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Stability result of a viscoelastic plate equation with past history and a logarithmic nonlinearity

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Abstract

In this paper, we are concerned with the decay rate of the solution of a viscoelastic plate equation with infinite memory and logarithmic nonlinearity. We establish an explicit and general decay rate results with imposing a minimal condition on the relaxation function. In fact, we assume that the relaxation function h satisfies

$$h'(t) \leq -\xi(t)H(h(t)), \quad t \geq 0,$$

where the functions ξ and H satisfy some conditions. Our proof is based on the multiplier method, convex properties, logarithmic inequalities, and some properties of integro-differential equations. Moreover, we drop the boundedness assumption on the history data, usually made in the literature. In fact, our results generalize, extend, and improve earlier results in the literature.

MSC: 35B35; 35L55; 75D05; 74D10; 93D20

Keywords: Stability; Logarithmic Sobolev inequalities; Infinite memory; Plate equation; Convexity

1 Introduction

In this work, we consider the following viscoelastic plate problem with velocity-dependent material density and logarithmic nonlinearity:

$$|u_t|^\rho u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^{+\infty} h(s)\Delta^2 u(t-s)\,ds = \alpha u \ln|u| \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

equipped with initial and boundary conditions

$$\begin{aligned} &u(x,t) = \frac{\partial u}{\partial n}(x,t) = 0 \quad \text{in } \partial\Omega \times (0,\infty), \\ &u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \end{aligned} \tag{2}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, n is the unit outer normal to $\partial\Omega$, and ρ and α are positive constants. The relaxation function h satisfies the following general condition:

$$h'(t) \leq -\xi(t)H\big(h(t)\big), \tag{3}$$

where the functions ξ and H satisfy some conditions specified later. To motivate our work, let us recall some results regarding problems with logarithmic nonlinearity.

1.1 Problems with logarithmic nonlinearity

The logarithmic nonlinearity has many applications in physics such as nuclear physics, optics, and geophysics [1–3]. For the problems with logarithmic nonlinearity, we start with the works of Birula and Mycielski [4] and [5], where they proved that the wave equations with logarithmic nonlinearity have stable and localized solutions. Cazenave and Haraux [6] considered the Cauchy problem

$$u_{tt} - \Delta u = u \ln |u|^{\alpha} \tag{4}$$

in $\mathbb{R}^3$ and established the existence and uniqueness of the solution. The corresponding one-dimensional problem of (4) was studied by Gorka [1], who established the global existence of weak solutions, provided that $(u_0, u_1) \in H_0^1 \times L^2$. Bartkowski and Gorka [2] investigated weak solutions and also proved the existence of classical solutions. Hiramatsu et al. [3] considered the problem

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u| \tag{5}$$

and investigated numerical solutions of this problem without theoretical analysis. Recently, Al-Gharabli et al. [7] considered the problem

$$u_{tt} + \Delta^2 u + u - \int_0^t h(t-s)\Delta^2 u(s)\,ds = \alpha u \ln |u| \quad \text{in } \Omega \times (0,\infty) \tag{6}$$

and proved existence and decay results of the solutions under the following condition on the relaxation function:

$$h'(t) \leq -\xi(t)h^p(t), \quad 1 \leq p < \frac{3}{2}. \tag{7}$$

Al-Gharabli et al. [8] considered the problem

$$|u_t|^{\rho} u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds = \alpha u \ln |u| \quad \text{in } \Omega \times (0,\infty)$$

and as in [7] proved the existence and decay results for the solutions with imposing the same condition (7). Very recently, Al-Gharabli [9] considered the same problem (6) and established a general decay result for which the relaxation function h satisfies $h'(t) \leq -\xi(t)H(h(t))$. For more results on some problems with logarithmic nonlinearity, we refer to the recent works [10–14].

1.2 Problems with infinite memory

Giorgi et al. [15] considered the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)\,ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}_+, \tag{8}$$

with $K(0), K(+\infty) > 0$ and $K' \leq 0$ and proved the existence of global attractors for the solutions. Conti and Pata [16] considered the following semilinear hyperbolic equation:

$$u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)\,ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}_+, \tag{9}$$

where the memory kernel is a convex decreasing smooth function such that $K(0) > K(+\infty) > 0$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonlinear term of at most cubic growth satisfying some conditions. They proved the existence of a regular global attractor. Appleby et al. [17] studied the linear integro-differential equation

$$u_{tt} + Au(t) + \int_{-\infty}^{t} K(t-s)Au(s)\,ds = 0 \quad \text{for } t > 0 \tag{10}$$

and established an exponential decay result for strong solutions in a Hilbert space. Pata [18] discussed the decay properties of the semigroup generated by the following equation:

$$u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} \mu(s)Au(t-s)\,ds = 0 \quad \text{for } t > 0, \tag{11}$$

where A is a strictly positive self-adjoint linear operator, $\alpha > 0$, $\beta \geq 0$, and the memory kernel μ is a decreasing function satisfying specific conditions. Subsequently, they established necessary and sufficient conditions for the exponential stability. Guesmia [19] considered the equation

$$u_{tt} + Au - \int_0^{+\infty} h(s)Bu(t-s)\,ds = 0 \quad \text{for } t > 0 \tag{12}$$

and introduced a new ingenuous approach for proving a more general decay result based on the properties of convex functions and the generalized Young inequality. He used a larger class of infinite history kernels satisfying the condition

$$\int_0^{+\infty} \frac{h(s)}{H^{-1}(-h'(s))}\,ds + \sup_{s\in\mathbb{R}_+} \frac{h(s)}{H^{-1}(-h'(s))} < +\infty \tag{13}$$

with

$$H(0) = H'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} H'(t) = +\infty, \tag{14}$$

where $H : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing strictly convex function. Using this approach, Guesmia and Messaoudi [20] later considered the equation

$$u_{tt} - \Delta u + \int_0^{t} h_1(t-s)div\big(a_1(x)\nabla u(s)\big)\,ds + \int_0^{+\infty} h_2(s)div\big(a_2(x)\nabla u(t-s)\big)\,ds = 0$$

in a bounded domain under suitable conditions on a_1 and a_2 for a wide class of relaxation functions h_1 and h_2, which are not necessarily decaying polynomially or exponentially, and established a general decay result such that the usual exponential and polynomial decay rates are only particular cases. Messaoudi and Al-Gharabli [7] considered the nonlinear wave equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^{+\infty} h(s)\Delta u(t-s)\,ds = 0 \quad \text{in } \Omega \times (0,+\infty),$$

in which the relaxation function g satisfies

$$h'(t) \le -\xi(t)h(t), \quad t \ge 0, \tag{15}$$

and they proved a general decay result on the solution energy using an approach different from that introduced by Guesmia [19]. Recently, Al-Mahdi and Al-Gharabli [21] considered the viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} h(s)\Delta u(t-s)\,ds + |u_t|^{m-2}u_t = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty), \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x) & \text{in } \Omega \times (0,+\infty), \end{cases} \tag{16}$$

established decay results in which the relaxation function h satisfies

$$h'(t) \le -\xi(t)h^p(t), \quad t \ge 0, 1 \le p < \frac{3}{2}, \tag{17}$$

and obtained a better decay rate than that in [19] and [22]. For more results on problems with infinite memory and finite memory, we refer the reader to [23–27]. Motivated by all these works, we intend to establish a three-fold objective:

(a) To extend many earlier works for the wave equations such as those discussed in [1, 3, 7, 28–30] to the plate equation with logarithmic nonlinearity.

