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Existence of positive solutions to negative power nonlinear integral equations with weights

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Abstract

This paper is devoted to the existence and non-existence of positive solutions to the following negative power nonlinear integral equation related to the sharp reversed Hardy–Littlewood–Sobolev inequality:

$$f^{q-1}(x)=\int_{\Omega}\frac{K(x)f(y)K(y)}{|x-y|^{n-\alpha}}\,dy+\lambda\int_{\Omega}\frac{G(x)f(y)G(y)}{|x-y|^{n-\alpha-\beta}}\,dy,\quad f\geq 0, x\in\overline{\Omega},$$

where $0<q<1$, $\alpha>n$, $0<\beta<\alpha-n$, $\lambda\in\mathbb{R}$, Ω is a smooth bounded domain, $K(x)$, $G(x)$ are positive continuous functions in $\overline{\Omega}$. For $K\equiv G\equiv 1$, the existence and non-existence of positive solutions to the equation have been studied by Dou–Guo–Zhu (2019). In this paper we consider the existence and non-existence of positive solutions to the above integral equation with the general weight functions $K(x)$, $G(x)$.

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Keywords: Reversed Sharp Hardy–Littlewood–Sobolev inequality; Positive solution; Integral equation

1 Introduction

In this paper we consider the existence and non-existence of positive solutions to the following negative power nonlinear integral equation:

$$f^{q-1}(x)=\int_{\Omega}\frac{K(x)f(y)K(y)}{|x-y|^{n-\alpha}}\,dy+\lambda\int_{\Omega}\frac{G(x)f(y)G(y)}{|x-y|^{n-\alpha-\beta}}\,dy,\quad f\geq 0, x\in\overline{\Omega},\tag{1.1}$$

where $0<q<1$, $\alpha>n$, $0<\beta<\alpha-n$, $\lambda\in\mathbb{R}$, Ω is a smooth bounded domain, $K(x)$, $G(x)$ are positive continuous functions in $\overline{\Omega}$.

For $0<\alpha<n$, $G(x)\equiv 1$, the existence and non-existence of positive solutions to (1.1) were studied by Dou–Zhu [2] and Guo–Wang [3] recently. Notice that when $0<\alpha<n$ this nonlinear integral equation is closely related to the sharp Hardy–Littlewood–Sobolev (HLS for short) inequality [4–7].

For $\alpha > n$, the existence and non-existence of positive solutions to (1.1) are also studied by Dou–Guo–Zhu [1] when $K(x) \equiv G(x) \equiv 1$. In this case the nonlinear integral equation is related to the sharp reversed HLS inequality obtained by Beckner [8] and Dou–Zhu [9], respectively. In fact, Eq. (1.1) (when $K(x) \equiv 1$, $\lambda = 0$) can be seen as the Euler–Lagrange equation of the following minimizing problem related to the reversed HLS inequality:

$$\xi_\alpha(\Omega) = \inf_{f \in L^{\frac{2n}{n+\alpha}}(\Omega), f \geq 0, f \neq 0} \frac{\int_\Omega \int_\Omega f(x)|x-y|^{-(n-\alpha)} f(y)\,dx\,dy}{\|f\|^2_{L^{\frac{2n}{n+\alpha}}(\Omega)}}.$$

On the other hand, for Eq. (1.1) with $K(x) \equiv 1$ and $\lambda = 0$, the blowup behavior of energy maximizing positive solutions as $q \to (\frac{2n}{n+\alpha})^+$ when $1 < \alpha < n$, and the blowup behavior of energy minimizing positive solution as $q \to (\frac{2n}{n+\alpha})^-$ when $\alpha > n$ are also analyzed by Guo [10].

In this paper we consider the integral equation (1.1) for general weight functions $K(x)$, $G(x)$ and $\alpha > n$.

The following condition is needed.

$(\mathcal{T})$. $K(x_*) - K(x) = o(|x - x_*|^\gamma)$ as $x \to x_*$, where $K(x_*) = \min_{x \in \overline{\Omega}} K(x)$, $\gamma > 0$.

Denote $G(\tilde{x}_*) = \max_{x \in \overline{\Omega}} G(x)$.

The main results are stated as follows.

Theorem 1.1 *Assume $\alpha > n$, $\beta \in (0, \alpha - n)$, Ω is a smooth bounded domain of diameter $d(\Omega)$.*

(i) *For $0 < q < \frac{2n}{n+\alpha}$ (subcritical case), $-\frac{K^2(x_*)}{d^\beta(\Omega) G^2(\tilde{x}_*)} < \lambda$, the positive functions $K(x), G(x) \in C^1(\overline{\Omega})$, then there is a positive solution $f \in C^1(\overline{\Omega})$ to Eq.* (1.1).

(ii) *For $q = \frac{2n}{n+\alpha}$ (critical case), $-\frac{K^2(x_*)}{d^\beta(\Omega) G^2(\tilde{x}_*)} < \lambda < 0$, the positive functions $K(x), G(x) \in C^1(\overline{\Omega})$, assume further that $\beta < n$ and $(\mathcal{T})$ holds, then there is a positive solution $f \in C^1(\overline{\Omega})$ to Eq.* (1.1).

(iii) *For $\frac{2n}{n+\alpha} \leq q < 1$ (critical case and supercritical case), $\lambda \geq 0$, the nonnegative functions $K(x), G(x) \in C^1(\overline{\Omega})$, if Ω is a star-shaped domain with respect to $\tilde{x}$, $(x - \tilde{x}, \nabla K(x)) \geq 0$ and $(x - \tilde{x}, \nabla G(x)) \geq 0$, then there is not any positive $C^1(\overline{\Omega})$ solution to Eq.* (1.1).

We use c, C throughout the paper to represent positive constants, which may vary from line to line.

