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Existence and concentration of positive solutions for a class of discontinuous quasilinear Schrödinger problems in $\mathbb{R}^N$

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Abstract

In this paper, a class of quasilinear Schrödinger equations with discontinuous nonlinearity is considered. After changing variables, by using nonsmooth critical point theory, we obtain the existence and concentration of positive solutions for this problem under suitable conditions. Our results cover and extend some results for these differentiable quasilinear Schrödinger problems.

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1 Introduction

Recently many papers [1–5] have focused on studying the existence of solutions for the following quasilinear Schrödinger equations:

$$i\epsilon\frac{\partial\psi}{\partial t} = -\epsilon^2\Delta\psi + W(x)\psi - \epsilon^2 k\Delta\big(h\big(|\psi|^2\big)\big)h'\big(|\psi|^2\big)\psi - g\big(|\psi|^2\big)\psi, \quad x\in\mathbb{R}^N, \qquad (1.1)$$

where $\epsilon > 0$, W is a given potential, $k\in\mathbb{R}$, g and h are real functions. Equation (1.1) with various types of h appears in several areas of physics. For example, in the case $h(s) = (1+s)^{\frac{1}{2}}$, problem (1.1) models the self-channeling of a high-power ultra-short laser in matter, the propagation of a high-irradiance laser in a plasma creates an optical index depending nonlinearly on the light intensity and this leads to interesting new nonlinear wave equations [6, 7]. For more applications, we can refer to [8–10] and the references therein. Here, we are interested in studying the case $h(s) = s$, which is used to model a superfluid film in plasma physics [4], especially the existence of standing wave solutions, that is, solutions of type $\psi = \exp(-iEt/\epsilon)u(x)$ with $E\in\mathbb{R}$ and function $u > 0$ [11–13]. After a direct computation, problem (1.1) is equivalent to

$$-\Delta u + V(\epsilon x)u - k\Delta\big(u^2\big)u = g(u), \quad x\in\mathbb{R}^N. \qquad (1.2)$$

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It is well known that there exist lots of results on discussing Eq. (1.2) with $k = 0$, i.e., the following semilinear case:

$$-\Delta u + V(\epsilon x)u = g(u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

In [14] Rabinowitz used the mountain pass theorem to prove the existence of positive solutions of (1.3) for $\epsilon > 0$ and V satisfying

(V0)　$V_\infty = \liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) = m > 0$.

Later, Alves and Figueiredo [15] extended (1.3) to the p-Laplace case with $2 \le p < N$ and proved that these solutions concentrate at global minimum points of $V(\epsilon x)$ as $\epsilon \to 0$. More results can be found in [16–20] and so on.

Compared to the semilinear case, the quasilinear case ($k \neq 0$) becomes much more complicated as there is no suitable space for the energy functional corresponding to problem (1.2) for $N \ge 2$. In order to overcome this difficulty, in [21], by changing of variables, the authors reduced the quasilinear equation (1.2) into the semilinear case. Based on this fact, problem (1.2) has been widely studied by assuming different hypotheses on V and f. Moameni [22] obtained the existence of a positive solution by assuming that f is a nonnegative function for $N \ge 2$, and the potential function V is radially symmetric. Miyagaki and Moreira [11] derived the existence and multiplicity of solutions for problem (1.2) when the nonlinearity is indefinite in sign. Liu et al. [12, 13] developed a perturbation method, the main idea of which is adding a regularizing term to recover the smoothness of the energy functional, so that the standard minimax theory can be used. Utilizing this method and a constrained minimization argument, they proved that problem (1.2) has a positive solution. Later, Wu [23] showed the existence of high energy solutions by employing the perturbation method for a general quasilinear problem. Recently, Carrião et al. [1] investigated the existence of a least energy solution for a class of nonhomogeneous asymptotically linear Schrödinger equations in $\mathbb{R}^N$ via the Pohozaev manifold. It is worth to point out that different from semilinear problems, the critical exponent of problem (1.2) is 22^*, not 2^*, where $22^* = \frac{4N}{N-2}$. This will lead to some difficulties. For example, some properties in the usual Sobolev space cannot be used directly. The behavior of h at infinity plays an important role when searching for a solution to problem (1.2), mainly supercritical, critical or subcritical cases, where h behaves at infinity as $|s|^{r-1}s$, with $r + 1 > 22^*$, $r + 1 = 22^*$ or $r + 1 < 22^*$, respectively. The critical case of (1.2) was considered in [24–27]. The supercritical results can be found in [28–39] and the references therein.

However, there seems to be little progress on the existence of positive solutions for general quasilinear elliptic equations with discontinuous nonlinearity. Based on this fact, we will study the quasilinear Schrödinger Eq. (1.2) from a discontinuous point of view. To some degree, the discontinuous case is more suitable to objective reality, and a smooth situation is usually just an ideal case. Hence, we consider the existence and concentration of solutions for the following problem:

$$\begin{cases} -\Delta u - \Delta(u^2)u + V(\epsilon x)u = H(u - a)u^p, \quad x \in \mathbb{R}^N, \\ u > 0, \end{cases} \tag{1.4}$$

where $\epsilon, \beta > 0$ are positive parameters, $p \in (3, 22^* - 2)$ if $N \ge 3$ or $p \in (3, +\infty)$ if $N = 1, 2$, $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ satisfying (V0).

As is well known, the interest in studying nonlinear partial differential equations with discontinuous nonlinearities has increased since many free boundary problems and obstacle problems may be reduced to partial differential equations with nonsmooth potentials. Among these problems, we have the seepage surface problem, the obstacle problem, and the Elenbaas equation, see [40–42]. The area of nonsmooth analysis is closely related with the development of critical point theory for nondifferentiable functionals, in particular, for locally Lipschitz continuous functionals based on Clarke's generalized gradient [43]. In 1981, Chang [40] extended the variational method to a class of nondifferentiable functionals, and directly applied the variational method to prove some existence of theorems for PDE with discontinuous nonlinearities. It provides an appropriate mathematical framework to extend the classic critical point theory for C^1-functionals in a natural way, and to meet specific needs in applications, such as nonsmooth mechanics and engineering. For a comprehensive understanding, we refer to Refs. [44–53].

This paper mainly discusses the existence of positive solutions to problem (1.4). Contrast to the previous results, our methods are totally different from those used in previous papers, since we are dealing with a discontinuous and non-convex problem. The main differences are the following:

(1) Unlike [1], the lack of differentiability of nonlinearities causes some technical difficulties. This means that variational methods for C^1 functionals are not suitable in our case, since in our case, the energy functional is only locally Lipschitz continuous. Therefore, we have to use another variational approach based on the nonsmooth critical point theory due to Clarke [43] and Chang [54]. In contrast to C^1 variational methods, this method is not adequately developed, and we need to improve it.

(2) In [1], if the energy functional associated to problem (1.2) is differentiable, it can be discussed on the Nehari manifold and the mountain pass level is equal to the minimum of the energy functional on Nehari manifolds, which is a key point in lots of papers. However, all these properties are not true for nondifferentiable problems. Hence, the arguments used in the above references cannot be directly repeated and we need to develop some new techniques to get over these difficulties.

(3) Due to the appearance of the non-convex term $\Delta(u^2)u$, some arguments used in standard semilinear problems cannot be used, therefore lots of estimates in this paper need to be reestablished.

(4) Since $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ ($p \in [2, 2^*]$) is not compact, and the compact embedding is very crucial to deduce (PS) sequences in variational methods, we have to use other means to overcome this difficulty.

The main result is the following.

Theorem 1.1 *If hypothesis* (V0) *holds, then there exist* $\epsilon^*, a^* > 0$ *such that problem* (1.4) *has a positive solution* $u_{\epsilon,a}$ *for* $\epsilon \in (0, \epsilon^*)$ *and* $a \in (0, a^*)$. *Furthermore, if* $y_{\epsilon,a} \in \mathbb{R}^N$ *denotes a maximum point of* $u_{\epsilon,a}$, *we have*

$$\lim_{(\epsilon,a)\to(0,0)} V(\epsilon y_{\epsilon,a}) = m.$$

Our paper is organized as follows. In Sect. 2, we give some basic results involving locally Lipschitz continuous functionals. In Sect. 3, we deal with the existence of solutions for an auxiliary problem. Then we prove Theorem 1.1 in Sect. 4.

