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Lower bound for the blow-up time for a general nonlinear nonlocal porous medium equation under nonlinear boundary condition

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Abstract

In this paper, we study the blow-up phenomenon for a general nonlinear nonlocal porous medium equation in a bounded convex domain ($\Omega \in \mathbb{R}^n, n \geq 3$) with smooth boundary. Using the technique of a differential inequality and a Sobolev inequality, we derive the lower bound for the blow-up time under the nonlinear boundary condition if blow-up does really occur.

Keywords: Lower bound; Blow-up time; Robin boundary condition; Nonlocal porous medium equation

1 Introduction

Liu in paper [1] studied the blow-up phenomena for the solution of the following problems:

$$\frac{\partial u}{\partial t} = \triangle u^m + u^p \int_\Omega u^q \, dx, \quad (x,t) \in \Omega \times (0, t^*), \tag{1.1}$$

$$u(x,0) = f(x) \geq 0, \quad x \in \Omega, \tag{1.2}$$

under the Robin boundary condition

$$\frac{\partial u}{\partial \nu} + ku = 0, \quad (x,t) \in \Omega \times (0, t^*). \tag{1.3}$$

He obtained a lower bound for the blow-up time of the system when the solution blows up.

In paper [2], the authors also studied equations (1.1) and (1.2) subject to either homogeneous Dirichlet boundary condition or homogeneous Neumann boundary condition. The lower bounds for the blow-up time under the above two boundary conditions were obtained. Equation (1.1) is used in the study of population dynamics (see [3]). For other systems in porous medium, one could see [4]. There have been a lot of papers in the literature on studying the question of blow-up for the solution of parabolic problems under a homogeneous Dirichlet boundary condition and Neumann boundary condition(one can see [5–12]). Some authors have started to consider the blow-up of these problems under Robin boundary conditions (see [13–17]). In papers [18–21], the authors studied the blow-up phenomena for the heat equation under nonlinear boundary conditions. Some new results about the nonlinear evolution equations may be founded in [22–24]. These papers have mainly focused on the bounded convex domain in $\mathbb{R}^3$. Recently, there have been some papers starting to study the blow-up problems in $\mathbb{R}^n$ $(n \geq 3)$ (see [25–29]). We continue the work of [2] for a more general equation. Until now, the authors have not found any paper dealing with lower bound for the blow-up time of a nonlinear nonlocal porous medium equation under nonlinear boundary condition in $\mathbb{R}^n$ $(n \geq 3)$. In this sense, the result obtained in this paper is new and interesting. In this paper, we consider the blow-up phenomena of the solution for the following equation:

$$\big(h(u)\big)_t = \triangle u^m + k_1(t)u^p \int_\Omega u^q \,dx, \quad (x,t) \in \Omega \times \big(0,t^*\big), \tag{1.4}$$

with the following boundary initial conditions:

$$u(x,0) = f(x) \geq 0, \quad x \in \Omega, \tag{1.5}$$

$$\frac{\partial u}{\partial \nu} = k_2(t) \int_\Omega g(u)\,dx, \quad (x,t) \in \partial\Omega \times \big(0,t^*\big), \tag{1.6}$$

where Ω is a bounded convex domain in $\mathbb{R}^n$, $n \geq 3$, with sufficiently smooth boundary, $\triangle$ is the Laplace operator, $\partial\Omega$ is the boundary of Ω, and t^* is the possible blow-up time, $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u. We assume $\frac{k_1'(t)}{k_1(t)} \leq \alpha$ and $\frac{dh(u)}{du} \geq M > 0$.

The function $g(\xi)$ satisfies

$$0 \leq g(\xi) \leq \xi^s, \quad \forall \xi > 0, \tag{1.7}$$

where $s > \max\{\frac{2n}{2n-1}, p+q+1-m\}$.

2 Some useful inequalities

We will use the following useful inequalities later in the proof.

Lemma 2.1 *We suppose that u is a nonnegative function and* σ, m *are positive constants, then we have the result as follows*:

$$\int_{\partial\Omega} u^{\sigma+m-2}\,dA \leq \frac{n}{\rho_0}\int_\Omega u^{\sigma+m-2}\,dx + \frac{(\sigma+m-2)d}{\rho_0}\int_\Omega u^{\sigma+m-3}|\nabla u|\,dx, \tag{2.1}$$

where $\rho_0 := \min_{\partial\Omega} |x \cdot \vec{\nu}|$, $\vec{\nu}$ *is the outward normal vector of* $\partial\Omega$ *and* $d := \max_{\partial\Omega} |x|$.

Proof Applying the divergence definition, we have

$$\operatorname{div}\big(u^{\sigma+m-2}x\big) = nu^{\sigma+m-2} + (\sigma+m-2)u^{\sigma+m-3}(x \cdot \nabla u). \tag{2.2}$$

Integrating (2.2), we deduce

$$\int_\Omega \operatorname{div}\big(u^{\sigma+m-2}x\big)\,dx \le n\int_\Omega u^{\sigma+m-2}\,dx + (\sigma+m-2)\int_\Omega u^{\sigma+m-3}|x\cdot\nabla u|\,dx.$$

Applying the divergence theorem, we obtain

$$\int_{\partial\Omega} u^{\sigma+m-2}x\cdot\vec{\nu}\,dA = n\int_\Omega u^{\sigma+m-2}\,dx + (\sigma+m-2)\int_\Omega u^{\sigma+m-3}|x\cdot\nabla u|\,dx.$$

