Affine-periodic solutions by asymptotic and homotopy equivalence

Jiamin Xing[1*] and Xue Yang[1,2]

*Correspondence:
xingjiamin1028@126.com
[1]School of Mathematics and
Statistics, and Center for
Mathematics and Interdisciplinary
Sciences, Northeast Normal
University, Changchun, P.R. China
Full list of author information is
available at the end of the article

Abstract

This paper studies the existence of affine-periodic solutions which have the form of $x(t + T) = Qx(t)$ with some nonsingular matrix Q. Depending on the structure of Q, they can be periodic, anti-periodic, quasi-periodic or even unbounded. Krasnosel'skii–Perov type existence theorem, asymptotic and homotopy equivalence approaches are given.

MSC: 34C14; 34C41

Keywords: Affine-periodic solutions; Krasnosel'skii–Perov type theorem; Asymptotic and homotopy equivalence

1 Introduction

For more than a century after Poincaré and Lyapunov, the existence theory of periodic solutions for a periodic system has been well developed; for example, see [7, 9, 12, 26, 29]. Besides periodicity, many systems may also have other symmetric structures. The anti-periodic system together with the existence of anti-periodic solutions, for example, is paid high attention to; see [1–5, 20, 25, 27].

If a system is subjected to an external force with a certain symmetry structure, a natural question is whether the system has a solution with the same symmetry structure. For example, one may ask whether the system under a spiral external force has a spiral form solution. Recently, the concept of affine periodicity, including the spiral symmetry was introduced. Some problems and methods concerning affine-periodic solutions, such as Levinson's problem, Lyapunov function type theorems, the dissipative second order rotating periodic systems, LaSalle type theorems, Hamiltonian systems and the averaging method of higher order perturbed systems were given; see [8, 16, 18, 19, 23, 24, 28].

Consider the following (Q, T)-affine-periodic system:

$$x' = f(t, x), \tag{1.1}$$

where $f(t, x) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $f(t + T, x) = Qf(t, Q^{-1}x)$ for every $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $Q \in GL(n)$ (all nonsingular $n \times n$ matrices). We want to find the solution of (1.1)

with

$$x(t + T) = Qx(t) \quad \forall t \in \mathbf{R}.$$

Such a solution is called a (Q, T)-affine-periodic solution.

According to the structure of Q, it will be seen that

  (i) $x(t)$ is a T-periodic solution if $Q = I$ (the identity matrix), and an anti-periodic one if $Q = -I$;

 (ii) If $Q \in O(n)$, that is, Q is an orthogonal matrix, $x(t)$ is a special quasi-periodic solution corresponding to the rotation of a rigid body;

(iii) A (Q, T)-affine-periodic solution $x(t)$ can be unbounded and $\frac{x(t)}{|x(t)|}$ is quasi-periodic, like a helical line, for example, $x(t) = e^{at}(\cos \omega t, \sin \omega t)$.

For periodic systems, Krasnosel'skii and Perov gave an interesting existence theorem of periodic solutions in [13, 14], which is well known today by using the method of topological degree. They proved that, if each solution starting from the boundary of a bounded region will not return to the initial point during a periodic time and the topological degree of $f(0, \cdot)$ is not equal to zero, then the system will have a periodic solution. In this paper, we give Krasnosel'skii–Perov type results for affine-periodic systems. When $I - Q$ is not invertible, we give a general result, which is comparable with Krasnosel'skii and Perov's theorem in the periodic case. It is well known that sometimes the conditions of the Krasnosel'skii–Perov type theorem are difficult to verify, but we give a more flexible condition. When $I - Q$ is invertible, we find that the existence of affine-periodic solutions can also be obtained without calculating the topological degree of $f(0, \cdot)$.

It is also meaningful to find the relationship for the existence of periodic solutions between asymptotically equivalent equations. It is well known that the existence of periodic solutions of a system is rather related to that of another asymptotically equivalent one; see [15, 17]. In the asymptotically linear case, more results have been obtained; for example see [6, 10]. In this paper, we use the method of asymptotically equivalent equation to study the existence of affine-periodic solutions. In our results, the asymptotically equivalent equation can be nonlinear, and the conditions are much easier to verify in linear case.

In the study of periodic solutions for differential equations, the alternative is an interesting phenomenon. Krasnosel'skii and Perov's theorem is a kind of alternative theorem. Another alternative result is achieved by the homotopy method. If the solutions of the auxiliary equations starting at the boundary are not periodic, then the system will have at least one periodic solution in the interior of the region; for example, see [11, 21, 22]. In this paper, we give a method to study the existence of affine-periodic solutions by using homotopy approach.

If there is a linear coordinate transformation $u = Bx$ with $B \in GL(n)$, then system (1.1) becomes

$$u' = g(t, u),$$

where $g(t, u) = Bf(t, B^{-1}u)$. According to the symmetry of $f(t, x)$, we have

$$g(t + T, u) = Bf\left(t + T, B^{-1}u\right)$$
$$= BQf\left(t, Q^{-1}B^{-1}u\right)$$

$$= BQB^{-1}Bf\left(t, B^{-1}BQ^{-1}B^{-1}u\right)$$

$$= \tilde{Q}g\left(t, \tilde{Q}^{-1}u\right),$$

where $\tilde{Q} = BQB^{-1}$. This means the affine periodicity is invariant under linear transformations.

In Sect. 2 and Sect. 4 of the paper, we assume Q has the following Jordan normal form:

$$J = \begin{pmatrix} I_{m \times m} & O \\ O & C_{(n-m) \times (n-m)} \end{pmatrix}, \tag{1.2}$$

where O denotes the zero matrix, $0 \leq m \leq n$, and $I_{(n-m) \times (n-m)} - C_{(n-m) \times (n-m)}$ is invertible. For example, Q is a symmetric or orthogonal matrix. Meanwhile in Sect. 3, we only need $Q \in GL(n)$.

The paper is organized as follows. In Sect. 2, we give some Krasnosel'skii–Perov type results. In Sect. 3, we study the existence of affine-periodic solutions by asymptotic equivalence. In Sect. 4, we give an existence theorem through the homotopy method. In Sect. 5, some examples are given to illustrate the characteristics of (Q, T)-affine-periodic systems and to show the effectiveness of the theorems.

2 Krasnosel'skii–Perov type results

Now we give our first main result.

