$ in $L^\infty(Q_T)$, respectively. Then, it is deduced from Corollary 2.3 in [12] that there exists a subsequence of $\{\Lambda(w_k)\}_{k=1}^\infty$, which converges in $L^2(0,T;H^1_\alpha(0,1))$, hence Λ is precompact.

Now we assume that $\{w_k\}_{k=1}^\infty$ converges to w in $L^2(0,T;H^1_\alpha(0,1))$, it follows from Lemma 2.2 that

$$\big(c\big(x,t,w_k(x,t)\big) - c\big(x,t,w(x,t)\big)\big)^2 \rightharpoonup 0 \text{ weakly } * \text{ in } L^\infty(Q_T),$$

$$\big(b\big(x,t,w_k(x,t)\big) - b\big(x,t,w(x,t)\big)\big)^2 \rightharpoonup 0 \text{ weakly } * \text{ in } L^\infty(Q_T).$$

From the convergence above and Corollary 2.4 in [12], $\Lambda(w_k)$ converges to $\Lambda(w)$ in $L^2(0,T;H^1_\alpha(0,1))$, therefore Λ is continuous.

According to the discussion above, we know that Λ is precompact and continuous on the closed and convex hull of its range, then Λ satisfies the hypotheses of the Schauder fixed point theorem. Therefore, there exists a function $u \in L^2(0,T;H^1_\alpha(0,1))$ such that $u = \Lambda(u) \in \mathcal{M}$ is the weak solution to the problem (2.5)–(2.8).

Step 2. Let us prove the uniqueness of the weak solution. Assume that u and v are two weak solutions to the problem (2.5)–(2.8) and set

$$w(x,t) = u(x,t) - v(x,t), \quad (x,t) \in Q_T.$$

Note that

$$
\begin{aligned}
g(x,t,u,u_x) - g(x,t,v,v_x) &= \int_0^1 \frac{\partial}{\partial \lambda} g\big(x,t,\lambda u + (1-\lambda)v, \lambda u_x + (1-\lambda)v_x\big)\, d\lambda \\
&= \int_0^1 (u-v)\frac{\partial}{\partial s} g\big(x,t,\lambda u + (1-\lambda)v, \lambda u_x + (1-\lambda)v_x\big)\, d\lambda \\
&\quad + \int_0^1 (u-v)_x \frac{\partial}{\partial p} g\big(x,t,\lambda u + (1-\lambda)v, \lambda u_x + (1-\lambda)v_x\big)\, d\lambda \\
&= w \int_0^1 \frac{\partial}{\partial s} g\big(x,t,\lambda u + (1-\lambda)v, \lambda u_x + (1-\lambda)v_x\big)\, d\lambda \\
&\quad + w_x \int_0^1 \frac{\partial}{\partial p} g\big(x,t,\lambda u + (1-\lambda)v, \lambda u_x + (1-\lambda)v_x\big)\, d\lambda \\
&= \theta(x,t)w + x^{\alpha/2}\psi(x,t)w_x, \quad (x,t) \in Q_T,
\end{aligned}
$$

where

$$\theta(x,t) = \int_0^1 \frac{\partial}{\partial s} g\big(x, t, \lambda u + (1-\lambda)v, \lambda u_x + (1-\lambda)v_x\big)\, d\lambda,$$

$$\psi(x,t) = x^{-\alpha/2} \int_0^1 \frac{\partial}{\partial p} g\big(x, t, \lambda u + (1-\lambda)v, \lambda u_x + (1-\lambda)v_x\big)\, d\lambda.$$

Then $w(x,t)$ is the solution to the following problem:

$$w_t - \big(x^\alpha w_x\big)_x + \theta(x,t)w + x^{\alpha/2}\psi(x,t)w_x = 0, \quad (x,t) \in Q_T,$$

$$w(0,t) = w(1,t) = 0 \quad \text{if } 0 < \alpha < 1, t \in (0,T),$$

$$x^\alpha w_x(0,t) = w(1,t) = 0 \quad \text{if } 1 \le \alpha < 2, t \in (0,T),$$

$$w(x,0) = 0, \quad x \in (0,1).$$

It follows from Proposition 2.1 that

$$w(x,t) = 0, \quad (x,t) \in Q_T,$$

which yields

$$u(x,t) = v(x,t), \quad (x,t) \in Q_T.$$

The proof is complete. □

3 Null controllability

In this section, we first consider the approximate null controllability of the linear problem

$$u_t - \big(x^\alpha u_x\big)_x + x^{\alpha/2}b(x,t)u_x + c(x,t)u = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{3.1}$$

$$u(0,t) = u(1,t) = 0 \quad \text{if } 0 < \alpha < 1, t \in (0,T), \tag{3.2}$$

$$x^\alpha u_x(0,t) = u(1,t) = 0 \quad \text{if } 1 \le \alpha < 2, t \in (0,T), \tag{3.3}$$

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{3.4}$$

where $b,c \in L^\infty(Q_T)$, $h \in L^2(Q_T)$, and $u_0 \in H^1_\alpha(0,1)$.

