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Response solution to complex Ginzburg–Landau equation with quasi-periodic forcing of Liouvillean frequency

Shimin Wang¹ and Jie Liu^{2*}

*Correspondence:

jyzliujie@hpu.edu.cn

²School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, P.R. China

Full list of author information is available at the end of the article

Abstract

In this paper, the existence of a response solution with the Liouvillean frequency vector to the quasi-periodically forced complex Ginzburg–Landau equation, whose linearized system is elliptic–hyperbolic, is obtained. The proof is based on constructing a modified KAM theorem for an infinite-dimensional dissipative system with Liouvillean forcing frequency.

Keywords: Complex Ginzburg–Landau equation; Response solution; Liouvillean frequency; KAM theory

1 Introduction and main result

The complex Ginzburg–Landau equation

$$u_t = ru + (b + iv)\partial_{xx}u + m\partial_xu - (1 + i\mu)|u|^2u \quad (1.1)$$

is extensively studied in the physics community. Here, the real parameter m depicts the group velocity, and the real parameters v and μ characterize linear and nonlinear dispersion, and b is real as is the control parameter r . It results from nonlinear stability theory and describes the evolution of complex amplitude coefficient $u = u(t, x)$ of a neutral plane wave. See [1–4] and the references therein for more details and physical and mathematical background.

The existence and stability of periodic or quasi-periodic solutions to (1.1) have been extensively investigated in many papers, for example [2, 5–8]. When $x \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$, there are some papers concerning the existence of KAM-type tori for (1.1). More concretely, Chung and Yuan [9] and Cong, Liu and Yuan [10] proved the existence of quasi-periodic solutions which are not traveling waves for $d = 1$ and $d \geq 2$ respectively in the case of the group velocity $m = 0$ by KAM-type theorems. See also [11–13].

In the present paper, we will prove the existence of response solution (i.e., quasi-periodic solution with the same frequency as the forcing) for the quasi-periodically forced complex

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Ginzburg–Landau equation

$$u_t = ru + (b + iv)\partial_{xx}u + m\partial_xu - (1 + i\mu)h(\omega t, x)|u|^2u + \varepsilon f(\omega t, x), \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad (1.2)$$

where $r > 0, b > 0, \mu \in \mathbb{R}, (v, m) \in \mathcal{O}$, and $\mathcal{O} \subseteq \mathbb{R}^2$ is a compact set with positive Lebesgue measure. We assume the basic frequency vector $\omega = (1, \alpha)$ ($\alpha \in \mathbb{R} \setminus \mathbb{Q}$), and ε is a small positive number.

As for (1.2), Cheng and Si [14] constructed the quasi-periodic solutions with $m = 0$ and the frequency $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, which is Diophantine, i.e., for $\gamma > 0, \tau > n - 1$,

$$|\langle k, \omega \rangle| \geq \gamma |k|^{-\tau}, \quad \forall k = (k_1, \dots, k_n) \in \mathbb{Z}^n \setminus \{0\},$$

where $\langle k, \omega \rangle = \sum_{i=1}^n k_i \omega_i$ and $|k| = \sum_{i=1}^n |k_i|$. Generally, the results for Diophantine frequency can be generalized to the case of Brjuno frequency, i.e.,

$$\sum_{m \geq 0} \frac{1}{2^m} \max_{0 < |k| \leq 2^m, k \in \mathbb{Z}^n} \ln \frac{1}{|\langle k, \omega \rangle|} < \infty.$$

However, it will involve more technique and work to obtain the quasi-periodic solution with Liouvillean frequency (a weaker assumption than Brjuno frequency), since the tori can be destroyed if the frequency is too near resonant.

In this paper, we assume the forcing frequency is $\omega = (1, \alpha)$, with $\alpha \in (0, 1)$ being any irrational numbers. Since we do not impose arithmetic condition (i.e., Diophantine or Brjuno condition) on ω , it can also be Liouvillean. There are some works addressing this frequency. More concretely, Avila, Fayad and Krikorian [15] developed a new KAM scheme for discrete $SL(2, \mathbb{R})$ co-cycles with one Liouvillean frequency by using the technique of CD bridge. Further, Hou and You [16] studied the reducibility problems for continuous two-dimensional quasi-periodic linear systems. For the nonlinear system, Wang, You and Zhou [17] and Lou and Geng [18] investigated the existence of response solutions for the quasi-periodically forced nonlinear harmonic oscillators in Hamiltonian and reversible case respectively.

For the infinite-dimensional case, Xu, You and Zhou [19] studied the nonlinear Schrödinger equation with the forcing frequency $\omega = (\bar{\omega}_1, \bar{\omega}_2)$, where $\bar{\omega}_1 = (1, \alpha)$ and $\bar{\omega}_2 \in \mathbb{R}^d$ satisfy

$$\begin{cases} \beta(\alpha) := \limsup_{n \rightarrow \infty} \frac{\ln \ln q_{n+1}}{\ln q_n} < \infty, \\ |\langle k, \bar{\omega}_1 \rangle| + |\langle l, \bar{\omega}_2 \rangle| \geq \frac{\gamma}{(|k| + |l|)^\tau}, \quad \forall k \in \mathbb{Z}^2, l \in \mathbb{Z}^d \setminus \{0\}, \end{cases}$$

and $\frac{p_n}{q_n}$ is the continued fraction approximating α (see Sect. 2.2). Recently, Wang, Cheng and Si [20] studied the quasi-periodically forced ill-posed Boussinesq equation with Liouvillean frequency $\omega = (1, \alpha)$ and obtained the existence of a response solution.

Note that the above work is all about the systems that possess a Hamiltonian or reversible structure. The question is that whether the systems without such structures still possess the response solution or not. Motivated by this question, in this paper, we consider quasi-periodically forced complex Ginzburg–Landau equation (1.2), which is a dis-

sipative system with the forcing frequency $\omega = (1, \alpha)$. The linearized equation of (1.2) is given by

$$u_t = ru + (b + iv)\partial_{xx}u + m\partial_xu, \quad x \in \mathbb{T},$$

and the linear operator $r + (b + iv)\partial_{xx} + m\partial_x$ possesses the eigenvalues

$$\lambda_n = r - bn^2 + i(mn - \nu n^2), \quad n \in \mathbb{Z}.$$

For any given $j \in \mathbb{N} \setminus \{0\}$, we can choose suitable $r, b \in \mathbb{R}$ such that $\text{Re } \lambda_{\pm j} = r - bj^2 = 0$ and $\text{Re } \lambda_l = r - bl^2 \neq 0$ for $|l| \neq j$. In this case, there are eigenvalues which are pure imaginary. Moreover, we assume the basic frequency ω is Liouvillean. Thus, the method in [9, 10, 14] cannot be directly applied since in these papers the frequency is Diophantine and the linear system is pure hyperbolic i.e., the real parts of all frequencies are not zero. In a Hamiltonian case like [17], one constructed the symplectic transformation by using the time-1-map of an auxiliary Hamiltonian flow to preserve the Hamiltonian structure in each KAM step. However, we deal with the infinite-dimensional dissipative system in this paper. So we directly construct the nearly identical coordinate transformation, which needs a more complicated computation. For the hyperbolic part, we only eliminate the terms depending only on the angle variables since $\text{Re } \lambda_n = \text{Re } \lambda_{-n}$ for $|n| \neq j$.

For Eq. (1.2), we always assume:

(H) $f(\omega t, x)$ and $h(\omega t, x)$ are quasi-periodic in t with frequency vector ω . Moreover, the functions $f(\theta, x)$ and $h(\theta, x)$ are analytic in $(\theta, x) \in \mathbb{T}^2 \times \mathbb{T}$ with the following Fourier expansions:

$$f(\theta, x) = \sum_{k \in \mathbb{Z}} f_k(\theta) e^{ikx},$$

$$h(\theta, x) = h_0 + \sum_{0 \neq \zeta \in \mathbb{Z}} h_\zeta(\theta) e^{i\zeta x}, \quad 0 \neq h_0 \in \mathbb{R}.$$

Now we state the main result of this paper.

Theorem 1.1 *Suppose that the assumption (H) holds, then, for any given $j \in \mathbb{N} \setminus \{0\}$, choosing r, b such that $\text{Re } \lambda_{\pm j} = r - bj^2 = 0$, and set $0 < \gamma \ll 1$, there exist a constant $\varepsilon_* > 0$ (depending on $r, b, \gamma, j, f, h, \mathcal{O}$) and a Cantor subset $\mathcal{O}_\gamma \subseteq \mathcal{O}$ with $\text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$ such that for $(\nu, m) \in \mathcal{O}_\gamma$, the complex Ginzburg–Landau equation (1.2) possesses a response solution provided $0 < \varepsilon < \varepsilon_*$.*

Our paper is organized as follows. In Sect. 2, we give some definitions and notations on vector field and continued fraction expansion. In Sect. 3, a modified infinite-dimensional KAM theorem for our dissipative equation with Liouvillean frequency is presented. In Sects. 4 and 5, we prove the KAM theorem, Theorem 3.1. In Sect. 6, we apply our KAM theorem 3.1 to the quasi-periodically forced complex Ginzburg–Landau equation (1.2) and prove Theorem 1.1.

2 Preliminary

2.1 Functional setting

Let $\mathbb{T}_c^2 = \mathbb{C}^2 / (2\pi\mathbb{Z}^2)$ be the two-dimensional complex torus. For $\delta > 0$, we denote the complex neighborhood of 2-torus \mathbb{T}^2 by

$$D(\delta) = \{\theta \in \mathbb{T}_c^2 : |\operatorname{Im} \theta| < \delta\},$$

where $|\cdot|$ is the supremum norm of the complex vector.

Suppose $\mathcal{O} \subseteq \mathbb{R}^2$ is a compact set. For a C_W^1 (C^1 smooth in the sense of Whitney) function $f : \mathcal{O} \rightarrow \mathbb{C}$, we define its norm as

$$\begin{aligned} \|f\|_{\mathcal{O}} &:= \sup_{\xi \in \mathcal{O}} \left(|f(\xi)| + \left| \frac{\partial f(\xi)}{\partial \xi} \right| \right) \\ &= \sup_{\xi \in \mathcal{O}} \left(|f(\xi)| + \left| \frac{\partial f(\xi)}{\partial \xi_1} \right| + \left| \frac{\partial f(\xi)}{\partial \xi_2} \right| \right). \end{aligned}$$

Given a function $f : D(\delta) \times \mathcal{O} \rightarrow \mathbb{C}$, which is analytic in $\theta \in D(\delta)$ and C_W^1 in $\xi \in \mathcal{O}$ with Fourier expansion $f(\theta; \xi) = \sum_{k \in \mathbb{Z}^2} \widehat{f}(k; \xi) e^{i\langle k, \theta \rangle}$, we define its norm as

$$\|f\|_{\delta, \mathcal{O}} := \sum_{k \in \mathbb{Z}^2} |\widehat{f}(k)|_{\mathcal{O}} e^{|k|\delta},$$

where $\langle k, \theta \rangle = k_1\theta_1 + k_2\theta_2$ and $|k| = |k_1| + |k_2|$.

For $K > 0$ and an analytic function f on $D(\delta) \times \mathcal{O}$, we define the truncation operator \mathcal{T}_K and projection operator \mathcal{R}_K as

$$\mathcal{T}_K f(\theta; \xi) := \sum_{|k| < K} \widehat{f}(k; \xi) e^{i\langle k, \theta \rangle}, \quad \mathcal{R}_K f(\theta; \xi) := \sum_{|k| \geq K} \widehat{f}(k; \xi) e^{i\langle k, \theta \rangle}.$$

The average $[f(\theta; \xi)]_{\theta}$ of $f(\theta; \xi)$ over \mathbb{T}^2 is defined as

$$[f(\theta; \xi)]_{\theta} := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(\theta; \xi) \, d\theta = \widehat{f}(0; \xi).$$

We denote the index sets by $\mathcal{J}_1 = \{j_i \in \mathbb{Z} \setminus \{0\} : 1 \leq i \leq d\}$ ($d \in \mathbb{N} \setminus \{0\}$) and $\mathcal{J}_2 = \mathbb{Z} \setminus \mathcal{J}_1$. Then we define the space $\ell_{a,p} := \{q = (\dots, q_j, \dots)_{n \in \mathcal{J}_2} : q_j \in \mathbb{C}\}$ of complex sequences equipped with the following norm:

$$\|q\|_{a,p} := \sum_{n \in \mathcal{J}_2} |q_j| e^{a|j|} \langle j \rangle^p < \infty,$$

where $\langle j \rangle := \max\{1, |j|\}$ and $a \geq 0, p > \frac{1}{2}$ are constants such that the Banach algebra property holds in this space.

Lemma 2.1 ([21]) *For $w, z \in \ell_{a,p}$, the convolution $w * z$ is defined by $(w * z)_j = \sum_{k \in \mathbb{Z}} w_{j-k} z_k$. For $a \geq 0, p > \frac{1}{2}$, then $\|w * z\|_{a,p} \leq c \|w\|_{a,p} \|z\|_{a,p}$ with a constant c depending only on p .*

For $\delta, s > 0$, we introduce a complex neighborhood of $\mathbb{T}^2 \times \{0\} \times \{0\}$ by

$$D(\delta, s) = \{(\theta, \rho, z) : |\operatorname{Im} \theta| < \delta, |\rho| < s, \|z\|_{a,p} < s\} \\ \subseteq \mathbb{C}^2 \times \mathbb{C}^d \times \ell_{a,p} =: \mathcal{P}_{a,p}.$$

Denote $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_d)$, $\alpha = (\dots, \alpha_j, \dots)_{j \in \mathcal{J}_2}$, with finitely many non-zero components $\mathbf{a}_j, \alpha_j \in \mathbb{N}$. Given a function $P : D(\delta, s) \times \mathcal{O} \rightarrow \mathbb{C}$, which is analytic in $(\theta, \rho, z) \in D(\delta, s)$ and C^1_W in $\xi \in \mathcal{O}$ and has Taylor–Fourier series expansion

$$P(\theta, \rho, z; \xi) = \sum_{\mathbf{a}, \alpha} P_{\mathbf{a}, \alpha}(\theta; \xi) \rho^{\mathbf{a}} z^{\alpha} = \sum_{k \in \mathbb{Z}^2, \mathbf{a}, \alpha} \widehat{P}_{\mathbf{a}, \alpha}(k; \xi) e^{i(k, \theta)} \rho^{\mathbf{a}} z^{\alpha},$$

where $\rho^{\mathbf{a}} = \prod_{j=1}^d \rho_j^{\mathbf{a}_j}$ and $z^{\alpha} = \prod_{j \in \mathcal{J}_2} z_j^{\alpha_j}$, we define the norm of P as

$$\|P\|_{D(\delta, s) \times \mathcal{O}} := \sup_{|\rho| < s, \|z\|_{a,p} < s} \sum_{\mathbf{a}, \alpha} \|P_{\mathbf{a}, \alpha}(\theta; \xi)\|_{\delta, \mathcal{O}} |\rho^{\mathbf{a}}| |z^{\alpha}| \\ = \sup_{|\rho| < s, \|z\|_{a,p} < s} \sum_{k, \mathbf{a}, \alpha} |\widehat{P}_{\mathbf{a}, \alpha}(k)|_{\mathcal{O}} e^{|k|\delta} |\rho^{\mathbf{a}}| |z^{\alpha}|.$$

For a finite-dimensional vector-valued function $P : D(\delta, s) \times \mathcal{O} \rightarrow \mathbb{C}^m$, ($m \in \mathbb{N} \setminus \{0\}$), i.e., $P = (P_1, \dots, P_m)$, we define its norm as

$$\|P\|_{D(\delta, s) \times \mathcal{O}} := \sum_{j=1}^m \|P_j\|_{D(\delta, s) \times \mathcal{O}}.$$

For an infinite-dimensional vector-valued function $P : D(\delta, s) \times \mathcal{O} \rightarrow \ell_{a,p}$, i.e., $P = (\dots, P_j, \dots)_{j \in \mathcal{J}_2}$, we define its weighted norm as

$$\|P\|_{a,p, D(\delta, s) \times \mathcal{O}} := \sum_{j \in \mathcal{J}_2} \|P_j\|_{D(\delta, s) \times \mathcal{O}} e^{a|j|} |j|^p.$$

Consider the dynamical system

$$\dot{w} = X(w), \quad w = (\theta, \rho, z) \in \mathcal{P}_{a,p},$$

where we have the vector field

$$X(w) = (X^{(\theta)}(w), X^{(\rho)}(w), X^{(z)}(w)) \in \mathcal{P}_{a,p}.$$

For the vector field $X : D(\delta, s) \times \mathcal{O} \rightarrow \mathcal{P}_{a,p}$, which is analytic in $(\theta, \rho, z) \in D(\delta, s)$ and depends C^1_W smoothly on parameter $\xi \in \mathcal{O}$, the weighted norm of X is defined as

$$\|X\|_{s, D(\delta, s) \times \mathcal{O}} := \|X^{(\theta)}\|_{D(\delta, s) \times \mathcal{O}} + \frac{1}{s} \|X^{(\rho)}\|_{D(\delta, s) \times \mathcal{O}} + \frac{1}{s} \|X^{(z)}\|_{a,p, D(\delta, s) \times \mathcal{O}}.$$

2.2 Continued fraction expansion

Let us recall some arithmetic properties of irrational number. Given an irrational number $\alpha \in (0, 1)$. We define

$$a_0 = 0, \quad \alpha_0 = \alpha,$$

and inductively for $k \geq 1$,

$$a_k = [\alpha_{k-1}^{-1}], \quad \alpha_k = \alpha_{k-1}^{-1} - a_k,$$

where $[\alpha] := \max\{m \in \mathbb{Z} : m \leq \alpha\}$.