Theorem 2.1 *Consider the (Q, T)-affine-periodic system (1.1), where $f(t, x)$ is continuous and locally Lipschitz continuous in the variable x. Assume there exists an open bounded subset $V \subset \mathbf{R}^n$ such that the following conditions hold*:

(i) *For every $y \in \overline{V}$, the solution $x(t, y)$ of system (1.1) exists at least on $[0, T]$.*

(ii) $\mathrm{Ker}(I - Q) \neq \{0\}$. *There exists a continuous matrix function $Q(t) : [0, T] \to GL(n)$, such that $Q(T) = Q$. For every $t \in [0, T]$, $Q(t)$ has the Jordan normal form (1.2) and $I - Q(t)$ has the same kernel space. If $y \in \partial V$, then*

$$x(\omega, y) \neq Q(\omega)y, \quad \forall \omega \in (0, T].$$

(iii) *Denote by $P : \mathbf{R}^n \to \mathrm{Ker}(I - Q)$ the orthogonal projection. For every $y \in \partial V \cap \mathrm{Ker}(I - Q)$, $Pf(0, y) \neq 0$ and*

$$\deg\left(Pf(0, \cdot), V \cap \mathrm{Ker}(I - Q), 0\right) \neq 0.$$

Then there exists at least one (Q, T)-affine-periodic solution of system (1.1).

Proof Suppose $x(t)$ is a solution of system (1.1) with boundary condition $x(T) = Qx(0)$. For $t \in [T, 2T]$, take $x(t) = Qx(t - T)$, we have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}Qx(t - T)}{\mathrm{d}(t - T)}$$

$$= Qf\left(t - T, x(t - T)\right)$$

$$= Qf\big(t - T, Q^{-1}x(t)\big)$$

$$= f\big(t, x(t)\big).$$

For $t \in \mathbf{R}$, let

$$x(t) = Q^m x(t - mT),$$

where m is an integer such that $t - mT \in [0, T]$, the solution $x(t)$ can be extended to the whole real line. So to prove the existence of (Q, T)-affine-periodic solutions of system (1.1), we just need to prove the existence of solutions of (1.1) with boundary condition

$$x(T) = Qx(0). \tag{2.1}$$

We define an operator $\Phi : \overline{V} \to \mathbf{R}^n$ by

$$\Phi(y) = (I - Q)y + (I - P) \int_0^T f\big(t, x(t, y)\big)\, dt + \frac{1}{T} \int_0^T Pf\big(t, x(t, y)\big)\, dt.$$

We claim that, for each zero y of Φ, $x(t, y)$ is a solution of (1.1) with boundary condition (2.1).

In fact, if y is a zero of Φ, we have

$$\frac{1}{T} \int_0^T Pf\big(t, x(t, y)\big)\, dt = 0, \tag{2.2}$$

$$(I - Q)y + (I - P) \int_0^T f\big(t, x(t, y)\big)\, dt = 0. \tag{2.3}$$

Then

$$(I - Q)y + \int_0^T f\big(t, x(t, y)\big)\, dt = 0, \tag{2.4}$$

and hence

$$x(T, y) = Qy.$$

Consider the homotopy operator $H : \overline{V} \times (0, 1] \to \mathbf{R}^n$:

$$H(y, \lambda) = \big(I - Q(\lambda T)\big)y + (I - P) \int_0^{\lambda T} f\big(t, x(t, y)\big)\, dt + \frac{1}{\lambda T} \int_0^{\lambda T} Pf\big(t, x(t, y)\big)\, dt.$$

For $(y, \lambda) \in \overline{V} \times (0, 1]$,

$$\lim_{\lambda \to 0} \frac{1}{\lambda T} \int_0^{\lambda T} Pf\big(t, x(t, y)\big)\, dt = Pf(0, y).$$

When $\lambda = 0$, denote

$$H(y, 0) = \big(I - Q(0)\big)y + Pf(0, y).$$

It is easy to prove the operator

$$H : \overline{V} \times [0,1] \to \mathbf{R}^n$$

is continuous, we omit the proof.

Now we prove that

$$0 \notin H\big(\partial V \times [0,1]\big).$$

Suppose on the contrary that there exists $(\tilde{y}, \tilde{\lambda}) \in \partial V \times [0,1]$, such that

$$H(\tilde{y}, \tilde{\lambda}) = 0.$$

(I): When $\tilde{\lambda} = 0$, we have

$$\big(I - Q(0)\big)\tilde{y} + Pf(0, \tilde{y}) = 0.$$

That is,

$$\big(I - Q(0)\big)\tilde{y} = 0 \tag{2.5}$$

and

$$Pf(0, \tilde{y}) = 0.$$

By (2.5), we get $\tilde{y} \in \partial \overline{V} \cap \mathrm{Ker}(I - Q)$, which contradicts assumption (iii).

(II): When $\tilde{\lambda} \in (0, 1]$, we have

$$\frac{1}{\tilde{\lambda} T} \int_0^{\tilde{\lambda} T} Pf\big(t, x(t, \tilde{y})\big) \, \mathrm{d}t = 0, \tag{2.6}$$

$$\big(I - Q(\tilde{\lambda} T)\big)\tilde{y} + (I - P) \int_0^{\tilde{\lambda} T} f\big(t, x(t, \tilde{y})\big) \, \mathrm{d}t = 0. \tag{2.7}$$

Then

$$\big(I - Q(\tilde{\lambda} T)\big)\tilde{y} + \int_0^{\tilde{\lambda} T} f\big(t, x(t, \tilde{y})\big) \, \mathrm{d}t = 0, \tag{2.8}$$

which implies

$$x(\tilde{\lambda} T, \tilde{y}) = Q(\tilde{\lambda} T)\tilde{y}.$$

This contradicts assumption (ii).

By (I) and (II), we obtain

$$0 \notin H\big(\partial V \times [0,1]\big).$$

Without loss of generality, we assume that $Q(0)$ has the form (1.2).

Let $Pf(0,y)$ be twice continuously differentiable and satisfy

$$0 \notin Pf(0,N_{f_m}),$$

where

$$N_{f_m} = \left\{ y : y \in \overline{V} \cap \mathrm{Ker}(I-Q), \det\left(\frac{\partial f_i(y)}{\partial y_j}; 1 \le i,j \le m \right) = 0 \right\}.$$

It is easy to see that $H(y*,0) = 0$ if and only if $y* \in \mathrm{Ker}(I-Q)$ and $Pf(0,y*) = 0$. Moreover,

$$\det\left(\frac{\partial H(y^*,0)}{\partial y} \right) = \det\left(I - Q(0) + \frac{\partial Pf(0,y^*)}{\partial y} \right)$$

$$= \det(I - C_{(n-m)\times(n-m)}) \cdot \det\left(\frac{\partial f_i(y^*)}{\partial y_j}; 1 \le i,j \le m \right).$$

Hence

$$\deg\big(H(\cdot,0), V, 0\big) = \gamma \deg\big(Pf(0,\cdot), V \cap \mathrm{Ker}(I-Q), 0\big),$$

where $\gamma = 1$ or $\gamma = -1$.