Because Ω is a convex domain, we have $\rho_0 := \min_{\partial\Omega}|x\cdot\vec{\nu}| > 0$. Then we derive

$$\int_{\partial\Omega} u^{\sigma+m-2}\,dA \le \frac{n}{\rho_0}\int_\Omega u^{\sigma+m-2}\,dx + \frac{(\sigma+m-2)d}{\rho_0}\int_\Omega u^{\sigma+m-3}|x\cdot\nabla u|\,dx. \qquad \square$$

Lemma 2.2 *Supposing that* $u\in W^{1,2}(\Omega)$ *and* $n\ge 3$, *we have*

$$\int_\Omega u^{\frac{(\sigma+m-1)n}{n-2}}\,dx \le C^{\frac{2n}{n-2}}2^{\frac{n}{n-2}-1}\left[\left(\int_\Omega u^{\sigma+m-1}\,dx\right)^{\frac{n}{n-2}} + \left(\int_\Omega \left|\nabla^{\frac{\sigma+m-1}{2}}u\right|^2 dx\right)^{\frac{n}{n-2}}\right], \tag{2.3}$$

where $C = C(n,\Omega)$ *is a Sobolev embedding constant depending on* n *and* Ω.

Proof In paper [30], we have $W^{1,2}(\Omega)\hookrightarrow L^{\frac{2n}{n-2}(\Omega)}$, $n\ge 3$. Then we deduce the Sobolev inequality as follows:

$$\left(\int_\Omega w^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \le C\left(\int_\Omega w^2\,dx + \int_\Omega |\nabla w|^2\,dx\right)^{\frac{1}{2}},$$

that is,

$$\left(\int_\Omega \big(u^{\frac{\sigma+m-1}{2}}\big)^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \le C\left(\int_\Omega \big(u^{\frac{\sigma+m-1}{2}}\big)^2\,dx + \int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx\right)^{\frac{1}{2}}.$$

We can get

$$\begin{aligned}\int_\Omega u^{\frac{(\sigma+m-1)n}{n-2}} &\le C^{\frac{2n}{n-2}}\left(\int_\Omega \big(u^{\frac{\sigma+m-1}{2}}\big)^2\,dx + \int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx\right)^{\frac{n}{n-2}}\\ &\le C^{\frac{2n}{n-2}}2^{\frac{n}{n-2}-1}\left[\left(\int_\Omega u^{\sigma+m-1}\,dx\right)^{\frac{n}{n-2}} + \left(\int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx\right)^{\frac{n}{n-2}}\right].\end{aligned} \qquad \square$$

Remark 2.1 For any nonnegative function u, the following Hölder inequality holds:

$$\int_\Omega u^{n_1+n_2}\,dx \le \left(\int_\Omega u^{\frac{n_1}{x_1}}\,dx\right)^{x_1}\left(\int_\Omega u^{\frac{n_2}{x_2}\,dx}\right)^{x_2}, \tag{2.4}$$

where n_1, n_2, x_1, x_2 are positive constants and x_1, x_2 satisfy $x_1+x_2=1$.

Remark 2.2 The fundamental inequality

$$(a+b)^l \le a^l + b^l, \tag{2.5}$$

where $a, b\ge 0$ and $0<l\le 1$, holds.

3 Lower bound for the blow-up time

In this section it is useful in the sequel to define an auxiliary function of the following form:

$$\phi(t) = k_1^n(t)\int_\Omega u^{2n(s-1)}\,dx = k_1^n(t)\int_\Omega u^\sigma\,dx, \quad 0 \le t < t^*. \tag{3.1}$$

We will derive a differential inequality for $\phi(t)$. From the inequality, we can establish the following theorem.

Theorem 3.1 *Let $u(x,t)$ be the classical nonnegative solution of problem (1.4)–(1.7) in a bounded convex domain Ω ($\Omega \in R^n$ ($n \ge 3$)). We assume that $m+s > p+q+1 > 2$, $m > 3$, $p > 0$, $q > 0$. Then the quantity $\phi(t)$ defined in (3.1) satisfies the differential inequality*

$$\phi'(t)\phi^{-5}(t) \le a(t)\phi^{-4}(t) + b(t), \tag{3.2}$$

from which it follows that the blow-up time t^* is bounded below. We have

$$t^* \ge \Theta^{-1}\left(\frac{1}{4\phi^4(0)}\right), \tag{3.3}$$

where Θ^{-1} is the inverse function of Θ, and $a(t)$, $b(t)$ are defined in (3.21), (3.22) respectively.

Proof Now we prove Theorem 3.1. For convenience, we set $\phi(t) = \phi$, $k_1(t) = k_1$, $k_2(t) = k_2$. First we compute

$$\begin{aligned}
\phi'(t) &= nk_1^{n-1}k_1'\int_\Omega u^\sigma\,dx + k_1^n\sigma\int_\Omega u^{\sigma-1}u_t\,dx \\
&= nk_1^{n-1}k_1'\int_\Omega u^\sigma\,dx + k_1^n\sigma\int_\Omega u^{\sigma-1}\frac{1}{h'(u)}\left[\triangle u^m + k_1u^p\int_\Omega u^q\,dx\right]dx \\
&\le n\alpha\phi + \frac{k_1^n\sigma}{M}\int_\Omega u^{\sigma-1}\left[\triangle u^m + k_1u^p\int_\Omega u^q\,dx\right]dx.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
\phi'(t) &\le n\alpha\phi + \frac{k_1^n\sigma}{M}\left[m\int_{\partial\Omega}u^{\sigma+m-2}\frac{\partial u}{\partial\nu}\,dA - m(\sigma-1)\int_\Omega u^{\sigma+m-3}|\nabla u|^2\,dx\right] \\
&\quad + \frac{k_1^{n+1}\sigma|\Omega|}{M}\int_\Omega u^{\sigma+p+q-1}\,dx \\
&\le n\alpha\phi + \frac{\sigma mk_1^nk_2}{M}\int_{\partial\Omega}u^{\sigma+m-2}\,dA\int_\Omega u^s\,dx - \frac{\sigma m(\sigma-1)k_1^n}{M}\int_\Omega u^{\sigma+m-3}|\nabla u|^2\,dx \\
&\quad + \frac{k_1^{n+1}\sigma|\Omega|}{M}\int_\Omega u^{\sigma+p+q-1}\,dx.
\end{aligned}$$