(b) To extend some general decay results, known for the case of finite history, to the case of infinite history where the relaxation function satisfies a wider class of relaxation functions instead of those considered in [7, 8, 12, 19, 21, 29, 31].

(c) To drop the boundedness assumptions on the history data considered in many earlier results in [7, 19, 21].

We obtain our results by using the multiplier method with some logarithmic inequalities and some properties of integro-differential equations and inequalities. Our decay result is based on ξ, H, and α. This paper is organized as follows. In Sect. 2, we present some notations, assumptions, and a local and global existence result of our problem. In Sect. 3, we establish some lemmas needed in the proof of our result. Stability results with an example are presented in Sect. 4. Some conclusions are given in Sect. 5.

2 Preliminaries

In this section, we introduce our assumptions and give some useful lemmas. We use c to denote a positive generic constant.

(*A*1) $h:\mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 nonincreasing function satisfying, for some $\beta_0 > 0$,

$$-\beta_0 h(s) \le h'(s), \qquad h(t) > 0 \quad \text{and} \quad 1 - \int_0^{+\infty} h(s)\,ds := \ell > 0, \tag{18}$$

(*A*2) $H:(0,\infty) \to (0,\infty)$ is a function in $C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*)$ that is increasing and strictly convex, with $H(0) = H'(0) = 0$ and $\lim_{s\to+\infty} H'(s) = +\infty$, $s \mapsto sH'(s)$ and $s \mapsto s(H')^{-1}(s)$ are convex on $(0,r]$, and there exists a nonincreasing function $\xi:\mathbb{R}_+ \to \mathbb{R}_+$ such that

$$h'(t) \le -\xi(t) H\big(h(t)\big), \quad t \ge 0. \tag{19}$$

(*A*3) The constant α in (1) is such that $0 < \alpha < \alpha_0 = \frac{2\pi\ell e^3}{c_p}$, where c_p is the smallest positive number satisfying $\|\nabla u\|_2^2 \le c_p \|\Delta u\|_2^2$ for $u \in H_0^2(\Omega)$, where $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$.

Remark 2.1 Assumption (*A*3) is needed for establishing the local existence of the solutions of problem (1). For more details, we refer to [8].

Remark 2.2 If H is a strictly increasing and strictly convex C^2 function on $(0,r]$ with $H(0) = H'(0) = 0$, then it has an extension $\overline{H}$ that is strictly increasing and strictly convex C^2 function on $(0,+\infty)$. For instance, if $H(r) = a$, $H'(r) = b$, and $H''(r) = C$, we can define $\overline{H}$ for $t > r$ by

$$\overline{H}(t) = \frac{C}{2}t^2 + (b - Cr)t + \left(a + \frac{C}{2}r^2 - br\right). \tag{20}$$

For simplicity, in the rest of this paper, we use H instead of $\overline{H}$.

Remark 2.3 Since H is strictly convex on $(0,r]$ and $H(0) = 0$, then

$$H(\theta t) \le \theta H(t), \quad 0 \le \theta \le 1 \text{ and } t \in (0,r]. \tag{21}$$

Remark 2.4 The function $g(s) = \sqrt{\frac{2\pi\ell}{c_p s}} - e^{-\frac{3}{2}}$ is a continuous decreasing function on $(0,\infty)$ with

$$\lim_{s\to 0^+} g(s) = \infty \quad \text{and} \quad \lim_{x\to\infty} g(x) = -e^{-\frac{3}{2}}.$$

Then there exists a unique $\alpha_0 > 0$ such that $g(\alpha_0) = 0$. Moreover,

$$e^{-\frac{3}{2}} < \sqrt{\frac{2\pi\ell}{c_p s}}, \quad s \in (0,\alpha_0), \tag{22}$$

which implies that the selection of α in (*A*3) is possible.

The modified energy functional associated with problem (1)–(2) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{\ell}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\Delta u_t\|_2^2 - \frac{\alpha}{2}\int_\Omega u^2 \ln|u|\,dx \\ & + \frac{\alpha}{4}\|u\|_2^2 + \frac{1}{2}(h \circ \Delta u), \end{aligned} \tag{23}$$

where

$$(h \circ \Delta u)(t) = \int_0^{+\infty} h(s) \big\| \Delta u(s) - \Delta u(t-s) \big\|_2^2 \, ds.$$

Direct differentiation of (23) using (1)–(2) leads to

$$E'(t) = \frac{1}{2} \big(h' \circ \Delta u\big)(t) \le 0. \tag{24}$$

Lemma 2.1 ([32, 33] (Logarithmic Sobolev inequality)) *Let u be any function in $H_0^1(\Omega)$, and let a be any positive real number. Then*

$$\int_\Omega u^2 \ln |u| \, dx \le \frac{1}{2} \|u\|_2^2 \ln \|u\|_2^2 + \frac{a^2}{2\pi} \|\nabla u\|_2^2 - (1 + \ln a) \|u\|_2^2. \tag{25}$$

Corollary 2.1 *Let u be any function in $H_0^2(\Omega)$, and let a be any positive real number. Then*

$$\int_\Omega u^2 \ln |u| \, dx \le \frac{1}{2} \|u\|_2^2 \ln \|u\|_2^2 + \frac{c_p a^2}{2\pi} \|\Delta u\|_2^2 - (1 + \ln a) \|u\|_2^2. \tag{26}$$

Lemma 2.2 *Let $\varepsilon_0 \in (0,1)$. Then there exists $d_{\varepsilon_0} > 0$ such that*

$$s |\ln s| \le s^2 + d_{\varepsilon_0} s^{1-\varepsilon_0}, \quad s > 0. \tag{27}$$

Proof Let $f(s) = s^{\varepsilon_0}(|\ln s| - s)$. Note that f is continuous on $(0,\infty)$, its limit at 0^+ is 0^+, and its limit at ∞ is $-\infty$. Then f has a maximum d_{ε_0} on $(0,\infty)$, so (27) holds. □

2.1 Existence results

In this subsection, we state without proof a local existence result of our problem (1)–(2).