Let $p_0 = 0, p_1 = 1, q_0 = 1, q_1 = a_1$, and inductively

$$p_k = a_k p_{k-1} + p_{k-2},$$

$$q_k = a_k q_{k-1} + q_{k-2}.$$

Then $\{q_n\}$ is the sequence of denominators of the best rational approximations for α . It satisfies

$$\|k\alpha\|_{\mathbb{T}} \geq \|q_{n-1}\alpha\|_{\mathbb{T}}, \quad \forall 1 \leq k < q_n,$$

and

$$\frac{1}{q_n + q_{n+1}} < \|q_n\alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}},$$

where $\|x\|_{\mathbb{T}} := \inf_{p \in \mathbb{Z}} |x - p|$.

In the sequence $\{q_n\}$, we will fix a special subsequence $\{q_{n_k}\}$. For simplicity, we denote the subsequences $\{q_{n_k}\}$ and $\{q_{n_{k+1}}\}$ by $\{Q_k\}$ and $\{\bar{Q}_k\}$, respectively. Next, we introduce the concept of CD bridge which was first given in [15].

Definition 1 (CD bridge, [15]) Let $0 < \mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$. We say that the pair of denominators (q_m, q_n) forms a CD($\mathcal{A}, \mathcal{B}, \mathcal{C}$) bridge if

- $q_{i+1} \leq q_i^{\mathcal{A}}, \forall i = m, \dots, n - 1;$
- $q_m^{\mathcal{C}} \geq q_n \geq q_m^{\mathcal{B}}.$

Lemma 2.2 (Lemma 3.2 in [15]) For any $\mathcal{A} \geq 1$, there exists a subsequence $\{Q_k\}$ such that $Q_0 = 1$ and for each $k \geq 0, Q_{k+1} \leq \bar{Q}_k^{\mathcal{A}^4}$, either $\bar{Q}_k \geq Q_k^{\mathcal{A}}$, or the pairs (\bar{Q}_{k-1}, Q_k) and (Q_k, Q_{k+1}) are both CD($\mathcal{A}, \mathcal{A}, \mathcal{A}^3$) bridge.

3 A modified KAM theorem

To prove Theorem 1.1, we give an abstract modified KAM theorem, which can be applied to the quasi-periodically forced complex Ginzburg–Landau equation (1.2). The proof of the KAM theorem will be finished by an iterative procedure in Sect. 5. Each step of the iterative procedure is set up by a finite Newton iteration.

Consider the following system:

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i\Omega(\xi)\rho + p(\theta, \rho, z; \xi), \\ \dot{z} = \Lambda(\xi)z + g(\theta, \rho, z; \xi), \end{cases} \tag{3.1}$$

on $D(\delta, s)$, where $\xi \in \mathcal{O}$ and $\mathcal{O} \subseteq \mathbb{R}^2$ is a compact set with positive Lebesgue measure. Here, $\Omega(\xi) = \text{diag}(\Omega_1(\xi), \dots, \Omega_d(\xi))$ with $\Omega_j(\xi) \in \mathbb{R}$, and $\Lambda(\xi) = \text{diag}(\dots, \lambda_j(\xi), \dots)_{j \in \mathcal{J}_2}$. We also identify $\Omega(\xi)$ and $\Lambda(\xi)$ as vectors

$$\Omega(\xi) = (\Omega_1(\xi), \dots, \Omega_d(\xi)) \in \mathbb{R}^d, \quad \Lambda(\xi) = (\dots, \lambda_j(\xi), \dots)_{j \in \mathcal{J}_2}.$$

When $p = g \equiv 0$, the system (3.1) admits an invariant torus $\mathbb{T}^2 \times \{0\} \times \{0\}$ for each parameter $\xi \in \mathcal{O}$.

Our goal is to show that if the perturbations p, g are small enough, the system (3.1) still admits invariant torus with Liouvillean frequency $\omega = (1, \alpha)$ for most of parameter $\xi \in \mathcal{O}$ (in Lebesgue measure sense) provided that Ω, Λ satisfy some non-degeneracy conditions.

Now we state our KAM theorem.

Theorem 3.1 *Let $\omega = (1, \alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\delta > \delta_* > 0, 1 > s > 0, \tau > 2, \varrho, \varrho_1, \varrho_2 > 0$. If the system (3.1) satisfies the non-degeneracy conditions*

$$\begin{aligned} \left| \frac{\partial}{\partial \xi} \langle l, \Omega(\xi) \rangle \right| &\geq \varrho, \quad \forall l \in \mathbb{Z}^d, 0 < |l| \leq 2, \\ |\text{Re } \lambda_j(\xi)| &\geq \varrho_1, \quad \left| \text{Re } \lambda_j(\xi) \right| \geq \varrho_2 \left| \frac{\partial \lambda_j(\xi)}{\partial \xi} \right|, \quad \forall j \in \mathcal{J}_2, \end{aligned} \tag{3.2}$$

then, for every sufficiently small $\gamma > 0$, there exists $\varepsilon_0 > 0$ depending on $\delta, \delta_*, s, \varrho, \varrho_1, \varrho_2, \tau, d$ but not on α , such that whenever

$$\|(0, p, g)\|_{s, D(\delta, s) \times \mathcal{O}} \leq \varepsilon_0,$$

there exist a subset $\mathcal{O}_\gamma \subseteq \mathcal{O}$ and an analytic transformation $\Phi : D(\delta_*, \frac{s}{2}) \times \mathcal{O}_\gamma \rightarrow D(\delta, s) \times \mathcal{O}_\gamma$, which transforms the system (3.1) into the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i(\Omega(\xi) + B_*(\theta; \xi))\rho + p_*(\theta, \rho, z; \xi), \\ \dot{z} = (\Lambda(\xi) + W_*(\theta; \xi))z + g_*(\theta, \rho, z; \xi), \end{cases}$$

where p_*, g_* are at least of order 2 with respect to ρ, z , and B_* is a diagonal matrix. Moreover, $\text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$.

4 Homological equation and its solution

The main idea of proving the KAM theorem, Theorem 3.1, is to construct a series of coordinate transformations $\{\Phi_l\}_{l=0}^\infty$ such that the perturbation of transformed system is smaller

and smaller. Because the system is dissipative, we construct the nearly identical transformation directly, which makes the proof more complicated. In this procedure, we need to solve a series of homological equations to construct the desired transformations. The idea of iterative procedure is detailed in Sect. 5.

4.1 Derivation of homological equation

Given a complex neighborhood $D(\delta, s)$ of $\mathbb{T}^2 \times \{0\} \times \{0\}$ in $\mathcal{P}_{a,p}$ and a compact subset $\mathcal{O} \subseteq \mathbb{R}^2$, we consider the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i(\Omega(\xi) + B(\theta; \xi) + b(\theta; \xi))\rho + p(\theta, \rho, z; \xi), \\ \dot{z} = (\Lambda(\xi) + W(\theta; \xi) + w(\theta; \xi))z + g(\theta, \rho, z; \xi), \end{cases} \tag{4.1}$$

defined on $D(\delta, s) \times \mathcal{O}$, where $\mathcal{T}_K \frac{\partial p_j}{\partial \rho_j}(\theta, 0, 0; \xi) = 0 (1 \leq j \leq d)$ for some integer K , which is given latter. Moreover, functions p and g are C^1_W in $\xi \in \mathcal{O}$. And $B(\theta; \xi)$, $b(\theta; \xi)$, $W(\theta; \xi)$ and $w(\theta; \xi)$ are C^1_W in $\xi \in \mathcal{O}$ and have the following form:

$$\begin{aligned} B(\theta; \xi) &= \text{diag}(B_1(\theta; \xi), \dots, B_d(\theta; \xi)), & W(\theta; \xi) &= (W_{ij}(\theta; \xi))_{i,j \in \mathcal{J}_2}, \\ b(\theta; \xi) &= \text{diag}(b_1(\theta; \xi), \dots, b_d(\theta; \xi)), & w(\theta; \xi) &= (w_{ij}(\theta; \xi))_{i,j \in \mathcal{J}_2}. \end{aligned}$$

We will construct a transformation Φ defined on a smaller domain such that the system (4.1) is transformed into

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho}_+ = i(\Omega(\xi) + B(\theta; \xi) + b(\theta; \xi))\rho_+ + p_+(\theta, \rho_+, z_+; \xi), \\ \dot{z}_+ = (\Lambda(\xi) + W(\theta; \xi) + w(\theta; \xi))z_+ + g_+(\theta, \rho_+, z_+; \xi), \end{cases} \tag{4.2}$$

where the norm of perturbation p_+ and g_+ in (4.2) on the small domain is smaller than that in (4.1) (see Lemma 5.3 for details).

For simplicity, we drop the parameter ξ in this section. Suppose that the coordinate transformation Φ has the following form:

$$\Phi : \begin{cases} \rho = \rho_+ + H_1(\theta) + H_2(\theta)\rho_+ + H_3(\theta)z_+, \\ z = z_+ + F_1(\theta) + F_2(\theta)\rho_+. \end{cases} \tag{4.3}$$

Let $\partial_\omega := \omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2}$. Insert (4.3) into (4.1), then we have

$$\begin{aligned} \dot{\rho}_+ &= i(\Omega + B(\theta) + b(\theta))\rho_+ \\ &\quad + i(\Omega + B(\theta) + b(\theta))H_1(\theta) + \tilde{p}^0(\theta) - \partial_\omega H_1(\theta) \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\quad + \left(i(\Omega + B(\theta) + b(\theta))H_2(\theta) - iH_2(\theta)(\Omega + B(\theta) + b(\theta)) + \frac{\partial \tilde{p}^0(\theta)}{\partial \rho} - \partial_\omega H_2(\theta) \right) \rho_+ \end{aligned} \tag{4.5}$$

$$+ \left(i(\Omega + B(\theta) + b(\theta))H_3(\theta) - H_3(\theta)(\Lambda + W(\theta) + w(\theta)) + \frac{\partial \tilde{p}^0(\theta)}{\partial z} - \partial_\omega H_3(\theta) \right) z_+ \tag{4.6}$$

$$+ p \circ \Phi - \left(\tilde{p}^0(\theta) + \frac{\partial \tilde{p}^0(\theta)}{\partial \rho} \rho_+ + \frac{\partial \tilde{p}^0(\theta)}{\partial z} z_+ \right) - H_2(\theta)p_+ - H_3(\theta)g_+, \tag{4.7}$$

$$\dot{z}_+ = (\Lambda + W(\theta) + w(\theta))z_+ + (\Lambda + W(\theta) + w(\theta))F_1(\theta) + \tilde{g}^0(\theta) - \partial_\omega F_1(\theta) \tag{4.8}$$

$$+ \left((\Lambda + W(\theta) + w(\theta))F_2(\theta) - iF_2(\theta)(\Omega + B(\theta) + b(\theta)) + \frac{\partial \tilde{g}^0(\theta)}{\partial \rho} - \partial_\omega F_2(\theta) \right) \rho_+ \tag{4.9}$$

$$+ g \circ \Phi - \left(\tilde{g}^0(\theta) + \frac{\partial \tilde{g}^0(\theta)}{\partial \rho} \rho_+ \right) - F_2(\theta)p_+, \tag{4.10}$$

where

$$\tilde{p}^0(\theta) = \mathcal{T}_K(p(\theta, 0, 0)), \quad \frac{\partial \tilde{p}^0}{\partial \rho}(\theta) = \mathcal{T}_K \left(\frac{\partial p}{\partial \rho} \Big|_{\rho=0, z=0} \right),$$

and the other expressions are similar.

Let Eqs. (4.4)–(4.6) and (4.8)–(4.9) be equal to 0, then we obtain the following homological equations:

$$\partial_\omega H_1(\theta) - i(\Omega + B(\theta) + b(\theta))H_1(\theta) = \tilde{p}^0(\theta), \tag{4.11}$$

$$\partial_\omega F_1(\theta) - (\Lambda + W(\theta) + w(\theta))F_1(\theta) = \tilde{g}^0(\theta), \tag{4.12}$$

$$\partial_\omega H_2(\theta) - i(\Omega + B(\theta) + b(\theta))H_2(\theta) + iH_2(\theta)(\Omega + B(\theta) + b(\theta)) = \frac{\partial \tilde{p}^0(\theta)}{\partial \rho}, \tag{4.13}$$

$$\partial_\omega H_3(\theta) - i(\Omega + B(\theta) + b(\theta))H_3(\theta) + H_3(\theta)(\Lambda + W(\theta) + w(\theta)) = \frac{\partial \tilde{p}^0(\theta)}{\partial z}, \tag{4.14}$$

$$\partial_\omega F_2(\theta) - (\Lambda + W(\theta) + w(\theta))F_2(\theta) + iF_2(\theta)(\Omega + B(\theta) + b(\theta)) = \frac{\partial \tilde{g}^0(\theta)}{\partial \rho}. \tag{4.15}$$

If we find the solutions H_j ($j = 1, 2, 3$) and F_j ($j = 1, 2$) of the homological equations (4.11)–(4.15), we will obtain a new system with another perturbation, which will be smaller on a small domain.

4.2 Solution to homological equation

In this subsection, we consider the homological equations with variable coefficients (4.11)–(4.15) and find their solutions. We only give the solutions to homological equations (4.13) and (4.15) in detail while omitting the other solutions, since the other equations can be dealt in the same way.

In the following, we assume that $\Omega(\xi) + [B(\theta; \xi)]_\theta \in DC_\omega(\gamma, \tau, K, \mathcal{O})$ for any given $\tau > 2$, $0 < \gamma < 1$ and $K > 0$, where

$$DC_\omega(\gamma, \tau, K, \mathcal{O}) := \left\{ \tilde{\Omega}(\xi) \mid \begin{array}{l} |\tilde{\Omega}(\xi)|_{\mathcal{O}} \leq 2 \text{ and} \\ | \langle k, \omega \rangle + \langle l, \tilde{\Omega}(\xi) \rangle | \geq \frac{\gamma}{(|k|+|l|)^\tau}, \forall 0 < |l| \leq 2, |k| < K \end{array} \right\}.$$

Moreover, we let $\mathcal{A} := \tau + 3$, $M := \frac{A^4}{2}$ and $\{Q_n\}$ be the selected sequence of α in Lemma 2.2 with respect to \mathcal{A} . In the process of solving the homological equations, we also use the following notations:

$$\begin{aligned} \eta &= \tilde{c}Q_n^{-\frac{1}{2A^4}}, & \mathcal{E} &= e^{-c_0\gamma\delta(\frac{\overline{Q}_n}{Q_n^M} + \overline{Q}_n^{-\frac{1}{M^{1/4}}})}, \\ K &= \left[\frac{\gamma}{4 \cdot 10^\tau} \max \left\{ \frac{\overline{Q}_{n+1}}{Q_{n+1}^\tau}, \overline{Q}_{n+1}^{\frac{3}{A}} \right\} \right], \end{aligned} \tag{4.16}$$

where $0 < \tilde{c} < 1$ is a constant which will be defined later and $c_0 := \frac{\tilde{c}}{4^5 \cdot 10^\tau}$.

The solutions to the homological equations (4.13) and (4.15) with estimates are given in Proposition 4.3 and 4.4, respectively. For the homological equation (4.13), we find an approximate solution with suitable small error term using idea in [16, 17]. We remove the non-resonance terms of $B(\theta; \xi)$ to eliminate relatively large $B(\theta; \xi)$ by solving the equation

$$\partial_\omega \mathcal{B}(\theta; \xi) = \mathcal{T}_{Q_{n+1}} B(\theta; \xi) + [B(\theta; \xi)]_\theta. \tag{4.17}$$

Due to the lack of Diophantine condition on ω , we will use the technique of the CD bridge introduced in Sect. 2.2 to obtain a good estimate of solution $\mathcal{B}(\theta; \xi)$ for (4.17) (see Lemma 4.1).