Using the result of Lemma 2.1, we obtain

$$\begin{aligned}\phi'(t) &\le n\alpha\phi + \frac{\sigma m k_1^n k_2}{M}\frac{n}{\rho_0}\int_\Omega u^{\sigma+m-2}\,dx\int_\Omega u^s\,dx \\ &\quad + \frac{\sigma m k_1^n k_2}{M}\frac{(\sigma+m-2)d}{\rho_0}\int_\Omega u^{\sigma+m-3}|\nabla u|\,dx\int_\Omega u^s\,dx \\ &\quad - \frac{\sigma m(\sigma-1)k_1^n}{M}\frac{4}{(\sigma+m-1)^2}\int_\Omega \bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx + \frac{k_1^{n+1}\sigma|\Omega|}{M}\int_\Omega u^{\sigma+p+q-1}\,dx \\ &\le n\alpha\phi + r_1k_1^nk_2\int_\Omega u^{\sigma+m+s-2}\,dx + r_2k_1^nk_2\int_\Omega u^{\sigma+m-3}|\nabla u|\,dx\int_\Omega u^s\,dx \\ &\quad - r_3k_1^n\int_\Omega \bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx + r_4k_1^{n+1}\int_\Omega u^{\sigma+p+q-1}\,dx, \end{aligned} \tag{3.4}$$

where $r_1 = \frac{\sigma m}{M}\frac{n|\Omega|}{\rho_0}$, $r_2 = \frac{\sigma m}{M}\frac{(\sigma+m-2)d}{\rho_0}$, $r_3 = \frac{\sigma m(\sigma-1)}{M}\frac{4}{(\sigma+m-1)^2}$, $r_4 = \frac{\sigma|\Omega|}{M}$.

Now we estimate the third term of the right-hand side of (3.4). Using Hölder's inequality, we have

$$\int_\Omega u^s\,dx \le \left(\int_\Omega u^\sigma\,dx\right)^{\frac{s}{\sigma}}|\Omega|^{\frac{\sigma-s}{\sigma}} = k_1^{-\frac{ns}{\sigma}}\phi^{\frac{s}{\sigma}}|\Omega|^{\frac{\sigma-s}{\sigma}}.$$

Then we obtain

$$\begin{aligned} &k_1^n\int_\Omega u^{\sigma+m-3}|\nabla u|\,dx\int_\Omega u^s\,dx \\ &\quad\le k_1^n\int_\Omega u^{\sigma+m-3}|\nabla u|dx k_1^{-\frac{ns}{\sigma}}\phi^{\frac{s}{\sigma}}|\Omega|^{\frac{\sigma-s}{\sigma}} \\ &\quad= k_1^{-\frac{ns}{\sigma}}|\Omega|^{\frac{\sigma-s}{\sigma}}\frac{2}{\sigma+m-1}\phi^{\frac{s}{\sigma}}k_1^n\int_\Omega u^{\frac{\sigma+m-3}{2}}\bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|\,dx \\ &\quad\le \left(\varepsilon_1^{-1}r_5k_1^n\phi^{\frac{2s}{\sigma}}\int_\Omega \bigl(u^{\frac{\sigma+m-3}{2}}\bigr)^2\,dx\right)^{\frac12}\left(\varepsilon_1k_1^n\int_\Omega \bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx\right)^{\frac12} \\ &\quad\le \frac12\varepsilon_1^{-1}r_5k_1^n\phi^{\frac{2s}{\sigma}}\int_\Omega u^{\sigma+m-3}\,dx + \frac12\varepsilon_1k_1^n\int_\Omega \bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx, \end{aligned}$$

where $r_5 = (k_1^{-\frac{ns}{\sigma}}|\Omega|^{\frac{\sigma-s}{\sigma}}\frac{2}{\sigma+m-1})^2$, ε_1 is a positive constant which will be defined later.

From the above deductions, we get

$$\begin{aligned} &r_2k_2k_1^n\int_\Omega u^{\sigma+m-3}|\nabla u|\,dx\int_\Omega u^s\,dx \\ &\quad\le \frac12 r_2k_2\varepsilon_1^{-1}r_5k_1^n\phi^{\frac{2s}{\sigma}}\int_\Omega u^{\sigma+m-3}\,dx + \frac12 r_2k_2\varepsilon_1k_1^n\int_\Omega \bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\begin{aligned}\phi'(t) &\le n\alpha\phi + r_1k_1^nk_2\int_\Omega u^{\sigma+m+s-2}\,dx + \frac12 r_2k_2\varepsilon_1^{-1}r_5k_1^n\phi^{\frac{2s}{\sigma}}\int_\Omega u^{\sigma+m-3}\,dx \\ &\quad + r_4k_1^{n+1}\int_\Omega u^{\sigma+p+q-1}\,dx + \left(\frac12 r_2k_2\varepsilon_1 - r_3\right)k_1^n\int_\Omega \bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx. \end{aligned} \tag{3.6}$$