2 Preliminaries

For simplicity, we denote $p_\alpha := \frac{2n}{n-\alpha}$, $q_\alpha := \frac{2n}{n+\alpha}$ throughout the paper. For $0 < q < 1$, we also denote $L^q(\Omega) := \{f \mid \int_\Omega |f|^q(x)\,dx < \infty\}$ for any domain $\Omega \subset \mathbb{R}^n$, $L^q_+(\Omega) := \{f \in L^q(\Omega) \setminus \{0\} : f \geq 0\}$ and define $\|f\|_{L^q(\Omega)} := (\int_\Omega |f|^q(x)\,dx)^{\frac{1}{q}}$ for $f \in L^q(\Omega)$. Notice that $\|f\|_{L^q(\Omega)}$ is not a norm if $0 < q < 1$.

We first recall the sharp reversed HLS inequality on $\mathbb{R}^n$.

Theorem A (see [8, 9]) *Let $\alpha > n$. Then*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-(n-\alpha)} g(y)\,dx\,dy \right| \geq N_\alpha \|f\|_{L^{q_\alpha}(\mathbb{R}^n)} \|g\|_{L^{q_\alpha}(\mathbb{R}^n)} \tag{2.1}$$

for all $f,g \in L^{q_\alpha}(\mathbb{R}^n)$, where $N_\alpha := \pi^{\frac{n-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n}{2}+\frac{\alpha}{2})}(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)})^{-\frac{\alpha}{n}}$. Moreover, the equality holds if and only if $f(x) = c_1 g(x) = c_2(\frac{1}{c_3+|x-x_0|^2})^{\frac{n+\alpha}{2}}$, where c_1, c_2, c_3 are any constants, $x_0 \in \mathbb{R}^n$.

3 Proofs of the main results

Here and hereafter we always assume $\alpha > n$.

3.1 Existence—subcritical case

We first prove the existence of positive solution to Eq. (1.1) in the subcritical case $0 < q < q_\alpha$. The following lemma from [2] is needed.

Lemma 3.1 (see [2]) *Let $q \in (0, q_\alpha)$. There exists a positive constant $C(n,q,\alpha,\Omega) > 0$ such that*

$$\int_\Omega \int_\Omega f(x)|x-y|^{-(n-\alpha)} f(y)\,dx\,dy \ge C(n,q,\alpha,\Omega)\|f\|^2_{L^q(\Omega)} \tag{3.1}$$

for any nonnegative function $f \in L^q(\Omega)$.

Now we prove the following lemma, which implies the existence result of part (i) in Theorem 1.1.

Lemma 3.2 *Assume the positive functions $K(x), G(x) \in C^1(\overline{\Omega})$. Then, for $0 < q < q_\alpha$, $\lambda > -\frac{K^2(x_*)}{d^\beta(\Omega)G^2(\tilde{x}_*)}$, the infimum*

$$Q_{\lambda,q}(\Omega) := \inf_{f \in L^q_+(\Omega)} \frac{\int_\Omega \int_\Omega f(x)(K(x)|x-y|^{-(n-\alpha)}K(y) + \lambda G(x)|x-y|^{-(n-\alpha-\beta)}G(y))f(y)\,dx\,dy}{\|f\|^2_{L^q(\Omega)}}$$

is attained by some nonnegative function in $L^q_+(\Omega)$.

Proof Notice that $\lambda > -\frac{K^2(x_*)}{d^\beta(\Omega)G^2(\tilde{x}_*)}$ and

$$K(x)K(y) + \lambda G(x)|x-y|^\beta G(y) \ge K^2(x_*) + \lambda d^\beta(\Omega)G^2(\tilde{x}_*) > 0, \quad x,y \in \Omega.$$

Then by Lemma 3.1, $Q_{\lambda,q}(\Omega) > 0$.

Now we can choose the minimizing positive sequence $\{f_j\}_{j=1}^\infty$ in $L^q_+(\Omega)$ and argue as Lemma 3.2 in [1]. We sketch it for the reader's convenience. Assume $f_j \in L^{q_\alpha}(\Omega)$ and $\|f_j\|_{L^{q_\alpha}(\Omega)} = 1$. Then, up to a subsequence,

$$f_j^q \rightharpoonup f_*^q \quad \text{in } L^{\frac{q_\alpha}{q}}(\Omega), \text{as } j \to \infty,$$

and

$$\int_\Omega f_j^q \to \int_\Omega f_*^q, \quad \text{as } j \to \infty.$$

As in [1], we have $\|f_j\|_{L^1(\Omega)} \le C$. Thus $\int_\Omega f_*^q > C > 0$ via an interpolation inequality and $f_j^q \rightharpoonup f_*^q$ weakly in $L^{\frac{1}{q}}(\Omega)$. For any fixed $x \in \overline{\Omega}$, $f_*^{1-q}(y)|x-y|^{\alpha-n}(K(x)K(y) + \lambda G(x)|x -$

$y|^{\beta}G(y)) \in L^{\frac{1}{1-q}}(\Omega)$. Therefore

$$\begin{aligned}&\int_{\Omega} f_j^q(y) f_*^{1-q}(y)|x-y|^{\alpha-n}\bigl(K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y)\bigr)\,dy\\&\quad\to \int_{\Omega} f_*(y)|x-y|^{\alpha-n}\bigl(K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y)\bigr)\,dy, \quad \text{as } j\to\infty.\end{aligned}$$