Lemma 4.1 (Lemma 3.1 in [17]) *Let $\delta > \delta_* > 0$ and f be an analytic finite-dimensional vector-valued function. Then there exists a positive constant $c_1(\delta_*, \tau, \tilde{c})$ such that the equation*

$$\partial_\omega \mathcal{B}(\theta; \xi) = -\mathcal{T}_{Q_{n+1}} f(\theta; \xi) + [f(\theta; \xi)]_\theta$$

has a solution $\mathcal{B}(\theta; \xi)$ with

$$\|\mathcal{B}(\theta; \xi)\|_{\delta(1-\eta), \mathcal{O}} \leq c_1(\delta_*, \tau, \tilde{c}) \delta \left(\frac{\overline{Q}_n}{Q_n^{A^4}} + \overline{Q}_n^{-\frac{1}{A}} \right) \|f(\theta; \xi)\|_{\delta, \mathcal{O}}.$$

The following lemma is about the estimate of small divisors.

Lemma 4.2 (Lemma 3.2 in [17]) *For $\tilde{K} = [\frac{\gamma}{4 \cdot 10^\tau} \max\{\frac{\overline{Q}_n}{Q_n^\tau}, \overline{Q}_n^{\frac{3}{A}}\}]$, there exists a positive constant $c_2(\tau)$ such that, for $|k| < \tilde{K}$, and $0 < |l| \leq 2$,*

$$|\langle k, \omega \rangle + \langle l, \Omega \rangle| \geq c_2(\tau) \gamma^{\frac{A\tau}{2} + 1} Q_n^{-3\tau},$$

provided $\Omega \in \text{DC}_\omega(\gamma, \tau, \tilde{K}, \mathcal{O})$.

Now we solve the homological equation (4.13) in the following proposition.

Proposition 4.3 *Write $H_2(\theta) = (H_{2lj})_{1 \leq l, j \leq d}$ and $\frac{\partial \tilde{p}^0(\theta)}{\partial \rho} = (\frac{\partial \tilde{p}_l^0(\theta)}{\partial \rho_j})_{1 \leq l, j \leq d} =: (R_{lj})_{1 \leq l, j \leq d}$, then the homological equation (4.13) becomes*

$$\partial_\omega H_{2lj} - i(\Omega_l + B_l(\theta) + b_l(\theta) - (\Omega_j + B_j(\theta) + b_j(\theta))) H_{2lj} = R_{lj}, \quad \forall 1 \leq l, j \leq d. \tag{4.18}$$

For $\delta_* > 0, 0 < \tilde{c} < 1$, there exist positive constants $c_3(\tau)$ and $\epsilon_1 = \epsilon_1(\delta_*, \tau, \tilde{c})$ such that for every $\sigma, \tilde{\delta}$ with $0 < \sigma < \delta_* < \tilde{\delta} \leq \delta(1 - \eta)$, if

$$\mathcal{R}_{Q_{n+1}} B = 0, \quad \|B\|_{\delta, \mathcal{O}} \cdot \left(\frac{\overline{Q}_n}{Q_n^{A^4}} + \overline{Q}_n^{\frac{1}{A}} \right) \leq \epsilon_1 \gamma \left(\frac{\overline{Q}_n}{Q_n^M} + \overline{Q}_n^{\frac{1}{M^{1/4}}} \right), \tag{4.19}$$

$$\|b\|_{\tilde{\delta}, \mathcal{O}} < \frac{\gamma^{A\tau+2}}{2c_3(\tau)Q_{n+1}^{6\tau}}, \tag{4.20}$$

then each equation in (4.18) has an approximate solution which can be estimated as follows: for any $1 \leq l, j \leq d$,

$$\|H_{2lj}\|_{\tilde{\delta}, \mathcal{O}} \leq c_3(\tau)\gamma^{-(A\tau+2)}Q_{n+1}^{6\tau}\mathcal{E}^{-\frac{1}{240}}\|R_{lj}\|_{\tilde{\delta}, \mathcal{O}}.$$

Moreover, the error term R_{lj}^e satisfies

$$\|R_{lj}^e\|_{\tilde{\delta}-\sigma, \mathcal{O}} \leq \mathcal{E}^{-\frac{1}{240}}e^{-K\sigma}(\|R_{lj}\|_{\tilde{\delta}, \mathcal{O}} + 2\|b\|_{\tilde{\delta}, \mathcal{O}}\|H_{2lj}\|_{\tilde{\delta}, \mathcal{O}}).$$

In the case of $B = 0$, the equation has an approximate solution H_{2lj} satisfying

$$\|H_{2lj}\|_{\tilde{\delta}, \mathcal{O}} \leq c_3(\tau)\gamma^{-(A\tau+2)}Q_{n+1}^{6\tau}\|R_{lj}\|_{\tilde{\delta}, \mathcal{O}},$$

and the error term R_{lj}^e satisfies

$$\|R_{lj}^e\|_{\tilde{\delta}-\sigma, \mathcal{O}} \leq e^{-K\sigma}(\|R_{lj}\|_{\tilde{\delta}, \mathcal{O}} + 2\|b\|_{\tilde{\delta}, \mathcal{O}}\|H_{2lj}\|_{\tilde{\delta}, \mathcal{O}}).$$

Proof We only prove the case of $B \neq 0$. The case of $B = 0$ is similar and easier.

Consider Eq. (4.18) for the unknown function H_{2lj} for any $1 \leq l, j \leq d$,

$$\partial_\omega H_{2lj}(\theta; \xi) - i(\Omega_{lj}(\xi) + B_{lj}(\theta; \xi) + b_{lj}(\theta; \xi))H_{2lj}(\theta; \xi) = R_{lj}(\theta; \xi),$$

where $\Omega_{lj}(\xi) = \Omega_l(\xi) - \Omega_j(\xi)$ and similarly for $B_{lj}(\theta; \xi)$ and $b_{lj}(\theta; \xi)$.

For $1 \leq l = j \leq d, H_{2lj} = 0$, by assumption $R_{lj} = 0$. Thus, we consider the case $1 \leq l \neq j \leq d$ in the following.

Let

$$\partial_\omega \mathcal{B}(\theta; \xi) = -B_{lj}(\theta; \xi) + [B_{lj}(\theta; \xi)]_\theta.$$

Then, by Lemma 4.1 and assumption (4.19), we have

$$\begin{aligned} \|\mathcal{B}(\theta; \xi)\|_{\delta(1-\eta), \mathcal{O}} &\leq c_1(\delta_*, \tau, \tilde{c})\delta \left(\frac{\overline{Q}_n}{Q_n^{A^4}} + \overline{Q}_n^{\frac{1}{A}} \right) \|B_{lj}(\theta; \xi)\|_{\delta, \mathcal{O}} \\ &\leq 2c_1(\delta_*, \tau, \tilde{c})\epsilon_1\gamma\delta \left(\frac{\overline{Q}_n}{Q_n^M} + \overline{Q}_n^{\frac{1}{M^{1/4}}} \right). \end{aligned}$$

Taking $0 < \epsilon_1 < \frac{c_0(\tau, \tilde{c})}{960c_1(\delta_*, \tau, \tilde{c})}$, together with the definition of \mathcal{E} in (4.16), we obtain

$$e^{\|\mathcal{B}\|_{\tilde{\delta}, \mathcal{O}}} \leq e^{2c_1(\delta_*, \tau, \tilde{c})\epsilon_1\gamma\delta \left(\frac{\overline{Q}_n}{Q_n^M} + \overline{Q}_n^{\frac{1}{M^{1/4}}} \right)} \leq \mathcal{E}^{-\frac{1}{480}}.$$

Let

$$\tilde{H}_{2lj}(\theta; \xi) = e^{-i\mathcal{B}(\theta; \xi)} H_{2lj}(\theta; \xi), \quad \tilde{R}_{lj}(\theta; \xi) = e^{-i\mathcal{B}(\theta; \xi)} R_{lj}(\theta; \xi).$$

Then Eq. (4.18) becomes

$$\partial_\omega \tilde{H}_{2lj}(\theta; \xi) - i(\tilde{\Omega}_{lj}(\xi) + b_{lj}(\theta; \xi)) \tilde{H}_{2lj}(\theta; \xi) = \tilde{R}_{lj}(\theta; \xi), \tag{4.21}$$

where $\tilde{\Omega}_{lj}(\xi) := \Omega_{lj}(\xi) + [B_{lj}(\theta; \xi)]_\theta$. We first solve the truncated equation of (4.21),

$$\mathcal{T}_K(\partial_\omega \tilde{H}_{2lj}(\theta; \xi) - i(\tilde{\Omega}_{lj}(\xi) + b_{lj}(\theta; \xi)) \tilde{H}_{2lj}(\theta; \xi)) = \mathcal{T}_K \tilde{R}_{lj}(\theta; \xi). \tag{4.22}$$

We write

$$\begin{aligned} \tilde{H}_{2lj}(\theta; \xi) &= \sum_{k \in \mathbb{Z}^2, |k| < K} \widehat{H}_{2lj}(k; \xi) e^{i\langle k, \theta \rangle}, \\ \tilde{R}_{lj}(\theta; \xi) &= \sum_{k \in \mathbb{Z}^2} \widehat{R}_{lj}(k; \xi) e^{i\langle k, \theta \rangle}, \\ b_{lj}(\theta; \xi) &= \sum_{k \in \mathbb{Z}^2} \widehat{b}_{lj}(k; \xi) e^{i\langle k, \theta \rangle}. \end{aligned}$$

By comparing the Fourier coefficients of Eq. (4.22), for $|k| < K$, we have

$$i(\langle k, \omega \rangle - \tilde{\Omega}_{lj}(\xi)) \widehat{H}_{2lj}(k; \xi) - \sum_{|k_1| < K} i \widehat{b}_{lj}(k - k_1; \xi) \widehat{H}_{2lj}(k_1; \xi) = \widehat{R}_{lj}(k; \xi).$$

This can be viewed as a vector equation:

$$(T + S)\mathfrak{H} = \mathfrak{R}, \tag{4.23}$$

where

$$\begin{aligned} T &= \text{diag}(\dots, i(\langle k, \omega \rangle - \tilde{\Omega}_{lj}(\xi)), \dots)_{|k| < K}, \\ S &= (-i \widehat{b}_{lj}(k_1 - k_2; \xi))_{|k_1|, |k_2| < K}, \\ \mathfrak{H} &= (\widehat{H}_{2lj}(k; \xi))_{|k| < K}^\top, \quad \mathfrak{R} = (\widehat{R}_{lj}(k; \xi))_{|k| < K}^\top. \end{aligned}$$

Let

$$E_\delta = \text{diag}(\dots, e^{|\delta|}, \dots)_{|k| < K}.$$

Then Eq. (4.23) is equivalent to

$$(T + E_\delta S E_\delta^{-1}) E_\delta \mathfrak{H} = E_\delta \mathfrak{R}.$$

It follows from $\Omega + [B]_\theta \in DC_\omega(\gamma, \tau, K, \mathcal{O})$ and Lemma 4.2 that

$$\begin{aligned} \|T^{-1}\|_{\mathcal{O}} &= \max_{|k| < K} \frac{1}{|\langle k, \omega \rangle + \widetilde{\Omega}_{lj}(\xi)|_{\mathcal{O}}} \\ &= \max_{|k| < K} \sup_{\xi \in \mathcal{O}} \left\{ \frac{1}{|\langle k, \omega \rangle + \widetilde{\Omega}_{lj}(\xi)|} + \frac{\left| \frac{\partial \widetilde{\Omega}_{lj}(\xi)}{\partial \xi} \right|}{|\langle k, \omega \rangle + \widetilde{\Omega}_{lj}(\xi)|^2} \right\} \\ &\leq \frac{1}{2} c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau}, \end{aligned}$$

where the matrix norm is defined by

$$\|A\|_{\mathcal{O}} = \max_i \sum_j |a_{ij}|_{\mathcal{O}}$$

with a_{ij} being the (i, j) element of the matrix A . Since

$$E_{\widetilde{\delta}} S E_{\widetilde{\delta}}^{-1} = i(e^{(|k_1|-|k_2|)\widetilde{\delta}}(l, \widehat{b}(k_1 - k_2; \xi)))_{|k_1|, |k_2| < K},$$

we have

$$\|E_{\widetilde{\delta}} S E_{\widetilde{\delta}}^{-1}\|_{\mathcal{O}} = \max_{|k_1| < K} \sum_{|k_2| < K} |e^{(|k_1|-|k_2|)\widetilde{\delta}} \widehat{b}_{lj}(k_1 - k_2; \xi)|_{\mathcal{O}} \leq 2 \|b\|_{\widetilde{\delta}, \mathcal{O}}.$$

Thus, if $\|b\|_{\widetilde{\delta}, \mathcal{O}} < \frac{\gamma^{A\tau+2}}{2c_3(\tau)Q_{n+1}^{6\tau}}$ (i.e., assumption (4.20) holds), we get

$$\|T^{-1} E_{\widetilde{\delta}} S E_{\widetilde{\delta}}^{-1}\|_{\mathcal{O}} \leq \frac{1}{2} c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau} 2 \|b\|_{\widetilde{\delta}, \mathcal{O}} < \frac{1}{2}.$$

This implies that $T + E_{\widetilde{\delta}} S E_{\widetilde{\delta}}^{-1}$ has a bounded inverse and

$$\begin{aligned} \|(T + E_{\widetilde{\delta}} S E_{\widetilde{\delta}}^{-1})^{-1}\|_{\mathcal{O}} &= \|(I + T^{-1} E_{\widetilde{\delta}} S E_{\widetilde{\delta}}^{-1})^{-1} T^{-1}\|_{\mathcal{O}} \\ &\leq \frac{1}{1 - \|T^{-1} E_{\widetilde{\delta}} S E_{\widetilde{\delta}}^{-1}\|_{\mathcal{O}}} \|T^{-1}\|_{\mathcal{O}} \\ &\leq c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\widetilde{H}_{2lj}\|_{\widetilde{\delta}, \mathcal{O}} &= \sum_{|k| < K} |\widehat{H}_{2lj}(k; \xi)|_{\mathcal{O}} e^{|k|\widetilde{\delta}} = \|E_{\widetilde{\delta}} \mathfrak{H}\|_{\mathcal{O}} \\ &= \|(T + E_{\widetilde{\delta}} S E_{\widetilde{\delta}}^{-1})^{-1} E_{\widetilde{\delta}} \mathfrak{H}\|_{\mathcal{O}} \\ &\leq c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau} \|\widetilde{R}_{lj}\|_{\widetilde{\delta}, \mathcal{O}}. \end{aligned}$$

Going back to $H_{2lj}(\theta; \xi) = e^{i\mathcal{B}(\theta; \xi)} \widetilde{H}_{2lj}(\theta; \xi)$, we get

$$\begin{aligned} \|H_{2lj}\|_{\widetilde{\delta}, \mathcal{O}} &\leq e^{\|\mathcal{B}\|_{\widetilde{\delta}, \mathcal{O}}} \|\widetilde{H}_{2lj}\|_{\widetilde{\delta}, \mathcal{O}} \leq c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau} e^{2\|\mathcal{B}\|_{\widetilde{\delta}, \mathcal{O}}} \|R_{lj}\|_{\widetilde{\delta}, \mathcal{O}} \\ &\leq c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau} \mathcal{E}^{-\frac{1}{240}} \|R_{lj}\|_{\widetilde{\delta}, \mathcal{O}}. \end{aligned}$$

For the error term R_{lj}^e , i.e.,

$$R_{lj}^e = e^{iB(\theta;\xi)} \mathcal{R}_K \left(e^{-iB(\theta;\xi)} (ib_{lj}(\theta; \xi)H_{2lj}(\theta; \xi) + R_{lj}(\theta; \xi)) \right),$$

we have the following estimate:

$$\begin{aligned} \|R_{lj}^e\|_{\tilde{\delta}-\sigma, \mathcal{O}} &\leq e^{2\|B\|_{\tilde{\delta}, \mathcal{O}}} e^{-K\sigma} (2\|b\|_{\tilde{\delta}, \mathcal{O}} \|H_{2lj}\|_{\tilde{\delta}, \mathcal{O}} + \|R_{lj}\|_{\tilde{\delta}, \mathcal{O}}) \\ &\leq \mathcal{E}^{-\frac{1}{240}} e^{-K\sigma} (2\|b\|_{\tilde{\delta}, \mathcal{O}} \|H_{2lj}\|_{\tilde{\delta}, \mathcal{O}} + \|R_{lj}\|_{\tilde{\delta}, \mathcal{O}}). \end{aligned}$$

Note that the case $B(\theta; \xi) = 0$ means that there is no need to define \tilde{H}_{2lj} or \tilde{R}_{lj} . In this case, we directly deal with the equations for H_{2lj} and R_{lj} to obtain the estimates for H_{2lj} and R_{lj}^e . \square

In the next proposition, we study the homological equation (4.15) using the non-degeneracy condition (3.2). Since the real part of $\lambda_j(\xi)$ satisfies $|\operatorname{Re} \lambda_j(\xi)| > \varrho_1$, there is no small divisor.