Theorem 2.1 *Let $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$. Assume that* (A1)–(A3) *hold and*

$$e^{-\frac{3}{2}} < a < \sqrt{\frac{2\pi \ell}{\alpha c_p}}. \tag{28}$$

Then problem (1)–(2) *has a weak solution on* $[0,T]$.

The proof of Theorem 2.1 can be obtained by following the same arguments as in [8] and adapting the finite history to the infinite case. For the global existence, we have the following:

Theorem 2.2 *Assume that* (A1)–(A3) *hold. Let $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$ be such that*

$$I(0) > 0 \quad \text{and} \quad \sqrt{54} \alpha c_*^3 \left(\frac{E(0)}{\ell} \right)^{\frac{1}{2}} < \ell, \tag{29}$$

where c_*^3 is a positive embedding constant. Then we have:

(i)

$$I(t) > 0, \quad t \in [0, T). \tag{30}$$

(ii) *Problem* (1)–(2) *has a global weak solution,*
where

$$I(t) := \ell \|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 + (h \circ \Delta u)(t) - 3\alpha \int_\Omega u^2 \ln |u| \, dx \tag{31}$$

and

$$J(t) := \frac{1}{3}\left[\ell \|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 + g \circ \Delta u\right] + \frac{k}{4}\|u\|_2^2 + \frac{1}{6} I(t). \tag{32}$$

The proof of Theorem 2.2 can be obtained by following the same arguments as in [8] by adapting the finite memory to infinite memory.

3 Technical lemmas

In this section, we start by establishing several lemmas needed for the proof of our main result.

Lemma 3.1 *There exists a positive constant* M_1 *such that*

$$\int_t^\infty h(s)\big(\triangle u(t) - \triangle u(t-s)\big)^2 \, ds \, dx \le M_1 h_1(t), \tag{33}$$

where $h_1(t) := \int_0^{+\infty} h(t+s)(1 + \|\triangle u_0(s)\|^2) \, ds$.

Proof The proof is based on some arguments in [30]. In fact, we have

$$\begin{aligned}
&\int_t^{+\infty} h(s) \big\| \triangle u(t) - \triangle u(t-s) \big\|^2 \, ds \\
&\quad \le 2 \big\| \triangle u(t) \big\|^2 \int_t^{+\infty} h(s) \, ds + 2 \int_t^{+\infty} h(s) \big\| \triangle u(t-s) \big\|^2 \, ds \\
&\quad \le 2 \sup_{s \ge 0} \big\| \triangle u(s) \big\|^2 \int_0^{+\infty} h(t+s) \, ds + 2 \int_0^{+\infty} g(t+s) \big\| \triangle u(-s) \big\|^2 \, ds \\
&\quad \le \left(\frac{4}{\ell} E(s) \right) \int_0^\infty h(t+s) \, ds + 2 \int_0^\infty h(t+s) \big\| \triangle u_0(s) \big\|^2 \, ds \\
&\quad \le \left(\frac{4}{\ell} E(0) \right) \int_0^{+\infty} h(t+s) \, ds + 2 \int_0^{+\infty} h(t+s) \big\| \triangle u_0(s) \big\|^2 \, ds \\
&\quad \le M_1 \int_0^{+\infty} h(t+s) \big(1 + \big\| \triangle u_0(s) \big\|^2 \big) \, ds,
\end{aligned} \tag{34}$$

where $M_1 = \max\{2, \frac{4E(0)}{\ell}\}$. □

Lemma 3.2 *Assume that h satisfies* $(A1)$. *Then, for* $u \in H_0^2(\Omega)$,

$$\int_\Omega \left(\int_0^{+\infty} h(s)\big(u(t)-u(t-s)\big)\,ds\right)^2 dx \le c(h \circ \Delta u)(t),$$
$$\int_\Omega \left(\int_0^{+\infty} h'(s)\big(u(t)-u(t-s)\big)\,ds\right)^2 dx \le -c\big(h' \circ \Delta u\big)(t).$$

Proof The proof can be easily obtained by applying the Cauchy–Schwarz and Poincaré inequalities. □

Lemma 3.3 *Assume that* $(A1)$–$(A3)$ *and* (29) *hold. Then the functionals*

$$\psi(t) := \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx + \int_\Omega \Delta u \Delta u_t\,dx,$$
$$\chi(t) := -\int_\Omega \left(\Delta^2 u_t + \frac{1}{\rho+1}|u_t|^\rho u_t\right)\int_0^{+\infty} h(s)\big(u(t)-u(t-s)\big)\,ds\,dx,$$

satisfy, along the solutions of (1)–(2), *the following estimates for any* $\delta, \delta_1, \delta_2 > 0$ *and* $\varepsilon_0 \in (0,1)$:

$$\psi'(t) \le -\frac{\ell}{2}\int_\Omega |\Delta u|^2\,dx + \int_\Omega |\Delta u_t|^2\,dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}\,dx + c(h \circ \Delta u)(t) + \alpha\int_\Omega u^2 \ln|u|\,dx, \tag{35}$$

$$\begin{aligned}\chi'(t) \le{}& \left[\big(1+2(1-\ell)^2\big)\delta_1 + \frac{\delta}{4}\right]\int_\Omega |\Delta u|^2\,dx - \frac{(1-\ell)}{\rho+1}\int_\Omega |u_t|^{\rho+2}\,dx \\ &+ c\left(\delta_1 + \frac{1}{\delta_1} + \frac{1}{\delta}\right)(h \circ \Delta u)(t) - \frac{c}{\delta_2}\big(h' \circ \nabla u\big)(t) \\ &+ \big[\delta_2 + c\delta_2\big(E(0)\big)^\rho - (1-\ell)\big]\int_\Omega |\Delta u_t|^2\,dx + c_{\varepsilon_0,\delta}(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t).\end{aligned} \tag{36}$$

Proof The proof of Lemma 3.3 is similar to that in [8] with some adjustments according to the infinite memory case. □