When $Pf(0,y)$ is only continuous, the same result can be obtained by selecting suitable twice continuously differentiable functions to approximate it.

From the homotopy invariance of topological degree, we have

$$\deg\big(H(\cdot,1), V, 0\big) \neq 0.$$

Then there exists a $y^* \in V$ such that

$$\Phi\big(y^*\big) = 0,$$

and $x(t,y^*)$ is a solution of equation (1.1) with boundary condition (2.1). Thus the existence of (Q,T)-affine-periodic solutions of system (1.1) is obtained.　　　　　　　　□

When $Q = I$, Theorem 2.1 is consistent with Krasnosel'skii and Perov's theorem. For a perturbed system, we give the following corollary.

Corollary 2.1 *Consider the system*

$$x' = \sum_{i=1}^{k} \varepsilon^i f_i(t,x) + \varepsilon^{k+1} r(t,x,\varepsilon), \tag{2.9}$$

where ε is a small parameter, $f_i(t,x)$ $(i = 1,\ldots,k)$ and $r(t,x,\varepsilon)$ are continuous, locally Lipschitz in x and (Q,T)-affine-periodic.

(i) *Assume $\det(I - Q) = 0$.*

(ii) *Let V be an open bounded subset of $\mathbf{R}^n$, and $P : \mathbf{R}^n \to \mathrm{Ker}(I - Q)$ the orthogonal projection, and denote $Pf_i(t, x) = ((Pf_i)^1(t, x), \ldots, (Pf_i)^n(t, x))^\top$. Suppose that, for every point $p \in \partial V \cap \mathrm{Ker}(I - Q)$, there exists a neighborhood U_p of p, a constant $\sigma_p > 0$ that are both independent of ε, and an integer $1 \le j \le n$, such that*

$$\left| \frac{1}{\varepsilon^k} \sum_{i=1}^{k} \varepsilon^i (Pf_i)^j(t, x) \right| \ge \sigma_p \tag{2.10}$$

for all $x \in U_p$, $t \in [0, T]$, and $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$, where ε_0 is a positive constant.

(iii) *Let*

$$F(y, \varepsilon) = \sum_{i=1}^{k} \varepsilon^i f_i(0, y). \tag{2.11}$$

Assume that, for each $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$,

$$\deg\big(PF(\cdot, \varepsilon), V \cap \mathrm{Ker}(I - Q), 0\big) \ne 0. \tag{2.12}$$

Then system (2.9) *has a (Q, T)-affine-periodic solution for $|\varepsilon| > 0$ small enough.*

Proof It is easy to prove that there exist positive constants r and σ, such that, for every $p \in \partial V \cap \mathrm{Ker}(I - Q)$ and every $y \in \overline{B}_r(p)$, one has

$$\left| \frac{1}{\varepsilon^k} \sum_{i=1}^{k} \varepsilon^i (Pf_i)^j(t, y) \right| \ge \sigma \tag{2.13}$$

for all $t \in [0, T]$, and $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$, where $B_r(p)$ is the open ball centered at p with radius r. Now we prove that $x(\omega, y) \ne Qy$ for every $y \in \partial V$ and $\omega \in (0, T]$. Denote $\Delta = \bigcup_{p \in \partial V \cap \mathrm{Ker}(I-Q)} \overline{B}_r(p)$.

When $y \in \partial V \setminus \Delta$, there exists a constant $\rho > 0$, such that

$$\big| (I - Q)y \big| \ge \rho. \tag{2.14}$$

At the same time, one has

$$x(\omega, y) = y + \int_0^\omega \left(\sum_{i=1}^{k} \varepsilon^i f_i\big(t, x(t, y)\big) + \varepsilon^{k+1} r\big(t, x(t, y), \varepsilon\big) \right) dt.$$

If $x(\omega, y) = Qy$, then

$$(I - Q)y + \int_0^\omega \left(\sum_{i=1}^{k} \varepsilon^i f_i\big(t, x(t, y)\big) + \varepsilon^{k+1} r\big(t, x(t, y), \varepsilon\big) \right) dt = 0.$$

This contradicts (2.14) for $|\varepsilon| > 0$ small enough.

When $y \in \Delta$ and $x(\omega, y) = Qy$, one has

$$\int_0^\omega \left(\sum_{i=1}^k \varepsilon^i P f_i(t, x(t, y)) + \varepsilon^{k+1} \Pr(t, x(t, y), \varepsilon) \right) dt = 0.$$

When $|\varepsilon| > 0$ small enough, this contradicts (2.13).

Consider the homotopy operator $H : \overline{V} \cap \mathrm{Ker}(I - Q) \times [0, 1] \to \mathbf{R}^m$:

$$H(y, \lambda) = PF(y, \varepsilon) + \lambda \varepsilon^{k+1} \Pr(0, y, \varepsilon).$$

By (2.13), $H(y, \lambda) \neq 0$ for every $y \in \partial V \cap \mathrm{Ker}(I - Q)$, $\lambda \in [0, 1]$ and $|\varepsilon| > 0$ small enough. Then from the homotopy invariance, one has

$$\deg(H(\cdot, 1), V, 0) \neq 0.$$

By Theorem 2.1, there exists a (Q, T)-affine-periodic solution of system (2.9) for $|\varepsilon| > 0$ small enough. $\qquad \square$

Theorem 2.1 depends on the existence of $Q(\omega)$. More generally if there exists a suitable nonlinear function $Q(\omega, y)$, we have the following theorem.

Theorem 2.2 *Consider the system* (1.1). *Assume $f(t, x)$ is locally Lipschitz continuous in the variable x and there exists an open bounded subset $V \subset \mathbf{R}^n$ which contains the origin, such that the following conditions hold*:
  (i) *For every $y \in \overline{V}$, the solution $x(t, y)$ of system* (1.1) *exists at least on $[0, T]$.*
  (ii) *There exists a continuous function $Q(t, y) : [0, T] \times \overline{V} \to \mathbf{R}^n$, such that $Q(T, y) = Qy$, and $x(\omega, y) \neq Q(\omega, y)$ for every $\omega \in (0, T]$ and $y \in \partial V$.*
  (iii) $\deg(\mathrm{id} - Q(0, \cdot), V, 0) \neq 0.$
Then there exists a (Q, T)-affine-periodic solution of system (1.1).