Proposition 4.4 *For $\delta_* > 0$, there exist positive constants $c_4 := c_4(\varrho_1, \varrho_2)$ and sufficiently small $\epsilon_2 > 0$ such that for every $\tilde{\delta}$ with $0 < \delta_* < \tilde{\delta} \leq \delta(1 - \eta)$. If*

$$\|B\|_{\tilde{\delta}, \mathcal{O}} \leq \epsilon_2, \quad \|b\|_{\tilde{\delta}, \mathcal{O}} \leq \epsilon_2, \quad \|W\|_{\tilde{\delta}, p, p, \mathcal{O}} \leq \epsilon_2, \quad \|w\|_{\tilde{\delta}, p, p, \mathcal{O}} \leq \epsilon_2, \quad (4.24)$$

where $\|\cdot\|_{\tilde{\delta}, p, p, \mathcal{O}}$ is the norm of linear operator from $\ell_{a, p}$ to $\ell_{a, p}$, then the homological equation (4.15) has an exact solution F_2 satisfying

$$\|F_2\|_{\tilde{\delta}, p, \mathcal{O}} \leq c_4 \left\| \frac{\partial \tilde{g}^0}{\partial \rho} \right\|_{\tilde{\delta}, p, \mathcal{O}},$$

where $\|\cdot\|_{\tilde{\delta}, p, \mathcal{O}}$ is the norm of linear operator from \mathbb{C}^d to $\ell_{a, p}$.

Proof Let matrix $U := \frac{\partial \tilde{g}^0}{\partial \rho} = \left(\frac{\partial \tilde{g}_l^0}{\partial \rho_j} \right)_{l \in \mathcal{J}_2, 1 \leq j \leq d}$ and set $\mathcal{F} := (F_{21}^T, F_{22}^T, \dots, F_{2d}^T)^T$, where F_{2j} is the j th column vector of F_2 for $1 \leq j \leq d$. We regard the infinite-dimensional matrices F_2 , U as vectors \mathcal{F} and \mathcal{U} , respectively. Then the homological equation (4.15) is equivalent to the vector-valued equation

$$\partial_\omega \mathcal{F} - (E_{\mathcal{J}_1} \otimes (\Lambda + W + w) - i(\Omega + B + b) \otimes E_{\mathcal{J}_2}) \mathcal{F} = \mathcal{U}, \quad (4.25)$$

where \otimes is the tensor product of two matrices (see [22] for details) and $E_{\mathcal{J}_1}$ ($E_{\mathcal{J}_2}$) is the identity matrix of $d(\infty)$ dimensions.

Let

$$\begin{aligned} \mathcal{F} &= \sum_{k \in \mathbb{Z}^2} \widehat{\mathcal{F}}(k) e^{i(k, \theta)}, & \mathcal{U} &= \sum_{k \in \mathbb{Z}^2} \widehat{\mathcal{U}}(k) e^{i(k, \theta)}, \\ W_{lj} &= \sum_{k \in \mathbb{Z}^2} \widehat{W}_{lj}(k) e^{i(k, \theta)}, & w_{lj} &= \sum_{k \in \mathbb{Z}^2} \widehat{w}_{lj}(k) e^{i(k, \theta)}, \quad l, j \in \mathcal{J}_2, \\ B_j &= \sum_{k \in \mathbb{Z}^2} \widehat{B}_j(k) e^{i(k, \theta)}, & b_j &= \sum_{k \in \mathbb{Z}^2} \widehat{b}_j(k) e^{i(k, \theta)}, \quad 1 \leq j \leq d, \end{aligned}$$

where W_{lj} is the (l, j) th element of matrix W .

By comparing the Fourier coefficients of Eq. (4.25), we obtain

$$(E_{\mathcal{J}_1} \otimes (i(k, \omega)E_{\mathcal{J}_2} - \Lambda) + i\Omega \otimes E_{\mathcal{J}_2})\widehat{\mathcal{F}}(k) + \sum_{k_1} \widehat{S}(k - k_1)\widehat{\mathcal{F}}(k_1) = \widehat{\mathcal{U}}(k),$$

where $\widehat{S}(k - k_1) := -E_{\mathcal{J}_1} \otimes (\widehat{W}(k - k_1) + \widehat{w}(k - k_1)) + i(\widehat{B}(k - k_1) + \widehat{b}(k - k_1)) \otimes E_{\mathcal{J}_2}$, which can be viewed as the vector equation

$$(T + S)\mathfrak{F} = \mathfrak{U}$$

with

$$\begin{aligned} T &= \text{diag}(E_{\mathcal{J}_1} \otimes (i(k, \omega)E_{\mathcal{J}_2} - \Lambda) + i\Omega \otimes E_{\mathcal{J}_2})_{k \in \mathbb{Z}^2}, \\ S &= (\widehat{S}(k_1 - k_2))_{k_1, k_2 \in \mathbb{Z}^2}, \\ \mathfrak{F} &= (\widehat{\mathcal{F}}(k))_{k \in \mathbb{Z}^2}^T, \quad \mathfrak{U} = (\widehat{\mathcal{U}}(k))_{k \in \mathbb{Z}^2}^T. \end{aligned}$$

Denote

$$E_{\tilde{\delta}} = \text{diag}(\dots, e^{|k|\tilde{\delta}} e^{a|l|} \langle l \rangle^p, \dots)_{l \in \mathcal{J}_2, k \in \mathbb{Z}^2} \otimes E_{\mathcal{J}_1},$$

then

$$(T + E_{\tilde{\delta}}SE_{\tilde{\delta}}^{-1})E_{\tilde{\delta}}\mathfrak{F} = E_{\tilde{\delta}}\mathfrak{U}.$$

It follows from the non-degeneracy condition for $\Lambda(\xi)$ in (3.2) that

$$\begin{aligned} \|T^{-1}\|_{\mathcal{O}} &= \max_{k \in \mathbb{Z}^2} \sup_{\xi \in \mathcal{O}} \max_{\substack{l \in \mathcal{J}_2 \\ 1 \leq j \leq d}} \left\{ \frac{1}{|\text{i}(\langle k, \omega \rangle + \Omega_j(\xi)) + \lambda_l(\xi)|} + \frac{|\frac{\partial \Omega_j(\xi)}{\partial \xi}| + |\frac{\partial \lambda_l(\xi)}{\partial \xi}|}{|\text{i}(\langle k, \omega \rangle + \Omega_j(\xi)) + \lambda_l(\xi)|^2} \right\} \\ &\leq \sup_{\xi \in \mathcal{O}} \max_{l \in \mathcal{J}_2} \left\{ \frac{1}{|\text{Re } \lambda_l(\xi)|} + \frac{2 + |\frac{\partial \lambda_l(\xi)}{\partial \xi}|}{|\text{Re } \lambda_l(\xi)|^2} \right\} \leq \frac{1}{8}c_4. \end{aligned}$$

Moreover,

$$\begin{aligned} \|E_{\tilde{\delta}}SE_{\tilde{\delta}}^{-1}\|_{\mathcal{O}} &= \max_{k \in \mathbb{Z}^2} \sum_{k_1 \in \mathbb{Z}^2} e^{(|k|-|k_1|)\tilde{\delta}} \|\widehat{W}(k - k_1; \xi) + \widehat{w}(k - k_1; \xi)\|_{\tilde{\delta}, p, \mathcal{O}} \\ &\quad + \max_{k \in \mathbb{Z}^2} \max_{1 \leq j \leq d} \sum_{k_1 \in \mathbb{Z}^2} |e^{(|k|-|k_1|)\tilde{\delta}} (\widehat{B}_j(k_1) + \widehat{b}_j(k_1))|_{\mathcal{O}} \\ &\leq \|B\|_{\tilde{\delta}, \mathcal{O}} + \|b\|_{\tilde{\delta}, \mathcal{O}} + \|W\|_{\tilde{\delta}, p, \mathcal{O}} + \|w\|_{\tilde{\delta}, p, \mathcal{O}}. \end{aligned}$$

If we take $\epsilon_2 \leq \frac{1}{c_4}$, by (4.24), then we get

$$\|T^{-1}E_{\tilde{\delta}}SE_{\tilde{\delta}}^{-1}\|_{\mathcal{O}} \leq \frac{1}{8}c_4 (\|B\|_{\tilde{\delta}, \mathcal{O}} + \|b\|_{\tilde{\delta}, \mathcal{O}} + \|W\|_{\tilde{\delta}, p, \mathcal{O}} + \|w\|_{\tilde{\delta}, p, \mathcal{O}}) < \frac{1}{2}.$$

This implies $T + E_{\tilde{\delta}}SE_{\tilde{\delta}}^{-1}$ has a bounded inverse. Therefore,

$$\begin{aligned} \|\mathcal{F}(\theta; \xi)\|_{\tilde{\delta}, \mathcal{O}} &= \sum_{j=1}^d \sum_{k \in \mathbb{Z}^2} \|\widehat{F}_{2j}(k; \xi)\|_{a,p, \mathcal{O}} e^{|k|\tilde{\delta}} = \|E_{\tilde{\delta}}\mathfrak{F}\|_{\mathcal{O}} \\ &\leq \|(T + E_{\tilde{\delta}}SE_{\tilde{\delta}}^{-1})^{-1}\|_{\mathcal{O}} \|E_{\tilde{\delta}}\mathcal{U}\|_{\mathcal{O}} \\ &\leq \frac{1}{4}c_4 \|\mathcal{U}(\theta; \xi)\|_{\tilde{\delta}, \mathcal{O}}. \end{aligned}$$

As a conclusion, we get

$$\|F_2\|_{\tilde{\delta}, p, \mathcal{O}} \leq c_4 \left\| \frac{\partial \tilde{g}^0}{\partial \rho} \right\|_{\tilde{\delta}, p, \mathcal{O}}. \quad \square$$

Similarly, we can deal with other homological equations and obtain the approximate or exact solutions with estimates respectively. After this, we can get a new system with a new perturbation as follows:

$$p_+ = (4.7) + R^{(pe)}, \quad g_+ = (4.10), \tag{4.26}$$

where $R^{(pe)}$ is the error term from solving the homological equations. In the proof of Proposition 5.1, we will prove that the above perturbation is smaller on a small domain.

5 Proof of Theorem 3.1

In this section, we will give the proof of Theorem 3.1. We will give one KAM step in detail, which needs finite Newton iteration. After one step of Newton iteration, the perturbation is smaller than that in previous step. Via finite steps of transformation, the perturbation is small enough to meet the KAM iterative requirement. So we can set up one cycle of KAM scheme. This is the essential difference from the classical KAM iteration with the Diophantine or Brjuno conditions.

For simplifying our notations, we drop the subscript n and write the symbol “+” for $(n + 1)$. Suppose at the n th step of the KAM scheme, we have the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i(\Omega(\xi) + B(\theta; \xi))\rho + p(\theta, \rho, z; \xi), \\ \dot{z} = (\Lambda(\xi) + W(\theta; \xi))z + g(\theta, \rho, z; \xi), \end{cases} \tag{5.1}$$

defined on $D(\delta, s) \times \mathcal{O}$, where the perturbation satisfies $\|(0, p, g)\|_{s, D(\delta, s) \times \mathcal{O}} \leq \varepsilon$ and B, W have the following form:

$$B(\theta; \xi) = \text{diag}(B_1(\theta; \xi), \dots, B_d(\theta; \xi)), \quad W(\theta; \xi) = (W_{ij}(\theta; \xi))_{i,j \in \mathcal{J}_2}.$$

Our goal is to find an analytic transformation $\Phi : D(\delta_+, s_+) \times \mathcal{O} \rightarrow D(\delta, s) \times \mathcal{O}$ such that the transformed system of (5.1) is of the form

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\rho}_+ = i(\Omega(\xi) + B_+(\theta; \xi))\rho_+ + p_+(\theta, \rho_+, z_+; \xi), \\ \dot{z}_+ = (\Lambda(\xi) + W_+(\theta; \xi))z_+ + g_+(\theta, \rho_+, z_+; \xi), \end{cases} \tag{5.2}$$

where the norm of new perturbation p_+ and g_+ in (5.2) on a small domain $D(\delta_+, s_+)$ is smaller than (5.1) (see Proposition 5.1).

In the following, we need some notations:

$$\begin{aligned} \delta_+ &= \delta(1 - \eta)^2, & \mathcal{E}_+ &= e^{-c_0 \gamma \delta_+ \left(\frac{\bar{Q}_{n+1}}{Q_{n+1}^M} + \bar{Q}_{n+1}^{\frac{1}{M/4}} \right)}, \\ \varepsilon_+ &= \mathcal{E}_+ \varepsilon, & s_+ &= \varepsilon^{\frac{7}{6}} \left[\left(\frac{8}{7} \right)^L - 1 \right]. \end{aligned}$$

where L is a positive integer satisfying

$$\log_{\frac{8}{7}} \left(1 + \frac{\ln \mathcal{E}_+}{\ln \varepsilon} \right) \leq L < 1 + \log_{\frac{8}{7}} \left(1 + \frac{\ln \mathcal{E}_+}{\ln \varepsilon} \right). \tag{5.3}$$

Proposition 5.1 *Consider the system (5.1) with $\mathcal{R}_{Q_{n+1}} B = 0$. For every $0 < \gamma < 1$, $\tau > 2$, $\delta > \delta_* > 0$, $1 > s > 0$, there exist positive constants $\epsilon_0 = \epsilon_0(\delta_*, \tau, \tilde{c}, d)$, $\epsilon_1 = \epsilon_1(\delta_*, \tau, \tilde{c})$, ϵ_2 and $J = J(\tau)$ such that if $\Omega + [B]_\theta \in DC_\omega(\gamma, \tau, K, \mathcal{O})$ and*

$$\begin{aligned} \|B\|_{\delta, \mathcal{O}} \cdot \left(\frac{\bar{Q}_n}{Q_n^{A^4}} + \bar{Q}_n^{\frac{1}{A}} \right) &\leq \epsilon_1 \gamma \left(\frac{\bar{Q}_n}{Q_n^M} + \bar{Q}_n^{\frac{1}{M/4}} \right), \\ \|W\|_{\delta, p, \mathcal{O}} \leq \epsilon_2, & \quad \|P\|_{s, D(\delta, s) \times \mathcal{O}} \leq \varepsilon \leq \epsilon_0 \gamma^J \mathcal{E}, \quad P := (0, p, g), \end{aligned} \tag{5.4}$$

then there is an analytic, nearly identity transformation

$$\Phi : D(\delta_+, s_+) \times \mathcal{O} \rightarrow D(\delta, s) \times \mathcal{O}$$

such that it transforms system (5.1) into system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho}_+ = i(\Omega(\xi) + B_+(\theta; \xi))\rho_+ + p_+(\theta, \rho_+, z_+; \xi), \\ \dot{z}_+ = (\Lambda(\xi) + W_+(\theta; \xi))z_+ + g_+(\theta, \rho_+, z_+; \xi), \end{cases}$$

with

$$\begin{aligned} \mathcal{R}_{Q_{n+2}} B_+ &= 0, \\ \|B_+ - B\|_{\delta_+, \mathcal{O}}, \|W_+ - W\|_{\delta_+, p, \mathcal{O}} &\leq 2\varepsilon, \\ \|P_+\|_{s_+, D(\delta_+, s_+) \times \mathcal{O}} \leq \varepsilon_+, & \quad P_+ := (0, p_+, g_+). \end{aligned}$$

Moreover, the transformation Φ satisfies

$$\|\Phi - id\|_{s_+, D(\delta_+, s_+) \times \mathcal{O}} \leq 4\varepsilon^{\frac{2}{3}}, \tag{5.5}$$

$$\|D\Phi - Id\|_{s_+, D(\delta_+, s_+) \times \mathcal{O}} \leq 4\varepsilon^{\frac{1}{2}}. \tag{5.6}$$

The above proposition plays an important role to run one KAM step. In the following, we use a Newtonian iterative procedure consisting of finite steps to prove Proposition 5.1.

Firstly, let

$$\tilde{\varepsilon}_0 = \varepsilon, \quad \tilde{\delta}_0 = \delta(1 - \eta), \quad \tilde{s}_0 = s.$$

Then, for $1 \leq j \leq L$, we define the following sequences:

$$\tilde{\varepsilon}_j = \tilde{\varepsilon}_{j-1}^{\frac{3}{2}}, \quad \tilde{\delta}_j = \tilde{\delta}_{j-1} - 2\tilde{\delta}_0\sigma_{j-1}, \quad \iota_j = \tilde{\varepsilon}_j^{\frac{1}{6}}, \quad \tilde{s}_j = \iota_{j-1}\tilde{s}_{j-1}, \tag{5.7}$$

where

$$\sigma_j = \begin{cases} \frac{\eta}{(2+\tilde{\varepsilon}_0^2)^{j+5}}, & j < j_0, \\ -\frac{\ln \tilde{\varepsilon}_j}{K\tilde{\delta}_0}, & j \geq j_0, \end{cases} \tag{5.8}$$

with

$$j_0 = \min\{j \in \mathbb{N} : K\tilde{\varepsilon}_j^{\frac{1}{20}} < 1\}. \tag{5.9}$$

In the following, we give some inequalities for the above sequences, which we use frequently in the proof.