Now we prove that the convergence is uniform for all $x \in \overline{\Omega}$. Firstly, as Lemma 3.2 in [1], we have $\int_{\Omega} f_j^q(y) f_*^{1-q}(y)|x-y|^{\alpha-n}(K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y))\,dy$ is uniformly bounded for $x \in \overline{\Omega}$. Now it is left to prove that $\int_{\Omega} f_j^q(y) f_*^{1-q}(y)|x-y|^{\alpha-n}(K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y))\,dy$ is equicontinuous in $\overline{\Omega}$. Notice that $K(x), G(x) \in C^1(\overline{\Omega})$ and for any $x_1, x_2, y \in \overline{\Omega}$,

$$\bigl||x_1-y|^{\alpha-n}-|x_2-y|^{\alpha-n}\bigr| \le \begin{cases} C|x_1-x_2|^{\alpha-n}, & 0<\alpha-n\le 1,\\ C|x_1-x_2|, & \alpha-n>1.\end{cases}$$

Then since $\int_{\Omega} f_j^q(y) f_*^{1-q}(y)K(y)\,dy$ is bounded, for any $x_1, x_2 \in \overline{\Omega}$,

$$\begin{aligned}&\biggl|\int_{\Omega} f_j^q(y) f_*^{1-q}(y)|x_1-y|^{\alpha-n}K(y)\,dy - \int_{\Omega} f_j^q(y) f_*^{1-q}(y)|x_2-y|^{\alpha-n}K(y)\,dy\biggr|\\&\quad\le \int_{\Omega} f_j^q(y) f_*^{1-q}(y)K(y)\bigl||x_1-y|^{\alpha-n}-|x_2-y|^{\alpha-n}\bigr|\,dy\\&\quad\le C\max\bigl(|x_1-x_2|^{\alpha-n}, |x_1-x_2|\bigr).\end{aligned}$$

So $\int_{\Omega} f_j^q(y) f_*^{1-q}(y)|x-y|^{\alpha-n}K(y)\,dy$ and, by a similar argument, $\lambda\int_{\Omega} f_j^q(y) f_*^{1-q}(y)|x-y|^{\alpha+\beta-n}G(y)\,dy$ are equicontinuous in $x \in \overline{\Omega}$. Thus we see that $\int_{\Omega} f_j^q(y) f_*^{1-q}(y)|x-y|^{\alpha-n} \times (K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y))\,dy$ is equicontinuous in $\overline{\Omega}$.

Now similar to Lemma 3.2 in [1], we can prove

$$\begin{aligned}&\liminf_{j\to\infty}\int_{\Omega}\int_{\Omega} f_j(x)|x-y|^{\alpha-n}\bigl(K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y)\bigr)f_j(y)\,dx\,dy\\&\quad\ge \int_{\Omega}\int_{\Omega} f_*(x)|x-y|^{\alpha-n}\bigl(K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y)\bigr)f_*(y)\,dx\,dy.\end{aligned}$$

By $\|f_j\|_{L^q(\Omega)} \to \|f_*\|_{L^q(\Omega)} > 0$ and the above,

$$\begin{aligned}&\liminf_{j\to\infty}\frac{\int_{\Omega}\int_{\Omega} f_j(x)|x-y|^{\alpha-n}(K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y))f_j(y)\,dx\,dy}{\|f_j\|^2_{L^q(\Omega)}}\\&\quad\ge \frac{\int_{\Omega}\int_{\Omega} f_*(x)|x-y|^{\alpha-n}(K(x)K(y)+\lambda G(x)|x-y|^{\beta}G(y))f_*(y)\,dx\,dy}{\|f_*\|^2_{L^q(\Omega)}}.\end{aligned}$$

That is, f_* is the minimizer. □

Again as that in [1], we obtain $u \in C^1(\overline{\Omega})$. Thus we complete the proof of Theorem 1.1 (i).

Remark 3.3 We assume $\lambda > -\frac{K^2(x_*)}{d^{\beta}(\Omega)G^2(\tilde{x}_*)}$ here to make sure that $Q_{\lambda,q}(\Omega)$ is positive.

3.2 Existence—critical case

Now we establish the existence and the regularity results for the weak solution to (1.1) with critical exponent for $\lambda < 0$. Consider

$$Q_{\lambda,q_\alpha}(\Omega) = \inf_{f\in L_+^{q_\alpha}(\Omega)} \frac{\int_\Omega\int_\Omega f(x)(K(x)|x-y|^{-(n-\alpha)}K(y)+\lambda G(x)|x-y|^{-(n-\alpha-\beta)}G(y))f(y)\,dy\,dx}{\|f\|^2_{L^{q_\alpha}(\Omega)}}.$$

Notice that the corresponding Euler–Lagrange equation for extremal functions, up to a constant multiplier, is the integral equation (1.1) with $q = q_\alpha$.

We first show the following lemma.

Lemma 3.4 *Assume that the positive functions $K(x), G(x) \in C^1(\overline{\Omega})$ and $(\mathcal{T})$ holds. Then $Q_{\lambda,q_\alpha}(\Omega) < K^2(x_*)N_\alpha$ for all $\lambda < 0$. Further, $0 < Q_{\lambda,q_\alpha}(\Omega) < K^2(x_*)N_\alpha$ for any $\lambda \in (-\frac{K^2(x_*)}{d^\beta(\Omega)G^2(\tilde{x}_*)}, 0)$, where $\beta > 0$.*

Proof We distinguish two cases: (I) $x_* \in \Omega$; (II) $x_* \in \partial\Omega$.