2 Preliminary results

In the sequel, we will use the following basic notations.

- $\rightharpoonup$ means weak convergence while $\to$ means strong convergence.
- C and C_i ($i = 1, 2, \ldots$) denote estimated constants (the exact value may be different from line to line). $o_n(1)$ denotes a sequence whose limit is 0 as $n \to \infty$.
- $(X, \| \cdot \|)$ denotes a (real) Banach space and $(X^*, \| \cdot \|_*)$ denotes its topological dual, $| \cdot |_r$ denotes the norm of $L^r(\mathbb{R}^N)$.

Definition 2.1 ([43]) A function $I : X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every $v, \eta \in U$

$$\left| I(v) - I(\eta) \right| \le L \| v - \eta \|.$$

Definition 2.2 ([43]) Let $I : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized derivative of I in u along the direction v is defined by

$$I^0(u; v) = \limsup_{\eta \to u, \tau \to 0^+} \frac{I(\eta + \tau v) - I(\eta)}{\tau},$$

where $u, v \in X$.

It is easy to see that the function $v \mapsto I^0(u; v)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial I(u) \subset X^*$, defined by

$$\partial I(u) = \left\{ u^* \in X^* : \left\langle u^*, v \right\rangle_X \le I^0(u; v) \text{ for all } v \in X \right\}.$$

If $I \in C^1(X)$, then

$$\partial I(u) = \left\{ I'(u) \right\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

Definition 2.3 ([46])

(i) I satisfies the nonsmooth (PS)$_c$ condition if every sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \to c \quad \text{and} \quad m^I(u_n) \to 0 \quad \text{as } n \to \infty,$$

has a strongly convergent subsequence, where $m^I(u_n) = \inf_{u_n^* \in \partial I(u_n)} \| u_n^* \|_{X^*}$.

(ii) I satisfies the nonsmooth C-condition if every sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \to c \quad \text{and} \quad \left(1 + \| u_n \| \right) m^I(u_n) \to 0,$$

has a strongly convergent subsequence, where $m^I(u_n) = \inf_{u_n^* \in \partial I(u_n)} \| u_n^* \|_{X^*}$.

Proposition 2.1 ([43])

(i) $(-h)^0(u; z) = h^0(u; -z)$ *for all* $u, z \in X$;

(ii) $h^0(u; z) = \max\{ \langle u^*, z \rangle_X : u^* \in \partial h(u) \}$ *for all* $u, z \in X$;

(iii) *Let $j : X \to \mathbb{R}$ be a continuously differentiable function. Then $\partial j(u) = \{j'(u)\}$, $j^0(u; z)$ coincides with $\langle j'(u), z \rangle_X$ and $(h + j)^0(u; z) = h^0(u; z) + \langle j'(u), z \rangle_X$ for all $u, z \in X$;*

(iv) *(Lebourg's mean value theorem) Let u and v be two points in X. Then there exists a point ξ in the open segment between u and v, and $u_\xi^* \in \partial h(\omega)$ such that*

$$h(u) - h(v) = \langle u_\xi^*, u - v \rangle_X;$$

(v) *(Second chain rule) Let Y be a Banach space and $j : Y \to X$ be a continuously differentiable function. Then $h \circ j$ is locally Lipschitz and*

$$\partial(h \circ j)(y) \subseteq \partial h\big(j(y)\big) \circ j'(y) \quad \text{for all } y \in Y;$$

(vi) *$m^h(u) = \inf_{u^* \in \partial h(u)} \|u^*\|_{X^*}$ is lower semicontinuous.*

Proposition 2.2 ([40, 42]) *Let $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ with $u_n^* \in \partial I(u_n)$. If $u_n \to u$ in X and $u_n^* \rightharpoonup u^*$ in X^*, then $u^* \in \partial I(u)$.*

Proposition 2.3 ([40, 42]) *Let $\Psi(u) = \int_{\mathbb{R}^N} G(u) \, dx$, where $G(t) = \int_0^t g(s) \, ds$. Then, $\Psi \in \mathrm{Lip}_{\mathrm{loc}}(L^{p+1}(\mathbb{R}^N), \mathbb{R})$, $\partial \Psi(u) \subset L^{\frac{p+1}{p}}(\mathbb{R}^N)$ and if $\rho \in \partial \Psi(u)$, it satisfies*

$$\rho(x) \in \big[\underline{g}\big(u(x)\big), \bar{g}\big(u(x)\big)\big] \quad \text{a.e. in } \mathbb{R}^N.$$

Lemma 2.1 ([55]) *Assume $\Phi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$. Let K be a compact metric space, $K_0 \subset K$, with $K_0 \neq \varnothing$ and $f_0 \in C(K_0, X)$. Set*

$$\Gamma \equiv \big\{ f \in C(K, X) | f|_{K_0} = f_0 \big\},$$
$$c \equiv \inf_{f \in \Gamma} \max_{t \in K} \Phi\big(f(t)\big).$$

Assume that for each $f \in \Gamma$, there is some $t_f \in K \setminus K_0$ such that

$$\max_{t \in K} \Phi\big(f(t)\big) = \Phi\big(f(t_f)\big).$$

Then there exists a sequence $u_n \in X$ satisfying

$$\Phi(u_n) \to c \quad \text{and} \quad \min_{u_n^* \in \partial \Phi(u_n)} \|u_n^*\|_{X^*} \to 0.$$

3 An auxiliary problem

In this section, we firstly discuss an auxiliary problem, which is very important in proving Theorem 1.1. Note that weak solutions of (1.4) are critical points of the following functional:

$$I_{\epsilon,a}(u) = \frac{1}{2} \int_{\mathbb{R}^N} \big(1 + 2|u|^2\big) |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(\epsilon x) u^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx, \tag{3.1}$$

where $G(u) = \int_0^t g(s)\, ds$, $g(t) = H(t-a)t^p$. While, in order to find critical points of (3.1), we need to study the existence of solutions to problem (1.4) with $\epsilon = 1$, i.e.,

$$
\begin{cases}
-\Delta u - \Delta(u^2)u + V(x)u = H(u-a)u^p, & x \in \mathbb{R}^N, \\
u > 0.
\end{cases}
\tag{3.2}
$$

The Euler–Lagrange functional corresponding to problem (3.2) $I_a : E \to \mathbb{R}$ is given by

$$
I_a(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(1 + 2|u|^2\right)|\nabla u|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\, dx - \int_{\mathbb{R}^N} G(u)\, dx,
\tag{3.3}
$$

where $E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2\, dx < \infty\}$ with the norm $\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\, dx$. However, from (3.3) we can see that I_a is not well defined in general in E. In order to overcome this difficulty, we adopt an method developed by Liu et al. [56] and Colin and Jeanjean [21]. Make the change of variables by $u = f(v)$, where f is defined by

$$
f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}} \quad \text{on } [0, +\infty)
$$

and

$$
f(-t) = -f(t) \quad \text{on } (-\infty, 0].
$$

From [21], one has the following lemma.