Proof Consider the homotopy operator $H : \overline{V} \times [0, 1] \to \mathbf{R}^n$

$$H(y, \lambda) = y - Q(\lambda T, y) + \int_0^{\lambda T} f(t, x(t, y)) \, dt.$$

By assumption (ii), we see $0 \notin H(\partial V \times [0, 1])$. From the homotopy invariance of topological degree, we have

$$\deg(H(\cdot, 1), V, 0) = \deg(H(\cdot, 0), V, 0).$$

Then, by assumption (iii),

$$\deg(H(\cdot, 1), V, 0) \neq 0.$$

Hence there exists a $y \in V$, such that

$$y - Q(T, y) + \int_0^T f(t, x(t, y)) \, dt = 0.$$

Since

$$x(T, y) = y + \int_0^T f\big(t, x(t, y)\big) \, \mathrm{d}t = 0,$$

we obtain

$$x(T, y) = Q(T, y) = Qy,$$

and there exists a (Q, T)-affine-periodic solutions of system (1.1).   □

Corollary 2.2 *Consider the (Q, T)-affine-periodic system (1.1), where $f(t, x)$ is continuous and locally Lipschitz continuous in the variable x. Assume there exists an open bounded subset $V \subset \mathbf{R}^n$ which contains the origin, such that the following conditions hold*:
  (i) *For every $y \in \overline{V}$, the solution $x(t, y)$ of system (1.1) exists at least on $[0, T]$*;
  (ii) *$x(\omega, y) \neq Qy$ for every $\omega \in (0, T]$ and $y \in \partial V$*;
  (iii) *$I - Q$ is invertible.*
Then there exists a (Q, T)-affine-periodic solution of system (1.1).

Denote by B_R the open ball centered at the origin with radius R in $\mathbf{R}^n$. By Theorem 2.1, we have the following invariant sphere principle.

Theorem 2.3 *Consider the system (1.1), where $f(t, x)$ is continuous and locally Lipschitz continuous in the variable x. Assume $Q \in O(n)$, $\mathrm{Ker}(I - Q) \neq \{0\}$ and there exists an $R > 0$, such that, for every $(t, y) \in [0, T] \times \partial B_R$, $f(t, y)$ is inward to B_R. Then there exists at least one (Q, T)-affine-periodic solution of system (1.1).*

Proof Since $f(t, y)$ is inward to B_R for every $(t, y) \in [0, T] \times \partial B_R$, we get

$$x(\omega, y) \neq Qy, \quad \forall \omega \in (0, T], y \in \partial B_R.$$

Now by Theorem 2.1, we only need to prove that

$$\deg\big(Pf(0, \cdot), B_R \cap \mathrm{Ker}(I - Q), 0\big) \neq 0.$$

It is easy to see that $Pf(t, y)$ is inward to $B_R \cap \mathrm{Ker}(I - Q)$ for every $(t, y) \in [0, T] \times (\partial B_R \cap \mathrm{Ker}(I - Q))$. Consider the following equation in $R^n \cap \mathrm{Ker}(I - Q)$:

$$u(t, y) = y + \int_0^t Pf\big(s, u(s, y)\big) \, \mathrm{d}s.$$

Note that

$$u\big(t, \overline{B_R \cap \mathrm{Ker}(I - Q)}\big) \subset B_R \cap \mathrm{Ker}(I - Q), \quad \forall 0 < t \leq T.$$

By Rothe's theorem we get

$$\deg\big(\mathrm{id} - u(t, \cdot), B_R \cap \mathrm{Ker}(I - Q), 0\big) = 1, \quad \forall 0 < t \leq T.$$

Consider the homotopy operator $H : \overline{B_R \cap \mathrm{Ker}(I - Q)} \times (0,1] \to \mathbf{R}^m$

$$H(y,\lambda) = \frac{y - u(\lambda T, y)}{\lambda T} = -\frac{1}{\lambda T} \int_0^{\lambda T} Pf(t, u(t,y)) \, dt.$$

When $\lambda = 0$, denote

$$H(y,0) = -Pf(0,y).$$

Obviously, the operator

$$H : \overline{B_R \cap \mathrm{Ker}(I - Q)} \times [0,1] \to \mathbf{R}^n$$

is continuous. By the homotopy invariance of topological degree, we get

$$\deg\big(Pf(0,\cdot), B_R \cap \mathrm{Ker}(I - Q), 0\big) = (-1)^m \deg\big(\mathrm{id} - u(T,\cdot), B_R \cap \mathrm{Ker}(I - Q), 0\big)$$
$$= (-1)^m. \qquad \Box$$

3 Asymptotic equivalence

In this section, we study the relationship for the existence of affine-periodic solutions between system (1.1) and the following system:

$$x' = A(t,x), \tag{3.1}$$

where $A(t,x) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous, and for every $t \in \mathbf{R}$, $A(t,x)$ is continuously differentiable in the variable x. Moreover, $A(t + T, x) = QA(t, Q^{-1}x)$ for every $(t,x) \in \mathbf{R} \times \mathbf{R}^n$,

Let

$$C_{Q,T} = \big\{ x : x \in C^1\big([0,T]; \mathbf{R}^n\big), x(T) = Qx(0) \big\},$$

and define the norm as $\|x\| = \sup_{t \in [0,T]} |x(t)|$. It is easy to see that $C_{Q,T}$ is a Banach space with the norm $\| \cdot \|$.

Theorem 3.1 *Consider the system* (1.1) *and system* (3.1). *Assume the following conditions hold*:

(i) $\lim_{M \to \infty} \frac{1}{M} \sup_{|x| \le M} \int_0^T |f(t,x) - A(t,x)| \, dt = 0$;

(ii) *Denote by* $\lambda(t,x)$ *the eigenvalue of* $A_x(t,x)$. *There exists a measurable function* $\beta : [0,T] \to \mathbf{R}$ *with* $\kappa := \exp(\int_0^T \beta(\tau) \, d\tau) < \infty$, *such that*

$$\sup_{x \in \mathbf{R}^n} |\lambda(t,x)| \le \beta(t).$$

(iii) *There exists a constant* $\sigma > 0$ *such that, for every* $\varphi \in C_{Q,T}$, *the solution* $y(t)$ *with* $\|y\| = 1$ *of the system*

$$y' = A_1\big(t, \varphi(t)\big)y \tag{3.2}$$

satisfies $|y(T) - Qy(0)| \ge \sigma$, *where* $A_1(t, \varphi(t)) = \int_0^1 A_x(t, \tau\varphi(t)) \, d\tau$.

Then there exists at least one (Q,T)-*affine-periodic solution of system* (1.1).