(I) Let $x_* \in \Omega$. By $(\mathcal{T})$, there exists small $R > 0$ such that $K(x) - K(x_*) \le c|x-x_*|^\gamma$ when $x \in B_R(x_*) \subset \Omega$. For small $\epsilon > 0$, we define

$$\widetilde{f}_\epsilon(x) = \begin{cases} f_\epsilon(x), & x \in B_R(x_*) \subset \Omega, \\ 0, & x \in \mathbb{R}^n \setminus B_R(x_*), \end{cases}$$

where $f_\epsilon(x) = \epsilon^{-\frac{n+\alpha}{2}} f_1(\frac{x-x_*}{\epsilon}) = (\frac{\epsilon}{\epsilon^2+|x-x_*|^2})^{\frac{n+\alpha}{2}}$, $f_1(x) = (\frac{1}{1+|x|^2})^{\frac{n+\alpha}{2}}$. Notice that f_1 and its conformal equivalent class f_ϵ are the extremal functions to the sharp reversed HLS inequality (2.1). Obviously, $\widetilde{f}_\epsilon \in L^{q_\alpha}(\mathbb{R}^n)$. By $(\mathcal{T})$, we have

$$\begin{aligned}
&\int_\Omega\int_\Omega \left(\frac{K(x)K(y)}{|x-y|^{n-\alpha}} + \frac{\lambda G(x)G(y)}{|x-y|^{n-\alpha-\beta}}\right)\widetilde{f}_\epsilon(x)\widetilde{f}_\epsilon(y)\,dx\,dy \\
&= \int_{B_R(x_*)}\int_{B_R(x_*)} \left(\frac{K(x)K(y)}{|x-y|^{n-\alpha}} + \frac{\lambda G(x)G(y)}{|x-y|^{n-\alpha-\beta}}\right) f_\epsilon(x)f_\epsilon(y)\,dx\,dy \\
&\le \int_{B_R(x_*)}\int_{B_R(x_*)} \frac{(K(x_*)+c|x-x_*|^\gamma)(K(x_*)+c|y-x_*|^\gamma)}{|x-y|^{n-\alpha}} f_\epsilon(x)f_\epsilon(y)\,dx\,dy \\
&\quad + \lambda\int_{B_R(x_*)}\int_{B_R(x_*)} \frac{G(x)G(y)f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha-\beta}}\,dx\,dy \\
&= \int_{B_R(x_*)}\int_{B_R(x_*)} \frac{K^2(x_*)+K(x_*)(c|x-x_*|^\gamma + c|y-x_*|^\gamma) + c^2|x-x_*|^\gamma|y-x_*|^\gamma}{|x-y|^{n-\alpha}} \\
&\quad \times f_\epsilon(x)f_\epsilon(y)\,dx\,dy \\
&\quad + \lambda\int_{B_R(x_*)}\int_{B_R(x_*)} \frac{G(x)G(y)f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha-\beta}}\,dx\,dy \\
&\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{K^2(x_*)f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy + \lambda\int_{B_R(x_*)}\int_{B_R(x_*)} \frac{G(x)G(y)f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha-\beta}}\,dx\,dy \\
&\quad + c^2\int_{B_R(x_*)}\int_{B_R(x_*)} \frac{|x-x_*|^\gamma|y-x_*|^\gamma f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy
\end{aligned}$$

$$\begin{aligned}&+2c\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{K(x_*)|x-x_*|^\gamma f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy\\&=K^2(x_*)N_\alpha\|f_\epsilon\|^2_{L^{q_\alpha}(\mathbb{R}^n)}+I_1+I_2+I_3,\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}I_1&:=\lambda\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{G(x)G(y)f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha-\beta}}\,dx\,dy,\\I_2&:=c^2\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{|x-x_*|^\gamma|y-x_*|^\gamma f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy,\\I_3&:=2c\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{K(x_*)|x-x_*|^\gamma f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy.\end{aligned}$$

For I_1, we have

$$\begin{aligned}I_1&=\lambda\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{G(x)G(y)}{|x-y|^{n-\alpha-\beta}}\left(\frac{\epsilon}{\epsilon^2+|x-x_*|^2}\right)^{\frac{n+\alpha}{2}}\left(\frac{\epsilon}{\epsilon^2+|y-x_*|^2}\right)^{\frac{n+\alpha}{2}}\,dx\,dy\\&\le C\lambda\epsilon^{-(n-\alpha-\beta)-(n+\alpha)}\int_{B_R(0)}\int_{B_R(0)}\left|\frac{x-y}{\epsilon}\right|^{-(n-\alpha-\beta)}\left(1+\left|\frac{x}{\epsilon}\right|^2\right)^{-\frac{n+\alpha}{2}}\left(1+\left|\frac{y}{\epsilon}\right|^2\right)^{-\frac{n+\alpha}{2}}\,dx\,dy\\&=C\lambda\epsilon^\beta\int_{B_{\frac{R}{\epsilon}}(0)}\int_{B_{\frac{R}{\epsilon}}(0)}|\xi-\eta|^{-(n-\alpha-\beta)}\left(1+|\xi|^2\right)^{-\frac{n+\alpha}{2}}\left(1+|\eta|^2\right)^{-\frac{n+\alpha}{2}}\,d\xi\,d\eta\\&\le C_1\lambda\epsilon^\beta.\end{aligned}$$

For I_2, we have

$$\begin{aligned}I_2&:=c^2\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{|x-x_*|^\gamma|y-x_*|^\gamma f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy\\&\le c^2R^{2\gamma}\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy\\&\le c^2R^{2\gamma}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy\\&=C_2R^{2\gamma},\end{aligned}$$

where $C_2:=c^2N_\alpha\|f_\epsilon\|^2_{L^{q_\alpha}(\mathbb{R}^n)}$. For I_3, we have

$$\begin{aligned}I_3&:=2c\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{K(x_*)|x-x_*|^\gamma f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy\\&\le 2cR^\gamma\int_{B_R(x_*)}\int_{B_R(x_*)}\frac{K(x_*)f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy\\&\le 2cR^\gamma\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{K(x_*)f_\epsilon(x)f_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy\\&=C_3R^\gamma,\end{aligned}$$

where $C_3 := 2cK(x_*)N_\alpha\|f_\epsilon\|^2_{L^{q_\alpha}(\mathbb{R}^n)}$. Therefore, for $\lambda < 0$, we can take s satisfying $\frac{\beta}{\gamma} < s$, and $R = \epsilon^s > 0$ for some $\epsilon > 0$ small enough, such that

$$\begin{aligned} I_1 + I_2 + I_3 &\le C_1\lambda\epsilon^\beta + C_2R^{2\gamma} + C_3R^\gamma \\ &= C_1\lambda\epsilon^\beta + C_2\epsilon^{2s\gamma} + C_3\epsilon^{s\gamma} < 0. \end{aligned}$$