Lemma 3.1 *The function $f(t)$ and its derivative satisfy the following properties*:

    (*f*1) *f is uniquely defined, $C^\infty(\mathbb{R})$ and invertible.*
    (*f*2) *$|f'(t)| \leq 1$ for all $t \in \mathbb{R}$.*
    (*f*3) *$|f(t)| \leq |t|$ for all $t \in \mathbb{R}$.*
    (*f*4) *$\frac{f(t)}{t} \to 1$ as $t \to 0$.*
    (*f*5) *$\frac{f(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ as $t \to +\infty$.*
    (*f*6) *$\frac{f(t)}{2} \leq tf'(t) \leq f(t)$ for all $t > 0$.*
    (*f*7) *$\frac{f^2(t)}{2} \leq tf'(t)f(t) \leq f^2(t)$ for all $t \in \mathbb{R}$.*
    (*f*8) *$|f(t)| \leq 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$ for all $t \geq \mathbb{R}$.*
    (*f*9) *There exists a positive constant C such that*

$$
|f(t)| \geq
\begin{cases}
C|t|, & |t| \leq 1, \\
C|t|^{\frac{1}{2}}, & |t| \geq 1.
\end{cases}
$$

    (*f*10) *For each $\alpha > 0$, there exists a positive constant $C(\alpha)$ such that*

$$
|f(\alpha t)|^2 \leq C(\alpha)|f(t)|^2.
$$

    (*f*11) *$|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}$.*
    (*f*12) *For each $\lambda > 1$ and all $t \in \mathbb{R}, f^2(\lambda t) \leq \lambda^2 f^2(t)$.*
    (*f*13) *For each $\lambda < 1$ and all $t \in \mathbb{R}, f^2(\lambda t) \geq \lambda^2 f^2(t)$.*

Therefore, after the change of variable, from $I_a(u)$ we have the following functional

$$J_a(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) \, dx - \int_{\mathbb{R}^N} G(f(v)) \, dx, \tag{3.4}$$

where J_a is well defined on the space E. Arguing as in [21], if v is a critical point of the functional J_a, then $u = f(v)$ is a critical point of the functional I_a, i.e., $u = f(v)$ is a solution of problem (3.2). Since we are looking for positive solutions to problem (3.2), we only need to require $f(v) > 0$, i.e., $v > 0$.

Lemma 3.2 *The functional J_a satisfies the mountain pass geometry.*

Proof We introduce the following notations for the functional J_a:

$$J_a(v) = Q_1(v) - Q_2(v),$$

where $Q_1(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) \, dx$ and $Q_2(v) = \int_{\mathbb{R}^N} G(f(v)) \, dx$. Since $Q_1(v)$ is a smooth continuous functions, we only need to show that $Q_2(v)$ is locally Lipschitz. Let $v_1, v_2 \in E$. Consider

$$
\begin{aligned}
\left| Q_2(v_2) - Q_2(v_1) \right| &\le \int_{\mathbb{R}^N} \left| \int_{v_1}^{v_2} f^p(t) f'(t) dt \right| dx \\
&\le \int_{\mathbb{R}^N} \left| \int_{v_1}^{v_2} f^p(t) dt \right| dx \\
&\le \int_{\mathbb{R}^N} \left(f^p(v_2) + f^p(v_1) \right) \left| v_2(x) - v_1(x) \right| dx \\
&\le 2^{\frac{p+4}{4}} \int_{\mathbb{R}^N} |w(x)|^{\frac{p}{2}} \left| v_2(x) - v_1(x) \right| dx \\
&\le 2^{\frac{p+4}{4}} |v_2 - v_1|_{\frac{p}{2}+1} |w|_{\frac{p}{2}+1}^{\frac{p}{2}} \\
&\le 2^{\frac{p+4}{4}} C \|w\|^{\frac{p}{2}} \|v_2 - v_1\|,
\end{aligned}
$$

where $w(x) = \max\{f^p(v_1(x)), f^p(v_2(x))\}$. Therefore, Q_2 is locally Lipschitz on E.

Setting $S(r) = \{v \in E : \|v\| = r\}$, we now show that there exist $r, \beta > 0$ such that

$$J(v) \ge \beta \quad \text{for all } v \in S(r). \tag{3.5}$$

By $(f3)$ in Lemma 3.1, we have

$$\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) f^2(v) \right) dx \le \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) v^2 \right) dx,$$

which means that $\|v\|_0^2 = \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) f^2(v)) \, dx$ is bounded. Using this condition, from Lemma [57, Lemma 2.4], we have

$$\|v\|_0^2 \ge C_1 \|v\|^2. \tag{3.6}$$

Hence, for $v \in S(r)$, it follows from the Sobolev embedding and $(f8)$ that

$$
\begin{aligned}
J_a(v) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) f^2(v) \right) dx - \int_{\mathbb{R}^N} G(f(v)) \, dx \\
&\geq \frac{C_1}{2} \|v\|^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} f^{p+1}(v) \, dx \\
&\geq \frac{C_1}{2} \|v\|^2 - \frac{2^{\frac{p+1}{4}}}{p+1} \int_{\mathbb{R}^N} |v|^{\frac{p+1}{2}} \, dx \\
&\geq \frac{C_1}{2} \|v\|^2 - C_2 \|v\|^{\frac{p+1}{2}}.
\end{aligned}
$$

Noting that $p > 3$, there exist $r, \beta > 0$ such that

$$
J_a(v) \geq \beta \quad \text{for } \|v\| = r, v \in E.
$$

Now, set $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi > 0$ and $K = \sup t\varphi \subset \mathbb{R}^N$. Then, for $t > 0$,

$$
\begin{aligned}
J_a(t\varphi) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(t\varphi) \, dx - \frac{1}{p+1} \int_{K \cap [t\varphi > a]} f^{p+1}(t\varphi) \, dx \\
&\quad + \frac{1}{p+1} \int_{K \cap [t\varphi > a]} a^{p+1} \, dx + C \operatorname{med}(K) \\
&\leq \frac{t^2}{2} \|\varphi\|^2 - \frac{1}{p+1} \int_{K \cap [t\varphi > a]} f^{p+1}(t\varphi) \, dx + \frac{1}{p+1} \int_{K \cap [t\varphi > a]} a^{p+1} \, dx + C \operatorname{med}(K) \\
&\leq \frac{t^2}{2} \|\varphi\|^2 - \frac{C}{p+1} t^{\frac{p+1}{2}} \int_{K \cap [t\varphi > \varphi_0]} \varphi_0^{\frac{p+1}{2}} \, dx + \frac{1}{p+1} \int_{K \cap [\varphi_0 > t\varphi > a]} \varphi_0^{p+1} \, dx + C \operatorname{med}(K) \\
&\to -\infty \quad \text{as } t \to +\infty,
\end{aligned}
$$

where $\varphi_0 = \max\{a, 1\}$. Hence for $t_0 > 0$ sufficiently large, we obtain $e = t_0 \varphi$ satisfying

$$
J_a(e) < 0 \quad \text{with } e \in E \setminus S_r(0).
$$

Note that $J_a(0) = 0$, then J_a satisfies the mountain pass geometry. It follows from the above lemma and Lemma 2.1 that there exists a sequence $\{v_n\} \subset E$ satisfying

$$
J_a(v_n) \to c_a \quad \text{and} \quad m^{J_a}(v_n) \to 0, \tag{3.7}
$$

where c_a is the mountain pass level of the functional J_a. □

Next, we will prove that $\{v_n\}$ given in (3.7) is bounded in E.

Lemma 3.3 *The sequence $\{v_n\}$ is bounded in E.*

Proof By (3.7) we have

$$
J_a(v_n) \to c_a \quad \text{and} \quad m^{J_a}(v_n) \to 0.
$$

Let $\{v_n^*\} \subset E^*$ satisfying $m^{J_a}(v_n) = \|v_n^*\|_{E^*}$ and

$$v_n^* = Q_1'(v_n) - \gamma_n,$$

where $\gamma_n \subset \partial Q_2(v_n)$. Then

$$\langle v_n^* + \gamma_n, v_n \rangle = \langle Q_1'(v_n), v_n \rangle = \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) v_n \, dx. \tag{3.8}$$

Once we have $0 \leq (p+1) G(f(v)) \leq v\underline{g}(f(v)) f'(v)$, it follows that

$$Q_2(v_n) = \int_{\mathbb{R}^N} G(f(v_n)) \, dx \leq \frac{1}{p+1} \int_{\mathbb{R}^N} v_n \underline{g}(f(v_n)) f'(v_n) \, dx.$$

By Proposition 2.2, one has

$$\underline{g}(f(v_n)) f'(v_n) \leq \gamma_n(x) \leq \bar{g}(f(v_n)) f'(v_n) \quad \text{a.e. in } \mathbb{R}^N$$

leading to

$$\underline{g}(f(v_n)) f'(v_n) v_n \leq \gamma_n(x) v_n \quad \text{a.e. in } \mathbb{R}^N,$$

which means that

$$\int_{\mathbb{R}^N} \underline{g}(f(v_n)) f'(v_n) v_n \, dx \leq \int_{\mathbb{R}^N} \gamma_n(x) v_n \, dx = \langle \gamma_n, v_n \rangle.$$