Using (2.3), (2.4), and (2.5), we obtain

$$\begin{aligned}\int_\Omega u^{\sigma+m+s-2}\,dx &\le \left(\int_\Omega u^{\frac{(\sigma+m-1)n}{n-2}}\,dx\right)^{x_1}\left(\int_\Omega u^\sigma\,dx\right)^{x_2}\\ &\le \left(C^{\frac{2n}{n-2}}2^{\frac{n}{n-2}-1}\right)^{x_1}\left[\left(\int_\Omega u^{\sigma+m-1}\,dx\right)^{\frac{x_1 n}{n-2}}\right.\\ &\quad\left.+\left(\int_\Omega\left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2dx\right)^{\frac{x_1 n}{n-2}}\right]\left(\int_\Omega u^\sigma\,dx\right)^{x_2}\\ &= r_6\left(\int_\Omega u^{\sigma+m-1}\,dx\right)^{\frac{x_1 n}{n-2}}\left(\int_\Omega u^\sigma\,dx\right)^{x_2}\\ &\quad+r_6\left(\int_\Omega\left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2dx\right)^{\frac{x_1 n}{n-2}}\left(\int_\Omega u^\sigma\,dx\right)^{x_2},\end{aligned}\tag{3.7}$$

where

$$x_1=\frac{(m+s-2)(n-2)}{(m-1)n+2\sigma},\qquad x_2=\frac{(m-1)n+2\sigma+(2-m-s)(n-2)}{(m-1)n+2\sigma},$$
$$r_6=\left(C^{\frac{2n}{n-2}}2^{\frac{n}{n-2}-1}\right)^{x_1}.$$

Using Hölder's and Young's inequalities, we have

$$\begin{aligned}&r_6\left(\int_\Omega u^{\sigma+m-1}\,dx\right)^{\frac{x_1 n}{n-2}}\left(\int_\Omega u^\sigma\,dx\right)^{x_2}\\ &\quad=\left(\frac{n-2}{x_1 n}\int_\Omega u^{\sigma+m-1}\,dx\right)^{\frac{x_1 n}{n-2}}\left\{\left[\left(\frac{n-2}{x_1 n}\right)^{-\frac{x_1 n}{n-2}}r_6\left(\int_\Omega u^\sigma\,dx\right)^{x_2}\right]^{\frac{n-2}{n-2-x_1 n}}\right\}^{\frac{n-2-x_1 n}{n-2}}\\ &\quad\le\int_\Omega u^{\sigma+m-1}\,dx+r_7\left(\int_\Omega u^\sigma\,dx\right)^{\frac{x_2(n-2)}{n-2-x_1 n}},\end{aligned}\tag{3.8}$$

where $r_7=\frac{n-2-x_1 n}{n-2}\left(\frac{n-2}{x_1 n}\right)^{-\frac{x_1 n}{n-2-x_1 n}}r_6^{\frac{n-2}{n-2-x_1 n}}$.

By Hölder's and Young's inequalities, we get

$$\begin{aligned}\int_\Omega u^{\sigma+m-1}\,dx&\le\left(\varepsilon_2\int_\Omega u^{\sigma+m+s-2}\,dx\right)^{x_{10}}\left(\varepsilon_2^{-\frac{x_{10}}{x_{20}}}\int_\Omega u^\sigma\,dx\right)^{x_{20}}\\ &\le x_{10}\varepsilon_2\int_\Omega u^{\sigma+m+s-2}\,dx+x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}\int_\Omega u^\sigma\,dx,\end{aligned}$$

where $x_{10}=\frac{m-1}{m+s-2}$, $n_{10}=\frac{(\sigma+m+s-2)(m-1)}{m+s-2}$, $x_{20}=\frac{s-1}{m+s-2}$, $n_{20}=\frac{(s-1)\sigma}{m+s-2}$.

If we choose ε_2 such that $x_{10}\varepsilon_2=\frac{1}{2}$, we have

$$\int_\Omega u^{\sigma+m-1}\,dx\le\frac{1}{2}\int_\Omega u^{\sigma+m+s-2}\,dx+x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}\int_\Omega u^\sigma\,dx.\tag{3.9}$$

Combining (3.7)–(3.9), we obtain

$$\int_\Omega u^{\sigma+m+s-2}\,dx \le 2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}\int_\Omega u^\sigma\,dx + 2r_7\left(\int_\Omega u^\sigma\,dx\right)^{\frac{x_2(n-2)}{n-2-x_1 n}} + 2r_6\left(\int_\Omega\left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx\right)^{\frac{x_1 n}{n-2}}\left(\int_\Omega u^\sigma\,dx\right)^{x_2}. \tag{3.10}$$

Then we can deduce

$$\begin{aligned}
&k_1^n\int_\Omega u^{\sigma+m+s-2}\,dx\\
&\le 2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}\phi + 2r_7k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}}\left(k_1^n\int_\Omega u^\sigma\,dx\right)^{\frac{x_2(n-2)}{n-2-x_1n}}\\
&\quad + 2r_6k_1^{n-\frac{x_1n^2}{n-2}-nx_2}\left(k_1^n\int_\Omega\left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx\right)^{\frac{x_1n}{n-2}}\left(k_1^n\int_\Omega u^\sigma\,dx\right)^{x_2}\\
&\le 2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}\phi + 2r_7k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}}\phi^{\frac{x_2(n-2)}{n-2-x_1n}} + 2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\left(k_1^n\int_\Omega\left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx\right)^{\frac{x_1n}{n-2}}\phi^{x_2}\\
&\le 2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}\phi + 2r_7k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}}\phi^{\frac{x_2(n-2)}{n-2-x_1n}} + 2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{x_1n}{n-2}\varepsilon_3k_1^n\int_\Omega\left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx\\
&\quad + 2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_3^{-\frac{x_1n}{n-2-x_1n}}\phi^{\frac{x_2(n-2)}{n-2-x_1n}}\\
&\le 2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}\phi + \left[2r_7k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}} + 2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_3^{-\frac{x_1n}{n-2-x_1n}}\right]\phi^{\frac{x_2(n-2)}{n-2-x_1n}}\\
&\quad + 2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{x_1n}{n-2}\varepsilon_3k_1^n\int_\Omega\left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx,
\end{aligned}\tag{3.11}$$

where ε_3 is a positive constant which will be defined later.

