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Pullback attractor of a non-autonomous order-2γ parabolic equation for an epitaxial thin film growth model

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Abstract

The non-autonomous order-2γ parabolic partial differential equation for an epitaxial thin film growth model with dimension $d = 3$ is investigated by the method of uniform estimates. The existence of a pullback attractor for the 3D model is proved for $\gamma > 3$.

Keywords: Pullback attractor; Asymptotic compactness; Non-autonomous parabolic equation

1 Introduction

In [1], Duan and Zhao showed the existence of pullback $\mathcal{D}$-attractor for a non-autonomous fourth-order parabolic equation. Inspired by [1], we consider the following non-autonomous order-2γ parabolic equation:

$$\begin{cases} h_t + \nabla \cdot (1 - |\nabla h|^{\gamma-2})\nabla h + (-\Delta)^\gamma h = g(x,t), & \text{in } \Omega \times [\tau, \infty), \\ h|_{t=\tau} = h_\tau(x), \end{cases} \tag{1}$$

where $h = h(x,t)$ is the scaled height of a thin film. $\Omega = [0,L]^3 \subseteq \mathbb{R}^3$ is the periodic domain. $\gamma > 3$ is a positive constant.

The system (1) is modified by equations in [1] and [2]. In [2], the global well-posedness of strong solution for a γ-order epitaxial growth model with $g \equiv 0$, $\gamma > 2$ and $d = 1,2$ or $g \equiv 0$, $\gamma \geq d \geq 3$ was studied by Fan–Alsaedi–Hayat–Zhou. Fan–Samet–Zhou [3] showed the global well-posedness of weak solutions and the regularity of strong solutions for an epitaxial growth model with $g \equiv 0$. In this note, we study the order-2γ non-autonomous equation of an epitaxial growth model.

Pullback attractors which form a family of compact sets that is bounded in phase space and has invariability under the dynamics system. And the pullback attractor plays a key role in the larger-time behavior of solutions. Compared with uniform attractors, the existence of pullback attractor is easy to get with the weak assumption of a force term. Since the pullback attractor appeared, it has aroused the interest of lots of authors and also has been made great progress. In [4–8], the pullback attractor was presented and proved. The pullback $\mathcal{D}$-attractor of systems for non-autonomous dynamics has been proposed in [9] and pullback $\mathcal{D}$-attractor of non-linear evolution equation was showed in [10]. In [11], the pullback attractors of the n-dimensional non-autonomous parabolic equation was considered. In [12], the norm-to-weak continuous process has been proposed and the proof of the existence of the pullback $\mathcal{D}$-attractor for non-autonomous equation in H_0^1 was showed by Li and Zhong. The continuity of pullback and uniform attractors was studied by Hoanga, Olsonb and Robinson [13]. And in [14], Cheskidov and Kavlie did many studies with pullback attractors. However, the existence of pullback $\mathcal{D}$-attractor for a non-autonomous order-2γ parabolic has not been studied for $\gamma > 3$ yet.

In this paper, we need to overcome the difficulty of the non-linear term $\nabla \cdot (1 - |\nabla h|^{\gamma-2})\nabla h$ and $(-\Delta)^\gamma h$. In [15], Park and Park put the condition of exponential growth on the external forcing term $g(x,t)$ and gave the proof of the existence of a modified non-autonomous equation. Inspired by [15], we also assume that the external forcing term $g(x,t)$ satisfies the condition of exponential growth, which we give in Sect. 2. To obtain an appropriate prior condition, we must limit the parameter $\gamma > 3$ and use the Sobolev inequality many times.

In this note, the pullback $\mathcal{D}$-attractor of an order-2γ non-autonomous model (1) with $\gamma > 3$ is studied. The paper is organized as follows. In Sect. 2, we do some preparatory work and give the main result. In Sect. 3, using the method of uniform estimates, the existence of a pullback $\mathcal{D}$-attractor for system (1) is proved for $\gamma > 3$.

In the following sections, let $(1)_1$ represent the first equation of (1). And note that constant C shows different data in different rows.

2 Preliminaries

Assume that X is a complete metric space and $\{h(t,\tau)\} = \{h(t,\tau) | t \geq \tau, \tau \in \mathbb{R}\}$ is a two-parameter family of mappings act on $X : h(t,\tau) : X \to X$, $t \geq \tau$, $\tau \in \mathbb{R}$.

Definition 2.1 (See [16]) A two-parameter family of mappings $\{h(t,\tau)\}$ is called a norm-to-weak continuous process in X if

(1) $h(t,s)h(s,\tau) = h(t,\tau)$ for all $t \geq s \geq \tau$,

(2) $h(t,\tau) = \mathrm{Id}$ is the identity operator for all $\tau \in \mathbb{R}$,

(3) $h(t,\tau)x_n \rightharpoonup h(t,\tau)x$, if $x_n \to x$ in X.

Assume $B(X)$ is the collection of bounded subsets for X and $A, B \subset X$. Then we denote the Kuratowski measure of non-compactness $\alpha(B)$ of B as follows:

$$\alpha(B) = \inf\{\delta > 0 : B \text{ has a finite open cover of sets of diameter} \leq \delta\}. \tag{2}$$

Let $\mathcal{D}$ be a nonempty class of parameterized sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset B(X)$.

Definition 2.2 (See [12]) A process $\{h(t,\tau)\}$ is said to be pullback ω-$\mathcal{D}$-limit compact if for any $\varepsilon > 0$ and $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(\widehat{D}, t) \leq t$ such that $\alpha(\bigcup_{\tau \geq \tau_0} h(t,\tau)D(\tau)) \leq \varepsilon$.