Hence

$$Q_2(v_n) \leq \frac{1}{p+1} \int_{\mathbb{R}^N} v_n \underline{g}(f(v_n)) f'(v_n) \, dx \leq \frac{1}{p+1} \langle \gamma_n, v_n \rangle. \tag{3.9}$$

From (3.8) and (3.9) we have

$$\begin{aligned}
c_a + o_n(1) &= J_a(v_n) - \frac{1}{p+1} \langle v_n^*, v_n \rangle \\
&= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) v_n \, dx \\
&\quad - \int_{\mathbb{R}^N} G(f(v_n)) \, dx + \frac{1}{p+1} \langle \gamma_n, v_n \rangle \\
&\geq \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x) f^2(v_n) \right) dx,
\end{aligned}$$

which means that $\|v_n\|_0$ is bounded. Using the same arguments used in [57, Lemma 2.1] we can obtain that $\|v_n\|$ is bounded in E, which completes the proof. $\qquad \square$

The following lemma is a key point in our analysis because the functional Q_2 is not compact. For each $R > 0$, let $Q_{2,R} : L^{p+1}(B_R(0)) \to \mathbb{R}$ be the function

$$Q_{2,R}(v) = \int_{B_R(0)} G\bigl(f(v)\bigr)\, dx.$$

Furthermore, for each $\varphi \in L^{p+1}(B_R(0))$, define the function $\tilde{\varphi} \in L^{p+1}(\mathbb{R}^N)$ by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \in B_R(0), \\ 0, & x \in B_R^c(0). \end{cases}$$

Lemma 3.4 *Let $\{v_n\} \subset E$ with $v_n \rightharpoonup v$ in E and $\gamma_n \subset \partial Q_2(v_n)$ with $\gamma_n \rightharpoonup \gamma_0$ in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Then*

$$\gamma_0(x) \in \bigl[\underline{g}\bigl(f(v)\bigr)f'(v), \bar{g}\bigl(f(v)\bigr)f'(v)\bigr] \quad a.e.\ in\ \mathbb{R}^N.$$

Proof Firstly, we denote by $v_{n,R}$, $\gamma_{n,R}$, v_R and $\gamma_{0,R}$ the restriction of the functions v_n, γ_n, v and γ_0 to $B_R(0)$. For $\forall \varphi \in L^{p+1}(B_R(0))$, from a simple computation one has

$$\int_{B_R(0)} \gamma_{n,R}\varphi\, dx = \int_{\mathbb{R}^N} \gamma_n\tilde{\varphi}\, dx$$

and

$$Q_{2,R}^0(v_{n,R}, \varphi) = Q_2^0(v_{n,R}, \tilde{\varphi}).$$

Noting that

$$\int_{\mathbb{R}^N} \gamma_n\tilde{\varphi}\, dx \le Q_2^0(v_{n,R}, \tilde{\varphi}),$$

we obtain

$$\int_{B_R(0)} \gamma_{n,R}\varphi\, dx \le Q_{2,R}^0(v_{n,R}, \varphi), \quad \forall \varphi \in L^{p+1}\bigl(B_R(0)\bigr),$$

which means

$$\gamma_{n,R} \in \partial Q_{2,R}(v_{n,R}).$$

Recalling that $v_{n,R} \to v_R$ in $L^{p+1}(B_R(0))$ and $\gamma_{n,R} \rightharpoonup \gamma_{0,R}$ in $L^{\frac{p+1}{p}}(B_R(0))$, from Proposition 2.2

$$\gamma_{0,R} \in \partial Q_{2,R}(v_R)$$

and so, from Proposition 2.3

$$\gamma_{0,R}(x) \in \bigl[\underline{g}\bigl(f\bigl(v_R(x)\bigr)\bigr)f'\bigl(v_R(x)\bigr), \bar{g}\bigl(f\bigl(v_R(x)\bigr)\bigr)f'\bigl(v_R(x)\bigr)\bigr] \quad a.e.\ in\ B_R(0),$$

or equivalently

$$\gamma_0(x) \in \left[\underline{g}(f(v(x)))f'(v(x)), \bar{g}(f(v(x)))f'(v(x))\right] \quad \text{a.e. in } B_R(0).$$

Employing the fact that $R > 0$ is arbitrary, we have

$$\gamma_0(x) \in \left[\underline{g}(f(v(x)))f'(v(x)), \bar{g}(f(v(x)))f'(v(x))\right] \quad \text{a.e. in } R^N. \qquad \square$$

Theorem 3.1 *Suppose that $c_a < c_\infty$, where c_∞ is the mountain pass level associated with the functional*

$$J_a(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V_\infty f^2(v) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} f^{p+1}(v) \, dx, \quad \forall v \in E.$$

Then, problem (3.2) *has at least one nontrivial solution.*

Proof From Lemma 3.2 and Lemma 2.1, there exists a sequence $\{v_n\} \subset E$ satisfying

$$J_a(v_n) \to c_a \quad \text{and} \quad m^{J_a}(v_n) \to 0.$$

By using standard arguments, we can assume, without loss of generality, that $\{v_n\}$ is bounded in E and $v_n(x) \geq 0$ for all $x \in \mathbb{R}^N$. Then there exists $v \in E$ such that, passing to a subsequence if necessary,

$$v_n \rightharpoonup v \quad \text{in } E \tag{3.10}$$

and

$$v_n \to v \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for } q \in [1, 2^*). \tag{3.11}$$

Claim 1 *The weak limit v is nontrivial.*

In fact, if $v \equiv 0$, the limit $v_n \to 0$ in E does not hold as $c_a > 0$. From Lions lemma [58], there exist $\{y_n\} \subset \mathbb{R}^N$ and $\alpha, r > 0$ satisfying

$$\liminf_{n \to \infty} \int_{B_r(y_n)} |v_n|^2 \, dx \geq \alpha > 0.$$

Since we are assuming $v = 0$, from the Sobolev embedding theorem we obtain that $\{y_n\}$ is unbounded. Now set

$$w_n(x) = v_n(x + y_n). \tag{3.12}$$

Employing the boundedness of $\{v_n\}$ in E, we infer that $\{w_n\}$ is bounded in E. Thus, there exist $w \in E \setminus \{0\}$ and subsequence of $\{w_n\}$, still denote by itself, such that

$$w_n \rightharpoonup w \quad \text{in } E \tag{3.13}$$

and

$$w_n \to w \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N),$$

$1 \le s$ if $N = 1, 2$ and $1 \le q < 2^*$ if $N \ge 3$.

Set $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfying $\psi(x) = 1$ for $x \in B_1(0)$, $\psi(x) = 0$ for $x \in B_2^c(0)$, $0 \le \psi(x) \le 1$ and $\psi_R(x) = \psi(\frac{x}{R})$ for $R > 0$. Then, there exists $v_n^* \in \partial J_a(v_n)$ such that

$$\langle v_n^*, (\psi_R w_n)(\cdot - y_n) \rangle = o_n(1)$$

as the sequence $\{(\psi_R w_n)(\cdot - y_n)\}$ is bounded in E. Hence

$$\int_{\mathbb{R}^N} \left(\nabla v_n \nabla (\psi_R w_n)(x - y_n) + V(x) f(v_n) f'(v_n)(\psi_R w_n)(x - y_n) \right) dx$$

$$= \int_{\mathbb{R}^N} \gamma_n (\psi_R w_n)(x - y_n) \, dx + o_n(1),$$

where $\gamma_n \in \partial Q_2(v_n)$, and so

$$\int_{B_{2R}} \left(|\nabla w_n|^2 \psi_R + V(x + y_n) f(w_n) f'(w_n) w_n \psi_R \right) dx$$

$$+ \int_{B_{2R}} w_n \nabla w_n \nabla \psi_R \, dx \le \int_{B_{2R}} f^p(w_n) f'(w_n) w_n \psi_R \, dx.$$