Lemma 3.4 *Assume that* $(A1)$–$(A3)$ *and* (29) *hold and let* $\varepsilon_0 \in (0,1)$. *Assume that*

$$0 < E(0) < \frac{e\ell\pi}{4c_p}. \tag{37}$$

Then, for α *small enough, there exist positive constants* ε *and* N *such that the functional*

$$L := NE + \varepsilon\psi + \chi$$

satisfies

$$L \sim E, \tag{38}$$

and, for any $t \geq 0$, there exists a positive constant m such that

$$L'(t) \leq -mE(t) + c(h \circ \Delta u)(t) + c_{\varepsilon_0}(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t). \tag{39}$$

Proof For the proof of (38), we refer to [8]. To prove (39), we let $\int_0^{+\infty} h(s)\,ds =: h_0$ and using (24), (35), and (36), for $t \geq 0$, we have

$$\begin{aligned}
L'(t) \leq{} & \left(\frac{N}{2} - \frac{c}{\delta_2}\right)\left(h' \circ \Delta u\right)(t) - \frac{h_0 - \varepsilon}{\rho + 1}\int_\Omega |u_t|^{\rho+2}\,dx \\
& - \left[\varepsilon\frac{\ell}{2} - \left(1 + 2(1-\ell)^2\right)\delta_1 - \frac{\delta}{4}\right]\|\Delta u\|_2^2 \\
& - \left[h_0 - \varepsilon - \delta_2 - c\delta_2\left(E(0)\right)^\rho\right]\|\Delta u_t\|_2^2 \\
& + c\left(\varepsilon + \delta_1 + \frac{1}{\delta_1} + \frac{1}{\delta}\right)(h \circ \Delta u)(t) \\
& + c_{\varepsilon_0,\delta}(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t) + \varepsilon\alpha\int_\Omega u^2 \ln|u|\,dx.
\end{aligned} \tag{40}$$

Using the definition of $E(t)$, we obtain, for any $m > 0$,

$$\begin{aligned}
L'(t) \leq{} & -mE(t) + \left(\frac{N}{2} - \frac{c}{\delta_2}\right)\left(h' \circ \Delta u\right)(t) - \left(\frac{h_0 - \varepsilon}{\rho + 1} - \frac{m}{\rho + 2}\right)\int_\Omega |u_t|^{\rho+2}\,dx \\
& - \left[\varepsilon\frac{\ell}{2} - \left(1 + 2(1-\ell)^2\right)\delta_1 - \frac{\delta}{4} - \frac{m(1-h_0)}{2}\right]\|\Delta u\|_2^2 \\
& - \left[h_0 - \varepsilon - \delta_2 - c\delta_2\left(E(0)\right)^\rho - \frac{m}{2}\right]\|\Delta u_t\|_2^2 \\
& + \left[c\left(\varepsilon + \delta_1 + \frac{1}{\delta_1} + \frac{1}{\delta}\right) + \frac{m}{2}\right](h \circ \Delta u)(t) \\
& + c_{\varepsilon_0,\delta}(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t) + \frac{m\alpha}{4}\|u\|_2^2 \\
& + \left(\varepsilon - \frac{m}{2}\right)\alpha\int_\Omega u^2 \ln|u|\,dx.
\end{aligned} \tag{41}$$

Using the logarithmic Sobolev inequality (26), we get, for $0 < m < 2\varepsilon$,

$$\begin{aligned}
L'(t) \leq{} & -mE(t) + \left[\frac{N}{2} - \frac{c}{\delta_2}\right]\left(h' \circ \Delta u\right)(t) - \left(\frac{h_0 - \varepsilon}{\rho + 1} - \frac{m}{\rho + 2}\right)\int_\Omega |u_t|^{\rho+2}\,dx \\
& - \left[\varepsilon\frac{\ell}{2} - \left(1 + 2(1-\ell)^2\right)\delta_1 - \frac{\delta}{4} - \frac{m(1-h_0)}{2} - \left(\varepsilon - \frac{m}{2}\right)\frac{\alpha c_p a^2}{2\pi}\right]\|\Delta u\|_2^2 \\
& - \left(h_0 - \varepsilon - \delta_2 - c\delta_2\left(E(0)\right)^\rho - \frac{m}{2}\right)\|\Delta u_t\|_2^2 \\
& + \left[c\left(\varepsilon + \delta_1 + \frac{1}{\delta_1} + \frac{1}{\delta}\right) + \frac{m}{2}\right](h \circ \Delta u)(t) + c_{\varepsilon_0,\delta}(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t) \\
& - \left(\varepsilon - \frac{m}{2}\right)\frac{\alpha}{2}\left(2(1 + \ln a) - \ln\|u\|_2^2\right)\|u\|_2^2 + \frac{m\alpha}{4}\|u\|_2^2.
\end{aligned} \tag{42}$$

At this point, we carefully choose our constant. First, we pick $0 < \varepsilon < h_0$. Then for δ_1, δ_2, and δ small enough, we have

$$k_1 := \varepsilon \frac{\ell}{2} - \bigl(1 + 2(1-\ell)^2\bigr)\delta_1 - \frac{\delta}{4} > 0$$

and

$$k_2 := h_0 - \varepsilon - \delta_2 - c\delta_2\bigl(E(0)\bigr)^{\rho} > 0.$$

Then, for N sufficiently large,

$$N > c(1+\varepsilon) \quad \text{and} \quad \frac{N}{2} - \frac{c}{\delta_2} \geq 0.$$

Consequently, we get

$$\begin{aligned} L'(t) \leq{}& -mE(t) - \left(\frac{h_0 - \varepsilon}{\rho+1} - \frac{m}{\rho+2}\right) \int_\Omega |u_t|^{\rho+2}\,dx \\ &- \left[k_1 - \frac{m(1-h_0)}{2} - \left(\varepsilon - \frac{m}{2}\right)\frac{\alpha c_p a^2}{2\pi}\right] \|\Delta u\|_2^2 \\ &- \left(k_2 - \frac{m}{2}\right)\|\Delta u_t\|_2^2 + \left(c + \frac{m}{2}\right)(h \circ \Delta u)(t) \\ &+ c_{\varepsilon_0}(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t) + \frac{m\alpha}{4}\|u\|_2^2 \\ &- \left(\varepsilon - \frac{m}{2}\right)\frac{\alpha}{2}\bigl(2(1+\ln a) - \ln \|u\|_2^2\bigr)\|u\|_2^2. \end{aligned} \tag{43}$$