If we choose $x_{11}=\frac{m-3}{m+s-2}$, $n_{11}=\frac{(\sigma+m+s-2)(m-3)}{m+s-2}$, $x_{21}=\frac{s+1}{m+s-2}$, $n_{21}=\frac{(s+1)\sigma}{m+s-2}$, using (2.4), we get

$$\begin{aligned}
\int_\Omega u^{\sigma+m-3}\,dx &\le \left(\int_\Omega u^{\sigma+m+s-2}\,dx\right)^{x_{11}}\left(\int_\Omega u^\sigma\,dx\right)^{x_{21}}\\
&\le x_{11}\int_\Omega u^{\sigma+m+s-2}\,dx + x_{21}\int_\Omega u^\sigma\,dx.
\end{aligned}$$

Then we obtain

$$k_1^n\phi^{\frac{2s}{\sigma}}\int_\Omega u^{\sigma+m-3}\,dx \le x_{11}\phi^{\frac{2s}{\sigma}}k_1^n\int_\Omega u^{\sigma+m+s-2}\,dx + x_{21}\phi^{\frac{2s}{\sigma}+1}. \tag{3.12}$$

Combining (3.10) and (3.12), we have

$$k_1^n\phi^{\frac{2s}{\sigma}}\int_\Omega u^{\sigma+m-3}\,dx \le \left(2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}x_{11}+x_{21}\right)\phi^{\frac{2s}{\sigma}+1} + x_{11}k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}}2r_7\phi^{\frac{2s}{\sigma}+\frac{x_2(n-2)}{n-2-x_1n}} + 2r_6x_{11}k_1^{nx_1-\frac{x_1n^2}{n-2}}\left(k_1^n\int_\Omega\left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 dx\right)^{\frac{x_1n}{n-2}}\phi^{\frac{2s}{\sigma}+x_2}$$

$$\le \bigl(2\varkappa_{20}\varepsilon_2^{-\frac{\varkappa_{10}}{\varkappa_{20}}}\varkappa_{11}+\varkappa_{21}\bigr)\phi^{\frac{2s}{\sigma}+1}+\varkappa_{11}k_1^{n-\frac{\varkappa_2(n-2)n}{n-2-\varkappa_1 n}}2r_7\phi^{\frac{2s}{\sigma}+\frac{\varkappa_2(n-2)}{n-2-\varkappa_1 n}}$$
$$+2r_6\varkappa_{11}k_1^{n\varkappa_1-\frac{\varkappa_1 n^2}{n-2}}\frac{\varkappa_1 n}{n-2}\varepsilon_4 k_1^n\int_\Omega\bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx$$
$$+2r_6\varkappa_{11}k_1^{n\varkappa_1-\frac{\varkappa_1 n^2}{n-2}}\frac{n-2-\varkappa_1 n}{n-2}\varepsilon_4^{-\frac{\varkappa_1 n}{n-2-\varkappa_1 n}}\phi^{\frac{(2s+\sigma\varkappa_2)(n-2)}{\sigma(n-2-\varkappa_1 n)}}, \tag{3.13}$$

where ε_4 is a positive constant which will be defined later.

Similarly, if we choose $\varkappa_{12}=\frac{p+q-1}{m+s-2}$, $n_{12}=\frac{(\sigma+m+s-2)(p+q-1)}{m+s-2}$, $\varkappa_{22}=\frac{m+s-(p+q+1)}{m+s-2}$, $n_{22}=\frac{\sigma[m+s-(p+q+1)]}{m+s-2}$, using (2.4), we get

$$\int_\Omega u^{\sigma+p+q-1}\,dx\le\Bigl(\int_\Omega u^{\sigma+m+s-2}\,dx\Bigr)^{\varkappa_{12}}\Bigl(\int_\Omega u^\sigma\,dx\Bigr)^{\varkappa_{22}}$$
$$\le\varkappa_{12}\int_\Omega u^{\sigma+m+s-2}\,dx+\varkappa_{22}\int_\Omega u^\sigma\,dx. \tag{3.14}$$

Combining (3.10) and (3.14), we obtain

$$k_1^{n+1}\int_\Omega u^{\sigma+p+q-1}\,dx$$
$$\le\varkappa_{12}k_1^{n+1}\int_\Omega u^{\sigma+m+s-2}\,dx+\varkappa_{22}k_1^{n+1}\int_\Omega u^\sigma\,dx$$
$$\le\bigl(2\varkappa_{20}\varepsilon_2^{-\frac{\varkappa_{10}}{\varkappa_{20}}}\varkappa_{12}k_1+\varkappa_{22}k_1\bigr)\phi$$
$$+\Bigl(2r_7\varkappa_{12}k_1^{n+1-\frac{\varkappa_2(n-2)n}{n-2-\varkappa_1 n}}+2r_6\varkappa_{12}k_1^{n\varkappa_1+1-\frac{\varkappa_1 n^2}{n-2}}\frac{n-2-\varkappa_1 n}{n-2}\varepsilon_5^{-\frac{\varkappa_1 n}{n-2-\varkappa_1 n}}\Bigr)$$
$$\cdot\phi^{\frac{\varkappa_2(n-2)}{n-2-\varkappa_1 n}}+2r_6\varkappa_{12}k_1^{n\varkappa_1+1-\frac{\varkappa_1 n^2}{n-2}}\frac{\varkappa_1 n}{n-2}\varepsilon_5 k_1^n\int_\Omega\bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx, \tag{3.15}$$

where ε_5 is a positive constant which will be defined later.