Definition 2.3 (See [12]) The family $\widehat{A} = \{A(t) : t \in \mathbb{R}\} \subset B(X)$ is called a pullback $\mathcal{D}$-attractor for $h(t,\tau)$ if

(1) A(t) is compact for all $t \in \mathbb{R}$,

(2) $\widehat{A}$ is invariant, i.e., $h(t,\tau)A(\tau) = A(t)$ for all $t \geq \tau$,
(3) $\widehat{A}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau\to-\infty} \operatorname{dist}\bigl(h(t,\tau)D(\tau), A(t)\bigr) = 0, \quad \forall \widehat{D} \in \mathcal{D}, t \in \mathbb{R}.$$

(4) If $\{C(t)\}_{t\in\mathbb{R}}$ is another family of closed attracting sets, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Next, we show Lemma 2.4 which was given in [12]. It is important for the existence proof of pullback attractor for a non-autonomous model.

Lemma 2.4 *Suppose $\{h(t,\tau)\}_{\tau\leq t}$ is a norm-to-weak continuous process such that $\{h(t,\tau)\}_{\tau\leq t}$ is pullback ω-$\mathcal{D}$-limit compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$, i.e., for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$ there is a $\tau_0(\widehat{\mathcal{D}}, t) \leq t$ such that $h(t,\tau)D(\tau) \subset B(t)$ for all $\tau \leq \tau_0$, then there is a pullback $\mathcal{D}$-attractor $\{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} h(t,\tau)D(\tau)}.$$

For convenience, assume that

$$g(x,t) \in L^2_{\text{loc}}\bigl(\mathbb{R}; L^2(\Omega)\bigr) \quad \text{and} \quad \sup_{t\in\mathbb{R}} \int_t^{t+1} \bigl|g(x,s)\bigr|^2 \, ds < \infty.$$

Assume that there exist $\beta \geq 0$ and $0 \leq \alpha \leq (\frac{\gamma+2}{4\gamma^2-9\gamma+6})^2\xi$ for any $t \in \mathbb{R}$, such that

$$\bigl\|g(t)\bigr\|^2 \leq \beta e^{\alpha|t|}, \tag{3}$$

where ξ is the first eigenvalue of $A = (-\Delta)^\gamma$. And in this paper, let $(-\Delta)^{\frac{1}{2}} = \Lambda$.
Using (3), we can obtain for any $t \in \mathbb{R}$

$$G_1(t) := \int_{-\infty}^t e^{\xi s} \bigl\|g(s)\bigr\|^2 \, ds < \infty, \tag{4}$$

$$G_2(t) := \int_{-\infty}^t \int_{-\infty}^s e^{\xi r} \bigl\|g(r)\bigr\|^2 \, dr \, ds < \infty, \tag{5}$$

$$\int_{-\infty}^t e^{-\frac{2(2\gamma^2-5\gamma+2)}{\gamma+2}\xi s} \Bigl[\bigl[G_1(s)\bigr]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}} + \bigl[G_2(s)\bigr]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\Bigr] \, ds < \infty. \tag{6}$$

By a slight modification of the classical results in the autonomous framework, mainly of the Faedo–Galerkin method [17]. In the following, we obtain the result on the existence and uniqueness of solution for system (1).

Lemma 2.5 *Assume that $g(x,t) \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$. There is a unique solution $h(x,t)$ such that*
(1) *if $h_0 = h_\tau \in L^2(\Omega) \Rightarrow h(x,t) \in C^0([\tau,\infty); L^2(\Omega))$;*
(2) *if $h_0 = h_\tau \in H^\gamma_0(\Omega) \Rightarrow h(x,t) \in C^0([\tau,\infty); H^\gamma_0(\Omega))$.*

By Lemma 2.5, we can see that the solution is continuous with initial condition h_τ in the space $H_0^\gamma(\Omega)$. Then we define $h(t,\tau): H_0^\gamma(\Omega) \to H_0^\gamma(\Omega)$ by $h(t,\tau)h_\tau$. We can find that the process $h(t,\tau)$ is a norm-to-weak continuous process in the space $H_0^\gamma(\Omega)$.

Next we give main result.

Theorem 2.6 *Assume that* $g(x,t) \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ *satisfies* (3) *and* $h_\tau \in H_0^\gamma(\Omega)$ *with* $\gamma > 3$. *Then there exists a unique pullback* $\mathcal{D}$*-attractor in space* $H_0^\gamma(\Omega)$ *which is in control of the process corresponding to system* (1).

3 Proof of main result

In this section, we prove the existence of pullback $\mathcal{D}$-attractors for (1). Firstly, we show uniform estimates of solutions for system (1). It has an important effect on the proof that the model (1) has an absorbing set of $\{h(t,\tau)\}$.

Lemma 3.1 *Assume that* $g(x,t) \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ *satisfies* (3) *and* $h_\tau \in H_0^\gamma(\Omega)$ *where* $\gamma > 3$. *Consider the system* (1), *we have, for all* $t \geq \tau$,

$$\|h(t)\|^2 \leq e^{-\xi(t-\tau)}\|h_\tau\|^2 + \frac{1}{\xi}e^{-\xi t}\int_{-\infty}^t e^{\xi s}\|g(s)\|^2\,ds + C \tag{7}$$

and

$$\int_\tau^t e^{\xi s}\|\Lambda^\gamma h(s)\|^2\,ds \leq \bigl[1+\xi(t-\tau)\bigr]e^{\xi\tau}\|h_\tau\|^2 + \frac{1}{\xi}G_1(t) + G_2(t) + Ce^{\xi t}. \tag{8}$$

Proof Multiplying $(1)_1$ by h and integrating it over Ω, we can deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega h^2(t)\,dx + \int_\Omega |\nabla h(t)|^\gamma\,dx + \int_\Omega \bigl(\Lambda^\gamma h(t)\bigr)^2\,dx \\
&\quad= \int_\Omega |\nabla h(t)|^2\,dx + \bigl(g(t),h(t)\bigr) \\
&\quad\leq \frac{1}{2}\|\nabla h(t)\|_\gamma^\gamma + \frac{1}{2}\xi^{-1}\|g(t)\|^2 + \frac{1}{2}\|\Lambda^\gamma h(t)\|^2 + C.
\end{aligned} \tag{9}$$