Finally, we choose m and α small enough so that $m \leq \varepsilon$ (so $\frac{m\alpha}{4} \leq (\varepsilon - \frac{m}{2})\frac{\alpha}{2}$),

$$\begin{aligned} &\frac{h_0 - \varepsilon}{\rho+1} - \frac{m}{\rho+2} > 0, \\ &k_1 - \frac{m(1-h_0)}{2} - \left(\varepsilon - \frac{m}{2}\right)\frac{\alpha c_p a^2}{2\pi} > 0, \end{aligned}$$

and

$$k_2 - \frac{m}{2} > 0,$$

and we get

$$\begin{aligned} L'(t) \leq{}& -mE(t) + c(h \circ \Delta u)(t) + c_{\varepsilon_0}(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t) \\ &- \left(\varepsilon - \frac{m}{2}\right)\frac{\alpha}{2}\bigl(1 + 2\ln a - \ln \|u\|_2^2\bigr)\|u\|_2^2. \end{aligned} \tag{44}$$

Using (23), (24), (30), (31), (32), and (37), we have

$$\ln \|u\|_2^2 \leq \ln\left(\frac{4}{\alpha}J(t)\right) \leq \ln\left(\frac{4}{\alpha}E(t)\right) \leq \ln\left(\frac{4}{\alpha}E(0)\right) \leq \ln\left(\frac{e\ell\pi}{\alpha c_p}\right). \tag{45}$$

By choosing a satisfying

$$\max\left\{e^{-\frac{3}{2}}, \sqrt{\frac{\ell\pi}{\alpha c_p}}\right\} < a < \sqrt{\frac{2\ell\pi}{\alpha c_p}} \tag{46}$$

we achieve that (28) is satisfied. This selection gives a guarantee that

$$1 + 2\ln a - \ln\|u\|_2^2 \geq 0,$$

which completes the proof of (39). □

Remark 3.1 Recalling (23), (24), (30), and (32), we have

$$E(0) \geq E(t) = J(t) + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} \geq J(t) \geq \frac{1}{3}(h \circ \Delta u)(t),$$

which gives

$$(h \circ \Delta u)(t) \leq 3E(0). \tag{47}$$

Using (47), for any $\varepsilon_0 \in (0,1)$, we obtain that

$$\begin{aligned}(h \circ \Delta u)(t) &= (h \circ \Delta u)^{\frac{\varepsilon_0}{1+\varepsilon_0}}(t)(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t)\\ &\leq c(h \circ \Delta u)^{\frac{1}{1+\varepsilon_0}}(t).\end{aligned} \tag{48}$$

Lemma 3.5 *If* (*A*1)–(*A*2) *are satisfied, then we have, for all* $t > 0$, *the estimate*

$$\int_0^t h(s)\left\|\Delta u(t) - \Delta u(t-s)\right\|_2^2 ds \leq \frac{t+1}{q_0}H^{-1}\left(\frac{q_0\mu(t+1)}{t\xi(t)}\right), \tag{49}$$

where $q_0 > 0$ *is small enough,* H *is defined in Remark* 2.2, *and*

$$\mu(t) := -\int_0^t h'(s)\left\|\Delta u(t) - \Delta u(t-s)\right\|_2^2 ds \leq -cE'(t). \tag{50}$$

Proof To establish (49), we introduce the functional

$$\lambda(t) := \frac{q_0}{t+1}\int_0^t \left\|\Delta u(t) - \Delta u(t-s)\right\|_2^2 ds. \tag{51}$$

Then since E is nonincreasing, by (23) we get

$$\begin{aligned}\lambda(t) &\le \frac{2q_0}{t+1}\left(\int_0^t \|\Delta u(t)\|_2^2 + \int_0^t \|\Delta u(t-s)\|_2^2\,ds\right)\\ &\le \frac{4q_0}{\ell(t+1)}\left(\int_0^t \big(E(t)+E(t-s)\big)\,ds\right)\\ &\le \frac{8q_0}{\ell(t+1)}\int_0^t E(s)\,ds\\ &\le \frac{8q_0}{\ell(t+1)}\int_0^t E(0)\,ds\\ &< +\infty.\end{aligned} \tag{52}$$

Thus q_0 can be chosen so small so that, for all $t>0$,

$$\lambda(t) < 1. \tag{53}$$

Without loss of generality, for all $t>0$, we assume that $\lambda(t)>0$; otherwise, we get an exponential decay from (39). Using Jensen's inequality, (2.3), (50), and (53) gives

$$\begin{aligned}\mu(t) &= \frac{1}{q_0\lambda(t)}\int_0^t \lambda(t)\big(-h'(s)\big)\int_\Omega q_0\big|\Delta u(t)-\Delta u(t-s)\big|^2\,dx\,ds\\ &\ge \frac{1}{q_0\lambda(t)}\int_0^t \lambda(t)\xi(s)H\big(h(s)\big)\int_\Omega q_0\big|\Delta u(t)-\Delta u(t-s)\big|^2\,dx\,ds\\ &\ge \frac{\xi(t)}{q_0\lambda(t)}\int_0^t H\big(\lambda(t)h(s)\big)\int_\Omega q_0\big|\Delta u(t)-\Delta u(t-s)\big|^2\,dx\,ds\\ &\ge \frac{(t+1)\xi(t)}{q_0}H\left(\frac{q_0}{(t+1)}\int_0^t h(s)\int_\Omega \big|\Delta u(t)-\Delta u(t-s)\big|^2\,dx\,ds\right)\\ &= \frac{(t+1)\xi(t)}{q_0}H\left(\frac{q_0}{(t+1)}\int_0^t h(s)\int_\Omega \big|\Delta u(t)-\Delta u(t-s)\big|^2\,dx\,ds\right),\end{aligned} \tag{54}$$

and hence (49) is established. □

4 Decay result

In this section, we state and prove our main result and provide an example to illustrate our decay results. Let us start introducing some functions and then establishing several lemmas needed for the proof of our main result. As in [30], we introduce the following functions:

$$G_1(t) := \int_t^1 \frac{1}{sG'(s)}\,ds, \tag{55}$$

$$G_2(t) = tG'(t), \qquad G_3(t) = t\big(G'\big)^{-1}(t), \qquad G_4(t) = G_3^*(t), \tag{56}$$

where $G^{-1}(t) = (H^{-1}(t))^{\frac{1}{1+\varepsilon_0}}$ and $\varepsilon_0 \in (0,1)$. Further, we introduce the class S of functions $\chi : \mathbb{R}_+ \to \mathbb{R}_+^*$ satisfying, for fixed $c_1, c_2 > 0$ (should be selected carefully in (76)),

$$\chi \in C^1(\mathbb{R}_+), \quad \chi \le 1, \chi' \le 0, \tag{57}$$

and

$$c_2 G_4\left[\frac{c}{d}q(t)h_0(t)\right] \leq c_1\left(G_2\left(\frac{G_5(s)}{\chi(s)}\right) - \frac{G_2(G_5(t))}{\chi(t)}\right), \tag{58}$$