Proof For each $\varphi \in C_{Q,T}$ and $\lambda \in [0,1]$, consider the following equation:

$$x' = A_1\big(t, \varphi(t)\big)x + \lambda\big(f\big(t, \varphi(t)\big) - A\big(t, \varphi(t)\big) + A(t, 0)\big). \tag{3.3}$$

It is easy to see the system (3.3) is also a (Q, T)-affine-periodic system, and by assumption (iii) it has a unique (Q, T)-affine-periodic solution $x_{\varphi,\lambda}(t)$. Consider the homotopy operator $H : C_{Q,T} \times [0,1] \to C_{Q,T}$:

$$H\big(\varphi(t), \lambda\big) = x_{\varphi,\lambda}(t).$$

Denote $\Omega_p = \{x \in C_{Q,T}, \|x\| \le p\}$. Now we prove that there exists a constant $p_0 > 0$ large enough, such that, for every $\varphi \in \partial\Omega_{p_0}$ and $\lambda \in [0,1]$,

$$\varphi - H(\varphi, \lambda) \ne 0.$$

Let $\Phi_\varphi(t)$ be a fundamental matrix solution of (3.2) such that $\Phi_\varphi(0) = I$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big|\Phi_\varphi(t)\big|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tr}\big(\Phi_\varphi(t)\Phi_\varphi^\top(t)\big) \\
&= 2\,\mathrm{tr}\big(\Phi_\varphi'(t)\Phi_\varphi^\top(t)\big) \\
&= 2\,\mathrm{tr}\big(A_1\big(t, \varphi(t)\big)\Phi_\varphi(t)\Phi_\varphi^\top(t)\big) \\
&\le 2\beta(t)\big|\Phi_\varphi(t)\big|^2,
\end{aligned}$$

for every $\varphi \in C_{Q,T}$ and $t \in [0, T]$, where $\mathrm{tr}(A)$ denote the trace of matrix A. Thus, we have

$$\big|\Phi_\varphi(t)\big| \le \kappa, \quad \forall \varphi \in C_{Q,T}, t \in [0, T].$$

Denote by $y_\varphi(t)$ the solution of (3.2) with initial value $y_\varphi(0) = x_\varphi(0)$. Then there exists a constant $p_1 > 0$, such that

$$\|y_\varphi\| \le \frac{p}{2}$$

for every $\varphi \in \Omega_p$ with $p \ge p_1$. If not, there would exist $\varphi_k \in \Omega_k$, $k = 1, 2, \dots$, such that $\|y_{\varphi_k}\| > \frac{k}{2}$. By the variation of constants formula, we get

$$\big|y_{\varphi_k}(T) - Qy_{\varphi_k}(0)\big| \le \kappa \int_0^T \big|f\big(s, \varphi_k(s)\big) - A\big(s, \varphi_k(s)\big)\big|\,\mathrm{d}s + \kappa \int_0^T \big|A(s, 0)\big|\,\mathrm{d}s.$$

Then

$$\frac{1}{\|y_{\varphi_k}\|}\big|y_{\varphi_k}(T) - Qy_{\varphi_k}(0)\big| \le \frac{2\kappa}{k} \int_0^T \big|f\big(s, \varphi_k(s)\big) - A\big(s, \varphi_k(s)\big)\big|\,\mathrm{d}s$$

$$+ \frac{2\kappa}{k} \int_0^T \big|A(s, 0)\big|\,\mathrm{d}s$$

$$\leq \frac{2\kappa}{k} \sup_{|x| \leq k} \int_0^T \big| f(s,x) - A(s,x) \big| \, ds$$

$$+ \frac{2\kappa}{k} \int_0^T \big| A(s,0) \big| \, ds.$$

By assumption (i), we get

$$\lim_{k \to \infty} \frac{1}{\|y_{\varphi_k}\|} \big| y_{\varphi_k}(T) - Q y_{\varphi_k}(0) \big| = 0,$$

which contradicts assumption (iii). Also by the variation of constant formula, for each $t \in [0, T]$ and $\lambda \in [0, 1]$, we have

$$\big| x_{\varphi,\lambda}(t) - y_\varphi(t) \big| \leq \kappa \int_0^T \big| f\big(s, \varphi(s)\big) - A\big(s, \varphi(s)\big) \big| \, ds + \kappa \int_0^T \big| A(s,0) \big| \, ds.$$

By assumption (i), there exists a $p_2 > 0$, such that

$$\big| x_{\varphi,\lambda}(t) - y_\varphi(t) \big| \leq \frac{p}{3}, \quad \forall \varphi \in \Omega_p, p \geq p_2.$$

Take $p_0 = \max\{p_1, p_2\}$, then

$$\|x_{\varphi,\lambda}\| \leq \|x_{\varphi,\lambda} - y_\varphi\| + \|y_\varphi\| \leq \frac{5p}{6}, \quad \forall \varphi \in \Omega_p, p \geq p_0. \tag{3.4}$$

Next we prove $H : \Omega_{p_0} \times [0, 1] \to C_{Q,T}$ is compact and continuous. By (3.3) and (3.4), it is easy to see that there exists a constant $M_0 > 0$, such that

$$\big\| x'_{\varphi,\lambda} \big\| \leq M_0, \quad \forall (\varphi, \lambda) \in \Omega_{p_0} \times [0, 1].$$

Then

$$\big| x_{\varphi,\lambda}(t) - x_{\varphi,\lambda}(s) \big| \leq M_0 |t - s|, \quad \forall s, t \in [0, T].$$

By Arzela–Ascoli's theorem, H is compact.

Take $(\varphi_k, \lambda_k) \in \Omega_{p_0} \times [0, 1]$, $(\tilde{\varphi}, \tilde{\lambda}) \in \Omega_{p_0} \times [0, 1]$, such that $\|\varphi_k - \tilde{\varphi}\| \to 0$, $|\lambda_k - \tilde{\lambda}| \to 0$ as $k \to \infty$. We claim that $\|x_{\varphi_k, \lambda_k} - x_{\tilde{\varphi}, \tilde{\lambda}}\| \to 0$. If not by the compactness of H, there would be a subsequence $\{x_{\varphi_{k_j}, \lambda_{k_j}}\}$ of $\{x_{\varphi_k, \lambda_k}\}$ and $u \in \Omega_{p_0}$, $u \neq x_{\tilde{\varphi}, \tilde{\lambda}}$, such that

$$\lim_{j \to \infty} \|x_{\varphi_{k_j}, \lambda_{k_j}} - u\| \to 0.$$

Let $w(t) = x_{\tilde{\varphi}, \tilde{\lambda}}(t) - u(t)$. Then $w(t)$ is a solution of (3.2) and $w(T) = Q w(0)$, this contradicts assumption (iii).

Now by the homotopy invariance of topological degree, we get

$$\deg\big(\mathrm{id} - H(\cdot, 1), \Omega_{p_0}, 0\big) = \deg\big(\mathrm{id} - H(\cdot, 0), \Omega_{p_0}, 0\big).$$

By assumption (iii), we see $\deg(\mathrm{id} - H(\cdot, 0), \Omega_{p_0}, 0) \neq 0$ which implies

$$\deg\big(\mathrm{id} - H(\cdot, 1), \Omega_{p_0}, 0\big) \neq 0.$$

Then there exists a $\varphi \in \Omega_{p_0}$, such that

$$\varphi(t) = x_{\varphi,1}(t), \quad \forall t \in [0, T],$$

which can be extended to a (Q, T)-affine-periodic solution of system (1.1). □

We give the asymptotically linear case as a corollary.