By Fatou's lemma, we have

$$\int_{B_{2R}} \left(|\nabla w|^2 \psi_R \right) + V_\infty f(w) f'(w) w \psi_R \, dx$$

$$+ \int_{B_{2R}} w \nabla w \nabla \psi_R \, dx \le \int_{B_{2R}} f^p(w) f'(w) w \psi_R \, dx.$$

Passing to the limit of $R \to +\infty$, from the above inequality one deduces that

$$\int_{\mathbb{R}^N} \left(|\nabla w|^2 + V_\infty f(w) f'(w) w \right) dx \le \int_{\mathbb{R}^N} f^p(w) f'(w) w \, dx. \tag{3.14}$$

Once we have $w \ne 0$, there exists $t > 0$ such that $tw \in \mathscr{N}_\infty$, where $\mathscr{N}_\infty$ is the Nehari manifold associated with J_∞ defined by

$$\mathscr{N}_\infty = \left\{ v \in E \setminus \{0\} : J'_\infty(v) v = 0 \right\}.$$

Then

$$\int_{\mathbb{R}^N} \left(t^2 |\nabla v|^2 + V_\infty f(tv) f'(tv) tv \right) dx = \int_{\mathbb{R}^N} f^p(tv) f'(tv) tv \, dx,$$

i.e.,

$$\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V_\infty \frac{f(tv) f'(tv)}{tv} v^2 \right) dx = \int_{\mathbb{R}^N} \frac{f^p(tv) f'(tv)}{tv} v^2 \, dx. \tag{3.15}$$

Note that

$$
\left[\frac{f^p(s)f'(s)}{s}\right]' = \frac{[pf^{p-1}(s)f'^2(s) + f^p(s)f''(s)]s - f^p(s)f'(s)}{s^2}
$$

$$
= \frac{[pf^{p-1}(s)f'^2(s) - 2f^{p+1}(s)f'^4(s)]s - f^p(s)f'(s)}{s^2}
$$

$$
= \frac{f^{p-1}(s)f'(s)[pf'(s)s - 2f^2(s)f'^3(s)s - f(s)]}{s^2}
$$

$$
\geq f^{p-1}(s)f'(s)\big[(p-1)f'(s)s - f(s)\big]
$$

$$
\geq f^p(s)f'(s)\left(\frac{p-1}{2} - 1\right)
$$

$$
> 0 \quad \text{for } s > 0
$$

and

$$
\left[\frac{f(s)f'(s)}{s}\right]' = \frac{sf'^2(s) + f(s)f''(s)s - f(s)f'(s)}{s^2}
$$

$$
= \frac{f'(s)\varphi_1(s)}{s^2},
$$

where $\varphi_1(s) = f'(s)s - 2f^2(s)f'^3(s)s - f(s)$. Since

$$
\varphi_1'(s) = f''(s)s + f'(s) - 4f(s)f'(s)s - 6f^2(s)f'^2(s)f''(s)s - 2f^2(s)f'^3(s) - f'(s)
$$

$$
= f(s)f'^3(s)\big[-6f'(s)s + 12f^2(s)f'^3(s)s - 2f(s)\big]
$$

$$
= f(s)f'^3(s)\left[-6f'(s)s + \frac{12f^2(s)}{1 + 2f^2(s)}f'(s)s - 2f(s)\right]
$$

$$
\leq -2f^2(s)f'^3(s)
$$

$$
< 0
$$

for $s > 0$, it demonstrates $(\frac{f(s)f'(s)}{s})' < 0$ for $s > 0$. The above inequalities mean that $\frac{f'(tv)f(tv)}{tv}$ is a decreasing function and $\frac{f^p(tv)f'(tv)}{tv}$ is an increasing function. Then from (3.14) and (3.15) we infer that $t \leq 1$.

By virtue of a result found in Willem [59, Theorem 4.2] we have

$$
c_\infty \leq \inf_{v \in \mathcal{N}_\infty} J_\infty(v),
$$

from which it follows that $c_\infty \leq J_\infty(tv)$. Consequently

$$
c_\infty \leq J_\infty(tv) - \frac{1}{p+1}J_\infty'(tv)tv
$$

$$
= \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}V_\infty f^2(tv)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N}f^{p+1}(tv)\,dx
$$

$$
- \frac{t^2}{p+1}\int_{\mathbb{R}^N}|\nabla v|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N}V_\infty f(tv)f'(tv)tv\,dx
$$

$$+ \frac{1}{p+1} \int_{\mathbb{R}^N} f^p(tv)f'(tv)tv\,dx$$

$$\leq \left(\frac{1}{2} - \frac{1}{p+1} \right) t^2 \int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2} \int_{\mathbb{R}^N} V_\infty f^2(tv)\,dx$$

$$- \frac{1}{p+1} \int_{\mathbb{R}^N} V_\infty f(tv)f'(tv)tv\,dx.$$

Set $A(s) = \frac{1}{2}f^2(s) - \frac{1}{p+1}f(s)f'(s)s$. Then

$$A'(s) = f(s)f'(s) - \frac{1}{p+1}\left[f'^2(s)s + f(s)f''(s)s + f(s)f'(s) \right]$$

$$= f(s)f'(s) - \frac{1}{p+1}\left[f'^2(s)s - 2f^2(s)f'^4(s)s + f(s)f'(s) \right]$$

$$= f'(s)\left[f(s) - \frac{1}{p+1}\left(f'(s)s - 2f^2(s)f'^4(s)s + f(s)f'(s) \right) \right]$$

$$\geq f'(s)\left[\left(1 - \frac{1}{p+1} \right) f(s) - \frac{1}{p+1}f'(s)s \right]$$

$$\geq f'(s)f(s)\left(1 - \frac{2}{p+1} \right)$$

$$> 0 \quad \text{for } p > 3, s > 0.$$

Since $t \leq 1$, we have

$$c_\infty \leq \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} |\nabla v|^2\,dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}^N} V_\infty f^2(v)\,dx - \frac{1}{p+1} \int_{\mathbb{R}^N} V_\infty f(v)f'(v)v\,dx.$$

According to Fatou's lemma and the inequality $\underline{g}(f(s))f'(s)s \geq (p+1)G(f(s))$ for all $s \geq 0$, we derive that

$$c_\infty \leq \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v_n)\,dx \right.$$

$$\left. - \frac{1}{p+1} \int_{\mathbb{R}^N} V(x)f(v_n)f'(v_n)v_n\,dx + \int_{\mathbb{R}^N} \left(\frac{\underline{g}(f(v_n))f'(v_n)v_n}{p+1} - G_H(f(v_n)) \right) dx \right]$$

$$\leq \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v_n)\,dx \right.$$

$$\left. - \frac{1}{p+1} \int_{\mathbb{R}^N} V(x)f(v_n)f'(v_n)v_n\,dx + \frac{1}{p+1} \int_{\mathbb{R}^N} \gamma_n v_n\,dx - \int_{\mathbb{R}^N} G(f(v_n))\,dx, \right.$$

that is,

$$c_\infty \leq \liminf_{n\to\infty}\left[J_a(v_n) - \frac{1}{p+1}\langle \gamma_n, v_n \rangle \right] = \lim_{n\to+\infty}\left[J_a(v_n) + o_n(1) \right] = c_a,$$

which is a contraction. Hence $v \geq 0$ and $v \neq 0$.