From (9), we can get

$$\frac{d}{dt}\|h(t)\|^2 + \|\nabla h(t)\|_\gamma^\gamma + \|\Lambda^\gamma h(t)\|^2 \leq \frac{1}{\xi}\|g(t)\|^2 + C \tag{10}$$

and

$$\frac{d}{dt}\|h(t)\|^2 + \xi\|h(t)\|^2 \leq \frac{1}{\xi}\|g(t)\|^2 + C. \tag{11}$$

Multiplying (11) by $e^{\xi(t-\tau)}$ and integrating it over (τ,t), we have

$$\|h(t)\|^2 \leq e^{-\xi(t-\tau)}\|h_\tau\|^2 + \frac{1}{\xi}e^{-\xi t}\int_{-\infty}^t e^{\xi s}\|g(s)\|^2\,ds + C. \tag{12}$$

Multiplying (12) by $e^{\xi t}$ and integrating it over (t,τ), we obtain

$$\int_\tau^t e^{\xi s}\|h(s)\|^2\,ds \leq (t-\tau)e^{\xi\tau}\|h_\tau\|^2 + \frac{1}{\xi}\int_{-\infty}^t\int_{-\infty}^s e^{\xi r}\|g(r)\|^2\,dr\,ds + Ce^{\xi t}. \tag{13}$$

Similarly, multiplying (10) by $e^{\xi t}$ and integrating it over (t,τ), using (13), it is easy to get

$$\begin{aligned}
&\int_\tau^t e^{\xi s}\|\nabla h(s)\|_\gamma^\gamma\,ds+\int_\tau^t e^{\xi s}\|\Lambda^\gamma h(s)\|^2\,ds\\
&\le e^{\xi\tau}\|h_\tau\|^2+\frac{1}{\xi}\int_\tau^t e^{\xi s}\|g(s)\|^2\,ds+\xi\int_\tau^t e^{\xi s}\|h(s)\|^2\,ds+Ce^{\xi t}\\
&\le\bigl[1+\xi(t-\tau)\bigr]e^{\xi\tau}\|h_\tau\|^2+\frac{1}{\xi}\int_\tau^t e^{\xi s}\|g(s)\|^2\,ds\\
&\quad+\int_{-\infty}^t\int_{-\infty}^s e^{\xi r}\|g(r)\|^2\,dr\,ds+Ce^{\xi t}.
\end{aligned}\tag{14}$$

The proof is complete. □

Lemma 3.2 *Let $h_\tau\in H_0^\gamma(\Omega)$, $g(x,t)\in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$ and $g(x,t)$ satisfy (3) with $\gamma>3$. In system (1) for any $t\ge\tau$, we have*

$$\|\nabla h(t)\|^2\le C\biggl\{e^{-\xi(t-\tau)}\bigl[\|\nabla h_\tau\|^2+\|h_\tau\|^2\bigr]+e^{-\xi t}\biggl[G_1(t)+\frac{1}{\xi}G_2(t)\biggr]+1\biggr\}.\tag{15}$$

Proof Multiplying $(1)_1$ by Δh and integrating it over Ω, we derive that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla h(t)\|^2+\|\Lambda^{\gamma+1}h(t)\|^2+\bigl(\nabla\cdot\bigl(|\nabla h(t)|^{\gamma-2}\nabla h(t)\bigr),\Delta h(t)\bigr)\\
&=\|\Delta h(t)\|^2-\bigl(g(t),\Delta h(t)\bigr).
\end{aligned}\tag{16}$$

Now we have

$$\begin{aligned}
&\bigl(\nabla\cdot\bigl(|\nabla h(t)|^{\gamma-2}\nabla h(t)\bigr),\Delta h(t)\bigr)\\
&=\int_\Omega|\nabla h(t)|^{\gamma-2}|\Delta h(t)|^2\,dx+\int_\Omega\nabla h(t)\nabla\bigl(|\nabla h(t)|^{\gamma-2}\bigr)\Delta h(t)\,dx\\
&=\int_\Omega|\nabla h(t)|^{\gamma-2}|\Delta h(t)|^2\,dx+\frac{\gamma-2}{2}\int_\Omega\nabla h(t)\nabla\bigl(|\nabla h(t)|^2\bigr)|\nabla h(t)|^{\gamma-4}\Delta h(t)\,dx\\
&=\int_\Omega|\nabla h(t)|^{\gamma-2}|\Delta h(t)|^2\,dx++(\gamma-2)\int_\Omega\Biggl(\sum_{i,j=3}^3h_{ij}h_ih_j\Biggr)\Biggl(\sum_{k=1}^3h_{kk}\Biggr)|\nabla h(t)|^{\gamma-4}\,dx\\
&=\int_\Omega|\nabla h(t)|^{\gamma-2}|\Delta h(t)|^2\,dx+(\gamma-2)\int_\Omega\Biggl(\sum_{i=1}^3h_{ii}h_i^2\Biggr)\Biggl(\sum_{k=1}^3h_{kk}\Biggr)|\nabla h(t)|^{\gamma-4}\,dx\\
&\quad+(\gamma-2)\int_\Omega\Biggl(\sum_{i,j=1,i\neq j}^3h_{ij}h_ih_j\Biggr)\Biggl(\sum_{k=1}^3h_{kk}\Biggr)|\nabla h(t)|^{\gamma-4}\,dx\\
&=\int_\Omega|\nabla h(t)|^{\gamma-2}|\Delta h(t)|^2\,dx+(\gamma-2)\sum_{i=1}^3\int_\Omega|\nabla h(t)|^{\gamma-4}h_i^2h_{ii}^2\,dx\\
&\quad+(\gamma-2)\sum_{i,j=1,i\neq j}^3\int_\Omega|\nabla h(t)|^{\gamma-4}h_i^2h_{ii}h_{jj}\,dx\\
&\quad+(\gamma-2)\int_\Omega|\nabla h(t)|^{\gamma-4}\Delta h(t)\Biggl(\sum_{i,j=1,i\neq j}^3h_{ij}h_ih_j\Biggr)\,dx.
\end{aligned}\tag{17}$$