Consider the system

$$x' = A(t)x, \tag{3.5}$$

where $A(t) : \mathbf{R} \to \mathbf{R}^n$ is continuous and $A(t + T) = QA(t)Q^{-1}$ for every $t \in \mathbf{R}$.

Corollary 3.1 *Assume that*

$$\lim_{M \to \infty} \frac{1}{M} \sup_{|x| \le M} \int_0^T |f(t, x) - A(t)x| \, dt = 0,$$

and system (3.5) *has only trivial* (Q, T)*-affine-periodic solution. Then there exists at least one* (Q, T)*-affine-periodic solution of system* (1.1).

4 Homotopy method

To investigate the affine-periodic solutions of system (1.1), in this section we consider the following auxiliary equation:

$$x' = \lambda f(t, x) \tag{4.1}$$

with $\lambda \in [0, 1]$.

Theorem 4.1 *Consider the system* (1.1), *where* $f(t, x)$ *is continuous and locally Lipschitz continuous in the variable* x. *Assume there exists an open bounded subset* V *of* $\mathbf{R}^n$ *such that the following conditions hold*:

(i) *For every* $y \in \overline{V}$ *and* $\lambda \in (0, 1]$, *the solution* $x_\lambda(t, y)$ *of system* (4.1) *exists at least on* $[0, T]$.

(ii) *If* $y \in \partial V$, *then*

$$x_\lambda(T, y) \ne Qy, \quad \forall \lambda \in (0, 1].$$

(iii) $\text{Ker}(I - Q) \ne \{0\}$. *Denote*

$$g(y) = -\frac{1}{T} \int_0^T Pf(t, y) \, dt,$$

where $P : \mathbf{R}^n \to \text{Ker}(I - Q)$ *is the orthogonal projection. For every* $y \in \partial V \cap \text{Ker}(I - Q)$, $g(y) \ne 0$ *and*

$$\deg\big(g(\cdot), V \cap \text{Ker}(I - Q), 0\big) \ne 0.$$

Then there exists at least one (Q, T)*-affine-periodic solution of system* (1.1).

Proof Consider the homotopy operator $H : \overline{V} \times (0,1] \to \mathbf{R}^n$:

$$H(y,\lambda) = Py + \frac{1}{T} \int_0^T Pf\big(t, x_\lambda(t,y)\big)\, dt + \lambda L_p^{-1} \int_0^T (I-P)f\big(t, x_\lambda(t,y)\big)\, dt,$$

where $L_p^{-1} := (I-Q)|_{\mathrm{Im}(I-Q)}$. Now we prove that

$$0 \notin (\mathrm{id} - H)\big(\partial V \times [0,1]\big).$$

Suppose on the contrary that there exists a $(\tilde{y}, \tilde{\lambda}) \in \partial V \times [0,1]$, such that

$$(\mathrm{id} - H)(\tilde{y}, \tilde{\lambda}) = 0.$$

When $\tilde{\lambda} = 0$, one has

$$(I - P)\tilde{y} = 0, \tag{4.2}$$

which implies $\tilde{y} \in \mathrm{Ker}(I - Q)$, and

$$\frac{1}{T} \int_0^T Pf(t, \tilde{y})\, dt = 0. \tag{4.3}$$

This contradicts assumption (iii).
When $\tilde{\lambda} \in (0,1]$, one has

$$\frac{1}{T} \int_0^T Pf\big(t, x_{\tilde{\lambda}}(t, \tilde{y})\big)\, dt = 0 \tag{4.4}$$

and

$$(I - P)\tilde{y} - \tilde{\lambda} L_p^{-1} \int_0^T (I - P)f\big(t, x_{\tilde{\lambda}}(t, \tilde{y})\big)\, dt = 0.$$

Then

$$(I - Q)\tilde{y} = \tilde{\lambda} \int_0^T (I - P)f\big(t, x_{\tilde{\lambda}}(t, \tilde{y})\big)\, dt. \tag{4.5}$$

By (4.4) and (4.5), one has

$$x_{\tilde{\lambda}}(T, \tilde{y}) = Q\tilde{y},$$

which contradicts assumption (ii).
From the homotopy invariance of topological degree, one gets

$$\deg\big(\mathrm{id} - H(\cdot, 1), V, 0\big) = \deg\big(\mathrm{id} - H(\cdot, 0), V, 0\big).$$

Since $H(0, y) \in \mathrm{Ker}(I - Q)$, one has

$$
\begin{aligned}
\deg\big(\mathrm{id} - H(\cdot, 0), V, 0\big) &= \deg\big(\mathrm{id} - H(\cdot, 0), V \cap \mathrm{Ker}(I - Q), 0\big) \\
&= \deg\big(g(\cdot), V \cap \mathrm{Ker}(I - Q), 0\big) \\
&\neq 0.
\end{aligned}
$$

Then there exists a $\tilde{y} \in V$, such that

$$
H(\tilde{y}, 1) = \tilde{y},
$$

which implies $x(T, \tilde{y}) = Q\tilde{y}$.　　　□

5 Examples

Using the results in this paper, we can obtain periodic, anti-periodic, quasi-periodic or general affine-periodic solutions. In this section we give some examples to show this.

Example 5.1　Consider the system

$$
\begin{aligned}
x' &= \varepsilon^2 x(\sin t + 5) + \varepsilon^3 r_1(t, x, y, z, \varepsilon), \\
y' &= \varepsilon\big(e^y - 1\big) + \varepsilon^3 r_2(t, x, y, z, \varepsilon), \\
z' &= \varepsilon y^2 z \sin t + \varepsilon^2 \frac{e^t}{2 + \sin t} xy + \varepsilon^3 r_3(t, x, y, z, \varepsilon),
\end{aligned}
$$

where r_i $(i = 1, 2, 3)$ are continuous and locally Lipschitz in the variable (x, y, z). Denote

$$
Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi} \end{pmatrix},
$$

$$
f_1(t, x, y, z) = \big(0, e^y - 1, y^2 z \sin t\big)^\top,
$$

$$
f_2(t, x, y, z) = \left(x(\sin t + 5), 0, \frac{e^t}{2 + \sin t} xy \right)^\top,
$$

$$
r(t, x, y, z, \varepsilon) = \big(r_1(t, x, y, z, \varepsilon), r_2(t, x, y, z, \varepsilon), r_3(t, x, y, z, \varepsilon)\big)^\top.
$$