Lemma 5.2 *There exist positive constants $J = J(\tau)$, $\epsilon_0 = \epsilon_0(\delta_*, \tau, \tilde{c}, d)$ and $T_0 = T_0(\delta_*, \tau, \tilde{c})$ such that if*

$$\tilde{\varepsilon}_0 \leq \epsilon_0\gamma^J \mathcal{E}, \quad \overline{Q}_{n+1} \geq T_0\gamma^{-A/2}, \tag{5.10}$$

then we have

$$\tilde{\varepsilon}_0 \leq \min\left\{Q_{n+1}^{-120\tau}, \left(\frac{\gamma^{-A\tau+2}}{16c_3(\tau)}\right)^{60}\right\}, \tag{5.11}$$

$$e^{-K\tilde{\delta}_0\sigma_j} \leq \tilde{\varepsilon}_j, \tag{5.12}$$

and

$$\tilde{\varepsilon}_j \leq \left(\frac{1}{4c_4}\tilde{\delta}_0\sigma_j\right)^{20}. \tag{5.13}$$

Proof For inequality (5.11), we only prove $\tilde{\varepsilon}_0 \leq Q_{n+1}^{-120\tau}$ since the other one can be verified similarly. Let $J(\tau) = [120\tau\mathcal{A}^5]$. By the definition of \mathcal{E} in (4.16), we can obtain

$$\mathcal{E} = e^{-c_0\gamma\delta\left(\frac{\overline{Q}_n}{Q_n^M} + \overline{Q}_n^{M^{1/4}}\right)} \leq e^{-c_0\gamma\delta\overline{Q}_n^{M^{1/4}}} \leq \frac{J!}{(c_0\gamma\delta\overline{Q}_n^{M-1/4})^J},$$

since $e^x \geq \frac{x^n}{n!}$ for $x > 0$ and $n \in \mathbb{N}$. Due to $Q_{n+1} \leq \overline{Q}_n^{A^4}$ in Lemma 2.2, there exists a constant $0 < \epsilon_3 = \epsilon_3(\delta_*, \tau, \tilde{c}, d)$ such that if $\epsilon_0 \leq \epsilon_3$,

$$\tilde{\varepsilon}_0 \leq \epsilon_0\gamma^J \mathcal{E} \leq \frac{\epsilon_3\gamma^J J!}{(c_0\gamma\delta\overline{Q}_n^{M-1/4})^J} \leq \frac{\epsilon_3 J!}{(c_0\delta_*)^J Q_{n+1}^{120\tau}} \leq Q_{n+1}^{-120\tau}.$$

Now consider the inequality (5.12). When $j \geq j_0$, it is obvious by the definition of σ_j in (5.8). When $0 \leq j < j_0$, we have

$$K \geq \left(\frac{1}{\tilde{\varepsilon}_j}\right)^{\frac{1}{20}} \geq \left(\frac{1}{\tilde{\varepsilon}_j}\right)^{\frac{1}{40}} \left(\ln \frac{1}{\tilde{\varepsilon}_j}\right)^2 \tag{5.14}$$

due to the smallness of $\tilde{\varepsilon}_0 \geq \tilde{\varepsilon}_j$ and the choice of j_0 . Moreover, by the definition of K in (4.16) and $\bar{Q}_{n+1} \geq T_0 \gamma^{-A/2}$, we can get

$$K^{\frac{1}{2}} \delta_* \geq \tilde{c}^{-1} Q_n^{\frac{1}{2A^4}} = \eta^{-1} \tag{5.15}$$

for sufficiently large T_0 . By the definition of $\tilde{\varepsilon}_j$, we have

$$\tilde{\varepsilon}_j^{-\frac{1}{80}} \geq (2 + \tilde{c}^2 \tilde{\delta}_0^2)^{j+5}.$$

It follows from (5.14) and (5.15) that

$$K \tilde{\delta}_0 \sigma_j \geq \frac{K \delta_* \eta}{(2 + \tilde{c}^2 \tilde{\delta}_0^2)^{j+5}} \geq \frac{K^{\frac{1}{2}}}{(2 + \tilde{c}^2 \tilde{\delta}_0^2)^{j+5}} \geq \frac{1}{(2 + \tilde{c}^2 \tilde{\delta}_0^2)^{j+5}} \left(\frac{1}{\tilde{\varepsilon}_j}\right)^{\frac{1}{80}} \ln \frac{1}{\tilde{\varepsilon}_j} \geq \ln \frac{1}{\tilde{\varepsilon}_j}.$$

Consider (5.13). When $j \geq j_0$, it is obvious from $K \tilde{\varepsilon}_j^{\frac{1}{20}} < 1$ by the choice of j_0 . In the case of $j < j_0$, by (5.10) and (5.11), there exists $0 < \epsilon_4 = \epsilon_4(\tau, \delta_*)$ such that if $\epsilon_0 \leq \min\{\epsilon_3, \epsilon_4\}$, then

$$\begin{aligned} \tilde{\varepsilon}_j^{\frac{1}{20}} &= \tilde{\varepsilon}_0^{\frac{1}{20} \cdot (\frac{8}{7})^j} \leq \epsilon_0^{\frac{1}{20} \cdot \frac{4}{5} \cdot (\frac{8}{7})^j} \cdot Q_{n+1}^{-120\tau \cdot \frac{1}{20} \cdot \frac{1}{5} \cdot (\frac{8}{7})^j} \leq \epsilon_0^{\frac{1}{25} \cdot (\frac{8}{7})^j} \cdot Q_{n+1}^{-\frac{6\tau}{5} \cdot (\frac{8}{7})^j} \\ &\leq \frac{\tilde{\delta}_0 \tilde{c}}{4c_4 \cdot (2 + \tilde{c}^2 \tilde{\delta}_0^2)^{j+5}} Q_n^{-\frac{1}{2A^4}} = \frac{1}{4c_4} \tilde{\delta}_0 \sigma_j. \end{aligned} \quad \square$$

5.1 A finite inductive lemma

We now give the following iterative lemma for a finite induction, which is used to prove Proposition 5.1.

In the following, we will denote by c a constant depending only on τ, d , but not on the iterative step number j .

Lemma 5.3 *Suppose that $\tilde{\varepsilon}_0$ satisfies assumptions in Lemma 5.2 and the system*

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i(\Omega(\xi) + B(\theta; \xi) + b_j(\theta; \xi))\rho + p_j(\theta, \rho, z; \xi), \\ \dot{z} = (\Lambda(\xi) + W(\theta; \xi) + w_j(\theta; \xi))z + g_j(\theta, \rho, z; \xi), \end{cases} \tag{5.16}$$

defined on $D(\tilde{\delta}_j, \tilde{\varepsilon}_j) \times \mathcal{O}$, where

$$b_j(\theta; \xi) = \text{diag}(b_{j,1}(\theta; \xi), \dots, b_{j,d}(\theta; \xi)), \quad w_j(\theta; \xi) = (w_{j,il}(\theta; \xi))_{i,l \in \mathcal{J}_2},$$

satisfies the conditions in Proposition 5.1 for B, W and

$$\mathcal{R}_{Q_{n+2}} b_j = 0,$$

$$\|b_j\|_{\tilde{\delta}_j, \mathcal{O}}, \|w_j\|_{\tilde{\delta}_j, p, p, \mathcal{O}} \leq \sum_{m=0}^{j-1} \tilde{\varepsilon}_m, \tag{5.17}$$

$$\|\tilde{P}_j\|_{\tilde{s}_j, D(\tilde{\delta}_j, \tilde{s}_j) \times \mathcal{O}} \leq \tilde{\varepsilon}_j, \quad \tilde{P}_j := (0, p_j, g_j).$$

Then there is an analytic transformation

$$\tilde{\Phi}_j : D(\tilde{\delta}_{j+1}, \tilde{s}_{j+1}) \times \mathcal{O} \rightarrow D(\tilde{\delta}_j, \tilde{s}_j) \times \mathcal{O}$$

satisfying

$$\|\tilde{\Phi}_j - id\|_{\tilde{s}_{j+1}, D(\tilde{\delta}_{j+1}, \tilde{s}_{j+1}) \times \mathcal{O}} \leq \tilde{\varepsilon}_j^{\frac{2}{3}}, \tag{5.18}$$

$$\|D\tilde{\Phi}_j - Id\|_{\tilde{s}_{j+1}, D(\tilde{\delta}_{j+1}, \tilde{s}_{j+1}) \times \mathcal{O}} \leq \tilde{\varepsilon}_j^{\frac{1}{2}}, \tag{5.19}$$

such that the transformed system of (5.16) is the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i(\Omega(\xi) + B(\theta; \xi) + b_{j+1}(\theta; \xi))\rho + p_{j+1}(\theta, \rho, z; \xi), \\ \dot{z} = (\Lambda(\xi) + W(\theta; \xi) + w_{j+1}(\theta; \xi))z + g_{j+1}(\theta, \rho, z; \xi), \end{cases} \tag{5.20}$$

defined on $D(\tilde{\delta}_{j+1}, \tilde{s}_{j+1}) \times \mathcal{O}$, and satisfies the same assumption (5.17) with $(j + 1)$ in place of j for $0 \leq j \leq L - 1$.

Proof (1) Firstly, we split the perturbation into three parts in the following way:

For the vector-valued function p_j ,

$$p_j = p_j^{(el)} + p_j^{(nf)} + p_j^{(pe)},$$

where

$$\begin{aligned} p_j^{(el)}(\theta, \rho, z; \xi) &= \sum_{|k| < K} \widehat{p}_j^0(k; \xi) e^{i(k, \theta)} + \left(\sum_{l=1, l \neq j}^d \sum_{|k| < K} \frac{\partial \widehat{p}_{j,l}^0}{\partial \rho_l}(k; \xi) e^{i(k, \theta)} \rho_l \right)_{1 \leq l \leq d} \\ &\quad + \sum_{k \in \mathbb{Z}^2} \frac{\partial \widehat{p}_j^0}{\partial z}(k; \xi) e^{i(k, \theta)} z, \\ p_j^{(nf)}(\theta, \rho, z; \xi) &= \left(\sum_{|k| < K} \frac{\partial \widehat{p}_{j,l}^0}{\partial \rho_l}(k; \xi) e^{i(k, \theta)} \rho_l \right)_{1 \leq l \leq d}, \\ p_j^{(pe)}(\theta, \rho, z; \xi) &= \sum_{|a| \leq 1, |k| \geq K} \widehat{p}_{j,a,0}(k; \xi) e^{i(k, \theta)} \rho^a + \sum_{|a| + |\alpha| \geq 2, k \in \mathbb{Z}^2} \widehat{p}_{j,a,\alpha}(k; \xi) e^{i(k, \theta)} \rho^a z^\alpha \\ &=: p_j^{(pe1)} + p_j^{(pe2)}, \end{aligned}$$

with

$$\widehat{p}_j^0(k; \xi) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} p(\theta, 0, 0; \xi) e^{-i(k, \theta)} d\theta,$$

and the other terms are similar. We still deal with g_j in the same way:

$$\begin{aligned}
 g_j^{(el)}(\theta, \rho, z; \xi) &= \sum_{k \in \mathbb{Z}^2, |\mathbf{a}| \leq 1} \widehat{g}_{j,\mathbf{a},0}(k; \xi) e^{i(k,\theta)} \rho^{\mathbf{a}}, \\
 g_j^{(nf)}(\theta, \rho, z; \xi) &= \sum_{k \in \mathbb{Z}^2, |\alpha|=1} \widehat{g}_{j,0,\alpha}(k; \xi) e^{i(k,\theta)} z^\alpha, \\
 g_j^{(pe)}(\theta, \rho, z; \xi) &= \sum_{k \in \mathbb{Z}^2, |\mathbf{a}|+|\alpha| \geq 2} \widehat{g}_{j,\mathbf{a},\alpha}(k; \xi) e^{i(k,\theta)} \rho^{\mathbf{a}} z^\alpha.
 \end{aligned}$$

Then we can rewrite the system (5.16) as

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i(\Omega(\xi) + B(\theta; \xi) + b_{j+1}(\theta; \xi))\rho + (p_j^{(el)} + p_j^{(pe)})(\theta, \rho, z; \xi), \\ \dot{z} = (\Lambda(\xi) + W(\theta; \xi) + w_{j+1}(\theta; \xi))z + (g_j^{(el)} + g_j^{(pe)})(\theta, \rho, z; \xi), \end{cases} \tag{5.21}$$

where

$$\begin{aligned}
 b_{j+1}(\theta; \xi) &:= b_j(\theta; \xi) - i \frac{\partial p_j^{(nf)}}{\partial \rho} \Big|_{\rho=0, z=0}, \\
 w_{j+1}(\theta; \xi) &:= w_j(\theta; \xi) + \frac{\partial g_j^{(nf)}}{\partial z} \Big|_{\rho=0, z=0}.
 \end{aligned}$$

Thus, $\mathcal{R}_{Q_{n+2}} b_{j+1} = 0$ since $K < Q_{n+2}$ by the choice of K in (4.16). Moreover, by (5.11) and (5.17), we have

$$\|b_{j+1}\|_{\widetilde{\delta}_j, \mathcal{O}}, \|w_{j+1}\|_{\widetilde{\delta}_j, p, p, \mathcal{O}} \leq \sum_{m=0}^j \widetilde{\varepsilon}_m \leq 2\widetilde{\varepsilon}_0 \leq \frac{\gamma^{-A\tau+2}}{2c_3(\tau)Q_{n+1}^{6\tau}}, \tag{5.22}$$

since $\widetilde{\varepsilon}_0$ can be chosen sufficiently small. It is obvious that the perturbation $\widetilde{P} := (0, p_j^{(el)} + p_j^{(pe)}, g_j^{(el)} + g_j^{(pe)})$ satisfies

$$\|\widetilde{P}\|_{\widetilde{\delta}_j, D(\widetilde{\delta}_j, \widetilde{\delta}_j) \times \mathcal{O}} \leq \|\widetilde{P}_j\|_{\widetilde{\delta}_j, D(\widetilde{\delta}_j, \widetilde{\delta}_j) \times \mathcal{O}} \leq \widetilde{\varepsilon}_j.$$

(2) Secondly, we construct the transformation $\widetilde{\Phi}_j$. Suppose that the desired change of variables $\widetilde{\Phi}_j$ has the form of (4.3):

$$\widetilde{\Phi}_j : \begin{cases} \rho = \rho_+ + H_1(\theta; \xi) + H_2(\theta; \xi)\rho_+ + H_3(\theta; \xi)z_+, \\ z = z_+ + F_1(\theta; \xi) + F_2(\theta; \xi)\rho_+. \end{cases} \tag{5.23}$$

It transforms the system (5.21) into the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho}_+ = i(\Omega(\xi) + B(\theta; \xi) + b_{j+1}(\theta; \xi))\rho_+ + p_{j+1}(\theta, \rho_+, z_+; \xi), \\ \dot{z}_+ = (\Lambda(\xi) + W(\theta; \xi) + w_{j+1}(\theta; \xi))z_+ + g_{j+1}(\theta, \rho_+, z_+; \xi). \end{cases} \tag{5.24}$$

Via the discussion in Sect. 4.1, the unknown functions in (5.23) can be obtained by solving the homological equations like (4.11)–(4.15).

By the assumptions in Lemma 5.3 and (5.22), B, W, w satisfy (4.24) and b satisfies the condition (4.20) due to the smallness of $\tilde{\epsilon}_0, \epsilon_1$ and ϵ_2 . Then, by Proposition 4.3 and Proposition 4.4, we can get the approximate or exact solutions to homological equations with estimates respectively. For convenience, we denote $\tilde{D}_j = D(\tilde{\delta}_j, \tilde{s}_j)$ in what follows.