Combining this with (3.2), for $\lambda < 0$, $\epsilon > 0$ small enough, we have

$$\int_\Omega\int_\Omega\left(\frac{K(x)K(y)}{|x-y|^{n-\alpha}} + \frac{\lambda G(x)G(y)}{|x-y|^{n-\alpha-\beta}}\right)\widetilde{f_\epsilon}(x)\widetilde{f_\epsilon}(y)\,dx\,dy < K^2(x_*)N_\alpha\|f_\epsilon\|^2_{L^{q_\alpha}(\mathbb{R}^n)}.$$

That is, for any $\lambda < 0$, $Q_{\lambda,q_\alpha}(\Omega) < K^2(x_*)N_\alpha$.

(II) Let $x_* \in \partial\Omega$. By $(\mathcal{T})$, there exists $\rho_1 > 0$ such that $K(x) - K(x_*) \le c|x-x_*|^\gamma$ when $x \in V := \overline{\Omega} \cap \overline{B(x_*, \rho_1)}$.

Let $0 < \rho_0 < \rho_1$, $x_0 \in V$ satisfying $B_{\rho_0}(x_0) \subset V - \partial V$, $|x_0 - x_*| = 2\rho_0$. Then, for any $x \in B_{\rho_0}(x_0)$, we have

$$K(x) - K(x_*) \le C\big(|x-x_0|^\gamma + |x_0-x_*|^\gamma\big).$$

We define

$$\widetilde{f_\epsilon}(x) = \begin{cases} \overline{f}_\epsilon(x), & x \in B_{\rho_0}(x_0) \subset \Omega, \\ 0, & x \in \mathbb{R}^n \setminus B_{\rho_0}(x_0), \end{cases}$$

where $\overline{f}_\epsilon(x) = \epsilon^{-\frac{n+\alpha}{2}}f_1(\frac{|x-x_0|}{\epsilon}) = (\frac{\epsilon}{\epsilon^2+|x-x_0|^2})^{\frac{n+\alpha}{2}}$.

Similar to (I),

$$\begin{aligned} &\int_\Omega\int_\Omega\left(\frac{K(x)K(y)}{|x-y|^{n-\alpha}} + \frac{\lambda G(x)G(y)}{|x-y|^{n-\alpha-\beta}}\right)\widetilde{f_\epsilon}(x)\widetilde{f_\epsilon}(y)\,dx\,dy \\ &\quad= \int_{B_{\rho_0}(x_0)}\int_{B_{\rho_0}(x_0)}\left(\frac{K(x)K(y)}{|x-y|^{n-\alpha}} + \frac{\lambda G(x)G(y)}{|x-y|^{n-\alpha-\beta}}\right)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)\,dx\,dy \\ &\quad\le \int_{B_{\rho_0}(x_0)}\int_{B_{\rho_0}(x_0)}\frac{(K(x_*)+C(|x-x_0|^\gamma+|2\rho_0|^\gamma))(K(x_*)+C(|y-x_0|^\gamma+|2\rho_0|^\gamma))}{|x-y|^{n-\alpha}} \\ &\qquad\times\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)\,dx\,dy \\ &\qquad+\lambda\int_{B_{\rho_0}(x_0)}\int_{B_{\rho_0}(x_0)}\frac{G(x)G(y)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha-\beta}}\,dx\,dy \\ &\quad\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{K^2(x_*)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy + \lambda\int_{B_{\rho_0}(x_0)}\int_{B_{\rho_0}(x_0)}\frac{G(x)G(y)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha-\beta}}\,dx\,dy \\ &\qquad+2C\int_{B_{\rho_0}(x_0)}\int_{B_{\rho_0}(x_0)}\frac{K(x_*)(|x-x_0|^\gamma+|2\rho_0|^\gamma)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy \\ &\qquad+C^2\int_{B_{\rho_0}(x_0)}\int_{B_{\rho_0}(x_0)}\frac{(|x-x_0|^\gamma+|2\rho_0|^\gamma)(|y-x_0|^\gamma+|2\rho_0|^\gamma)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy \\ &\quad= K^2(x_*)N_\alpha\|\overline{f}_\epsilon\|^2_{L^{q_\alpha}(\mathbb{R}^n)} + J_1 + J_2 + J_3, \end{aligned} \tag{3.3}$$

where

$$J_1 := \lambda \int_{B_{\rho_0}(x_0)} \int_{B_{\rho_0}(x_0)} \frac{G(x)G(y)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha-\beta}}\,dx\,dy,$$

$$J_2 := 2C \int_{B_{\rho_0}(x_0)} \int_{B_{\rho_0}(x_0)} \frac{K(x_*)(|x-x_0|^\gamma + |2\rho_0|^\gamma)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy,$$

$$J_3 := C^2 \int_{B_{\rho_0}(x_0)} \int_{B_{\rho_0}(x_0)} \frac{(|x-x_0|^\gamma + |2\rho_0|^\gamma)(|y-x_0|^\gamma + |2\rho_0|^\gamma)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy.$$