Then, for $i = 1, 2, f_i(t + 2\pi, x, y, z) = Q f_i(t, Q^{-1}(x, y, z)^\top)$. Suppose

$$
r(t + 2\pi, x, y, z, \varepsilon) = Q r\big(t, Q^{-1}(x, y, z)^\top, \varepsilon\big).
$$

Denote by V the unit sphere

$$
V = \big\{ (x, y, z)^\top \in \mathbf{R}^3 : x^2 + y^2 + z^2 < 1 \big\}.
$$

It is easy to see

$$
\mathrm{Ker}(I - Q) = \big\{ (x, y, 0)^\top ; (x, y) \in \mathbf{R}^2 \big\},
$$

$$\partial\left(V \cap \mathrm{Ker}(I - Q)\right) = \left\{(x, y, 0)^\top; x^2 + y^2 = 1\right\}.$$

Note that

$$Pf_1 = \left(0, e^y - 1, 0\right)^\top,$$

$$Pf_2 = \left(x(\sin t + 5), 0, 0\right)^\top.$$

Then, for $(x_0, y_0, 0) \in \partial(V \cap \mathrm{Ker}(I - Q))$, if (x, y, z) tends to $(x_0, y_0, 0)$, we have $|x| \geq \frac{1}{2}$ or $|y| \geq \frac{1}{2}$.

 If $|x| \geq \frac{1}{2}$, we have

$$\left|x(\sin t + 5)\right| \geq 2.$$

 If $|y| \geq \frac{1}{2}$, for $|\varepsilon| > 0$ sufficiently small, we have

$$\left|\frac{1}{\varepsilon}(e^y - 1)\right| \geq 1.$$

Denote

$$F(x, y, z, \varepsilon) = \varepsilon f_1(0, x, y, z) + \varepsilon^2 f_2(0, x, y, z).$$

Then

$$PF(x, y, z, \varepsilon)|_{\mathrm{Ker}(I-Q)} = \left(5\varepsilon^2 x, \varepsilon\left(e^y - 1\right), 0\right)^\top.$$

By simple calculations, we have

$$\deg\left(PF(\cdot, \varepsilon), V \cap \mathrm{Ker}(I - Q), 0\right) \neq 0.$$

By Corollary 2.1, when $|\varepsilon| > 0$ is small enough, the system has a (Q, T)-affine-periodic solution $(x(t), y(t), z(t))$, such that

$$\left(x(t + 2\pi), y(t + 2\pi), z(t + 2\pi)\right)^\top = Q\left(x(t), y(t), z(t)\right)^\top.$$

Clearly, it is unbounded.

Example 5.2 Consider the system

$$x' = \varepsilon x^3 + \varepsilon^2 x \cos t + \varepsilon^3\left(y^2 + z^2\right)^3,$$

$$y' = \varepsilon x \sin \frac{\sqrt{3}t}{2} + \varepsilon^2\left(y^2 + z^2\right)\cos \frac{\sqrt{3}t}{2},$$

$$z' = \varepsilon x \cos \frac{\sqrt{3}t}{2} - \varepsilon^2\left(y^2 + z^2\right)\sin \frac{\sqrt{3}t}{2}.$$

Denote

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\sqrt{3}\pi & \sin\sqrt{3}\pi \\ 0 & -\sin\sqrt{3}\pi & \cos\sqrt{3}\pi \end{pmatrix},$$

$$f_1(t,x,y,z) = \left(x^3, x\sin\frac{\sqrt{3}t}{2}, x\cos\frac{\sqrt{3}t}{2} \right)^\top,$$

$$f_2(t,x,y,z) = \left(x\cos t, (y^2+z^2)\cos\frac{\sqrt{3}t}{2}, -(y^2+z^2)\sin\frac{\sqrt{3}t}{2} \right)^\top,$$

$$r(t,x,y,z,\varepsilon) = \left((y^2+z^2)^3, 0, 0 \right)^\top.$$

Then, for $i = 1, 2$,

$$f_i(t + 2\pi, x, y, z) = Qf_i\left(t, Q^{-1}(x,y,z)^\top\right),$$

$$r(t + 2\pi, x, y, z, \varepsilon) = Qr\left(t, Q^{-1}(x,y,z)^\top, \varepsilon\right).$$

Denote by V the unit sphere

$$V = \left\{ (x,y,z)^\top \in \mathbf{R}^3 : x^2 + y^2 + z^2 < 1 \right\}.$$

Then we have

$$\text{Ker}(I - Q) = \left\{ (x, 0, 0)^\top ; x \in \mathbf{R} \right\},$$

$$\partial\left(V \cap \text{Ker}(I - Q) \right) = \left\{ (1,0,0)^\top, (-1,0,0)^\top \right\}.$$

Note that

$$Pf_1 = \left(x^3, 0, 0 \right)^\top,$$

$$Pf_2 = (x\cos t, 0, 0)^\top,$$

where $P : \mathbf{R}^3 \to \text{Ker}(I - Q)$ is the orthogonal projection. Obviously there exists a neighborhood U_p of $p \in \{(1,0,0)^\top, (-1,0,0)^\top\}$, such that, for $|\varepsilon| > 0$ small enough,

$$\left| \frac{1}{\varepsilon}x^3 + x\cos t \right| \geq 1, \quad \forall(x,y,z) \in U_p.$$

Let

$$F(x,y,z,\varepsilon) = \varepsilon f_1(0,x,y,z) + \varepsilon^2 f_2(0,x,y,z).$$

Then

$$PF(x,y,z,\varepsilon)|_{\text{Ker}(I-Q)} = \left(\varepsilon x^3 + \varepsilon^2 x, 0, 0 \right)^\top.$$

It is easy to see that

$$\deg\big(PF(\cdot,\varepsilon), V \cap \mathrm{Ker}(I-Q),0\big) \neq 0.$$

By Corollary 2.1, for $|\varepsilon| > 0$ small enough, the system has a quasi-periodic solution $(x(t), y(t), z(t))$ such that

$$\big(x(t+2\pi), y(t+2\pi), z(t+2\pi)\big)^{\top} = Q\big(x(t), y(t), z(t)\big)^{\top}.$$

Example 5.3 Consider the following system in $\mathbf{R}^n$:

$$x' = -|x|^{2\alpha}x + e(t)x + h(t),$$

where $\alpha > 0$ is a constant, $e : \mathbf{R} \to \mathbf{R}^{n\times n}$ and $h : \mathbf{R} \to \mathbf{R}^n$ are continuous. Moreover,

$$e(t+T) = Qe(t)Q^{-1}, \qquad h(t+T) = Qh(t), \quad \forall t \in \mathbf{R},$$

with $Q \in O(n)$. Since

$$\big\langle -|x|^{2\alpha}x + e(t)x + h(t), x \big\rangle < 0$$

for $(t,x) \in [0,T] \times \partial B_R$ with $R > 0$ large enough, we see that the vector field is inward to B_R. By Theorem 2.3, the system has a (Q,T)-affine-periodic solution.