In the following, we will prove that v is a solution of problem (3.2). With this aim in mind, we need to show

$$-\Delta v(x) \in \frac{1}{\sqrt{1+2f^2(v)}}\left(\left[\underline{g}(f(v)),\bar{g}(f(v))\right] - V(x)f(v)\right) \quad \text{a.e. in } \mathbb{R}^N.$$

Noting that $\{v_n\} \subset E$ is a $(PS)_{c_a}$ sequence, there exist $v_n^* \in \partial J_a(v_n)$ and $\gamma_n \subset \partial Q_2(v_n)$ satisfying

$$\left\|v_n^*\right\|_{E^*} \to 0 \tag{3.16}$$

and

$$\langle v_n^*, y \rangle = \int_{\mathbb{R}^N} \left(\nabla v_n \nabla y + V(x)f(v_n)f'(v_n)y\right)dx - \int_{\mathbb{R}^N} \gamma_n y\,dx, \quad \forall y \in E, \tag{3.17}$$

where $\gamma_n(x) \in [\underline{g}(f(v_n))f'(v_n), \bar{g}(f(v_n))f'(v_n)]$ a.e. in $\mathbb{R}^N$. The boundedness of $\{v_n\}$ combined with (3.17) means that $\{\gamma_n\}$ is bounded in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Hence, there exist $\gamma_0 \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ and a subsequence of $\{\gamma_n\}$, still denoted the same, such that

$$\gamma_n \rightharpoonup \gamma_0 \quad \text{in } L^{\frac{p+1}{p}}(\mathbb{R}^N). \tag{3.18}$$

It follows from (3.13) and (3.18) that

$$\int_{\mathbb{R}^N} \left(\nabla v \nabla y + V(x)f(v)f'(v)y\right)dx = \int_{\mathbb{R}^N} \gamma_0 y\,dx \quad \forall y \in E.$$

Furthermore, by Lemma 3.4 we have

$$\gamma_0(x) \in \left[\underline{g}(f(v))f'(v), \bar{g}(f(v))f'(v)\right] \quad \text{a.e. in } \mathbb{R}^N, \tag{3.19}$$

which means that v is a nonnegative weak solution of the following problem:

$$-\Delta v(x) = \frac{1}{\sqrt{1+2f^2(v)}}\left(\gamma_0 - V(x)f(v)\right). \tag{3.20}$$

Hence (3.19) and (3.20) mean that v is a weak solution of problem (3.2).   □

Remark 3.1 Due to the fact that $V(x) \geq m$ for all $x \in \mathbb{R}^N$, it is easily to verify, by using the Stampacchia theorem, that $\{x \in \mathbb{R}^N : f(v(x)) = a\}$ has null measure for a small enough. Thus the weak solution v satisfies

$$-\Delta v(x) = \frac{1}{\sqrt{1+2f^2(v)}}\left(H(f(v)-a)f^p(v) - V(x)f(v)\right) \quad \text{a.e. in } \mathbb{R}^N.$$

This is very important in many applications.

4 Existence and concentration of solution for (1.4)

In this part, we define the space

$$E_\epsilon = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)|u|^2\, dx < \infty \right\}$$

endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(\epsilon x)u^2 \right) dx.$$

Similar to (3.2), the dual energy functional associated with (1.4) is defined by

$$J_{\epsilon,a}(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(\epsilon x) f^2(v)\, dx - \int_{\mathbb{R}^N} G(f(v))\, dx,$$

and $c_{\epsilon,a}$ denotes its mountain pass level. Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Divide the proof into two steps.

   Step 1. We firstly show the existence of solutions to problem (1.4). Let $\tilde{v} \in H^1(\mathbb{R}^N)$ be a positive ground state solution of the problem

$$\begin{cases} -\Delta\tilde{v} = \frac{1}{\sqrt{1+2f^2(\tilde{v})}}[f^p(\tilde{v}) - mf(\tilde{v})] & \text{in } \mathbb{R}^N, \\ \tilde{v} > 0. \end{cases} \tag{4.1}$$

If $J_0 : H^1(\mathbb{R}^N) \to \mathbb{R}$ is the energy functional associated with (4.1) given by

$$J_0(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2\, dx + \frac{m}{2} \int_{\mathbb{R}^N} f^2(v)\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} f^p(v)\, dx,$$

we have $J_0(\tilde{v}) = c_0$ and $J_0'(\tilde{v}) = 0$, where c_0 is the mountain pass level of J_0. Define $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$0 \leq \varphi(x) \leq 1, \quad \varphi(x) = 1 \quad \text{for } \forall x \in B_1(0) \quad \text{and} \quad \varphi(x) = 0 \quad \text{for } \forall x \in B_2^c(0).$$

For each $R > 1$, we denote by φ_R and $\tilde{v}_R$ the functions

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right) \quad \text{and} \quad \tilde{v}_R(x) = \varphi_R(x)\tilde{v}(x).$$

A direct computation shows that

$$\tilde{v}_R \to \tilde{v} \quad \text{in } H^1(\mathbb{R}^N) \text{ as } R \to +\infty.$$

Thus $\tilde{v}_R \neq 0$ for R sufficiently large. By this, there exists $t_R > 0$ such that

$$J_0(t_R\tilde{v}_R) = \max_{t \geq 0} J_0(t\tilde{v}_R)$$

and so

$$\int_{B_{2R}} \left[|\nabla \tilde{v}_k|^2 + \frac{mf(t\tilde{v}_k)f'(t\tilde{v}_k)\tilde{v}_k^2}{t\tilde{v}_k} \right] dx = \int_{B_{2R}} \frac{f^p(t\tilde{v}_k)f'(t\tilde{v}_k)\tilde{v}_k^2}{t\tilde{v}_k} \, dx$$

and

$$\lim_{R \to \infty} t_R = 1.$$

These facts mean that

$$\hat{v}_k = \tilde{v}_k t_k \to \tilde{v} \quad \text{in } H^1(\mathbb{R}^N) \text{ as } R \to \infty.$$

Once that $c_0 < c_\infty$ (see [14]), we can choose $\delta, R > 0$ such that

$$c_0 + \delta < c_\infty \quad \text{and} \quad J_0(\hat{v}_k) < c_0 + \frac{\delta}{2},$$

and $t > 0$ satisfying $J_{\epsilon,a}(t^*\hat{v}_k) < 0$ uniformly for $\epsilon, a > 0$ small enough.

Next, we consider $\hat{\gamma}(t) = t(t^*\hat{v}_k)$ for $t \in [0,1]$, where $\hat{\gamma} \in \Gamma$. By the definition of $c_{\epsilon,a}$ one has

$$c_{\epsilon,a} \leq \max_{t\in[0,1]} J_{\epsilon,a}(t\hat{v}_k) = J_{\epsilon,a}(\hat{t}\hat{v}_k)$$

for some $\hat{t} = \hat{t}(\epsilon, a, R) > 0$.

For each given $R > 0$, it is obvious that there exist positive constants A_1 and A_2 such that $A_1 \leq \hat{t} \leq A_2$ for $\epsilon, a > 0$ small enough. Note that $m \leq V(x)$ for all $x \in \mathbb{R}^N$. Then

$$c_0 \leq c_{\epsilon,a} \leq \max_{t \geq 0} J_{\epsilon,a}(t\hat{v}_k). \tag{4.2}$$

Without loss of generality, we suppose that $V(0) = m$. Hence, for each $\zeta > 0$, there exists $\epsilon_0 > 0$ such that

$$0 < V(\epsilon x) - m < \zeta \quad \text{for } \epsilon \in (0, \epsilon_0) \text{ and } x \in \sup t\hat{v}_k = B_{2R}(0)$$

from which one deduces that

$$\int_{\mathbb{R}^N} V(\epsilon x) f^2(\hat{v}_k) \, dx < \int_{\mathbb{R}^N} (m + \zeta) f^2(\hat{v}_k) \, dx.$$

By the above inequality we have

$$c_{\epsilon,a} \leq J_0(\hat{t}\hat{v}_k) + \frac{\zeta}{2} \int_{B_{2R}} f^2(\hat{t}\hat{v}_k) \, dx + \frac{1}{p+1} \int_{B_{2k} \cap [\hat{t}\hat{v}_k \leq a]} f^{p+1}(\hat{t}\hat{v}_k) \, dx$$

$$+ \frac{1}{p+1} \int_{B_{2k} \cap [\hat{t}\hat{v}_k > a]} a^{p+1} \, dx,$$

which implies

$$c_{\epsilon,a} \leq c_0 + \zeta C_1 + \frac{\delta}{2} + C_2 a^{p+1},$$

where C_1, C_2 do not depend on $\epsilon, a > 0$. Hence for $\epsilon, a > 0$ small enough we have

$$c_{\epsilon,a} \leq c_0 + \frac{\delta}{4} + \frac{\delta}{2} + \frac{\delta}{4} \leq c_0 + \delta < c_\infty. \tag{4.3}$$

It follows from Theorem 3.1 that problem (1.4) has at least one nontrivial solution for ϵ, $a > 0$ sufficiently small.