As in case (I), we know $J_1 \leq C_4\lambda\epsilon^\beta$. For J_2, we have

$$\begin{aligned} J_2 &= 2C \int_{B_{\rho_0}(x_0)} \int_{B_{\rho_0}(x_0)} \frac{K(x_*)(|x-x_0|^\gamma + |2\rho_0|^\gamma)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy \\ &\leq CK(x_*)\rho_0^\gamma \int_{B_{\rho_0}(x_0)} \int_{B_{\rho_0}(x_0)} \frac{\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy \\ &\leq C_5\rho_0^\gamma. \end{aligned}$$

For J_3, we have

$$\begin{aligned} J_3 &= C^2 \int_{B_{\rho_0}(x_0)} \int_{B_{\rho_0}(x_0)} \frac{(|x-x_0|^\gamma + |2\rho_0|^\gamma)(|y-x_0|^\gamma + |2\rho_0|^\gamma)\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy \\ &\leq C^2\rho_0^{2\gamma} \int_{B_{\rho_0}(x_0)} \int_{B_{\rho_0}(x_0)} \frac{\overline{f}_\epsilon(x)\overline{f}_\epsilon(y)}{|x-y|^{n-\alpha}}\,dx\,dy \\ &\leq C_6\rho_0^{2\gamma}. \end{aligned}$$

Taking s with $\frac{\beta}{\gamma} < s$ and $\rho_0 = \epsilon^s > 0$, then

$$\begin{aligned} J_1 + J_2 + J_3 &\leq C_4\lambda\epsilon^\beta + C_5\rho_0^\gamma + C_6\rho_0^{2\gamma} \\ &= C_4\lambda\epsilon^\beta + C_5\epsilon^{s\gamma} + C_6\epsilon^{2s\gamma} < 0 \end{aligned}$$

for $\epsilon > 0$ small enough. Thus, combining this with (3.3), for $\lambda < 0$, $\epsilon > 0$ small enough, we have

$$\int_\Omega \int_\Omega \left(\frac{K(x)K(y)}{|x-y|^{n-\alpha}} + \frac{\lambda G(x)G(y)}{|x-y|^{n-\alpha-\beta}}\right)\widetilde{f}_\epsilon(x)\widetilde{f}_\epsilon(y)\,dx\,dy < K^2(x_*)N_\alpha \|\overline{f}_\epsilon\|^2_{L^{q_\alpha}(\mathbb{R}^n)}.$$

That is, for any $\lambda < 0$, we have $Q_{\lambda,q_\alpha}(\Omega) < K^2(x_*)N_\alpha$.

On the other hand, for any $\lambda \in (-\frac{K^2(x_*)}{d^\beta(\Omega)G^2(\tilde{x}_*)}, 0)$, we also have $Q_{\lambda,q_\alpha}(\Omega) > 0$. So we complete the proof. □

In order to prove the existence of weak solutions, we need to prove that the minimal energy $Q_{\lambda,q_\alpha}(\Omega)$ is attained.

Proposition 3.5 *For given* $\lambda \in (-\frac{K^2(x_*)}{d^\beta(\Omega)G^2(\tilde{x}_*)}, 0)$, $Q_{\lambda,q_\alpha}(\Omega)$ *is attained by some positive function* $f_* \in L^{q_\alpha}(\Omega)$.

For $q < q_\alpha$, we consider

$$Q_{\lambda,q}(\Omega) = \inf_{f\in L_+^q(\Omega)} \frac{\int_\Omega \int_\Omega f(x)(K(x)|x-y|^{-(n-\alpha)}K(y) + \lambda G(x)|x-y|^{-(n-\alpha-\beta)}G(y))f(y)\,dx\,dy}{\|f\|^2_{L^q(\Omega)}}.$$

By Lemma 3.2, the infimum is attained by the positive function f_q which satisfies the integral equation with the subcritical exponent

$$Q_{\lambda,q}(\Omega)f^{q-1}(x) = \int_\Omega \frac{K(x)f(y)K(y)}{|x-y|^{n-\alpha}}\,dy + \lambda \int_\Omega \frac{G(x)f(y)G(y)}{|x-y|^{n-\alpha-\beta}}\,dy, \quad x \in \overline{\Omega}. \tag{3.4}$$

That is, f_q is the minimal energy solution to Eq. (3.4). It is easy to see that $\|f_q\|_{L^q(\Omega)} = 1$, $f_q \in C(\overline{\Omega})$ and $Q_{\lambda,q} \to Q_\lambda$ as $q \to (q_\alpha)^-$.

Lemma 3.6 *For* $\lambda \in (-\frac{K^2(x_*)}{d^\beta(\Omega)G^2(\tilde{x}_*)}, 0)$, $q \in (0, q_\alpha)$. *Let* $f_q > 0$ *be a minimal energy solution to* (3.4), *where* $\|f_q\|_{L^q(\Omega)} = 1$. *If* $0 < Q_{\lambda,q}(\Omega) \le K^2(x_*)N_\alpha - \epsilon$ *for some* $\epsilon > 0$, *then there exists* $C > 0$ *such that* $\frac{1}{C} \le f_q(x) \le C$ *uniformly for all* $x \in \overline{\Omega}$, $q \in (0, q_\alpha)$.

Proof We prove it by modifying the argument of Lemma 4.3 in [1].

For any $x \in \overline{\Omega}$, $q \in (0, q_1)$, we see that $\max_{\overline{\Omega}} f_q(x) = f_q(x_q) \le C < \infty$ holds uniformly, where $0 < q_1 < q_\alpha$.