Combining (3.6), (3.11), (3.13), and (3.15), we have

$$\phi'(t)\le\bigl(n\alpha+2r_1k_2\varkappa_{20}\varepsilon_2^{-\frac{\varkappa_{10}}{\varkappa_{20}}}+2r_4\varkappa_{20}\varepsilon_2^{-\frac{\varkappa_{10}}{\varkappa_{20}}}\varkappa_{12}k_1+r_4\varkappa_{22}k_1\bigr)\phi$$
$$+\Bigl(2r_1k_2r_7k_1^{n-\frac{\varkappa_2(n-2)n}{n-2-\varkappa_1 n}}+2r_1k_2r_6k_1^{n\varkappa_1-\frac{\varkappa_1 n^2}{n-2}}\frac{n-2-\varkappa_1 n}{n-2}\varepsilon_3^{-\frac{\varkappa_1 n}{n-2-\varkappa_1 n}}$$
$$+2r_4r_7\varkappa_{12}k_1^{n+1-\frac{\varkappa_2(n-2)n}{n-2-\varkappa_1 n}}+2r_4r_6\varkappa_{12}k_1^{n\varkappa_1+1-\frac{\varkappa_1 n^2}{n-2}}\frac{n-2-\varkappa_1 n}{n-2}\varepsilon_5^{-\frac{\varkappa_1 n}{n-2-\varkappa_1 n}}\Bigr)\phi^{\frac{\varkappa_2(n-2)}{n-2-\varkappa_1 n}}$$
$$+\frac{1}{2}r_2k_2\varepsilon_1^{-1}r_5\bigl(2\varkappa_{20}\varepsilon_2^{-\frac{\varkappa_{10}}{\varkappa_{20}}}\varkappa_{11}+\varkappa_{21}\bigr)\phi^{\frac{2s}{\sigma}+1}$$
$$+\frac{1}{2}r_2k_2\varepsilon_1^{-1}r_5\varkappa_{11}k_1^{n-\frac{\varkappa_2(n-2)n}{n-2-\varkappa_1 n}}2r_7\phi^{\frac{2s}{\sigma}+\frac{\varkappa_2(n-2)}{n-2-\varkappa_1 n}}$$
$$+r_2k_2\varepsilon_1^{-1}r_5r_6\varkappa_{11}k_1^{n\varkappa_1-\frac{\varkappa_1 n^2}{n-2}}\frac{n-2-\varkappa_1 n}{n-2}\varepsilon_4^{-\frac{\varkappa_1 n}{n-2-\varkappa_1 n}}\phi^{\frac{(2s+\sigma\varkappa_2)(n-2)}{\sigma(n-2-\varkappa_1 n)}}$$
$$+\Bigl(2r_1k_2r_6k_1^{n\varkappa_1-\frac{\varkappa_1 n^2}{n-2}}\frac{\varkappa_1 n}{n-2}\varepsilon_3+\frac{1}{2}r_2k_2\varepsilon_1+r_2k_2\varepsilon_1^{-1}r_5r_6\varkappa_{11}k_1^{n\varkappa_1-\frac{\varkappa_1 n^2}{n-2}}\frac{\varkappa_1 n}{n-2}\varepsilon_4$$
$$+2r_4r_6\varkappa_{12}k_1^{n\varkappa_1+1-\frac{\varkappa_1 n^2}{n-2}}\frac{\varkappa_1 n}{n-2}\varepsilon_5-r_3\Bigr)k_1^n\int_\Omega\bigl|\nabla u^{\frac{\sigma+m-1}{2}}\bigr|^2\,dx. \tag{3.16}$$

If we choose suitable ε_1, ε_3, ε_4, ε_5 such that

$$2r_1k_2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{x_1n}{n-2}\varepsilon_3+\frac{1}{2}r_2k_2\varepsilon_1+r_2k_2\varepsilon_1^{-1}r_5r_6x_{11}k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{x_1n}{n-2}\varepsilon_4 + 2r_4r_6x_{12}k_1^{nx_1+1-\frac{x_1n^2}{n-2}}\frac{x_1n}{n-2}\varepsilon_5-r_3=0. \tag{3.17}$$

Substituting (3.17) into (3.16), we derive

$$\begin{aligned}\phi'(t)\le{}&\left(n\alpha+2r_1k_2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}+2r_4x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}x_{12}k_1+r_4x_{22}k_1\right)\phi\\&+\Bigg(2r_1k_2r_7k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}}+2r_1k_2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_3^{-\frac{x_1n}{n-2-x_1n}}\\&+2r_4r_7x_{12}k_1^{n+1-\frac{x_2(n-2)n}{n-2-x_1n}}+2r_4r_6x_{12}k_1^{nx_1+1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_5^{-\frac{x_1n}{n-2-x_1n}}\Bigg)\phi^{1+\frac{2x_1}{n-2-x_1n}}\\&+\frac{1}{2}r_2k_2\varepsilon_1^{-1}r_5\left(2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}x_{11}+x_{21}\right)\phi^{1+\frac{2s}{\sigma}}\\&+\frac{1}{2}r_2k_2\varepsilon_1^{-1}r_5x_{11}k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}}2r_7\phi^{1+(\frac{2s}{\sigma}+\frac{2x_1}{n-2-x_1n})}\\&+r_2k_2\varepsilon_1^{-1}r_5r_6x_{11}k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_4^{-\frac{x_1n}{n-2-x_1n}}\phi^{1+\frac{2s(n-2)+2x_1\sigma}{\sigma(n-2-x_1n)}}.\end{aligned} \tag{3.18}$$