Step 2. Now, we begin to prove the concentration of the solution. Denote by $v_{\epsilon,a}$ the solution given by step 1. Thus, there is $\gamma_{\epsilon,a} \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ such that

$$-\Delta v_{\epsilon,a}(x) = \frac{1}{\sqrt{1 + 2f^2(v_{\epsilon,a})}} \big(\gamma_{\epsilon,a} - V(x)f\big(v_{\epsilon,a}(x)\big)\big) \quad \text{a.e. in } \mathbb{R}^N \tag{4.4}$$

with $\gamma_{\epsilon,a}(x) \in [\underline{g}(f(v_{\epsilon,a}(x)))f'(v_{\epsilon,a}(x)), \bar{g}(f(v_{\epsilon,a}(x)))f'(v_{\epsilon,a}(x))]$ a.e. in $\mathbb{R}^N$.

Now, fix $\epsilon_n \to 0$, $a_n \to 0$. $v_n = v_{\epsilon_n,a_n}$ and $\gamma_n = \gamma_{\epsilon_n,a_n}$. We are ready to discuss the behavior of the maximum points related to $\{v_n\}$, more precisely, if $y_n \in \mathbb{R}^N$ denotes a maximum point of v_n, we will show that

$$\lim_{n \to \infty} V(\epsilon_n y_n) = m.$$

By just the same method as used in (4.2) and (4.3), we obtain

$$\lim_{n \to \infty} c_{\epsilon_n,a_n} = c_0 > 0. \tag{4.5}$$

Claim 2 *There exist $\{z_n\} \subset \mathbb{R}^N$ and $\eta, r > 0$ such that*

$$\liminf_{n \to \infty} \int_{B_r(z_n)} |v_n|^2 \, dx \geq \eta > 0.$$

In fact, if the claim does not hold, from a result due to Lions, one has

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^q \, dx = 0$$

for $q \in (2, 2^*)$. This limit combined with the fact that v_n is a solution of (1.4) with $\epsilon = \epsilon_n$ and $a = a_n$ means that

$$\lim_{n \to \infty} c_{\epsilon_n,a_n} = \lim_{n \to \infty} J_{\epsilon_n,a_n}(v_n) = 0,$$

which contradicts (4.5).

Claim 3 *The sequence $w_n = v_n(\cdot - z_n)$ is strongly convergent in $H^1(\mathbb{R}^N)$. Furthermore,*

$$\lim_{|x| \to \infty} w_n(x) = 0$$

uniformly in $n \in \mathbb{N}$, that is, for $\forall \eta > 0$, there exists $R > 0$ such that

$$\big|w_n(x)\big| < \eta \quad \forall x \in \mathbb{R}^N \setminus B_R(0).$$

Using the same arguments in Claim 1, we can assume that $\{\epsilon_n z_n\}$ is a convergent sequence in $\mathbb{R}^N$ with $\epsilon_n z_n \to z^* \in V^{-1}(m)$. Moreover, we obtain that if w is the weak limit of $\{w_n\}$, then

$$w_n \to w \quad \text{in } H^1(\mathbb{R}^N).$$

In the following, we prove that

$$\lim_{|x|\to\infty} w_n(x) = 0. \tag{4.6}$$

The main idea is borrowed from [15]. For $\forall R > 0$, $0 < r \le \frac{R}{2}$. Set $\varphi \in C^\infty(\mathbb{R}^N)$, $\varphi \in [0,1]$ with $\varphi(x) = 1$ if $|x| \ge R$ and $\varphi = 0$ if $|x| \le R - r$ and $|\nabla\varphi| \le \frac{2}{r}$. Note that

$$\left|g(f(v))f'(v)\right| \le \left|f(v)\right|^p \le 2^{\frac{p}{4}} |v|^{\frac{p}{2}} \le \xi |v|^2 + C_\xi |v|^{2^*-1}. \tag{4.7}$$

For each $n \in \mathbb{N}$ and $L > 0$, set

$$v_{L,n}(x) = \begin{cases} v_n(x), & v_n(x) \le L, \\ L, & v_n(x) \ge L, \end{cases}$$

$$y_{L,n} = \varphi^2 v_{L,n}^{2(\beta-1)} v_n \quad \text{and} \quad w_{L,n} = \varphi v_n v_{L,n}^{\beta-1},$$

where $\beta > 1$ is to be determined later.

Take $y_{L,n}$ as a test function in (4.4), then

$$\int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx = -2(\beta-1)\int_{\mathbb{R}^N} \varphi^2 v_n v_{L,n}^{2\beta-3} \nabla v_n \nabla v_{L,n} \, dx$$

$$-2\int_{\mathbb{R}^N} \varphi v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla\varphi \, dx + \int_{\mathbb{R}^N} \gamma_n \varphi^2 v_{L,n}^{2(\beta-1)} v_n \, dx$$

$$-\int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) \varphi^2 v_{L,n}^{2(\beta-1)} v_n \, dx.$$

For ξ sufficiently small, (4.7) and $\gamma_n(x) \le f^p(v_n) f'(v_n)$ yield that

$$\int_{\mathbb{R}^N} \varphi v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx \le -2\int_{\mathbb{R}^N} \varphi v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla\varphi \, dx$$

$$+ \int_{\mathbb{R}^N} f^p(v_n) f'(v_n) \varphi^2 v_{L,n}^{2(\beta-1)} v_n \, dx$$

$$\le -2\int_{\mathbb{R}^N} \varphi v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla\varphi \, dx + C_3 \int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} \, dx.$$

For each $\epsilon > 0$, by Young's inequality we have

$$\int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx \le C_\xi \int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} \, dx + 2\xi \int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx$$

$$+ 2C_\xi \int_{\mathbb{R}^N} v_n^2 v_{L,n}^{2(\beta-1)} |\nabla\varphi|^2 \, dx.$$

Taking $\xi > 0$ sufficiently small, the above inequality becomes

$$\int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx \leq C \int_{\mathbb{R}^N} v_n^{2^*} \varphi^2 v_{L,n}^{2(\beta-1)} \, dx + C \int_{\mathbb{R}^N} v_n^2 v_{L,n}^{2(\beta-1)} |\nabla \varphi|^2 \, dx. \qquad (4.8)$$

By Hölder's inequality and a Sobolev embedding, we conclude that

$$
\begin{aligned}
|w_{L,n}|_{2^*}^2 &\leq C |\nabla w_{L,n}|_2^2 \\
&= C \int_{\mathbb{R}^N} \left(\nabla \varphi v_n v_{L,n}^{\beta-1} + \varphi \nabla v_n v_{L,n}^{\beta-1} + (\beta-1) \varphi v_n v_{L,n}^{\beta-2} \nabla v_{L,n} \right)^2 dx \\
&\leq C\beta^2 \left[\int_{\mathbb{R}^N} v_n^2 |\nabla \varphi|^2 v_{L,n}^{2(\beta-1)} \, dx + \int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx \right]. \qquad (4.9)
\end{aligned}
$$

It follows from (4.8) and (4.9) that

$$|w_{L,n}|_{2^*}^2 \leq C\beta^2 \left[\int_{\mathbb{R}^N} v_n^2 |\nabla \varphi|^2 v_{L,n}^{2(\beta-1)} \, dx + \int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} \, dx \right]. \qquad (4.10)$$

We assert that $v_n \in L^{\frac{2^* \cdot 2}{2}}(|x| \geq R)$ for R large enough and uniformly in n. In fact, set $\beta = \frac{2^*}{2}$. By virtue of (4.10) one has

$$|w_{L,n}|_{2^*}^2 \leq C\beta^2 \left[\int_{\mathbb{R}^N} v_n^2 |\nabla \varphi|^2 v_{L,n}^{(2^*-2)} \, dx + \int_{\mathbb{R}^N} \varphi^2 v_{L,n}^{(2^*-2)} v_n^{2^*} \, dx \right],$$

or equivalently

$$|w_{L,n}|_{2^*}^2 \leq C\beta^2 \left[\int_{\mathbb{R}^N} v_n^2 |\nabla \varphi|^2 v_{L,n}^{(2^*-2)} \, dx + \int_{\mathbb{R}^N} v_n^2 \varphi^2 v_{L,n}^{(2^*-2)} v_n^{(2^*-2)} \, dx \right].$$