where $d > 0$, c is a generic positive constant that may change from line to line, h_2 and q will be defined later in the proof of our main result, and

$$G_5(t) = G_1^{-1}\left(c_1\int_0^t \xi(s)\,ds\right). \tag{59}$$

Remark 4.1 According to the properties of H introduced in $(A2)$ and the definition of G, we can see that $G' > 0$ and $G'' > 0$ on $(0, r]$, G_2 is convex increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, G_1 is decreasing and defines a bijection from $(0, 1]$ to $\mathbb{R}_+$, and G_3 and G_4 are convex increasing functions on $(0, r]$. Then the set S is not empty because it contains $\chi(s) = \varepsilon G_5(s)$ with $0 < \varepsilon \leq 1$ small enough. Indeed, (57) is satisfied (since (55) and (59)).

Theorem 4.1 *Assume that* $(A1)$–$(A3)$ *and* (29) *hold. Then for any* χ *satisfying* (57) *and* (58) *and for any* $\varepsilon_0 \in (0, 1)$, *there exists a strictly positive constant* C *such that the solution of* (1)–(2) *satisfies, for all* $t \geq 0$,

$$E(t) \leq \frac{CG_5(t)}{\chi(t)q(t)}, \tag{60}$$

where G_5 *and* χ *are defined in* (55) *and* (57), *respectively, and* q *will be defined later in the proof.*

Proof Using (39), (48), and (49), for some positive constant m, $\varepsilon_0 \in (0, 1)$, and any $t \geq 0$, we get

$$L'(t) \leq -mE(t) + c\left(\frac{t+1}{q_0}\right)^{\frac{1}{1+\varepsilon_0}}\left(H^{-1}\left(\frac{q_0\mu(t)}{(t+1)\xi(t)}\right)\right)^{\frac{1}{1+\varepsilon_0}}(t) + ch_1^{\frac{1}{1+\varepsilon_0}}(t). \tag{61}$$

Combining the strict increasing of H and the inequality $\frac{1}{t+1} < 1$ for $t > 0$, we obtain

$$H^{-1}\left(\frac{q_0\mu(t)}{(t+1)\xi(t)}\right) \leq H^{-1}\left(\frac{q_0\mu(t)}{(t+1)^{\frac{1}{1+\varepsilon_0}}\xi(t)}\right), \tag{62}$$

and, then (61) becomes, for any $t \geq 0$ and $\varepsilon_0 \in (0, 1)$,

$$L'(t) \leq -mE(t) + c_{\varepsilon_0}\frac{(t+1)^{\frac{1}{1+\varepsilon_0}}}{q_0}\left(H^{-1}\left(\frac{q_0\mu(t)}{(t+1)^{\frac{1}{1+\varepsilon_0}}\xi(t)}\right)\right)^{\frac{1}{1+\varepsilon_0}}(t) + ch_1^{\frac{1}{1+\varepsilon_0}}(t). \tag{63}$$

For simplicity, we let $q(t) := q_0(t+1)^{\frac{-1}{1+\varepsilon_0}}$ and $h_2(t) := ch_1^{\frac{1}{1+\varepsilon_0}}(t)$. Then (63) becomes

$$L'(t) \leq -mE(t) + \frac{c_{\varepsilon_0}}{\gamma(t)}\left(H^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right)\right)^{\frac{1}{1+\varepsilon_0}}(t) + ch_2(t). \tag{64}$$

Further, letting $G^{-1}(t) = (H^{-1}(t))^{\frac{1}{1+\varepsilon_0}}$ we reduce (64) to

$$L'(t) \le -mE(t) + \frac{c_{\varepsilon_0}}{\gamma(t)} G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right) + ch_2(t), \quad t \ge 0. \tag{65}$$

For $\varepsilon_1 < r$, let the functional $\mathcal{F}$ be defined by

$$\mathcal{F}(t) := G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) L(t),$$

which satisfies $\mathcal{F} \sim E$. Noting that $G'' \ge 0$, $q' \le 0$, and $E' \le 0$, we get

$$\begin{aligned}
\mathcal{F}'(t) &= \varepsilon_1 \frac{(qE)'(t)}{E(0)} G''\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) L(t) + G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) L'(t) \\
&\le -mE(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) + \frac{c}{q(t)} G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) G^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right) \\
&\quad + ch_2(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right).
\end{aligned} \tag{66}$$

Let G^* be the convex conjugate of G in the sense of Young (see [34]). Then

$$G^*(s) = s\bigl(G'\bigr)^{-1}(s) - G\bigl[\bigl(G'\bigr)^{-1}(s)\bigr] \quad \text{for } s \in \bigl(0, G'(r)\bigr], \tag{67}$$

and G^* satisfies the generalized Young inequality

$$AB \le G^*(A) + G(B) \quad \text{if } A \in \bigl(0, G'(r)\bigr], B \in (0, r]. \tag{68}$$

So, with $A = G'(\varepsilon_1 \frac{E(t)q(t)}{E(0)})$ and $B = G^{-1}(\frac{q(t)\mu(t)}{\xi(t)})$, using (24) and (66)–(68), we arrive at

$$\begin{aligned}
\mathcal{F}'(t) &\le -mE(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) + \frac{c}{q(t)} G^*\left(G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right)\right) + c\left(\frac{\mu(t)q(t)}{\xi(t)}\right) \\
&\quad + ch_2(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) \\
&\le -mE(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) + c\varepsilon_1 \frac{E(t)}{E(0)} G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) + c\left(\frac{\mu(t)q(t)}{\xi(t)}\right) \\
&\quad + ch_2(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right).
\end{aligned} \tag{69}$$

Multiplying (69) by $\xi(t)$, using (50), and the facts that $\varepsilon_1 \frac{E(t)q(t)}{E(0)} < r$ and $G'(\varepsilon_1 \frac{E(t)q(t)}{E(0)}) = G'(\varepsilon_1 \frac{E(t)q(t)}{E(0)})$, we get

$$\begin{aligned}
\xi(t)\mathcal{F}'(t) &\le -m\xi(t)E(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) + c\xi(t)\varepsilon_1 \frac{E(t)}{E(0)} G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) \\
&\quad + c\mu(t)q(t) + c\xi(t)h_2(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) \\
&\le -\left(\frac{mE(0)}{\varepsilon_1} - c\right) \xi(t)\varepsilon_1 \frac{E(t)}{E(0)} G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right) \\
&\quad - cE'(t) + c\xi(t)h_2(t) G'\left(\varepsilon_1 \frac{E(t)q(t)}{E(0)}\right).
\end{aligned} \tag{70}$$