For example, by Proposition 4.3, we can obtain an approximate solution $H_2 = (H_{2il})_{1 \leq i, l \leq d}$ with estimates

$$\|H_{2il}\|_{\tilde{\delta}_j, \mathcal{O}} \leq c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau} \mathcal{E}^{-\frac{1}{240}} \left\| \frac{\partial p_{j,i}^{(el)}}{\partial \rho_l} \right\|_{\tilde{\delta}_j, \mathcal{O}}$$

and the corresponding error term $R_2^{(pe)} = (R_{2il}^{(pe)})_{1 \leq i, l \leq d}$ satisfies

$$\|R_{2il}^{(pe)}\|_{\tilde{\delta}_j - \tilde{\delta}_0 \sigma_j, \mathcal{O}} \leq 2\mathcal{E}^{-\frac{1}{240}} e^{-K\tilde{\delta}_0 \sigma_j} \left\| \frac{\partial p_{j,i}^{(el)}}{\partial \rho_l} \right\|_{\tilde{\delta}_j, \mathcal{O}}.$$

Thus, we can obtain

$$\begin{aligned} \|H_2\|_{\tilde{\delta}_j, \mathcal{O}} &\leq c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau} \mathcal{E}^{-\frac{1}{240}} \left\| \frac{\partial p_j^{(el)}}{\partial \rho} \right\|_{\tilde{\delta}_j, \mathcal{O}}, \\ \|R_2^{(pe)}\|_{\tilde{\delta}_j - \tilde{\delta}_0 \sigma_j, \mathcal{O}} &\leq 2\mathcal{E}^{-\frac{1}{240}} e^{-K\tilde{\delta}_0 \sigma_j} \left\| \frac{\partial p_j^{(el)}}{\partial \rho} \right\|_{\tilde{\delta}_j, \mathcal{O}}. \end{aligned}$$

Similarly, we can get

$$\|H_1\|_{\tilde{\delta}_j, \mathcal{O}} \leq c_3(\tau) \gamma^{-(A\tau+2)} Q_{n+1}^{6\tau} \mathcal{E}^{-\frac{1}{240}} \|p_j^0\|_{\tilde{\delta}_j, \mathcal{O}},$$

where $p_j^0 := p_j(\theta, 0, 0; \xi)$. Moreover, the estimates of the error terms are

$$\|R_1^{(pe)}\|_{\tilde{\delta}_j, \mathcal{O}} \leq 2\mathcal{E}^{-\frac{1}{240}} e^{-K\tilde{\delta}_0 \sigma_j} \|p_j^0\|_{\tilde{\delta}_j, \mathcal{O}}.$$

By Proposition 4.4, we can get exact solutions for H_3 and F_i ($i = 1, 2$) with estimates

$$\begin{aligned} \|H_3\|_{\tilde{\delta}_j, \mathcal{O}} &\leq c \left\| \frac{\partial p^{(el)}}{\partial z} \right\|_{\tilde{\delta}_j, \mathcal{O}}, & \|F_2\|_{\tilde{\delta}_j, p, \mathcal{O}} &\leq c \left\| \frac{\partial g^{(el)}}{\partial \rho} \right\|_{\tilde{\delta}_j, \mathcal{O}}, \\ \|F_1\|_{a, p, \tilde{\delta}_j, \mathcal{O}} &\leq c \|g^0\|_{a, p, \tilde{\delta}_j, \mathcal{O}}, & g^0 &:= g(\theta, 0, 0; \xi). \end{aligned}$$

Since $\|\tilde{P}_j\|_{\tilde{s}_j, D(\tilde{\delta}_j, \tilde{s}_j) \times \mathcal{O}} \leq \tilde{\epsilon}_j$, by the weighted norm of vector field, we have

$$\|p_j\|_{\tilde{D}_j \times \mathcal{O}} \leq \tilde{s}_j \tilde{\epsilon}_j, \quad \|g_j\|_{a, p, \tilde{D}_j \times \mathcal{O}} \leq \tilde{s}_j \tilde{\epsilon}_j.$$

Thus by Cauchy’s estimate and the inequalities in Lemma 5.2, we have

$$\begin{aligned} \|H_1\|_{\tilde{\delta}_{j+1}, \mathcal{O}} &\leq c_3(\tau)\gamma^{-(A\tau+2)}Q_{n+1}^{6\tau}\mathcal{E}^{-\frac{1}{240}}\tilde{s}_j\tilde{\varepsilon}_j \leq \tilde{\varepsilon}_j^{\frac{5}{6}}\tilde{s}_j, \\ \|H_2\rho_+\|_{\tilde{D}_{j+1}\times\mathcal{O}} &\leq 2c_3(\tau)\gamma^{-(A\tau+2)}Q_{n+1}^{6\tau}\mathcal{E}^{-\frac{1}{240}}\tilde{s}_{j+1}\tilde{\varepsilon}_j \leq \tilde{\varepsilon}_j\tilde{s}_j, \\ \|H_3z_+\|_{\tilde{D}_{j+1}\times\mathcal{O}} &\leq \tilde{c}\tilde{s}_{j+1}\tilde{\varepsilon}_j. \end{aligned}$$

For F_1, F_2, F_3 , we can obtain

$$\|F_1\|_{a,p,\tilde{D}_{j+1}\times\mathcal{O}} \leq \tilde{c}\tilde{s}_j\tilde{\varepsilon}_j, \quad \|F_2\rho_+\|_{a,p,\tilde{D}_{j+1}\times\mathcal{O}} \leq \tilde{c}\tilde{s}_{j+1}\tilde{\varepsilon}_j.$$

Thus, we have

$$\|(\tilde{\Phi}_j - id)_\rho\|_{\tilde{D}_{j+1}\times\mathcal{O}} \leq \tilde{s}_j\tilde{\varepsilon}_j^{\frac{4}{3}}, \quad \|(\tilde{\Phi}_j - id)_z\|_{a,p,\tilde{D}_{j+1}\times\mathcal{O}} \leq \tilde{s}_j\tilde{\varepsilon}_j^{\frac{4}{5}}.$$

That implies $\tilde{\Phi}_j(D(\tilde{\delta}_{j+1}, \tilde{s}_{j+1})) \subseteq D(\tilde{\delta}_j, \tilde{s}_j)$. And we obtain

$$\begin{aligned} \|\tilde{\Phi}_j - id\|_{\tilde{s}_{j+1}, \tilde{D}_{j+1}\times\mathcal{O}} &= \frac{1}{\tilde{s}_{j+1}}\|(\tilde{\Phi}_j - id)_\rho\|_{\tilde{D}_{j+1}\times\mathcal{O}} + \frac{1}{\tilde{s}_{j+1}}\|(\tilde{\Phi}_j - id)_z\|_{a,p,\tilde{D}_{j+1}\times\mathcal{O}} \\ &\leq \tilde{\varepsilon}_j^{\frac{2}{3}}. \end{aligned}$$

Similarly, by Cauchy’s estimate, we can also obtain the tangent map $D(\tilde{\Phi}_j - id)$ of $\tilde{\Phi}_j - id$ satisfying

$$\|D(\tilde{\Phi}_j - Id)\|_{\tilde{s}_{j+1}, \tilde{D}_{j+1}\times\mathcal{O}} \leq \tilde{\varepsilon}_j^{\frac{1}{2}}.$$

(3) Finally, we give the estimate of the new perturbation in detail. From the (4.26) in Sect. 4.2, we can get the new perturbation:

$$\begin{aligned} p_{j+1} &= p_j \circ \tilde{\Phi}_j - \left(p_j^{(el)} + \frac{\partial p_j^{(el)}}{\partial \rho} \rho_+ + \frac{\partial p_j^{(el)}}{\partial z} z_+ \right) + R^{(pe)} - H_2 p_{j+1} - H_3 g_{j+1}, \\ g_{j+1} &= g_j \circ \tilde{\Phi}_j - \left(g_j^{(el)} + \frac{\partial g_j^{(el)}}{\partial \rho} \rho_+ \right) - F_2 p_{j+1}. \end{aligned}$$

We mainly focus on the term p_{j+1} since all others can be dealt with in the same way. For the form of p_{j+1} , we decompose it into five parts:

$$p_{j+1} = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned} I_1 &= p_j \circ \tilde{\Phi}_j - p_j(\theta, \rho_+, z_+; \xi), \\ I_2 &= p_j(\theta, \rho_+, z_+; \xi) - \left(p_j^0 + \frac{\partial p_j^0}{\partial \rho} \rho_+ + \frac{\partial p_j^0(\theta)}{\partial z} z_+ \right), \\ I_3 &= \mathcal{R}_K \left(p_j^0 + \frac{\partial p_j^0}{\partial \rho} \rho_+ + \frac{\partial p_j^0}{\partial z} z_+ \right), \end{aligned}$$

$$I_4 = -H_2(\theta; \xi)p_{j+1} - H_3(\theta; \xi)g_{j+1},$$

$$I_5 = R^{(pe)} := R_1^{(pe)} + R_2^{(pe)}\rho_+.$$

By the mean value theorem, Cauchy’s estimate and the inequalities in Lemma 5.2, we can get

$$\begin{aligned} \|I_1\|_{\tilde{D}_{j+1} \times \mathcal{O}} &\leq c \frac{\tilde{\varepsilon}_j \tilde{s}_j}{\tilde{s}_j} \cdot \tilde{s}_j \tilde{\varepsilon}_j^{\frac{4}{5}} = c \tilde{\varepsilon}_j^{\frac{9}{5}} \tilde{s}_j, \\ \|I_2\|_{\tilde{D}_{j+1} \times \mathcal{O}} &\leq c \frac{\tilde{\varepsilon}_j \tilde{s}_j}{\tilde{s}_j^2} \cdot \tilde{s}_{j+1}^2 = c \tilde{\varepsilon}_j^{\frac{4}{3}} \tilde{s}_j, \\ \|I_3\|_{\tilde{D}_{j+1} \times \mathcal{O}} &\leq e^{-K\tilde{r}_0\sigma_j} \left\| \left(p_j^0 + \frac{\partial p_j^0}{\partial \rho} \rho_+ + \frac{\partial p_j^0}{\partial z} z_+ \right) \right\|_{\tilde{D}_{j+1} \times \mathcal{O}} \\ &\leq ce^{-K\tilde{r}_0\sigma_j} \tilde{\varepsilon}_j \tilde{s}_j \leq c \tilde{\varepsilon}_j^2 \tilde{s}_j, \\ \|I_4\|_{\tilde{D}_{j+1} \times \mathcal{O}} &\leq \|H_2 p_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}} + \|H_3 g_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}} \\ &\leq c \tilde{\varepsilon}_j^{\frac{5}{6}} (\|p_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}} + \|g_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}}), \\ \|I_5\|_{\tilde{D}_{j+1} \times \mathcal{O}} &\leq \|R_1^{(pe)}\|_{\tilde{D}_{j+1} \times \mathcal{O}} + \|R_2^{(pe)}\rho_+\|_{\tilde{D}_{j+1} \times \mathcal{O}} \leq c \tilde{\varepsilon}_j^2 \tilde{s}_j, \end{aligned}$$

where c is a constant independent of j .

Hence, to summarize, we obtain

$$\|p_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}} \leq c \tilde{\varepsilon}_j^{\frac{4}{3}} \tilde{s}_j + c \tilde{\varepsilon}_j^{\frac{5}{6}} (\|p_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}} + \|g_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}}).$$

Similarly, we can obtain the estimate of $\|g_{j+1}\|_{a,p,\tilde{D}_{j+1} \times \mathcal{O}}$.

Thus, we conclude that

$$\frac{1}{\tilde{s}_{j+1}} \|p_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}} + \frac{1}{\tilde{s}_{j+1}} \|g_{j+1}\|_{\tilde{D}_{j+1} \times \mathcal{O}} < \tilde{\varepsilon}_j^{\frac{8}{7}}$$

since $c\tilde{\varepsilon}_j \ll 1$ for sufficiently small $\tilde{\varepsilon}_0$. The proof in the case of $B = 0$ is similar when we assume that $\tilde{\varepsilon}_0 \leq \min\{\varepsilon_0 \gamma^J, Q_{n+1}^{-120\tau}\}$. □

5.2 One KAM step

In this subsection, we complete the proof of Proposition 5.1 by using Lemma 5.3 inductively. Thus, we need to construct a transformation Φ , which transforms the system (5.1) into (5.2), at the n th KAM step.

We point out that if $\tilde{\varepsilon}_0 > \mathcal{E}_+^7$, then $\tilde{\varepsilon}_0^{\frac{8}{7}} > \varepsilon_+ = \mathcal{E}_+ \tilde{\varepsilon}_0$. This means that just via one transformation like (5.23), the perturbation of the transformed system may be bigger than size $O(\mathcal{E}_+)$, which is used to control the solution of the homological equation. Therefore, in order to run one cycle of KAM scheme, we need $L (\geq 2)$ times of induction such that the size of perturbation is smaller than ε_+ . By the choose of L in (5.3), we have $\tilde{\varepsilon}_L \leq \varepsilon_+ < \tilde{\varepsilon}_{L-1}$. Therefore, we terminate the finite induction at L th step.

Now we need to verify $\delta_+ \leq \tilde{\delta}_L$ for $\delta_+ = \tilde{\delta}_0(1 - \eta)$. It is sufficient to show that $\sum_{m=0}^{L-1} \sigma_m \leq \eta/2$. By the definition of σ_j in (5.8), we obtain

$$\sum_{j=0}^{j_0-1} \sigma_j = \sum_{j=0}^{j_0-1} \frac{\eta}{(2 + \tilde{c}\tilde{\delta}_0^2)^{j+5}} \leq \eta \sum_{j=0}^{j_0-1} \frac{1}{2^{j+5}} \leq \frac{\eta}{4}.$$

And if $L > j_0$,

$$\begin{aligned} \sum_{j=j_0}^{L-1} \sigma_j &= -\frac{8 \ln \tilde{\varepsilon}_{L-1}}{K\tilde{r}_0} + \frac{7 \ln \tilde{\varepsilon}_{j_0}}{K\tilde{r}_0} \leq -\frac{8 \ln(\mathcal{E}_+ \tilde{\varepsilon}_0)}{K\tilde{r}_0} + \frac{7 \ln \tilde{\varepsilon}_{j_0}}{K\tilde{r}_0} \\ &\leq -\frac{8 \ln \mathcal{E}_+}{K\tilde{r}_0} - \frac{\ln \tilde{\varepsilon}_0}{K\tilde{r}_0} \leq -\frac{15 \ln \mathcal{E}_+}{K\tilde{r}_0} \\ &\leq \frac{\tilde{c}}{8} \left(Q_{n+1}^{\tau-M} + \overline{Q}_{n+1}^{-\frac{1}{A}} \right) \leq \frac{\tilde{c}}{4} Q_n^{-\frac{1}{2A^4}} = \frac{\eta}{4} \end{aligned}$$

by $\tilde{\varepsilon}_{L-1} > \varepsilon_+ = \mathcal{E}_+ \tilde{\varepsilon}_0$ provided $\tilde{\varepsilon}_0 > \mathcal{E}_+^7$. Therefore,

$$\sum_{m=0}^{L-1} \sigma_m = \sum_{j=0}^{j_0-1} \sigma_j + \sum_{j=j_0}^{L-1} \sigma_j \leq \frac{\eta}{2}.$$

As a consequence,

$$\tilde{\delta}_L = \tilde{\delta}_0 - 2\tilde{\delta}_0 \sum_{j=0}^{L-1} \sigma_j \geq \tilde{\delta}_0(1 - \eta) = \delta_+.$$

Proof of Proposition 5.1 The proof of Proposition 5.1 is an immediate result of Lemma 5.3.