Using Hölder's and Young's inequalities, we have

$$\phi^{1+\gamma}\le\left(1-\frac{\gamma}{4}\right)\phi+\frac{\gamma}{4}\phi^5. \tag{3.19}$$

Applying (3.19) to $\phi^{1+\frac{2x_1}{n-2-x_1n}}$, $\phi^{1+\frac{2s}{\sigma}}$, $\phi^{1+(\frac{2s}{\sigma}+\frac{2x_1}{n-2-x_1n})}$, $\phi^{1+\frac{2s(n-2)+2x_1\sigma}{\sigma(n-2-x_1n)}}$ in (3.18), respectively, we obtain

$$\phi'(t)\le a(t)\phi(t)+b(t)\phi^5(t), \tag{3.20}$$

where

$$\begin{aligned}a(t)={}&\left(n\alpha+2r_1k_2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}+2r_4x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}x_{12}k_1+r_4x_{22}k_1\right)\\&+\Bigg(2r_1k_2r_7k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}}+2r_1k_2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_3^{-\frac{x_1n}{n-2-x_1n}}\\&+2r_4r_7x_{12}k_1^{n+1-\frac{x_2(n-2)n}{n-2-x_1n}}\\&+2r_4r_6x_{12}k_1^{nx_1+1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_5^{-\frac{x_1n}{n-2-x_1n}}\Bigg)\left[1-\frac{x_1}{2(n-2-x_1n)}\right]\\&+\frac{1}{2}r_2k_2\varepsilon_1^{-1}r_5\left(2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}x_{11}+x_{21}\right)\left(1-\frac{s}{2\sigma}\right)\\&+\frac{1}{2}r_2k_2\varepsilon_1^{-1}r_5x_{11}k_1^{n-\frac{x_2(n-2)}{n-2-x_1n}}2r_7\left[1-\frac{s(x_2n-2)+x_1\sigma}{2\sigma(n-2-x_1n)}\right]\\&+r_2k_2\varepsilon_1^{-1}r_5r_6x_{11}k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_4^{-\frac{x_1n}{n-2-x_1n}}\left[1-\frac{s(n-2)+x_1\sigma}{2\sigma(n-2-x_1n)}\right]\end{aligned} \tag{3.21}$$

and

$$b(t) = \left(2r_1k_2r_7k_1^{n-\frac{x_2(n-2)n}{n-2-x_1n}} + 2r_1k_2r_6k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_3^{-\frac{x_1n}{n-2-x_1n}} + 2r_4r_7x_{12}k_1^{n+1-\frac{x_2(n-2)n}{n-2-x_1n}} + 2r_4r_6x_{12}k_1^{nx_1+1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_5^{-\frac{x_1n}{n-2-x_1n}}\right)\frac{x_1}{2(n-2-x_1n)} + \frac{1}{2}r_2k_2\varepsilon_1^{-1}r_5\left(2x_{20}\varepsilon_2^{-\frac{x_{10}}{x_{20}}}x_{11} + x_{21}\right)\frac{s}{2\sigma} + \frac{1}{2}r_2k_2\varepsilon_1^{-1}r_5x_{11}k_1^{n-\frac{x_2(n-2)}{n-2-x_1n}}2r_7\left[\frac{s(x_2n-2)+x_1\sigma}{2\sigma(n-2-x_1n)}\right] + r_2k_2\varepsilon_1^{-1}r_5r_6x_{11}k_1^{nx_1-\frac{x_1n^2}{n-2}}\frac{n-2-x_1n}{n-2}\varepsilon_4^{-\frac{x_1n}{n-2-x_1n}}\frac{s(n-2)+x_1\sigma}{2\sigma(n-2-x_1n)}. \tag{3.22}$$

Multiplying both sides of (3.20) by $\phi^{-5}(t)$, we obtain

$$\phi'(t)\phi^{-5}(t) \le a(t)\phi^{-4}(t) + b(t). \tag{3.23}$$

That is,

$$-\left(\phi^{-4}(t)\right)' \le 4a(t)\phi^{-4}(t) + 4b(t). \tag{3.24}$$

Setting $H(t) = \int_0^t a(\tau)\,d\tau$, (3.24) can be rewritten as

$$\left(\phi^{-4}(t)e^{4H(t)}\right)' \ge -4b(t)e^{4H(t)}. \tag{3.25}$$

Integrating (3.25) from 0 to t, we have

$$\phi^{-4}(t)e^{4H(t)} - \phi^{-4}(0) \ge -4\int_0^t b(\tau)e^{4H(\tau)}\,d\tau. \tag{3.26}$$

That is to say,

$$\frac{e^{4H(t)}}{\phi^4(t)} - \frac{1}{\phi^4(0)} \ge -4\Theta(t), \tag{3.27}$$

where $\Theta(t) = \int_0^t b(\tau)e^{4H(\tau)}\,d\tau$.