Theorem 3.1 *The problem* (3.1)–(3.4) *is approximately null controllable, which means that, for any $\varepsilon > 0$, there exists a function $h_\varepsilon \in L^2(Q_T)$ such that*

$$\int_0^T \int_\omega h_\varepsilon^2 \, dx\, dt \le C \int_0^1 u_0^2(x)\, dx, \quad \big\| u_\varepsilon(x,T) \big\|_{L^2(0,1)} \le \varepsilon, \tag{3.5}$$

where $C > 0$ is a constant independent of ε and u_ε is the solution of (3.1)–(3.4) *with $h = h_\varepsilon$.*

Proof Define a functional

$$J_\varepsilon(h) = \frac{1}{2} \int_0^T \int_\omega h^2 \, dx\, dt + \frac{1}{2\varepsilon} \int_0^1 u^2(x,T)\, dx, \quad h \in L^2(Q_T),$$

where u is the solution to problem (3.1)–(3.4). It is not hard to prove that the functional has a unique minimum point

$$h_\varepsilon = -\varphi_\varepsilon \chi_\omega, \tag{3.6}$$

where φ_ε is the solution to the conjugate problem

$$(\varphi_\varepsilon)_t + \left(x^\alpha (\varphi_\varepsilon)_x\right)_x + \left(x^{\alpha/2} b\varphi_\varepsilon\right)_x - c\varphi = 0, \quad (x,t) \in Q_T, \tag{3.7}$$

$$\varphi_\varepsilon(0,t) = \varphi_\varepsilon(1,t) = 0 \quad \text{if } 0 < \alpha < 1, t \in (0,T), \tag{3.8}$$

$$x^\alpha(\varphi_\varepsilon)_x(0,t) = \varphi_\varepsilon(1,t) = 0 \quad \text{if } 1 \le \alpha < 2, t \in (0,T), \tag{3.9}$$

$$\varphi_\varepsilon(x,T) = \frac{1}{\varepsilon} u_\varepsilon(x,T), \quad x \in (0,1). \tag{3.10}$$

Multiplying (3.7) by u_ε and then integrating by parts, one can get that

$$\int_0^1 u_\varepsilon(x,T)\varphi_\varepsilon(x,T)\,dx - \int_0^1 u_0(x)\varphi_\varepsilon(x,0)\,dx$$

$$= \iint_{Q_T} \left((u_\varepsilon)_t - \left(x^\alpha(u_\varepsilon)_x\right)_x + x^{\alpha/2} b(x,t)(u_\varepsilon)_x + c(x,t)u_\varepsilon\right)\varphi_\varepsilon\,dx\,dt$$

$$= \int_0^T \int_\omega h_\varepsilon \varphi_\varepsilon\,dx\,dt. \tag{3.11}$$

A combination of (3.6), (3.10) and (3.11) implies that

$$\int_0^T \int_\omega h_\varepsilon^2\,dx\,dt + \frac{1}{\varepsilon}\int_0^1 u_\varepsilon^2(x,T)\,dx = \int_0^1 u_0(x)\varphi_\varepsilon(x,0)\,dx. \tag{3.12}$$

As shown in Lemma 3.1 [19], there exists a constant C such that

$$\int_0^1 \varphi_\varepsilon^2(x,0)\,dx \le C \int_0^T \int_\omega (\varphi_\varepsilon)^2(x,t)\,dx\,dt. \tag{3.13}$$