By Young's inequality, we obtain

$$
\begin{aligned}
&(\gamma-2)\sum_{i,j=1,i\neq j}^{3}\int_{\Omega}|\nabla h|^{\gamma-4}h_i^2h_{ii}h_{jj}\,dx\\
&\geq -\frac{\gamma-2}{2}\sum_{i,j=1,i\neq j}^{3}\int_{\Omega}|\nabla h|^{\gamma-4}h_i^2\left(h_{ii}^2+h_{jj}^2\right)dx\\
&= -\frac{\gamma-2}{2}\sum_{i=1}^{3}\int_{\Omega}|\nabla h|^{\gamma-4}h_i^2h_{ii}^2\,dx\\
&\quad -\frac{\gamma-2}{2}\sum_{i,j=1,i\neq j}^{3}\int_{\Omega}|\nabla h|^{\gamma-4}h_i^2h_{jj}^2\,dx.
\end{aligned}
\tag{18}
$$

On account of the regularity theorem of elliptic equations, we have

$$
\begin{aligned}
&(\gamma-2)\int_{\Omega}|\nabla h|^{\gamma-4}\Delta h\left(\sum_{i,j=1,i\neq j}^{3}h_{ij}h_ih_j\right)dx\\
&\geq -\frac{\gamma-2}{2}\int_{\Omega}|\nabla h|^{\gamma-2}|\Delta h|^2\,dx.
\end{aligned}
\tag{19}
$$

Using (17), (18) and (19), we get

$$
\begin{aligned}
&\left(\nabla\cdot\left(\left|\nabla h(t)\right|^{\gamma-2}\nabla h(t)\right),\Delta h(t)\right)\\
&\geq \int_{\Omega}\left|\nabla h(t)\right|^{\gamma-2}\left|\Delta h(t)\right|^2dx+\frac{\gamma-2}{2}\sum_{i=1}^{3}\int_{\Omega}|\nabla h|^{\gamma-4}h_i^2h_{ii}^2\,dx\\
&\quad -\frac{\gamma-2}{2}\int_{\Omega}|\nabla h|^{\gamma-2}|\Delta h|^2\,dx\geq 0.
\end{aligned}
\tag{20}
$$

Using the Sobolev inequality, we deduce

$$
\left\|\Delta h(t)\right\|\leq C\left\|\Lambda^{\gamma+1}h(t)\right\|^{\frac{2}{\gamma+1}}\left\|h(t)\right\|^{\frac{\gamma-1}{\gamma+1}}.
\tag{21}
$$

Combining (16), (20) and (21) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla h(t)\right\|^2+\left\|\Lambda^{\gamma+1}h(t)\right\|^2\\
&\leq \left\|\Delta h(t)\right\|^2-\left(g(t),h(t)\right)\\
&\leq \left\|\Delta h(t)\right\|^2+\left\|g(t)\right\|\left\|\Delta h(t)\right\|\\
&\leq C\left\|\Lambda^{\gamma+1}h(t)\right\|^{\frac{4}{\gamma+1}}\left\|h(t)\right\|^{\frac{2(\gamma-1)}{\gamma+1}}+C\left\|g(t)\right\|\left\|\Lambda^{\gamma+1}h(t)\right\|^{\frac{2}{\gamma+1}}\left\|h(t)\right\|^{\frac{\gamma-1}{\gamma+1}}\\
&\leq \frac{1}{2}\left\|\Lambda^{\gamma+1}h(t)\right\|^2+C\left(\left\|h(t)\right\|^2+\left\|g(t)\right\|^2\right),
\end{aligned}
\tag{22}
$$

that is,

$$
\frac{d}{dt}\left\|\nabla h(t)\right\|^2+\left\|\Lambda^{\gamma+1}h(t)\right\|^2\leq C\left(\left\|h(t)\right\|^2+\left\|g(t)\right\|^2\right)
\tag{23}
$$

and

$$\frac{d}{dt}\left\|\nabla h(t)\right\|^2+\xi\left\|\nabla h(t)\right\|^2\le C\left(\left\|h(t)\right\|^2+\left\|g(t)\right\|^2\right). \tag{24}$$

Multiplying (24) by $e^{\xi(t-\tau)}$ and integrating over (t,τ), we derive that

$$\left\|\nabla h(t)\right\|^2\le e^{-\xi(t-\tau)}\|\nabla h_\tau\|^2+C\left\{e^{-\xi t}\int_\tau^t e^{\xi s}\left\|h(s)\right\|^2\,ds+e^{-\xi t}\int_\tau^t e^{\xi s}\left\|g(s)\right\|^2\,ds\right\}. \tag{25}$$

By (13) and (25), we get

$$\left\|\nabla h(t)\right\|^2\le C\left\{e^{-\xi(t-\tau)}\left[\|\nabla h_\tau\|^2+\|h_\tau\|^2\right]+e^{-\xi t}\left[G_1(t)+\frac{1}{\xi}G_2(t)\right]+1\right\}. \tag{26}$$

Hence, the proof is complete. □

Lemma 3.3 *Suppose that $h_\tau\in H_0^\gamma(\Omega)$, $g(x,t)\in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$ and $g(x,t)$ satisfies (3) with $\gamma>3$. Consider the system (1), we have for all $t\ge\tau$*

$$\begin{aligned}\left\|\Lambda^\gamma h(t)\right\|^2\le{}& C\left\{\left[1+(t-\tau)+\frac{1}{t-\tau}\right]e^{-\xi(t-\tau)}\left[\|h_\tau\|^2+\|h_\tau\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}\right.\right.\\&\left.+\|\nabla h_\tau\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}\right]+\left(1+\frac{1}{t-\tau}\right)\left\{1+e^{-\xi t}\left[G_1(t)+G_2(t)\right]\right\}\\&+e^{-\xi t}\int_{-\infty}^t e^{-\frac{2(2\gamma^2-5\gamma+2)}{\gamma+2}\xi s}\left[\left[G_1(s)\right]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\right.\\&\left.\left.+\left[G_2(s)\right]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\right]ds\right\}.\end{aligned} \tag{27}$$