Example 5.4 Consider the following system in $\mathbf{R}^n$:

$$x' = A(t)x + \big(1 + |x|^2\big)^{\alpha}x + g(t), \tag{5.1}$$

where $\alpha < 0$ is a constant, $A : \mathbf{R} \to \mathbf{R}^{n\times n}$ and $g : \mathbf{R} \to \mathbf{R}^n$ are continuous. Moreover,

$$A(t+T) = A(t), \qquad g(t+T) = -g(t), \quad \forall t \in \mathbf{R}.$$

Denote by $\Phi(t)$ the fundamental matrix solution of

$$y' = A(t)y, \tag{5.2}$$

such that $\Phi(0) = I$. We claim that if $\Phi(T) + I$ is invertible, the system (5.1) would have a T-anti-periodic solution. In fact, the system (5.1) is a $(-I, T)$-affine-periodic system. Since $\Phi(T) + I$ is invertible, the system (5.2) has only a trivial T-anti-periodic solution. It is easy to see

$$\lim_{M\to\infty} \frac{1}{M} \sup_{|x|\leq M} \int_0^T \big|\big(1 + |x|^2\big)^{\alpha}x + g(t)\big|\, dt = 0.$$

By Corollary 3.1, the system has a T-anti-periodic solution.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work described was an original research, which has not been published previously and is not under consideration for publication elsewhere. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, and Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun, P.R. China. [2]College of Mathematics, Jilin University, Changchun, P.R. China.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
 1. Aizicovici, S., McKibben, M., Reich, S.: Anti-periodic solutions to nonmonotone evolution equations with discontinuous nonlinearities. Nonlinear Anal. **43**(2), 233–251 (2001)
 2. Aizicovici, S., Pavel, N.: Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space. J. Funct. Anal. **99**(2), 387–408 (1991)
 3. Aizicovici, S., Reich, S.: Anti-periodic solutions to difference inclusions in Banach spaces. Dyn. Syst. Appl. **1**(2), 121–130 (1992)
 4. Aizicovici, S., Reich, S.: Anti-periodic solutions to a class of non-monotone evolution equations. Discrete Contin. Dyn. Syst. **5**(1), 35–42 (1999)
 5. Belley, J., Bondo, É.: Anti-periodic solutions of Liénard equations with state dependent impulses. J. Differ. Equ. **261**, 4164–4187 (2016)
 6. Benci, V., Fortunato, D.: Periodic solutions of asymptotically linear dynamical systems. NoDEA Nonlinear Differ. Equ. Appl. **1**, 267–280 (1994)
 7. Burton, T.A.: Stability and Periodic Solutions of Ordinary and Functional-Differential Equations. Mathematics in Science and Engineering, vol. 178. Academic Press, Orlando (1985)
 8. Chang, X., Li, Y.: Rotating periodic solutions of second order dissipative dynamical systems. Discrete Contin. Dyn. Syst. **36**, 643–652 (2016)
 9. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Applied Mathematical Sciences, vol. 42. Springer, New York (1983)
10. Guo, Y.: Nontrivial periodic solutions for asymptotically linear Hamiltonian systems with resonance. J. Differ. Equ. **175**, 71–87 (2001)
11. Hale, J.K., Mawhin, J.: Coincidence degree and periodic solutions of neutral equations. J. Differ. Equ. **15**, 295–307 (1974)
12. Hartman, P.: Ordinary Differential Equations. Birkhäuser, Boston (1982). Reprint of the second edition
13. Krasnosel'skii, M., Perov, A.: On a certain principle for the existence of bounded, periodic and almost periodic solutions of systems of ordinary differential equations. Dokl. Akad. Nauk SSSR **123**, 235–238 (1958)
14. Krasnosel'skii, M., Perov, A.: Some Criteria for Existence of Periodic Solutions of a System of Ordinary Differential Equations. Izdat Akad. Nauk Ukr. SSR, Kiev (1963)
15. Lakshmikantham, V.: Notes on a variety of problems of differential systems. Arch. Ration. Mech. Anal. **10**, 119–126 (1962)
16. Li, Y., Huang, F.: Levinson's problem on affine-periodic solutions. Adv. Nonlinear Stud. **15**(1), 241–252 (2015)
17. Li, Y., Wang, H., Lü, X., Lu, X.: Periodic solutions for functional-differential equations with infinite lead and delay. Appl. Math. Comput. **7**(1), 1–28 (1995)
18. Liu, G., Li, Y., Yang, X.: Rotating periodic solutions for asymptotically linear second-order Hamiltonian systems with resonance at infinity. Math. Methods Appl. Sci. **40**(18), 7139–7150 (2017)
19. Liu, G., Li, Y., Yang, X.: Existence and multiplicity of rotating periodic solutions for resonant Hamiltonian systems. J. Differ. Equ. **265**, 1324–1352 (2018)
20. Liu, Z.: Anti-periodic solutions to nonlinear evolution equations. J. Funct. Anal. **258**, 2026–2033 (2010)
21. Mawhin, J.: Periodic solutions of nonlinear functional differential equations. J. Differ. Equ. **10**, 240–261 (1971)
22. Mawhin, J., Gaines, R.: Coincidence Degree and Nonlinear Differential Equations. Springer, Berlin (1977)
23. Wang, C., Yang, X., Li, Y.: Affine-periodic solutions for nonlinear differential equations. Rocky Mt. J. Math. **46**(5), 1717–1737 (2016)
24. Wang, H., Yang, X., Li, Y., Li, X.: LaSalle type stationary oscillation theorems for affine-periodic systems. Discrete Contin. Dyn. Syst., Ser. B **22**, 2907–2921 (2017)
25. Wang, K., Li, Y.: A note on existence of (anti-)periodic and heteroclinic solutions for a class of second-order odes. Nonlinear Anal. **70**(4), 1711–1724 (2009)

26. Wiggins, S.: Introduction to Applied Nonlinear Dynamical Systems and Chaos. Texts in Applied Mathematics, vol. 2. Springer, New York (1990)
27. Wu, R., Cong, F., Li, Y.: Anti-periodic solutions for second order differential equations. Appl. Math. Lett. **24**(6), 860–863 (2011)
28. Xing, J., Yang, X., Li, Y.: Affine-periodic solutions by averaging methods. Sci. China Math. **61**(3), 439–452 (2018)
29. Yoshizawa, T.: Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions. Applied Mathematical Sciences, vol. 14. Springer, New York (1975)