Consequently, recalling the definition of G_2 and choosing ε_1 such that $k = (\frac{mE(0)}{\varepsilon_1} - c) > 0$, we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}\mathcal{F}_1'(t) &\le -k\varepsilon_1\xi(t)\left(\frac{E(t)}{E(0)}\right)G'\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right) + c\xi(t)h_2(t)G'\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right)\\ &= -k\frac{\xi(t)}{q(t)}G_2\left(\frac{E(t)q(t)}{E(0)}\right) + c\xi(t)h_2(t)G'\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right),\end{aligned} \tag{71}$$

where $\mathcal{F}_1 = \xi\mathcal{F} + cE \sim E$ satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1\mathcal{F}_1(t) \le E(t) \le \alpha_2\mathcal{F}_1(t). \tag{72}$$

Since $G_2'(t) = G'(t) + tG''(t)$, using the strict convexity of G on $(0,r]$, we find that $G_2'(t), G_2(t) > 0$ on $(0,r]$. Applying the general Young inequality (68) to the last term in (71) with $A = G'(\varepsilon_1\frac{E(t)q(t)}{E(0)})$ and $B = [\frac{c}{d}h_2(t)]$, we have

$$\begin{aligned}ch_2(t)G'\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right) &= \frac{d}{q(t)}\left[\frac{c}{d}q(t)h_2(t)\right]\left(G'\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right)\right)\\ &\le \frac{d}{q(t)}G_3\left(G'\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right)\right) + \frac{d}{q(t)}G_3^*\left[\frac{c}{d}q(t)h_2(t)\right]\\ &\le \frac{d}{q(t)}\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right)\left(G'\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right)\right) + \frac{d}{q(t)}G_4\left[\frac{c}{d}q(t)h_2(t)\right]\\ &\le \frac{d}{q(t)}G_2\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)}G_4\left[\frac{c}{d}q(t)h_2(t)\right].\end{aligned} \tag{73}$$

Now, combining (71) and (73) and choosing d small enough so that $k_1 = (k-d) > 0$, we arrive at

$$\begin{aligned}\mathcal{F}_1'(t) &\le -k\frac{\xi(t)}{q(t)}G_2\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_2\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_4\left[\frac{c}{d}q(t)h_2(t)\right]\\ &\le -k_1\frac{\xi(t)}{q(t)}G_2\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_4\left[\frac{c}{d}q(t)h_2(t)\right].\end{aligned} \tag{74}$$

Using the equivalent property in (72) and the nonincrease of G_2, we have, for some $d_0 = \frac{\alpha_1}{E(0)} > 0$,

$$G_2\left(\varepsilon_1\frac{E(t)q(t)}{E(0)}\right) \ge G_2\big(d_0\mathcal{F}_1(t)q(t)\big).$$

Letting $\mathcal{F}_2(t) := d_0\mathcal{F}_1(t)q(t)$ and recalling that $q' \le 0$, we arrive at,

$$\mathcal{F}_2'(t) \le d_0q(t)\left(-k_1\frac{\xi(t)}{q(t)}\Psi_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right]\right). \tag{75}$$

Then (75) becomes, for some constants $c_1 = d_0k_1 > 0$ and $c_2 = d_0d > 0$,

$$\mathcal{F}_2'(t) \le -c_1\xi(t)G_2\big(\mathcal{F}_2(t)\big) + c_2\xi(t)G_4\left[\frac{c}{d}q(t)h_2(t)\right]. \tag{76}$$

Since $d_0 q(t)$ is nonincreasing. Using the equivalent property $\mathcal{F}_1 \sim E$ implies that there exists $b_0 > 0$ such that $\mathcal{F}_2(t) \geq b_0 E(t) q(t)$. Since $\chi(t)$ satisfies (57) and (58), if $b_0 q(t) E(t) \leq 2\frac{G_5(t)}{\chi(t)}$, then we get

$$E(t) \leq \frac{2}{b_0} \frac{G_5(t)}{\chi(t) q(t)}. \tag{77}$$

If $b_0 q(t) E(t) > 2\frac{G_5(t)}{\chi(t)}$, then since $q(t)E(t)$ is a nonincreasing function, for any $0 \leq s \leq t$, we have $b_0 q(s) E(s) > 2\frac{G_5(t)}{\chi(t)}$. Therefore, for any $0 \leq s \leq t$,

$$\mathcal{F}_2(s) > 2\frac{G_5(t)}{\chi(t)}. \tag{78}$$

Using (21), $0 < \chi \leq 1$, and the convexity of G_2, we have, for any $0 < \varepsilon_2 \leq 1$,

$$\begin{aligned} G_2\big(\varepsilon_2 \chi(s) \mathcal{F}_2(s) - \varepsilon_2 G_5(s)\big) &= G_2\left(\varepsilon_2 \chi(s) \mathcal{F}_2(s) - \frac{\varepsilon_2 \chi(s) G_5(s)}{\chi(s)}\right) \\ &\leq \varepsilon_2 \chi(s) G_2\left(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\right). \end{aligned} \tag{79}$$

Recalling the definition of G_2, that is, $G_2(t) = tG'(t)$, (79) becomes

$$\begin{aligned} G_2\big(\varepsilon_2 \chi(s) \mathcal{F}_2(s) - \varepsilon_2 G_5(s)\big) &\leq \varepsilon_2 \chi(s)\left(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\right) G'\left(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\right) \\ &\leq \varepsilon_2 \chi(s) \mathcal{F}_2(s) G'\left(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\right) \\ &\quad - \varepsilon_2 \chi(s) \frac{G_5(s)}{\chi(s)} G'\left(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\right). \end{aligned} \tag{80}$$

Now, using (78) and the increase of G', for any $0 \leq s \leq t$, we have

$$G'\left(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\right) < G'\big(\mathcal{F}_2(s)\big), G'\left(\mathcal{F}_2(s) - \frac{G_5(s)}{\chi(s)}\right) > G'\left(\frac{G_5(s)}{\chi(s)}\right). \tag{81}$$

Combining (81) and (80), we arrive at

$$G_2\big(\epsilon_1 \chi(s) \mathcal{F}_2(s) - \varepsilon_2 G_5(s)\big) \leq \varepsilon_2 \chi(s) \mathcal{F}_2(s) G'\big(\mathcal{F}_2(s)\big) - \varepsilon_2 \chi(s) \frac{G_5(s)}{\chi(s)} G'\left(\frac{G_5(s)}{\chi(s)}\right). \tag{82}$$