Proof Multiplying $(1)_1$ by $(-\Delta)^\gamma h$ and integrating over Ω, then using the Sobolev inequality we can get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\Lambda^\gamma h(t)\right)^2dx+\int_\Omega\left((-\Delta)^\gamma h(t)\right)^2dx\\&\quad=\left\|\Lambda^{\gamma+1}h(t)\right\|^2+\left(\nabla\cdot\left(\left|\nabla h(t)\right|^{\gamma-2}\nabla h(t)\right),(-\Delta)^\gamma h(t)\right)+\left(g(t),(-\Delta)^\gamma h(t)\right)\\&\quad\le\frac{1}{8}\left\|(-\Delta)^\gamma h(t)\right\|^2+\left\|\Lambda^{\gamma+1}h(t)\right\|^2+C\left(\left\|\nabla\cdot\left(\left|\nabla h(t)\right|^{\gamma-2}\nabla h(t)\right)\right\|^2+\left\|g(t)\right\|^2\right)\\&\quad\le\frac{1}{8}\left\|(-\Delta)^\gamma h(t)\right\|^2+C\left(\left\|(-\Delta)^\gamma h(t)\right\|^{\frac{2}{\gamma}}\left\|\Lambda^\gamma h(t)\right\|^{\frac{2(\gamma-1)}{\gamma}}+\left\|g(t)\right\|^2\right.\\&\qquad\left.+\left\|\left|\nabla h(t)\right|^{\gamma-2}\Delta h(t)\right\|^2\right)\\&\quad\le\frac{1}{4}\left\|(-\Delta)^\gamma h(t)\right\|^2+C\left(\left\|\Lambda^\gamma h(t)\right\|^2+\left\|g(t)\right\|^2+\left\|\nabla h(t)\right\|_{2(\gamma-2)}^{2(\gamma-2)}\left\|\Delta h(t)\right\|_\infty^2\right)\\&\quad\le\frac{1}{4}\left\|(-\Delta)^\gamma h(t)\right\|^2+C\left(\left\|\Lambda^\gamma h(t)\right\|^2+\left\|g(t)\right\|^2\right.\\&\qquad\left.+\left\|(-\Delta)^\gamma h(t)\right\|^{\frac{3(\gamma-3)}{2\gamma-1}}\left\|\nabla h(t)\right\|^{\frac{4\gamma^2-13\gamma+13}{2\gamma-1}}\left\|(-\Delta)^\gamma h(t)\right\|^{\frac{5}{2\gamma-1}}\left\|\nabla h(t)\right\|^{\frac{4\gamma-7}{2\gamma-1}}\right)\\&\quad\le\frac{1}{2}\left\|(-\Delta)^\gamma h(t)\right\|^2+C\left(\left\|\Lambda^\gamma h(t)\right\|^2+\left\|g(t)\right\|^2+\left\|\nabla h(t)\right\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}\right),\end{aligned} \tag{28}$$

that is,

$$\frac{d}{dt}\left\|\Lambda^{\gamma}h(t)\right\|^{2}+\left\|(-\Delta)^{\gamma}h(t)\right\|^{2}\leq C\left(\left\|\Lambda^{\gamma}h(t)\right\|^{2}+\left\|g(t)\right\|^{2}+\left\|\nabla h(t)\right\|^{\frac{2(4\gamma^{2}-9\gamma+6)}{\gamma+2}}\right) \tag{29}$$

and

$$\frac{d}{dt}\left\|\Lambda^{\gamma}h(t)\right\|^{2}+\xi\left\|\Lambda^{\gamma}h(t)\right\|^{2}\leq C\left(\left\|\Lambda^{\gamma}h(t)\right\|^{2}+\left\|g(t)\right\|^{2}+\left\|\nabla h(t)\right\|^{\frac{2(4\gamma^{2}-9\gamma+6)}{\gamma+2}}\right). \tag{30}$$

Multiplying (30) by $(t-\tau)e^{\xi t}$ and integrating it over (t,τ), we get

$$\begin{aligned}
&(t-\tau)e^{\xi t}\left\|\Lambda^{\gamma}h(t)\right\|^{2}\\
&\quad\leq C\left\{\int_{\tau}^{t}\left[1+(s-\tau)\right]e^{\xi s}\left\|\Lambda^{\gamma}h(s)\right\|^{2}ds+\int_{\tau}^{t}(s-\tau)e^{\xi s}\left\|g(s)\right\|^{2}ds\right.\\
&\qquad\left.+\int_{\tau}^{t}(s-\tau)e^{\xi s}\left\|\nabla h(s)\right\|^{\frac{2(4\gamma^{2}-9\gamma+6)}{\gamma+2}}ds\right\}.
\end{aligned} \tag{31}$$

Hence,

$$\begin{aligned}
\left\|\Lambda^{\gamma}h(t)\right\|^{2}&\leq C\left(1+\frac{1}{t-\tau}\right)e^{-\xi t}\int_{\tau}^{t}e^{\xi s}\left\|\Lambda^{\gamma}h(s)\right\|^{2}ds+Ce^{-\xi t}G_{1}(t)\\
&\qquad+Ce^{-\xi t}\int_{\tau}^{t}e^{\xi s}\left\|\nabla h(s)\right\|^{\frac{2(4\gamma^{2}-9\gamma+6)}{\gamma+2}}ds\\
&\quad=:I_{1}+I_{2}+I_{3}.
\end{aligned} \tag{32}$$