By applying Lemma 5.3 for L times inductively, we get a sequence of transformations

$$\tilde{\Phi}^j := \tilde{\Phi}_0 \circ \tilde{\Phi}_1 \circ \dots \circ \tilde{\Phi}_{j-1} : D(\tilde{\delta}_j, \tilde{s}_j) \times \mathcal{O} \rightarrow D(\delta, s) \times \mathcal{O}, \quad \forall 1 \leq j \leq L.$$

Let $\Phi := \tilde{\Phi}^L$, which maps $D(\delta_+, s_+) \times \mathcal{O}$ into $D(\delta, s) \times \mathcal{O}$. Then, via transformation Φ , we get the new system (5.2) with $B_+ = B + b_L$, $W_+ = W + w_L$, satisfying

$$\begin{aligned} \mathcal{R}_{Q_{n+2}} B_+ &= 0, \\ \|B_+ - B\|_{\delta_+, \mathcal{O}} &\leq \|b_L\|_{\tilde{\delta}_L, \mathcal{O}} \leq 2\varepsilon, \\ \|W_+ - W\|_{\delta_+, p, p, \mathcal{O}} &\leq \|w_L\|_{\tilde{\delta}_L, p, p, \mathcal{O}} \leq 2\varepsilon, \end{aligned}$$

and the perturbation $P_+ = \tilde{P}_L$ satisfying

$$\|P_+\|_{s_+, D(\delta_+, s_+) \times \mathcal{O}} \leq \|\tilde{P}_L\|_{\tilde{s}_L, D(\tilde{\delta}_L, \tilde{s}_L) \times \mathcal{O}} \leq \tilde{\varepsilon}_L \leq \varepsilon_+.$$

Next, we verify that the transformation Φ satisfies (5.5) and (5.6). By the chain rule and (5.19), one has

$$\|D\tilde{\Phi}^j\|_{\tilde{s}_j, D(\tilde{\delta}_j, \tilde{s}_j) \times \mathcal{O}} \leq \prod_{i=0}^{j-1} \|D\tilde{\Phi}_i\|_{\tilde{s}_{i+1}, D(\tilde{\delta}_{i+1}, \tilde{s}_{i+1}) \times \mathcal{O}} \leq \prod_{i=0}^j (1 + \tilde{\varepsilon}_i^{\frac{1}{2}}) \leq 2.$$

Then, by the mean value theorem and (5.18),

$$\begin{aligned} & \|\tilde{\Phi}^{j+1} - \tilde{\Phi}^j\|_{\tilde{s}_{j+1}, D(\tilde{\delta}_{j+1}, \tilde{s}_{j+1}) \times \mathcal{O}} \\ & \leq \|D\tilde{\Phi}^j\|_{\tilde{s}_j, D(\tilde{\delta}_j, \tilde{s}_j) \times \mathcal{O}} \|\tilde{\Phi}^j - id\|_{\tilde{s}_{j+1}, D(\tilde{\delta}_{j+1}, \tilde{s}_{j+1}) \times \mathcal{O}} \leq 2\tilde{\varepsilon}_j^{\frac{2}{3}}. \end{aligned}$$

As a consequence,

$$\begin{aligned} & \|\Phi - id\|_{\tilde{s}_+, D(\tilde{\delta}_+, \tilde{s}_+) \times \mathcal{O}} \\ & \leq \sum_{j=1}^{L-1} \|\tilde{\Phi}^{j+1} - \tilde{\Phi}^j\|_{\tilde{s}_{j+1}, D(\tilde{\delta}_{j+1}, \tilde{s}_{j+1}) \times \mathcal{O}} + \|\tilde{\Phi}_0 - id\|_{\tilde{s}_1, D(\tilde{\delta}_1, \tilde{s}_1) \times \mathcal{O}} \leq 4\tilde{\varepsilon}_0^{\frac{2}{3}}. \end{aligned}$$

Similarly,

$$\|D\Phi - Id\|_{\tilde{s}_L, D(\tilde{\delta}_L, \tilde{s}_L) \times \mathcal{O}} \leq 4\tilde{\varepsilon}_0^{\frac{1}{2}}. \quad \square$$

5.3 Iterative lemma for KAM scheme

We define infinitely many successive steps of KAM iteration using Proposition 5.1. For given $1 \geq \delta > \delta_* > 0$, $\tau > 2$, $1 > \gamma > 0$, $d \in \mathbb{N}_+$ and \mathcal{A}, M are defined in Sect. 4.2. There exists a constant $\tilde{c}_0 = \frac{1}{2}(\frac{\delta}{\delta_*} + 1) > 1$ such that $\delta > \delta_* \tilde{c}_0$. Let $\tilde{c} = \frac{1}{12}(1 - \frac{1}{\tilde{c}_0}) < 1$ and

$$T = \max \left\{ \left(\frac{4^5 \cdot 10^\tau}{\gamma \delta_*^2 \tilde{c}^2} \right)^{\mathcal{A}}, T_0 \left(\frac{\gamma}{2} \right)^{-\mathcal{A}/2}, 4^{\mathcal{A}^4} \right\}, \tag{5.25}$$

where $J = J(\tau)$, $\epsilon_0 = \epsilon_0(\delta_*, \tau, \tilde{c}, d) > 0$, $\epsilon_1 = \epsilon_1(\delta_*, \tau, \tilde{c}) > 0$, $\epsilon_2 > 0$ are small positive constants given in Lemma 5.2, Proposition 4.3 and 4.4.

By the discussion in [17], for T defined above, we can choose $n_0 \in \mathbb{N}$ such that $Q_{n_0} \geq T$. Then we choose sufficiently small ε depending on the constants $\delta, \delta_*, \tau, \gamma$, such that

$$\varepsilon \leq \min \left\{ \epsilon_0 \left(\frac{\gamma}{2} \right)^J, \epsilon_1 \frac{\gamma}{8}, \frac{\epsilon_2}{4}, Q_{n_0}^{-120\tau} \right\}. \tag{5.26}$$

We define the iterative sequences depending on $\varepsilon, \delta, s, \gamma$ by

$$\begin{aligned} \varepsilon_0 &= \varepsilon, & \delta_0 &= \delta, & \gamma_0 &= \gamma, & \gamma_n &= \gamma_0 \left(\frac{1}{2} + \frac{1}{2^{n+1}} \right), \\ \mathcal{E}_{n+1} &= e^{-c_0 \gamma_n \delta_{n+1} \left(\frac{\overline{Q}_{n_0+n}}{Q_{n_0+n}} + \frac{1}{\overline{Q}_{n_0+n}^{1/4}} \right)}, & \mathcal{E}_{n+1} &= \mathcal{E}_{n+1} \varepsilon_n, \\ \eta_n &= \tilde{c} Q_{n_0+n-1}^{-\frac{1}{4}}, & \delta_{n+1} &= \delta_n (1 - \eta_n)^2, \\ K_n &= \left[\frac{\gamma_n}{4 \cdot 10^\tau} \max \left\{ \frac{\overline{Q}_{n_0+n}}{Q_{n_0+n}^\tau}, \overline{Q}_{n_0+n}^{\frac{3}{\mathcal{A}}} \right\} \right]. \end{aligned} \tag{5.27}$$

Let L_n be the unique positive integer satisfying

$$\log_{\frac{7}{8}} \left(\frac{\ln \varepsilon_{n+1}}{\ln \varepsilon_n} \right) \leq L_n < 1 + \log_{\frac{7}{8}} \left(\frac{\ln \varepsilon_{n+1}}{\ln \varepsilon_n} \right).$$

That is,

$$\varepsilon_n^{(\frac{8}{7})^{L_n}} \leq \varepsilon_{n+1} < \varepsilon_n^{(\frac{8}{7})^{L_{n-1}}}.$$

For given $1 \geq s > 0$, we also define

$$s_0 = s, \quad s_{n+1} = \varepsilon_n^{\frac{7}{6}[(\frac{8}{7})^{L_n} - 1]} s_n.$$

Firstly, the sequence $\{s_n\}_{n \geq 0}$ is decreasing and goes to 0 as $n \rightarrow \infty$. For the sequences $\{\delta_n\}_{n \geq 0}$ defined in (5.27), we show that $\delta_n > \delta_*$ for every $n \geq 0$. Indeed, by $Q_{n_0+1} \geq T \geq 4^{A^4}$ in (5.25), one has

$$\prod_{k=2}^{\infty} (1 - 2\eta_k) \geq 1 - \sum_{k=2}^{\infty} 4\eta_k \geq 1 - 8\tilde{c}Q_{n_0+1}^{\frac{-1}{2A^4}} \geq 1 - 8\tilde{c}.$$

Then

$$\delta_n = \delta_0 \prod_{k=0}^{n-1} (1 - \eta_k)^2 > \delta_0(1 - 2\eta_0)(1 - 2\eta_1) \prod_{k=2}^{\infty} (1 - 2\eta_k) > \delta_0(1 - 12\tilde{c}) > \delta_*.$$

According to the analysis in Sect. 5.2, we can conclude to the following iterative lemma.

Lemma 5.4 (Iterative lemma) *For integer $n \geq 0$, suppose we have a family of systems defined on $D(r_n, s_n) \times \mathcal{O}_{n-1}$,*

$$\begin{cases} \dot{\theta}_n = \omega, \\ \dot{\rho}_n = i(\Omega(\xi) + B_n(\theta_n; \xi))\rho_n + p_n(\theta_n, \rho_n, z_n; \xi), \\ \dot{z}_n = (\Lambda(\xi) + W_n(\theta_n; \xi))z_n + g_n(\theta_n, \rho_n, z_n; \xi), \end{cases} \tag{5.28}$$

at n th KAM step satisfying $\mathcal{R}_{Q_{n_0+n}} B_n = 0$,

$$\|B_n - B_{n-1}\|_{\delta_n, \mathcal{O}_{n-1}}, \|W_n - W_{n-1}\|_{\delta_n, p, p, \mathcal{O}_{n-1}} \leq 2\varepsilon_{n-1} \tag{5.29}$$

and

$$\|P_n\|_{s_n, D(\delta_n, s_n) \times \mathcal{O}_{n-1}} \leq \varepsilon_n, \quad P_n := (0, p_n, g_n),$$

where we set $\varepsilon_{-1} = B_0 = W_0 = 0$, $|\Omega(\xi)|_{\mathcal{O}} \leq 1$ and the \mathcal{O}_{n-1} is defined as

$$\mathcal{O}_{n-1} = \left\{ \xi \in \mathcal{O} : \left| (k, \omega) + \langle l, \Omega(\xi) \rangle + [B_{n-1}(\theta_{n-1}; \xi)]_{\theta_{n-1}} \right| \geq \frac{\gamma_{n-1}}{(|k| + |l|)^{\tau}}, \right. \\ \left. \forall 0 < |l| \leq 2, |k| < K_{n-1} \right\}.$$

Then there exist a subset $\mathcal{O}_n \subseteq \mathcal{O}_{n-1}$ with

$$\mathcal{O}_n = \mathcal{O}_{n-1} \setminus \bigcup_{K_{n-2} \leq |k| < K_n} \Gamma_k^n(\gamma_n), \tag{5.30}$$

where

$$\Gamma_k^n(\gamma_n) = \left\{ \xi \in \mathcal{O}_{n-1} : \left| \langle k, \omega \rangle + \langle l, \Omega + [B_n(\theta_n; \xi)]_{\theta_n} \rangle \right| < \frac{\gamma_n}{(|k| + |l|)^\tau}, \right. \\ \left. \forall 0 < |l| \leq 2 \right\}, \tag{5.31}$$

and an analytic coordinate transformation

$$\Phi_n : D(\delta_{n+1}, s_{n+1}) \times \mathcal{O}_n \rightarrow D(\delta_n, s_n) \times \mathcal{O}_n$$

of the form

$$\begin{cases} \theta_n = \theta_{n+1}, \\ \rho_n = V_n(\theta_{n+1}, \rho_{n+1}, z_{n+1}; \xi), \\ z_n = U_n(\theta_{n+1}, \rho_{n+1}, z_{n+1}; \xi), \end{cases} \tag{5.32}$$

where V_n and U_n are affine in ρ_{n+1}, z_{n+1} , such that by the coordinate transformation Φ_n , the system (5.28) is changed into

$$\begin{cases} \dot{\theta}_{n+1} = \omega, \\ \dot{\rho}_{n+1} = i(\Omega(\xi) + B_{n+1}(\theta_{n+1}; \xi))\rho_{n+1} + p_{n+1}(\theta_{n+1}, \rho_{n+1}, z_{n+1}; \xi), \\ \dot{z}_{n+1} = (\Lambda(\xi) + W_{n+1}(\theta_{n+1}; \xi))z_{n+1} + g_{n+1}(\theta_{n+1}, \rho_{n+1}, z_{n+1}; \xi), \end{cases} \tag{5.33}$$

which satisfies the above assumptions with $n + 1$ replacing n . Furthermore, we have the estimate

$$\|\Phi_n - id\|_{s_{n+1}, D(\delta_{n+1}, s_{n+1}) \times \mathcal{O}_n} \leq 4\varepsilon_n^{\frac{2}{3}}, \tag{5.34}$$

$$\|D\Phi_n - Id\|_{s_{n+1}, D(\delta_{n+1}, s_{n+1}) \times \mathcal{O}_n} \leq 4\varepsilon_n^{\frac{1}{2}}. \tag{5.35}$$

Proof Lemma 5.4 can be proved immediately by applying Proposition 5.1. It is sufficient to verify conditions in Proposition 5.1.

Firstly, we need to check that $\Omega + [B_n]_\theta \in DC_\omega(\gamma_n, \tau, K_n, \mathcal{O}_n)$. From (5.29) and $B_0 = 0$, we can get

$$\|B_n\|_{\delta_n, \mathcal{O}_{n-1}} \leq \sum_{j=1}^n \|B_j - B_{j-1}\|_{\delta_j, \mathcal{O}_{j-1}} \leq 2 \sum_{j=1}^n \varepsilon_{j-1} \leq 4\varepsilon_0. \tag{5.36}$$

This implies that $|\Omega + [B_n]_\theta|_{\mathcal{O}_{n-1}} \leq 2$ due to the smallness of ε_0 . Moreover, for $\xi \in \mathcal{O}_{n-1}$, we have

$$\left| \langle k, \omega \rangle + \langle l, \Omega + [B_{n-1}]_\theta \rangle \right| \geq \frac{\gamma_{n-1}}{(|k| + |l|)^\tau}, \quad \forall 0 < |l| \leq 2, |k| < K_{n-1}.$$

Then, for $0 < |l| \leq 2$, $|k| < K_{n-2}$, it follows that

$$\begin{aligned} |\langle k, \omega \rangle + \langle l, \Omega + [B_n]_\theta \rangle| &\geq |\langle k, \omega \rangle + \langle l, \Omega + [B_{n-1}]_\theta \rangle| - |\langle l, [B_n - B_{n-1}]_\theta \rangle| \\ &\geq \frac{\gamma_{n-1}}{(|k| + |l|)^\tau} - 4\varepsilon_{n-1} \geq \frac{\gamma_n}{(|k| + |l|)^\tau}, \quad \forall \xi \in \mathcal{O}_{n-1}. \end{aligned}$$

The last inequality is obvious, when $|k| < K_{n-2}$, by the choice of K_n, ε_n in (5.27). Therefore, it is verified that $\Omega + [B_n]_\theta \in \text{DC}_\omega(\gamma_n, \tau, K_n, \mathcal{O}_n)$ by the definition of \mathcal{O}_n in (5.31).

Secondly, we need to show B_n and W_n are small enough. From (5.36) and $M = \frac{A^4}{2}$, we see that B_n satisfies

$$\|B_n\|_{\delta_n, \mathcal{O}_{n-1}} \cdot \left(\frac{\overline{Q}_{n_0+n-1}}{Q_{n_0+n-1}^{A^4}} + \overline{Q}_{n_0+n-1}^{\frac{1}{A}} \right) \leq \varepsilon_1 \gamma_n \left(\frac{\overline{Q}_{n_0+n-1}}{Q_{n_0+n-1}^M} + \overline{Q}_{n_0+n-1}^{\frac{1}{M^{1/4}}} \right),$$

since

$$\|B_n\|_{\delta_n, \mathcal{O}_{n-1}} \leq 4\varepsilon_0 < \varepsilon_1 \frac{\gamma}{2} < \varepsilon_1 \gamma_n.$$

From (5.29) and $W_0 = 0$, we get

$$\|W_n\|_{\delta_n, \mathcal{O}_{n-1}} \leq \sum_{j=1}^n \|W_j - W_{j-1}\|_{\delta_j, \mathcal{O}_{j-1}} \leq 2 \sum_{j=1}^n \varepsilon_{j-1} \leq 4\varepsilon_0 \leq \varepsilon_2.$$

Finally, we prove

$$\|P_n\|_{s_n, D(\delta_n, s_n) \times \mathcal{O}_{n-1}} \leq \varepsilon_n \leq \varepsilon_0 \gamma_n^J \mathcal{E}_n.$$

The definition of ε_n in (5.27) and condition (5.26) show that

$$\varepsilon_n = \mathcal{E}_n \cdots \mathcal{E}_1 \varepsilon_0 \leq \varepsilon_0 \left(\frac{\gamma}{2} \right)^J \mathcal{E}_n \leq \varepsilon_0 \gamma_n^J \mathcal{E}_n, \quad \forall n \geq 1.$$

When $n = 0$, it suffices to take ε_0 satisfying (5.26).

Therefore, by applying Proposition 5.1, there exists an analytic transformation Φ_n which is of the form (5.32) such that the transformed system (5.33) has the same properties as the system (5.28) at the n th KAM step. Moreover, the transformation Φ_n satisfies the estimates (5.34) and (5.35) by Proposition 5.1 again. □

5.4 Convergence and measure estimates

We begin with the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i\Omega(\xi)\rho + p(\theta, \rho, z; \xi), \\ \dot{z} = \Lambda(\xi)z + g(\theta, \rho, z; \xi), \end{cases} \tag{5.37}$$

on $D(\delta, s) \times \mathcal{O}$. Since \mathcal{O} is a compact subset and $\Omega(\xi)$ is C_W^1 in $\xi \in \mathcal{O}$, we can suppose $|\Omega(\xi)|_{\mathcal{O}} < 1$ without loss of generality. Then the non-resonance condition $\Omega(\xi) \in$

$DC_\omega(\gamma, \tau, K_0, \mathcal{O}_0)$ is satisfied by setting

$$\mathcal{O}_0 = \left\{ \xi \in \mathcal{O} : |\langle k, \omega \rangle + \langle l, \Omega(\xi) \rangle| \geq \frac{\gamma_0}{(|k| + |l|)^\tau}, \forall 0 < |l| \leq 2, |k| < K_0 \right\}.$$

Since $|\operatorname{Re} \lambda_j(\xi)| > \varrho_1$ for some positive constant ϱ_1 , we do not encounter a small divisor when solving the homological equations with respect to z . Therefore, one does not need any non-resonant condition for $\Lambda(\xi)$.