Taking the limit to (3.27) as $t \to t^*$, we get

$$\Theta\left(t^*\right) \ge \frac{1}{4\phi^4(0)}.$$

From the definition of $\Theta(t)$, we have $\frac{d\Theta(t)}{dt} = b(t)e^{4H(t)} > 0$. We get $\Theta(t)$ is a strictly increasing function. So we can get

$$t^* \ge \Theta^{-1}\left(\frac{1}{4\phi^4(0)}\right),$$

from which we complete the proof of Theorem 3.1. □

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References

1. Liu, Y.: Blow up phenomena for the nonlinear nonlocal porous medium equation under Robin boundary condition. Comput. Math. Appl. **66**, 2092–2095 (2013)
2. Liu, D.M., Mu, C.L., Qiao, X.: Lower bounds estimate for the blow up time of a nonlinear nonlocal porous medium equation. Acta Math. Sci. Ser. B Engl. Ed. **32**, 1206–1212 (2012)
3. Galaktionov, V.A.: On asymptotic self-similar behavior for a quasilinear heat equation: single point blow-up. SIAM J. Math. Anal. **26**, 675–693 (1995)
4. Bögelein, V., Duzaar, F., Korte, R., Scheven, C.: The higher integrability of weak solutions of porous medium systems. Adv. Nonlinear Anal. **8**(1), 1004–1034 (2019)
5. Payne, L.E., Philippin, G.A., Schaefer, P.W.: Blow-up phenomena for some nonlinear parabolic problems. Nonlinear Anal. **69**, 3495–3502 (2008)
6. Payne, L.E., Philippin, G.A., Schaefer, P.W.: Bounds for blow-up time in nonlinear parabolic problems. J. Math. Anal. Appl. **338**, 438–447 (2008)
7. Payne, L.E., Schaefer, P.W.: Lower bounds for blow-up time in parabolic problems under Dirichlet conditions. J. Math. Anal. Appl. **328**, 1196–1205 (2007)
8. Payne, L.E., Schaefer, P.W.: Lower bounds for blow-up time in parabolic problems under Neumann conditions. Appl. Anal. **85**, 1301–1311 (2006)
9. Payne, L.E., Song, J.C.: Lower bounds for blow-up time in a nonlinear parabolic problem. J. Math. Anal. Appl. **354**, 394–396 (2009)
10. Payne, L.E., Song, J.C.: Lower bounds for the blow-up time in a temperature dependent Navier–Stokes flow. J. Math. Anal. Appl. **335**, 371–376 (2007)
11. Schaefer, P.W.: Blow up phenomena in some porous medium problems. Preprint 394-396
12. Song, J.C.: Lower bounds for blow-up time in a non-local reaction-diffusion problem. Appl. Math. Lett. **5**, 793–796 (2011)
13. Li, Y.F., Liu, Y., Lin, C.H.: Blow-up phenomena for some nonlinear parabolic problems under mixed boundary conditions. Nonlinear Anal., Real World Appl. **11**, 3815–3823 (2010)
14. Payne, L.E., Schaefer, P.W.: Blow-up in parabolic problems under Robin boundary conditions. Appl. Anal. **87**, 699–707 (2008)
15. Papageorgiou, N.S., Radulescu, V.D., Repovs, D.D.: Parametric nonlinear resonant Robin problems. Math. Nachr. **292**(11), 2456–2480 (2019)
16. Pikula, M., Vladicic, V., Vojvodic, B.: Inverse spectral problems for Sturm–Liouville operators with a constant delay less than half the length of the interval and 2 Robin boundary conditions. Results Math. **74**(1), Article ID UNSP 45 (2019)
17. Gala, S., Liu, Q., Ragusa, M.A.: A new regularity criterion for the nematic liquid crystal flows. Appl. Anal. **91**(9), 1741–1747 (2012)
18. Liu, Y., Luo, S.G., Ye, Y.H.: Blow-up phenomena for a parabolic problem with a gradient nonlinearity under nonlinear boundary conditions. Comput. Math. Appl. **65**, 1194–1199 (2013)
19. Payne, L.E., Philippin, G.A., Vernier Piro, S.: Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition. Z. Angew. Math. Phys. **61**, 999–1007 (2010)
20. Payne, L.E., Schaefer, P.W.: Bounds for the blow-up time for the heat equation under nonlinear boundary conditions. Proc. R. Soc. Edinb. **139**, 1289–1296 (2009)
21. Tang, G.S.: Blow-up phenomena for a parabolic system with gradient nonlinearity under nonlinear boundary conditions. Comput. Math. Appl. **74**, 360–368 (2017)
22. Ghisi, M., Gobbino, M., Haraux, A.: Quantization of energy and weakly turbulent profiles of solutions to some damped second-order evolution equations. Adv. Nonlinear Anal. **8**(1), 902–927 (2019)

23. Papageorgiou, N.S., Radulescu, V.D., Repovs, D.D.: Nonlinear second order evolution inclusions with noncoercive viscosity term. J. Differ. Equ. **264**(7), 4749–4763 (2018)
24. Papageorgiou, N.S., Radulescu, V.D., Repovs, D.D.: Nonlinear Analysis—Theory and Methods. Springer Monographs in Mathematics. Springer, Cham (2019)
25. Ding, J.T., Shen, X.H.: Blow-up analysis in quasilinear reaction–diffusion problems with weighted nonlocal source. Comput. Math. Appl. **75**, 1288–1301 (2018)
26. Shen, X.H., Ding, J.T.: Blow-up phenomena in porous medium equation systems with nonlinear boundary conditions. Comput. Math. Appl. **77**, 3250–3263 (2019)
27. Ding, J.T., Kou, W.: Blow-up solutions for reaction diffusion equations with nonlocal boundary conditions. J. Math. Anal. Appl. **470**, 1–15 (2019)
28. Ding, J.T., Hu, H.J.: Blow-up and global solutions for a class of nonlinear reaction diffusion equations under Dirichlet boundary conditions. J. Math. Anal. Appl. **433**, 1718–1735 (2016)
29. Chen, W.H., Liu, Y.: Lower bound for the blow up time for some nonlinear parabolic equations. Bound. Value Probl. **2016**, 161 (2016)
30. Brezis, H.: Functional analysis, Sobolev spaces and partial differential equations. In: Universitext. Springer, New York (2011)