We first prove that $\max_{\overline{\Omega}} f_q(x) = f_q(x_q) \le C < \infty$ holds uniformly as $q \to (q_\alpha)^-$. Otherwise, $f_q(x_q) \to \infty$, where $x_q \to \tilde{x}$, up to a subsequence, as $q \to (q_\alpha)^-$. Denote

$$\mu_q := f_q^{-\frac{2-q}{\alpha}}(x_q), \qquad \Omega_\mu := \frac{\Omega - x_q}{\mu_q} = \left\{ z \,\middle|\, z = \frac{x - x_q}{\mu_q}, x \in \Omega \right\}.$$

We define

$$g_q(z) = \mu_q^{\frac{\alpha}{2-q}} f_q(\mu_q z + x_q), \quad z \in \overline{\Omega}_\mu.$$

Thus g_q satisfies

$$\begin{aligned} Q_{\lambda,q}(\Omega)g_q^{q-1}(z) &= \int_{\Omega_\mu} \frac{K(\mu_q z + x_q)g_q(y)K(\mu_q y + x_q)}{|z-y|^{n-\alpha}}\,dy \\ &\quad + \lambda \mu_q^\beta \int_{\Omega_\mu} \frac{G(\mu_q z + x_q)g_q(y)G(\mu_q y + x_q)}{|z-y|^{n-\alpha-\beta}}\,dy, \quad z \in \overline{\Omega}_\mu, \end{aligned} \tag{3.5}$$

and $g_q(0) = 1$, $g_q(z) \in (0, 1]$.

For convenience, we define $h_q(z) := g_q^{q-1}(z)$. So (3.5) is equivalent to

$$\begin{aligned} Q_{\lambda,q}(\Omega)h_q(z) &= \int_{\Omega_\mu} \frac{K(\mu_q z + x_q)h_q^{p-1}(y)K(\mu_q y + x_q)}{|z-y|^{n-\alpha}}\,dy \\ &\quad + \lambda \mu_q^\beta \int_{\Omega_\mu} \frac{G(\mu_q z + x_q)h_q^{p-1}(y)G(\mu_q y + x_q)}{|z-y|^{n-\alpha-\beta}}\,dy, \quad z \in \overline{\Omega}_\mu, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $h_q(0) = 1$, $h_q(z) \ge 1$.

Claim: There exist $C_1, C_2 > 0$ such that

$$0 < C_1\left(1 + |z|^{\alpha-n}\right) \le h_q(z) \le C_2\left(1 + |z|^{\alpha-n}\right), z \in \tilde{\Omega}, \tag{3.6}$$

holds uniformly for any domain $\tilde{\Omega} \subset \Omega_\mu$ as $q \to (q_\alpha)^-$.

The claim can be verified by a similar argument to that in [1], we omit it here. Thus $h_q(z)$ is equicontinuous in any bounded domain $\widehat{\Omega} \subset \Omega_\mu$ as $q \to (q_\alpha)^-$. In fact, for $R > 0$,

$$\begin{aligned}
&Q_{\lambda,q}(\Omega)h_q(z) \\
&\quad= \int_{\Omega_\mu \setminus B(0,R)} \frac{K(\mu_q z + x_q)h_q^{p-1}(y)K(\mu_q y + x_q)}{|z-y|^{n-\alpha}}\,dy \\
&\qquad+ \int_{\Omega_\mu \cap B(0,R)} \frac{K(\mu_q z + x_q)h_q^{p-1}(y)K(\mu_q y + x_q)}{|z-y|^{n-\alpha}}\,dy \\
&\qquad+ \lambda\mu_q^\beta \int_{\Omega_\mu \setminus B(0,R)} \frac{G(\mu_q z + x_q)h_q^{p-1}(y)G(\mu_q y + x_q)}{|z-y|^{n-\alpha-\beta}}\,dy \\
&\qquad+ \lambda\mu_q^\beta \int_{\Omega_\mu \cap B(0,R)} \frac{G(\mu_q z + x_q)h_q^{p-1}(y)G(\mu_q y + x_q)}{|z-y|^{n-\alpha-\beta}}\,dy.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\int_{\Omega_\mu \setminus B(0,R)} \frac{h_q^{p-1}(y)}{|z-y|^{n-\alpha}}\left(K(\mu_q z + x_q)K(\mu_q y + x_q) + \lambda\mu_q^\beta G(\mu_q z + x_q)G(\mu_q y + x_q)|z-y|^\beta\right)dy \\
&\quad\ge \left(K^2(x_*) - |\lambda| d^\beta(\Omega)G^2(\tilde{x}_*)\right)\int_{\Omega_\mu \setminus B(0,R)} \frac{h_q^{p-1}(y)}{|z-y|^{n-\alpha}}\,dy \ge 0.
\end{aligned}$$

For any $\epsilon > 0$ small enough, by (3.6), we have

$$\begin{aligned}
0 &\le \int_{\Omega_\mu \setminus B(0,R)} \frac{h_q^{p-1}(y)}{|z-y|^{n-\alpha}}\left(K(\mu_q z + x_q)K(\mu_q y + x_q)\right. \\
&\quad\left. + \lambda\mu_q^\beta G(\mu_q z + x_q)|z-y|^\beta G(\mu_q y + x_q)\right)dy \\
&\le C\int_{\Omega_\mu \setminus B(0,R)} \frac{h_q^{p-1}(y)}{|y|^{n-\alpha}}\,dy \\
&\le C\int_R^\infty r^{(\alpha-n)(p-1)+\alpha-1}\,dr \\
&= CR^{(\alpha-n)(p-1)+\alpha} < \epsilon,
\end{aligned} \tag{3.7}$$

where $R > 0$ is large enough, $q \to (q_\alpha)^-$, $z \in \widehat{\Omega}$. Since $\beta < n$, by using the same argument as above, we also have

$$\left|\lambda\mu_q^\beta \int_{\Omega_\mu \cap B(0,R)} \frac{G(\mu_q z + x_q)h_q^{p-1}(y)G(\mu_q y + x_q)}{|z-y|^{n-\alpha-\beta}}\,dy\right| < \epsilon, \tag{3.8}$$

when $R > 0$ large enough, $q \to (q_\alpha)^-$.