It the follows from (8) that

$$\begin{aligned}
I_{1}&\leq C\left(1+\frac{1}{t-\tau}\right)e^{-\xi t}\left\{\left[1+\xi(t-\tau)\right]e^{\xi\tau}\|h_{\tau}\|^{2}+\frac{1}{\xi}G_{1}(t)+G_{2}(t)+Ce^{\xi t}\right\}\\
&\leq C\left[1+(t-\tau)+\frac{1}{t-\tau}\right]e^{-\xi(t-\tau)}\|h_{\tau}\|^{2}+C\left(1+\frac{1}{t-\tau}\right)e^{-\xi t}\left[G_{1}(t)\right.\\
&\qquad\left.+G_{2}(t)\right]+C\left(1+\frac{1}{t-\tau}\right).
\end{aligned} \tag{33}$$

On the other hand, by Hölder's inequality and (15), we get

$$\begin{aligned}
I_{3}&\leq Ce^{-\xi t}\int_{\tau}^{t}e^{\xi s}\left\{e^{-\xi(s-\tau)}\left(\|\nabla h_{\tau}\|^{2}+\|h_{\tau}\|^{2}\right)\right.\\
&\qquad\left.+e^{-\xi s}\left[G_{1}(s)+G_{2}(s)\right]+1\right\}^{\frac{4\gamma^{2}-9\gamma+6}{\gamma+2}}ds\\
&\leq Ce^{-\xi t}\int_{\tau}^{t}e^{\xi s}ds+ce^{-\xi t}\int_{\tau}^{t}e^{\xi s}e^{-\frac{4\gamma^{2}-9\gamma+6}{\gamma+2}\xi(s-\tau)}\left(\|h_{\tau}\|^{2}\right.\\
&\qquad\left.+\|\nabla h_{\tau}\|^{2}\right)^{\frac{4\gamma^{2}-9\gamma+6}{\gamma+2}}ds\\
&\qquad+Ce^{-\xi t}\int_{\tau}^{t}e^{\xi s}e^{-\frac{4\gamma^{2}-9\gamma+6}{\gamma+2}\xi s}\left\{\left[G_{1}(s)\right]^{\frac{4\gamma^{2}-9\gamma+6}{\gamma+2}}+\left[G_{2}(s)\right]^{\frac{4\gamma^{2}-9\gamma+6}{\gamma+2}}\right\}ds\\
&\leq Ce^{-\xi t}\left[e^{\xi t}-e^{\xi\tau}\right]+ce^{-\xi(t-\tau)}\left(\|h_{\tau}\|^{\frac{2(4\gamma^{2}-9\gamma+6)}{\gamma+2}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\|\nabla h_\tau\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}\Big)\int_\tau^t e^{-\frac{4\gamma^2-9\gamma+6}{\gamma+2}\xi(s-\tau)}\,ds\\
&+Ce^{-\xi t}\int_\tau^t e^{-\frac{2(2\gamma^2-5\gamma+2)}{\gamma+2}\xi s}\Big\{\big[G_1(s)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}+\big[G_2(s)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\Big\}\,ds\\
\le{}& C+C(t-\tau)e^{-\xi(t-\tau)}\Big(\|h_\tau\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}+\|\nabla h_\tau\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}\Big)\\
&+Ce^{-\xi t}\int_\tau^t e^{-\frac{2(2\gamma^2-5\gamma+2)}{\gamma+2}\xi s}\Big\{\big[G_1(s)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}+\big[G_2(s)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\Big\}\,ds.
\end{aligned}
\tag{34}
$$

Combining (33) and (34) with (32), we complete the proof. □

Remark 3.4 By Sobolev inequality and (28), we can get the exponent of $\|(-\Delta)^\gamma h(t)\|$ is $\frac{3(\gamma-3)}{2\gamma-1}>0$, which is in the seventh row in (28). Hence, we limit $\gamma>3$ with $d=3$. Inspired by [12] and [15], the existence of pullback $\mathcal{D}$-attractor for system (1) is proved as follows.

Firstly, assume that $\mathfrak{R}$ is the set of all function $r:\mathbb{R}\to(0,\infty)$ such that

$$
\lim_{t\to\infty} te^{\delta t}r^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}(t)=0.
$$

Let $\widehat{D}:=\{D(t):t\in\mathbb{R}\}\subset B(H_0^\gamma(\Omega))$, then $D(t)\subset\overline{B_0}(r(t))$ for some $r(t)\in\mathfrak{R}$. Here $\overline{B_0}(r(t))\subseteq H_0^\gamma(\Omega)$ is a closed sphere and its radius is r(t). Assume

$$
\begin{aligned}
r_0^2(t)={}&2C\Big\{1+e^{-\xi t}G_1(t)+e^{-\xi t}G_2(t)\\
&+e^{-\xi t}\int_{-\infty}^t e^{-\frac{2(2\gamma^2-5\gamma+2)}{\gamma+2}\xi s}\Big[\big[G_1(s)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\\
&+\big[G_2(s)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\Big]\,ds\Big\}.
\end{aligned}
\tag{35}
$$

Using the continuity of embedding $H_0^\gamma(\Omega)\hookrightarrow L^2(\Omega)$, for any $\widehat{D}\in\mathcal{D}$ and $t\in\mathbb{R}$ there exists $\tau_0(\widehat{D},t)<t$, such that

$$
\big\|\Lambda^\gamma h(t)\big\|\le r_0(t),\quad \forall\tau<\tau_0.
\tag{36}
$$

In addition, since $0\le\alpha\le(\frac{\gamma+2}{4\gamma^2-9\gamma+6})^2\xi$, we have $\overline{B_0}(r_0(t))\in\mathcal{D}$. Hence, by Lemma 2.4, there is a family of pullback $\mathcal{D}$-absorbing sets $\overline{B_0}(r_0(t))$ in $H_0^\gamma(\Omega)$.