Using Hölder's inequality with (3.12) and (3.13), one has

$$\int_0^T \int_\omega h_\varepsilon^2 \le \int_0^1 u_0(x)\varphi_\varepsilon(x,0)\,dx$$

$$\le \left(\int_0^1 u_0^2(x)\,dx\right)^{1/2}\left(\int_0^1 \varphi_\varepsilon^2(x,0)\,dx\right)^{1/2}$$

$$\le \left(\int_0^1 u_0^2(x)\,dx\right)^{1/2}\left(C\int_0^T \int_\omega \varphi_\varepsilon^2(x,t)\,dx\,dt\right)^{1/2}$$

$$\le C^{1/2}\left(\int_0^1 u_0^2(x)\,dx\right)^{1/2}\left(\int_0^T \int_\omega h_\varepsilon^2(x,t)\,dx\,dt\right)^{1/2}$$

and

$$\int_0^1 u_\varepsilon^2(x, T)\, dx \le \varepsilon \int_0^1 u_0(x)\varphi_\varepsilon(x, 0)\, dx$$

$$\le \varepsilon \left(\int_0^1 u_0^2(x)\, dx \right)^{1/2} \left(\int_0^1 \varphi_\varepsilon^2(x, 0)\, dx \right)^{1/2}$$

$$\le \varepsilon \left(\int_0^1 u_0^2(x)\, dx \right)^{1/2} \left(C \int_0^T \int_\omega h_\varepsilon^2\, dx \right)^{1/2},$$

thus (3.5) holds and the proof is complete.   □

Theorem 3.2 *If $u_0 \in H_\alpha^1(0, 1)$, then the semilinear system (1.1)–(1.4) is approximately null controllable, it means that, for any $\varepsilon > 0$, there exists a control function $h_\varepsilon \in L^2(Q_T)$ such that*

$$\int_0^T \int_\omega h_\varepsilon^2\, dx\, dt \le C \int_0^1 u_0^2(x)\, dx, \quad \left\| u_\varepsilon(x, T) \right\|_{L^2(0,1)} \le \varepsilon, \tag{3.14}$$

where $C > 0$ is a constant independent of ε and u_ε is the solution of (1.1)–(1.4) with $h = h_\varepsilon$.

Proof For any $w \in \mathcal{M}$, we first consider the following problem:

$$u_t - \left(x^\alpha u_x \right)_x + x^{\alpha/2} b\big(x, t, w(x, t)\big) u_x + c\big(x, t, w(x, t)\big) u = h\chi_\omega, \quad (x, t) \in Q_T, \tag{3.15}$$

$$u(0, t) = u(1, t) = 0 \quad \text{if } 0 < \alpha < 1, t \in (0, T), \tag{3.16}$$

$$x^\alpha u_x(0, t) = u(1, t) = 0 \quad \text{if } 1 \le \alpha < 2, t \in (0, T), \tag{3.17}$$

$$u(x, 0) = u_0(x), \quad x \in (0, 1). \tag{3.18}$$

For any $h \in L^2(Q_T)$, we denote $u[w]$ to be the solution to the problem (3.15)–(3.18). Theorem 3.1 shows that, for any $\varepsilon > 0$, there exists a function $h_\varepsilon[w] = \min\limits_{h \in L^2(Q_T)} J_\varepsilon[w](h)$ such that

$$\int_0^T \int_\omega \big(h_\varepsilon[w] \big)^2\, dx\, dt \le C \int_0^1 u_0^2(x)\, dx, \quad \left\| u_\varepsilon[w](x, T) \right\|_{L^2(0,1)} \le \varepsilon, \tag{3.19}$$

where

$$J_\varepsilon[w](h) = \frac{1}{2} \int_0^T \int_\omega h^2\, dx\, dt + \frac{1}{2\varepsilon} \int_0^1 u^2[w](x, T)\, dx, \quad h \in L^2(Q_T)$$

and $u_\varepsilon[w]$ is the solution to problem (3.15)–(3.18) with $h = h_\varepsilon$. Define an operator as follows:

$$\Gamma : w \in \mathcal{M} \mapsto u_\varepsilon[w] \in \mathcal{M}.$$

It is easy to prove that Γ is a bounded and compact operator from Proposition 2.1 and Lemma 2.1.

Now we will focus on proving the continuity of Γ. If $\{w_k\}_{k=1}^\infty$ converges to w in $\mathcal{M}$, then we have (2.10) and (2.11) from Lemma 2.2. Since $h_\varepsilon[w_k]$ is bounded due to (3.19), then

there exists a subsequence of $\{h_\varepsilon[w_k]\}_{k=1}^\infty$, denoted by itself for convenience, such that $\{h_\varepsilon[w_k]\}_{k=1}^\infty$ converges weakly to $\bar{h}_\varepsilon$ in $L^2(Q_T)$. Moreover, it follows from Proposition 2.1 and Lemma 2.1 that there exists a subsequence of $\{u_\varepsilon[w_k]\}_{k=1}^\infty$, denoted by itself for convenience, such that $\{u_\varepsilon[w_k]\}_{k=1}^\infty$ converges to $\bar{u}_\varepsilon$ in $\mathcal{M}$. Similarly, we can get that, for any $h \in L^2(Q_T)$, $\{u[w_k]\}_{k=1}^\infty$ converges to $u[w]$ in $\mathcal{M}$. From Definition 2.1, we know that, for any function $\varphi \in \mathcal{M}$ with $\varphi_t \in L^2(Q_T)$ and $\varphi(\cdot, T)|_{(0,1)} = 0$, the following integral equality holds:

$$\iint_{Q_T} \left(-u_\varepsilon[w_k]\varphi_t + x^\alpha \left(u_\varepsilon[w_k]\right)_x \varphi_x + x^{\alpha/2} b(x,t,w_k)\left(u_\varepsilon[w_k]\right)_x \varphi \right.$$
$$\left. + c(x,t,w_k)u_\varepsilon[w_k]\varphi\right) dx\, dt$$
$$= \iint_{Q_T} h_\varepsilon[w_k]\chi_\omega \varphi \, dx\, dt + \int_0^1 u_0(x)\varphi(x,0)\, dx. \tag{3.20}$$

Letting $k \to \infty$ in (3.20), we have

$$\iint_{Q_T} \left(-\bar{u}_\varepsilon \varphi_t + x^\alpha (\bar{u}_\varepsilon)_x \varphi_x + x^{\alpha/2} b(x,t,w)(\bar{u}_\varepsilon)_x \varphi + c(x,t,w)\bar{u}_\varepsilon \varphi\right) dx\, dt$$
$$= \iint_{Q_T} \bar{h}_\varepsilon \chi_\omega \varphi \, dx\, dt + \int_0^1 u_0(x)\varphi(x,0)\, dx,$$

which means that $\bar{u}_\varepsilon$ is the weak solution to the problem (3.15)–(3.18) with $h = \bar{h}_\varepsilon$. To prove the continuity of Γ, we only need to prove $\bar{h}_\varepsilon = h_\varepsilon[w]$. Since $h_\varepsilon[w_k]$ is the minimum of $J_\varepsilon[w_k]$, then for all $h \in L^2(Q_T)$, it is obvious that

$$\frac{1}{2}\int_0^T \int_\omega \left(h_\varepsilon[w_k]\right)^2 dx\, dt + \frac{1}{2\varepsilon}\int_0^1 \left(u_\varepsilon[w_k]\right)^2(x,T)\, dx$$
$$\leq \frac{1}{2}\int_0^T \int_\omega h^2 \, dx\, dt + \frac{1}{2\varepsilon}\int_0^1 \left(u[w_k]\right)^2(x,T)\, dx. \tag{3.21}$$

Note that

$$h_\varepsilon[w_k] \rightharpoonup \bar{h}_\varepsilon \text{ in } L^2(Q_T), \quad u_\varepsilon[w_k] \to \bar{u}_\varepsilon \text{ in } \mathcal{M}, \quad u[w_k] \to u[w] \text{ in } \mathcal{M}, \quad \text{as } k \to \infty.$$

Let $k \to \infty$ in (3.21), we obtain that

$$\frac{1}{2}\int_0^T \int_\omega (\bar{h}_\varepsilon)^2 \, dx\, dt + \frac{1}{2\varepsilon}\int_0^1 (\bar{u}_\varepsilon)^2(x,T)\, dx$$
$$\leq \frac{1}{2}\int_0^T \int_\omega h^2 \, dx\, dt + \frac{1}{2\varepsilon}\int_0^1 \left(u[w]\right)^2(x,T)\, dx.$$

Thus, $\bar{h}_\varepsilon = h_\varepsilon[w] = \min_{h \in L^2(Q_T)} J_\varepsilon[w](h)$, so Γ is continuous.

From the discussion above, one can get that Γ satisfies the hypotheses of the Schauder fixed point theorem. Therefore, there exists a fixed point $u \in \mathcal{M}$ such that $\Gamma(u) = u$, it means that u is the solution to problem (1.1)–(1.4) and satisfies (3.14). The proof is complete. □

Inspired by the proof of Theorem 3.6 and Theorem 3.8 in [3], one can prove the null controllability of system (1.1)–(1.4).

Theorem 3.3 *The problem* (1.1)–(1.4) *is null controllable. More precisely, for any* $u_0 \in L^2(0,1)$, *there exists a control function* $h \in L^2(Q_T)$ *such that the solution* u *to the problem* (1.1)–(1.4) *satisfies*

$$u(x, T) = 0, \quad a.e. \ x \in (0, 1).$$

Availability of data and materials
No applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed to each part of this study equally, read and approved the final version of the manuscript.

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