On the other hand, it is easy to see that, for $z \in \widehat{\Omega}$, $\int_{\Omega_\mu \cap B(0,R)} \frac{K(\mu_q z+x_q)h_q^{p-1}(y)K(\mu_q y+x_q)}{|z-y|^{n-\alpha}}\,dy \in C^1(\widehat{\Omega})$. Combining this with (3.7) and (3.8), we conclude that $h_q(z)$ is equicontinuous in any bounded domain $\widehat{\Omega} \subset \mathbb{R}^n$ as $q \to (q_\alpha)^-$.

When $q \to (q_\alpha)^-$, we distinguish two cases:

Case 1. $\Omega_\mu \to \mathbb{R}^n_T := \{(z_1, z_2, \ldots, z_n) \mid z_n > -T\}$ for some $T \geq 0$, and $h_q(z) \to h(z) \in C(\mathbb{R}^n_T)$ holds uniformly on any compact sets of $\mathbb{R}^n_T$, where $h(z)$ satisfying

$$Q_\lambda h(z) = K^2(\widetilde{x}) \int_{\mathbb{R}^n_T} \frac{h^{p_\alpha - 1}(y)}{|z-y|^{n-\alpha}}\,dy, \quad h(0) = 1.$$

Then, similar to Lemma 4.3 in [1], we obtain a contradiction.

Case 2. $\Omega_\mu \to \mathbb{R}^n$, $h_q(z) \to h(z) \in C(\mathbb{R}^n)$ holds uniformly on any compact sets of $\mathbb{R}^n$, where $h(z)$ satisfying

$$Q_\lambda h(z) = K^2(\widetilde{x}) \int_{\mathbb{R}^n} \frac{h^{p_\alpha - 1}(y)}{|z-y|^{n-\alpha}}\,dy, \quad h(0) = 1.$$

Again similar to Lemma 4.3 in [1], we obtain a contradiction.

Thus we conclude that there exists $C > 0$ such that $f_q(y) \leq C$ uniformly for $y \in \overline{\Omega}$, $q \in (0, q_\alpha)$.

On the other hand, if $\min_{\overline{\Omega}} f_q(x) := f_q(\widetilde{x}_q) \to 0$ as $q \to (q_\alpha)^-$. Then

$$\infty \leftarrow f_q^{q-1}(\widetilde{x}_q) = \int_\Omega \frac{K(\widetilde{x}_q)f_q(y)K(y)}{|\widetilde{x}_q - y|^{n-\alpha}}\,dy + \lambda \int_\Omega \frac{G(\widetilde{x}_q)f_q(y)G(y)}{|\widetilde{x}_q - y|^{n-\alpha-\beta}}\,dy \leq C < \infty,$$

as $q \to (q_\alpha)^-$, which implies a contradiction. □

Proof of Proposition 3.5 By Lemma 3.6, $\{f_q\}$ are uniformly bounded above and bounded below by a positive constant. Thus the $\{f_q\}$ are equicontinuous due to Eq. (3.4). It follows that $f_q \to f_*$ as $q \to (q_\alpha)^-$ in $C(\overline{\Omega})$, and f_* is the energy minimizer for Q_λ. □

Proof of Theorem 1.1 (*ii*) Lemma 3.4 and Proposition 3.5 imply the existence of a positive solution $f \in L^{q_\alpha}(\Omega) \cap C(\overline{\Omega})$ to Eq. (1.1) for $q = \frac{2n}{n+\alpha}$, $\lambda \in (-\frac{K^2(x_*)}{d^\beta(\Omega)G^2(\tilde{x}_*)}, 0)$. It is also easy to see that $f \in C^1(\overline{\Omega})$. □

3.3 Nonexistence—critical and supercritical case

We first state a Pohozaev type identity.

Lemma 3.7 *Assume that the origin is in Ω and the domain is star-shaped with respect to the origin. If $u \in C^1(\overline{\Omega})$ is a nonnegative solution to*

$$u(x) = \int_\Omega \frac{K(x)u^{p-1}(y)K(y)}{|x-y|^{n-\alpha}}\,dy + \lambda \int_\Omega \frac{G(x)u^{p-1}(y)G(y)}{|x-y|^{n-\alpha-\beta}}\,dy, \quad x \in \overline{\Omega},$$

where $p \neq 0$, $\lambda \in \mathbb{R}$, $K(x), G(x) \in C^1(\overline{\Omega})$, then

$$\left(\frac{n}{p} + \frac{\alpha - n}{2}\right) \int_\Omega u^p(x)\,dx$$

$$
\begin{aligned}
&= -\frac{\lambda\beta}{2}\int_{\Omega}\int_{\Omega}\frac{G(x)u^{p-1}(x)u^{p-1}(y)G(y)}{|x-y|^{n-\alpha-\beta}}\,dy\,dx + \frac{1}{p}\int_{\partial\Omega}(x\cdot\nu)u^{p}(x)\,d\sigma \\
&\quad - \int_{\Omega}\int_{\Omega}\frac{(x,\nabla K(x))u^{p-1}(x)u^{p-1}(y)K(y)}{|x-y|^{n-\alpha}}\,dy\,dx \\
&\quad - \lambda\int_{\Omega}\int_{\Omega}\frac{(x,\nabla G(x))u^{p-1}(x)u^{p-1}(y)G(y)}{|x-y|^{n-\alpha-\beta}}\,dy\,dx,
\end{aligned}
$$

where ν is the outward unit normal vector to $\partial\Omega$.

Proof The argument is standard. We omit it here. □

Proof of Theorem 1.1(*iii*) We can prove by using Lemma 3.7 and a similar argument to that used in [1]. We omit it here. □

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