Then the main idea is proved.

Proof of Theorem 2.6 By Lemma 2.4, we need only to demonstrate that the process $h(t,\tau)$ is pullback ω-$\mathcal{D}$-limit compact. Then we can obtain the existence of pullback $\mathcal{D}$-attractor. The operator $A^{-1}\in L^2(\Omega)$ is continuous and compact, so there is a sequence $\{\xi_j\}_{j=1}^\infty$ satisfying

$$
\xi_1\le\xi_2\le\cdots\le\xi_j\le\cdots,\quad \xi_j\to\infty,\text{as } j\to\infty,
$$

and there is a family $\{w_j\}_{j=1}^\infty$, where $w_j\in H_0^\gamma(\Omega)$, $j=1,2,\ldots$. The elements in $\{w_j\}_{j=1}^\infty$ are orthonormal in $L^2(\Omega)$ such that

$$
Aw_j=\xi_j w_j,\quad \text{for } j=1,2,\ldots.
$$

Let $X_n = \operatorname{span}\{w_1, w_2, \dots, w_n\} \subset H_0^\gamma(\Omega)$ and $P_n : H_0^\gamma(\Omega) \to X_n$ be an orthogonal projector. Therefore

$$h = P_n h + (I - P_n)h := h_1 + h_2. \tag{37}$$

Testing $(1)_1$ by $(-\Delta)^\gamma h_2$, using Young's inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\Lambda^\gamma h_2(t)\right\|^2 + \left\|(-\Delta)^\gamma h_2(t)\right\|^2\\&\quad\le \frac{1}{2}\left\|(-\Delta)^\gamma h_2(t)\right\|^2 + C\left(\left\|\Lambda^\gamma h(t)\right\|^2 + \left\|g(t)\right\|^2 + \left\|\nabla h(t)\right\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}\right),\end{aligned} \tag{38}$$

which means

$$\frac{d}{dt}\left\|\Lambda^\gamma h_2(t)\right\|^2 + \xi_n\left\|\Lambda^\gamma h_2(t)\right\|^2 \le C\left(\left\|\Lambda^\gamma h(t)\right\|^2 + \left\|g(t)\right\|^2 + \left\|\nabla h(t)\right\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}\right). \tag{39}$$

Multiplying (39) by $(t-\tau)e^{\xi_n t}$ and integrating it over (τ, t), we obtain

$$\begin{aligned}&(t-\tau)e^{\xi_n t}\left\|\Lambda^\gamma h_2(t)\right\|^2\\&\quad\le \int_\tau^t e^{\xi_n s}\left\|\Lambda^\gamma h_2(s)\right\|^2 ds + C\int_\tau^t (s-\tau)e^{\xi_n s}\left\|\Lambda^\gamma h(s)\right\|^2 ds\\&\qquad + C\int_\tau^t (s-\tau)e^{\xi_n s}\left\|g(s)\right\|^2 ds + C\int_\tau^t (s-\tau)e^{\xi_n s}\left\|\nabla h(s)\right\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}} ds\\&\quad\le \int_\tau^t e^{\xi_n s}\left\|\Lambda^\gamma h(s)\right\|^2 ds + C(t-\tau)\left\{\int_\tau^t e^{\xi_n s}\left\|\Lambda^\gamma h(s)\right\|^2 ds\right.\\&\qquad\left. + \int_\tau^t e^{\xi_n s}\left\|g(s)\right\|^2 ds + \int_\tau^t e^{\xi_n s}\left\|\nabla h(s)\right\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}} ds\right\}.\end{aligned} \tag{40}$$

Combining (36) and (40) gives for any $\tau \le \tau_0$

$$\begin{aligned}&\left\|\Lambda^\gamma h_2(t)\right\|^2\\&\quad\le (t-\tau)^{-1}e^{-\xi_n t}\int_\tau^t e^{\xi_n s}\left\|\Lambda^\gamma h(s)\right\|^2 ds + Ce^{-\xi_n t}\int_\tau^t e^{\xi_n s}\left\|\Lambda^\gamma h(s)\right\|^2 ds\\&\qquad + Ce^{-\xi_n t}\int_\tau^t e^{\xi_n s}\left\|g(s)\right\|^2 ds + Ce^{-\xi_n t}\int_\tau^t e^{\xi_n s}\left\|\nabla h(s)\right\|^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}} ds\\&\quad\le (t-\tau)^{-1}e^{-\xi_n t}\int_\tau^t e^{\xi_n s}r_0^2(s)\,ds + Ce^{-\xi_n t}\int_\tau^t e^{\xi_n s}r_0^2(s)\,ds\\&\qquad + Ce^{-\xi_n t}\int_\tau^t e^{\xi_n s}\left\|g(s)\right\|^2 ds + Ce^{-\xi_n t}\int_\tau^t e^{\xi_n s}r_0^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}(s)\,ds.\end{aligned} \tag{41}$$

Note that

$$\begin{aligned}e^{-\xi_n t}\int_\tau^t e^{\xi_n s}r_0^2(s)\,ds &\le Ce^{-\xi_n t}\int_\tau^t e^{\xi_n s}\left[1 + e^{-\xi s}G_1(S) + e^{-\xi s}G_2(s)\right] ds\\&\le C\xi_n^{-1} + C(\xi_n - \xi)^{-1}e^{-\xi t}\left[G_1(t) + G_2(t)\right]\end{aligned} \tag{42}$$