Denote $B_0 = 0, P_0 = P := (0, p, g)$, then

$$\|P_0\|_{\delta_0, D(\delta_0, s_0) \times \mathcal{O}_0} \leq \varepsilon_0 \leq \min \{ \varepsilon_0 \gamma_0^J, Q_{n_0}^{-120\tau} \},$$

due to assumption (5.26). Thus, we are able to apply the iterative lemma, Lemma 5.4, inductively to get a sequence of subsets \mathcal{O}_n and transformations

$$\Phi_n : D(\delta_{n+1}, s_{n+1}) \times \mathcal{O}_n \rightarrow D(\delta_n, s_n) \times \mathcal{O}_n$$

satisfying estimate (5.34) and (5.35) for each $n \in \mathbb{N}$. Let

$$\Phi^n := \Phi_0 \circ \dots \circ \Phi_{n-1} : D(\delta_n, s_n) \times \mathcal{O}_{n-1} \rightarrow D(\delta, s) \times \mathcal{O}_{n-1},$$

then the transformed system of (5.37) by transformation Φ^n still satisfies the properties in Lemma 5.4 for each $n \geq 1$.

Convergence: Now we give the uniform convergence of transformation Φ^n . Let

$$\mathcal{O}_\gamma = \bigcap_{n=0}^\infty \mathcal{O}_n.$$

Then $\Phi^n, D\Phi^n$ converge uniformly to $\Phi, D\Phi$ on the domain $D(\delta_*, 0) \times \mathcal{O}_\gamma$ as in [23]. Moreover, $\Phi, D\Phi$ can be defined on the domain $D(\delta_*, \frac{\varepsilon}{2}) \times \mathcal{O}_\gamma$ following the analysis in [23] since it is affine in the variables ρ, z .

It follows from the estimates (5.29) of B_n and W_n that B_n and W_n converge uniformly to limits B_* and W_* on domain $D(\delta_*, \frac{\varepsilon}{2}) \times \mathcal{O}_\gamma$ with

$$\begin{aligned} \|B_*\|_{\delta_*, \mathcal{O}_\gamma} &\leq \sum_{n=1}^\infty \|B_n - B_{n-1}\|_{\delta_n, \mathcal{O}_{n-1}} \leq 4\varepsilon_0, \\ \|W_*\|_{\delta_*, p, p, \mathcal{O}_\gamma} &\leq \sum_{n=1}^\infty \|W_n - W_{n-1}\|_{\delta_n, p, p, \mathcal{O}_{n-1}} \leq 4\varepsilon_0. \end{aligned}$$

Moreover, the sequence $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ by the definition of ε_n provided that ε_0 is sufficiently small. Thus, the final transformed system of (5.37) by coordinate transformation Φ is

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i(\Omega(\xi) + B_*(\theta; \xi))\rho + p_*(\theta, \rho, z; \xi), \\ \dot{z} = (\Lambda(\xi) + W_*(\theta; \xi))z + g_*(\theta, \rho, z; \xi), \end{cases}$$

defined on $D(\delta_*, \frac{\varepsilon}{2}) \times \mathcal{O}_\gamma$ and p_*, g_* are at least of order 2 with respect to ρ, z .

Measure estimates: During the procedure of KAM iteration, we obtain a decreasing sequence of closed subsets $\mathcal{O}_0 \supseteq \mathcal{O}_1 \supseteq \dots$. It is crucial to prove that the Lebesgue measure of their intersection \mathcal{O}_γ is positive in KAM theory for small enough $\gamma > 0$.

According to Lemma 5.4, we have the set

$$\mathcal{O} \setminus \mathcal{O}_\gamma = \left(\bigcup_{0 \leq |k| < K_0} \Gamma_k^0(\gamma_0) \right) \cup \left(\bigcup_{n \geq 1} \bigcup_{K_{n-2} \leq |k| < K_n} \Gamma_k^n(\gamma_n) \right),$$

where

$$\Gamma_k^0(\gamma_0) = \left\{ \xi \in \mathcal{O} : |\langle k, \omega \rangle + \langle l, \Omega(\xi) \rangle| < \frac{\gamma_0}{(|k| + |l|)^\tau}, \forall 0 < |l| \leq 2 \right\},$$

and, for $n \geq 1$,

$$\Gamma_k^n(\gamma_n) = \left\{ \xi \in \mathcal{O}_{n-1} : |\langle k, \omega \rangle + \langle l, \Omega(\xi) + [B_n(\theta; \xi)]_\theta \rangle| < \frac{\gamma_0}{(|k| + |l|)^\tau}, \forall 0 < |l| \leq 2 \right\}.$$

Then using the non-degeneracy condition (3.2) and the analysis in Sect. 4.2 of [17], we have the following lemma for the measure of the parameter set \mathcal{O}_γ .

Lemma 5.5 *For $\tau > 2$ and sufficiently small $\gamma > 0$, we have*

$$\text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma).$$

As a conclusion, we complete the proof of Theorem 3.1.

6 Proof of Theorem 1.1

Firstly, we rescale (1.2) via $u \mapsto \varepsilon^{\frac{1}{2}} u$ to obtain the following equation:

$$u_t = ru + (b + i\nu)\partial_{xx}u + m\partial_xu - (1 + i\mu)\varepsilon h(\omega t, x)|u|^2u + \varepsilon^{\frac{1}{2}}f(\omega t, x), \quad x \in \mathbb{T}. \tag{6.1}$$

Then the linearized equation of (6.1) is

$$u_t = ru + (b + i\nu)\partial_{xx}u + m\partial_xu.$$

And the linear operator $r + (b + i\nu)\partial_{xx} + m\partial_x$ under periodic boundary condition possesses the eigenvalues

$$\lambda_n = r - bn^2 + i(mn - \nu n^2), \quad n \in \mathbb{Z},$$

and the corresponding eigenfunctions $\phi_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$.

For any given $j \in \mathbb{N} \setminus \{0\}$, choose the parameters $r > 0, b > 0$ such that $\text{Re } \lambda_{\pm j} = r - bj^2 = 0$. Then, another eigenvalue $\lambda_n, n \neq \pm j$, satisfies $|\text{Re } \lambda_n| \neq 0$.

We will find the solution to (6.1) of the form

$$u(t, x) = \sum_{n \in \mathbb{Z}} q_n(t)\phi_n(x). \tag{6.2}$$

Substituting (6.2) into (6.1), one gets a lattice formulation of the problem

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{q}_n = \lambda_n q_n - \varepsilon(1 + i\mu) \sum_{i-k+l-n+\zeta=0} T_{ikln\zeta} h_\zeta(\theta) q_i \bar{q}_k q_l + \varepsilon^{\frac{1}{2}} P_n(\theta), \end{cases}$$

where

$$\begin{aligned} T_{ikln\zeta} &= \sqrt{2\pi} \int_0^{2\pi} \phi_i(x) \bar{\phi}_k(x) \phi_j(x) \bar{\phi}_n(x) \phi_\zeta(x) \, dx \\ &= \begin{cases} (2\pi)^{-2}, & i - k + l - n + \zeta = 0, \\ 0, & i - k + l - n + \zeta \neq 0, \end{cases} \end{aligned}$$

and

$$P_n(\theta) = \int_0^{2\pi} f(\theta, x) \bar{\phi}_n(x) \, dx = \sqrt{2\pi} f_n(\theta).$$

Let $\tilde{\rho}_1 = q_j$ and $\tilde{\rho}_2 = q_{-j}$. Then

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\tilde{\rho}}_1 = i\Omega_1 \rho_1 - \varepsilon(1 + i\mu) \sum_{i-k+l-j+\zeta=0} T_{iklj\zeta} h_\zeta(\theta) q_i \bar{q}_k q_l + \varepsilon^{\frac{1}{2}} P_j(\theta), \\ \dot{\tilde{\rho}}_2 = i\Omega_2 \rho_2 - \varepsilon(1 + i\mu) \sum_{i-k+l+j+\zeta=0} T_{ikl(-j)\zeta} h_\zeta(\theta) q_i \bar{q}_k q_l + \varepsilon^{\frac{1}{2}} P_{-j}(\theta), \\ \dot{q}_n = \lambda_n q_n - \varepsilon(1 + i\mu) \sum_{i-k+l-n+\zeta=0} T_{ikln\zeta} h_\zeta(\theta) q_i \bar{q}_k q_l + \varepsilon^{\frac{1}{2}} P_n(\theta), \quad n \neq \pm j, \end{cases}$$

where $\Omega_1 = -i\lambda_j = mj - vj^2 \in \mathbb{R}$ and $\Omega_2 = -i\lambda_{-j} = -mj - vj^2 \in \mathbb{R}$.

Denote the parameter $\xi := (v, m) \in \mathcal{O}$, and $\mathcal{O} \subseteq \mathbb{R}^2$ is a compact set with positive measure. Let

$$\Omega(\xi) = \text{diag}\{\Omega_1(\xi), \Omega_2(\xi), -\Omega_1(\xi), -\Omega_2(\xi)\}, \quad \Lambda(\xi) = \text{diag}\{\Lambda_1(\xi), \bar{\Lambda}_1(\xi)\}$$

with $\Lambda_1(\xi) = \text{diag}\{\lambda_n(\xi) : n \neq \pm j\}$

and $\rho = (\tilde{\rho}_1, \tilde{\rho}_2, \bar{\rho}_1, \bar{\rho}_2)$, $z = ((q_n)_{n \neq \pm j}, (\bar{q}_n)_{n \neq \pm j})$. Then we get the system

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i\Omega(\xi)\rho + p(\theta, \rho, z) \\ \dot{z} = \Lambda(\xi)z + g(\theta, \rho, z), \end{cases} \tag{6.3}$$

where the nonlinear terms are

$$p = (p_1, p_2, \bar{p}_1, \bar{p}_2), \quad g = ((g_n)_{n \neq \pm j}, (\bar{g}_n)_{n \neq \pm j}),$$

with

$$\begin{aligned}
 p_1 &= -\varepsilon(1 + i\mu) \sum_{i-k+l-j+\varsigma=0} T_{iklj\varsigma} h_\varsigma(\theta) q_i \bar{q}_k q_l + \varepsilon^{\frac{1}{2}} P_j(\theta), \\
 p_2 &= -\varepsilon(1 + i\mu) \sum_{i-k+l+j+\varsigma=0} T_{ikl(-j)\varsigma} h_\varsigma(\theta) q_i \bar{q}_k q_l + \varepsilon^{\frac{1}{2}} P_{-j}(\theta), \\
 g_n &= -\varepsilon(1 + i\mu) \sum_{i-k+l-n+\varsigma=0} T_{ikln\varsigma} h_\varsigma(\theta) q_i \bar{q}_k q_l + \varepsilon^{\frac{1}{2}} P_n(\theta), \quad n \neq \pm j.
 \end{aligned}$$

It is obvious that p and g are independent of the parameter $\xi \in \mathcal{O}$.

Note the the dimension of the vector z in (6.3) is different from the one in (3.1) even though both are infinite. Actually, the latter is the double of the former. So to apply Theorem 3.1, we need to redefine the sets \mathcal{J}_2 and $D(\delta, s)$. To simplify this work, we omit these discussions.

Now we apply the abstract KAM theorem 3.1 obtained in Sect. 3 for system (6.3) to prove Theorem 1.1. So we need to check that the frequencies $\Omega(\xi)$ and $\Lambda(\xi)$ satisfy non-degeneracy condition (3.2), and the perturbations p, g satisfy the smallness condition for sufficiently small ε .

It is obvious that for $\mathbb{Z} \ni n \neq \pm j$,

$$\left| \operatorname{Re} \lambda_n(\xi) \right| = \left| r - bn^2 \right| \geq \left| r - b(j - 1)^2 \right| \geq |b| > 0, \quad \left| \operatorname{Re} \lambda_n(\xi) \right| \geq \varrho_2 \left| \frac{\partial \lambda_n}{\partial \xi}(\xi) \right|,$$

where $\varrho_2 > 0$ is a constant independent of n, ξ and

$$\left| \frac{\partial \langle l, \Omega \rangle}{\partial \xi} \right| \geq j \geq 1, \quad \forall 0 < |l| \leq 2, l \in \mathbb{Z}^2.$$

Therefore, the non-degeneracy condition (3.2) holds for $\Omega(\xi)$ and $\Lambda(\xi)$.

Since f and h satisfy assumption **(H)**, the norms of f and h are

$$\begin{aligned}
 \|f\|_{a,p,D(\delta)} &:= \sum_{n \in \mathbb{Z}} \|f_n(\theta)\|_\delta e^{a|n|} \langle n \rangle^p < \infty, \\
 \|h\|_{a,p,D(\delta)} &:= \sum_{n \in \mathbb{Z}} \|h_n(\theta)\|_\delta e^{a|n|} \langle n \rangle^p < \infty
 \end{aligned}$$

for some $1 > \delta > 0, a > 0$ and $p \geq \frac{1}{2}$.

For some $1 > s > 0$, on domain $D(\delta, s) = \{(\theta, \rho, z) \in \mathcal{P}_{a,p} : |\operatorname{Im} \theta| < \delta, |\rho| < s, \|z\|_{a,p} < s\}$, we have

$$\begin{aligned}
 \|p\|_{D(\delta,s)} &\leq c_1 \varepsilon (1 + i\mu) s^3 \|h\|_{a,p,D(\delta)} + 2\varepsilon^{\frac{1}{2}} \|P_j\|_\delta \\
 &\leq c\varepsilon^{\frac{1}{2}} (\|f\|_{a,p,D(\delta)} + s^3 \|h\|_{a,p,D(\delta)}), \\
 \|g\|_{a,p,D(\delta,s)} &\leq c_1 \varepsilon (1 + i\mu) s^3 \|h\|_{a,p,D(\delta)} + c_1 \varepsilon^{\frac{1}{2}} \|f\|_{a,p,D(\delta)} \\
 &\leq c\varepsilon^{\frac{1}{2}} (\|f\|_{a,p,D(\delta)} + s^3 \|h\|_{a,p,D(\delta)}),
 \end{aligned}$$

where we use Lemma 2.1 for sequence $\{\|f_j(\theta)\|_{D(\delta)}\}_{j \in \mathbb{Z}} \in \ell_{a,p}$ and $\{\|h_j(\theta)\|_{D(\delta)}\}_{j \in \mathbb{Z}} \in \ell_{a,p}$ and c, c_1 are positive constants.

Hence, for nonlinear term $P := (0, p, g)$, we have

$$\|P\|_{s,D(\delta,s) \times \mathcal{O}} \leq c\varepsilon^{\frac{1}{2}} \left(s^{-1} \|f\|_{a,p,D(\delta)} + s^2 \|h\|_{a,p,D(\delta)} \right) \leq C_0 \varepsilon^{\frac{1}{2}},$$

where $C_0 > 0$ is a constant depending on δ, s and functions f, h .

By Theorem 3.1, for $\tau > 2$, sufficiently small $0 < \gamma \ll 1$ and $\delta_* := \frac{\delta}{2}$, there exists a constant $\varepsilon_* := \frac{1}{C_0^2} \varepsilon_0^2$ depending on $j, r, b, \mu, \gamma, \tau, \mathcal{O}, f, h$ such that if $\varepsilon < \varepsilon_*$, there exist a subset \mathcal{O}_γ with $\text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$ and a family of analytic transformations

$$\Phi_\xi : D\left(\frac{\delta}{2}, \frac{s}{2}\right) \rightarrow D(\delta, s), \quad \forall \xi \in \mathcal{O}_\gamma,$$

which transforms system (6.3) into

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\rho} = i(\Omega(\xi) + B_*(\theta; \xi))\rho + p_*(\theta, \rho, z; \xi), \\ \dot{z} = -(\Lambda(\xi) + W_*(\theta; \xi))z + g_*(\theta, \rho, z; \xi). \end{cases} \tag{6.4}$$

where p_* and g_* are at least of order 2 with respect to variables ρ and z . Therefore, for any $\xi \in \mathcal{O}_\gamma$, the transformed system (6.4) admits a special solution $(\theta(0) + \omega t, 0, 0)$.

Let

$$(\theta(t), \rho(t), z(t)) = \Phi_\xi(\theta(0) + \omega t, 0, 0),$$

then

$$(\theta(t), \rho(t), z(t)) =: (\theta(0) + \omega t, V_\xi(\theta(0) + \omega t), U_\xi(\theta(0) + \omega t))$$

is a analytic quasi-periodic solution to system (6.3) for $\xi \in \mathcal{O}_\gamma$. As a conclusion, the complex Ginzburg–Landau equation (6.1) possesses a quasi-periodic solution with the form of

$$u(t, x) = V_{1,\xi}(\theta(0) + \omega t)\phi_j(x) + V_{2,\xi}(\theta(0) + \omega t)\phi_{-j}(x) + \sum_{n \neq \pm j} U_{n,\xi}(\theta(0) + \omega t)\phi_n(x).$$

Then, the forcing complex Ginzburg–Landau equation (1.2) has a response solution $\varepsilon^{\frac{1}{2}} u$ when the parameters of the coefficients $(\nu, m) \in \mathcal{O}_\gamma$.

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Availability of data and materials

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

SMW and JL contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Author details

¹School of Mathematics, Shandong University, Jinan, P.R. China. ²School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, P.R. China.

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