Using Hölder's inequality with the exponent $\frac{2^*}{2}$ and $\frac{2^*}{2^*-2}$, we see that

$$
\begin{aligned}
|w_{L,n}|_{2^*}^2 \leq C\beta^2 \Bigg[&\int_{\mathbb{R}^N} v_n^2 |\nabla \varphi|^2 v_{L,n}^{(2^*-2)} \, dx \\
&+ \left(\int_{\mathbb{R}^N} \left(v_n \varphi v_{L,n}^{\frac{2^*-2}{2}} \right)^{2^*} dx \right)^{\frac{2}{2^*}} \left(\int_{|x| \geq R/2} v_n^{2^*} \, dx \right)^{\frac{2^*-2}{2^*}} \Bigg].
\end{aligned}
$$

From the definition of $w_{L,n}$, we obtain

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^N} \left(v_n \varphi v_{L,n}^{\frac{2^*-2}{2}} \right)^{2^*} dx \right)^{\frac{2}{2^*}} \\
&\leq C\beta^2 \left[\int_{\mathbb{R}^N} v_n^2 |\nabla \varphi|^2 v_{L,n}^{(2^*-2)} \, dx + \left(\int_{\mathbb{R}^N} \left(v_n \varphi v_{L,n}^{\frac{2^*-2}{2}} \right)^{2^*} dx \right)^{\frac{2}{2^*}} \left(\int_{|x| \geq R/2} v_n^{2^*} \, dx \right)^{\frac{2^*-2}{2^*}} \right].
\end{aligned}
$$

Observing that $v_n \to v$ in $H^1(\mathbb{R}^N)$, for R sufficiently large, we infer that

$$\int_{|x| \geq R/2} v_n^{2^*} \, dx \leq \varepsilon \quad \text{uniformly in } n.$$

Hence

$$\left(\int_{|x|\geq R} \left(v_n v_{L,n}^{\frac{2^*-2}{2}} \right)^{2^*} dx \right)^{\frac{2}{2^*}} \leq C\beta^2 \int_{\mathbb{R}^N} v_n^2 v_{L,n}^{(2^*-2)} \, dx,$$

or equivalently

$$\left(\int_{|x|\geq R} \left(v_n v_{L,n}^{\frac{2^*-2}{2}} \right)^{2^*} dx \right)^{\frac{2}{2^*}} \leq C\beta^2 \int_{\mathbb{R}^N} v_n^{2^*} \, dx \leq M < \infty.$$

By Fatou's lemma in the variable L, one derives

$$\int_{|x|\geq R} v_n^{\frac{2^{*2}}{2}} \, dx < \infty,$$

which proves the claim.

Notice that if $\beta = \frac{2^*(t-1)}{2t}$ with $t = \frac{2^{*2}}{2(2^*-2)}$, then $\beta > 1$, $\frac{2t}{t-1} < 2^*$ and $v_n \in L^{\frac{2\beta t}{t-1}}(|x| \geq R - r)$. By (4.10) one has

$$|w_{L,n}|_{2*}^2 \leq c\beta^2 \left(\int_{R-r\leq|x|\leq R} v_n^2 v_{L,n}^{2(\beta-1)} \, dx + \int_{R-r\leq|x|} v_n^{2^*} v_{L,n}^{2(\beta-1)} \, dx \right),$$

or equivalently

$$|w_{L,n}|_{2*}^2 \leq c\beta^2 \left(\int_{R-r\leq|x|\leq R} v_n^{2\beta} \, dx + \int_{R-r\leq|x|} v_n^{2^*-2} v_{L,n}^{2\beta} \, dx \right).$$

Hölder's inequality with exponent $\frac{t}{t-1}$ and t shows that

$$|w_{L,n}|_{2*}^2 \leq c\beta^2 \left(\int_{R-r\leq|x|\leq R} v_n^{\frac{2\beta t}{t-1}} \, dx \right)^{\frac{t-1}{t}} \left(\int_{R-r\leq|x|\leq R} 1 \, dx \right)^{\frac{1}{t}}$$
$$+ c\beta^2 \left(\int_{R-r\leq|x|} v_n^{(2^*-2)t} \, dx \right)^{\frac{1}{t}} \left(\int_{R-r\leq|x|} v_n^{\frac{2\beta t}{t-1}} \, dx \right)^{\frac{t-1}{t}}.$$

Since $(2^* - 2)t = 2^{*2}$, we infer that

$$|w_{L,n}|_{2*}^2 \leq C\beta^2 \left(\int_{R-r\leq|x|} v_n^{\frac{2\beta t}{t-1}} \, dx \right)^{\frac{t-1}{t}}.$$

Note that

$$|v_{L,n}|_{2^*\beta(|x|\geq R)}^{2\beta} \leq \left(\int_{|x|\geq R-r} v_{L,n}^{2^*\beta} \, dx \right)^{\frac{2}{2^*}} \leq \left(\int_{\mathbb{R}^N} \varphi^2 v_n^{2^*} v_{L,n}^{2^*(\beta-1)} \, dx \right)^{\frac{2}{2^*}}$$
$$= |w_{L,n}|_{2*}^2 \leq c\beta^2 \left(\int_{|x|\geq R-r} v_n^{\frac{2\beta t}{t-1}} \right)^{\frac{t-1}{t}} = C\beta^2 |v_n|_{\frac{2\beta t}{t-1}(|x|\geq R-r)}^{2\beta},$$

and therefore, from Fatou's lemma, we obtain

$$|v_n|_{2^*\beta(|x|\geq R)}^{2\beta} \leq C\beta^2 |v_n|_{\frac{2\beta t}{t-1}(|x|\geq R-r)}^{2\beta}.$$

Choosing $\theta = \frac{2^*(t-1)}{2t}$, $s = \frac{2t}{t-1}$, we can show that

$$|v_n|_{\theta^{m+1} s(|x| \geq R)} \leq C^{\sum_{i=1}^{m} \theta^{-i}} \theta^{\sum_{i=1}^{m} i\theta^{-i}} |v_n|_{2^*} (|x| \geq R - r),$$

which means $\|v_n\|_{L^\infty(|x| \geq R)} \leq C|v_n|_{2^*(|x| \geq R-r)}$. Applying the convergence of $v_n \to v$ in H, given $\epsilon > 0$, there is $R > 0$ such that

$$\|v_n\|_{L^\infty(|x| \geq R)} < \epsilon \quad \forall n \in \mathbb{N}.$$

Hence

$$\lim_{|x| \to \infty} w_n(x) = 0 \quad \text{uniformly in } n.$$

Furthermore, from (4.5) we infer that $\lim_{n \to \infty} \|w_n\|_{\infty, \mathbb{R}^N} > 0$ and there exist $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|w_n\|_{\infty, \mathbb{R}^N} \geq \delta^*, \quad \forall n \geq n_0.$$

Choose $\eta = \frac{\delta^*}{2}$ and $R > 0$ such that

$$w_n(x) < \frac{\delta^*}{2} \quad \forall x \in \mathbb{R}^N \setminus B_R(0) \text{ and } n \in \mathbb{N},$$

and so, if y_n denotes a maximum point of w_n, we derive

$$w_n(y_n) \geq \delta^* \quad \text{and} \quad y_n \in B_R(0).$$

Setting $\hat{y}_n$ to be the maximum point of v_n, we have $\hat{y}_n = y_n + z_n$, which means $\epsilon_n \hat{y}_n = \epsilon_n y_n + \epsilon_n z_n \to z^*$. From the continuity of the function V one derives

$$\lim_{n \to \infty} V(\epsilon_n \hat{y}_n) = V(z^*) = m.$$

Thus the proof is completed. □

Availability of data and materials
Not applicable.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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