and

$$
\begin{aligned}
& e^{-\xi_n t}\int_\tau^t e^{\xi_n s} r_0^{\frac{2(4\gamma^2-9\gamma+6)}{\gamma+2}}(s)\,ds \\
&\quad = Ce^{-\xi_n t}\int_\tau^t e^{\xi_n s}\bigg\{1+e^{-\xi s}G_1(s)+e^{-\xi s}G_2(s) \\
&\qquad + e^{-\xi s}\int_{-\infty}^s e^{-\frac{2(2\gamma^2-5\gamma+2)}{\gamma+2}\xi r}\Big[\big[G_1(r)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}+\big[G_2(r)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\Big]\,dr\bigg\}^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\,ds \\
&\quad \le C\xi_n^{-1} \\
&\qquad + C\Big(\xi_n-\frac{4\gamma^2-9\gamma+6}{\gamma+2}\xi\Big)^{-1}e^{-\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\Big\{\big[G_1(t)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}+\big[G_2(t)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\Big\} \\
&\qquad + C\Big(\xi_n-\frac{4\gamma^2-9\gamma+6}{\gamma+2}\xi\Big)^{-1}e^{-\frac{4\gamma^2-9\gamma+6}{\gamma+2}} \\
&\qquad \cdot\bigg\{\Big(\int_{-\infty}^t e^{-\frac{2(2\gamma^2-5\gamma+2)}{\gamma+2}\xi s}\big[G_1(s)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\,ds\Big)^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}} \\
&\qquad + \Big(\int_{-\infty}^t e^{-\frac{2(2\gamma^2-5\gamma+2)}{\gamma+2}\xi s}\big[G_1(s)\big]^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\,ds\Big)^{\frac{4\gamma^2-9\gamma+6}{\gamma+2}}\bigg\}.
\end{aligned}
\tag{43}
$$

On the other hand, by simple calculations, we get

$$
\begin{aligned}
e^{-\xi_n t}\int_\tau^t e^{\xi_n s}\|g(s)\|^2\,ds &\le \begin{cases} \beta e^{-\xi_n t}\int_\tau^t e^{\xi_n s}e^{-\alpha s}\,ds, & t\le 0, \\ \beta e^{-\xi_n t}\int_\tau^t e^{\xi_n s}e^{\alpha|s|}\,ds, & t\ge 0\end{cases} \\
&\le \begin{cases} \frac{\beta e^{-\alpha t}}{\xi_n-\alpha}, & t\le 0, \\ \frac{\beta e^{\alpha t}}{\xi_n+\alpha}+\frac{\beta e^{-\xi_n t}}{\xi_n-\alpha}, & t\ge 0.\end{cases}
\end{aligned}
\tag{44}
$$

By (41), (42), (43) and (44), we see that, for any $\varepsilon > 0$, there exist $\tau_0 < t$ and $N \in \mathbb{N}$ such that for every $n > N$ and every $\tau < \tau_0$

$$
\|\Lambda^\gamma h_2(t)\| \le \varepsilon.
$$

It shows that the process $\{h(t,\tau)\}$ is pullback ω-$\mathcal{D}$-limit compact.

Then, using Lemma 2.4, the proof of Theorem 2.6 is complete. □

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References

1. Duan, N., Zhao, X.P.: Pullback attractor of a nonautonomous fourth-order parabolic equation modeling epitaxial thin film growth. Electron. J. Differ. Equ. **2015**, 272 (2015)
2. Fan, J., Alsaedi, A., Hayat, T., Zhou, Y.: Global well-posedness and regularity criteria for epitaxial growth models. Comput. Math. Appl. **74**, 459–465 (2017)
3. Fan, J., Samet, B., Zhou, Y.: Global well-posedness of weak solution and a regularity criterion of strong solutions for an epitaxial growth model. Appl. Math. Lett. **80**, 8–11 (2018)
4. Schmalfuß, B.: Backward cocycles and attractors of stochastic differential equations. In: Reitmann, V., Riedrich, T., Koksch, N. (eds.) International Seminar on Applied Mathematics—Nonlinear Dynamics: Attractor Approximation and Global Behaviour, TU Dresden, pp. 185–192 (1992)
5. Flandoli, F., Schmalfus, B.: Random attractors for the stochastic 3-D Navier–Stokes equation with multiplicative white noise. Stoch. Stoch. Rep. **59**, 21–45 (1996)
6. Kloeden, P.E., Schmalfuß, B.: Nonautonomous systems, cocycle attractors and variable time-step discretization. Numer. Algorithms **14**, 141–152 (1997)
7. Kloeden, P.E., Simsem, J.: Pullback attractors for non-autonomous evolution with spatially variable exponents. Commun. Pure Appl. Anal. **13**, 2543–2557 (2014)
8. Ochs, G.: Weak random attractors. Technical report, Institut für Dvnamische Systeme, Universität Bremen (1999)
9. Caraballo, T., Lukaszewicz, G., Real, J.: Pullback attractor for asymptotically compact nonautonomous dynamical systems. Nonlinear Anal. **64**, 484–498 (2006)
10. Caraballo, T., Colucci, R.: Pullback attractor for a non-linear evolution equation in elasticity. Nonlinear Anal., Real World Appl. **15**, 80–88 (2014)
11. Krause, A., Wang, B.X.: Pullback attractors of non-autonomous stochastic degenerate parabolic equations on unbounded domains. arXiv:1309.1211 [math.AP]
12. Li, Y., Zhong, C.K.: Pullback attractors for the norm-to-weak continuous process and application to the nonautonomous reaction–diffusion equations. Appl. Math. Comput. **190**, 1020–1029 (2007)
13. Hoang, L.T., Olson, E.J., Robinson, J.C.: Continuity of pullback and uniform attractors. arXiv:1601.07436 [math.DS]
14. Cheskidov, A., Kavlie, L.: Pullback attractors for generalized evolutionary systems. Discrete Contin. Dyn. Syst., Ser. B **20**(3), 749–779 (2015)
15. Park, S.H., Park, J.Y.: Pullback attractors for nonautonomous modified Swift–Hohenberg equation. Comput. Math. Appl. **67**, 542–548 (2014)
16. Li, Y., Wang, S., Wu, H.: Pullback attractors for non-autonomous reaction–diffusion equations in L^p. Appl. Math. Comput. **207**, 373–379 (2009)
17. Temam, R.: Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd edn. Springer, Berlin (1997)