Now we let

$$\mathcal{F}_3(s) = \varepsilon_2 \chi(s) \mathcal{F}_2(s) - \varepsilon_2 G_5(s), \tag{83}$$

where ε_2 is small enough such that $\mathcal{F}_3(0) \leq 1$. Recalling the definition of G_2, (82) becomes, for any $0 \leq s \leq t$,

$$G_2\big(\mathcal{F}_3(s)\big) \leq \varepsilon_2 \chi(t) G_2\big(\mathcal{F}_2(s)\big) - \varepsilon_2 \chi(t) G_2\left(\frac{G_5(s)}{\chi(s)}\right). \tag{84}$$

Further, we have

$$\mathcal{F}_3'(t) = \varepsilon_2\chi'(t)\mathcal{F}_2(t) + \varepsilon_2\chi(s)\mathcal{F}_2'(t) - \varepsilon_2 G_5'(t). \tag{85}$$

Since $\chi' \leq 0$, using (76), for any $0 \leq s \leq t$ and $0 < \varepsilon_2 \leq 1$, we obtain

$$\begin{aligned}\mathcal{F}_3'(t) &\leq \varepsilon_2\chi(s)\mathcal{F}_2'(t) - \varepsilon_2 G_5'(t)\\ &\leq -c_1\varepsilon_2\xi(t)\chi(t)G_2\big(\mathcal{F}_2(t)\big) + c_2\varepsilon_2\xi(t)\chi(s)G_4\left[\frac{c}{d}q(t)h_2(t)\right] - \varepsilon_2 G_5'(t).\end{aligned} \tag{86}$$

Then, using (58) and (84), we get

$$\begin{aligned}\mathcal{F}_3'(t) \leq{}& -c_1\xi(t)G_2\big(\mathcal{F}_3(t)\big) + c_2\varepsilon_2\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)h_2(t)\right]\\ &- c_1\varepsilon_2\xi(t)\chi(t)G_2\left(\frac{G_5(s)}{\chi(s)}\right) - \varepsilon_2 G_5'(t).\end{aligned} \tag{87}$$

From the definitions of G_1 and G_5 we have

$$G_1\big(G_5(s)\big) = c_1\int_0^s \xi(\tau)\,d\tau,$$

and hence

$$G_5'(s) = -c_1\xi(s)G_2\big(G_5(s)\big). \tag{88}$$

Now we have

$$\begin{aligned}&c_2\varepsilon_2\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)h_2(t)\right] - c_1\varepsilon_2\xi(t)\chi(t)G_2\left(\frac{G_5(s)}{\chi(s)}\right) - \varepsilon_2 G_5'(t)\\ &\quad= c_2\varepsilon_2\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)h_2(t)\right] - c_1\varepsilon_2\xi(t)\chi(t)G_2\left(\frac{G_5(s)}{\chi(s)}\right) + c\varepsilon_2\xi(t)G_2\big(G_5(t)\big)\\ &\quad= \varepsilon_2\xi(t)\chi(t)\left(c_2G_4\left[\frac{c}{d}q(t)h_2(t)\right] - c_1G_2\left(\frac{G_5(s)}{\chi(s)}\right)\right) + \frac{G_2(G_5(t))}{\chi(t)}.\end{aligned} \tag{89}$$

Then, according to (58), we get

$$\varepsilon_2\xi(t)\chi(t)\left(c_2G_4\left[\frac{c}{d}q(t)h_2(t)\right] - c_1G_2\left(\frac{G_5(s)}{\chi(s)}\right)\right) - \frac{G_2(G_5(t))}{\chi(t)} \leq 0.$$

Then (87) gives

$$\mathcal{F}_3'(t) \leq -c_1\xi(t)G_2\big(\mathcal{F}_3(t)\big). \tag{90}$$

Thus from (90) and the definitions of G_1 and G_2 in (55) and (56) we obtain

$$\big(G_1\big(\mathcal{F}_3(t)\big)\big)' \geq c_1\xi(t). \tag{91}$$

Integrating (91) over $[0,t]$, we get

$$G_1(\mathcal{F}_3(t)) \geq c_1 \int_0^t \xi(s)\,ds + G_1(\mathcal{F}_3(0)). \tag{92}$$

Since G_1 is decreasing, $\mathcal{F}_3(0) \leq 1$, and $G_1(1) = 0$, we have

$$\mathcal{F}_3(t) \leq G_1^{-1}\left(c_1 \int_0^t \xi(s)\,ds\right) = G_5(t). \tag{93}$$

Recalling that $\mathcal{F}_3(t) = \varepsilon_2 \chi(t)\mathcal{F}_2(t) - \varepsilon_2 G_5(t)$, we have

$$\mathcal{F}_2(t) \leq \frac{(1+\varepsilon_2)}{\varepsilon_2}\frac{G_5(t)}{\chi(t)}. \tag{94}$$

Similarly, recalling that $\mathcal{F}_2(t) := d_0\mathcal{F}_1(t)q(t)$, we get

$$\mathcal{F}_1(t) \leq \frac{(1+\varepsilon_2)}{d_0\varepsilon_2}\frac{G_5(t)}{\chi(t)q(t)}. \tag{95}$$

Since $\mathcal{F}_1 \sim E$, for some $b > 0$, we have $E(t) \leq b\mathcal{F}_1$, which gives

$$E(t) \leq \frac{b(1+\varepsilon_2)}{d_0\varepsilon_2}\frac{G_5(t)}{\chi(t)q(t)}. \tag{96}$$

From (77) and (96) we obtain the estimate

$$E(t) \leq c_3\left(\frac{G_5(t)}{\chi(t)q(t)}\right), \tag{97}$$

where $c_3 = \max\{\frac{2}{b_0}, \frac{b(1+\varepsilon_2)}{d_0\varepsilon_2}\}$. □

In the following example, we illustrate our decay result.

Example 4.2 Let $h(t) = \frac{a}{(1+t)^\nu}$, where $\nu > 1$ and $0 < a < \nu - 1$, so that $(A1)$ is satisfied. In this case, $\xi(t) = \nu a^{\frac{-1}{\nu}}$, $H(t) = t^{\frac{\nu+1}{\nu}}$, and $G^{-1}(t) = (H^{-1}(t))^{\frac{1}{1+\varepsilon_0}}$. Then for any $\varepsilon_0 \in (0,1)$, we have $G(t) = t^\lambda$, where $\lambda := \frac{(\varepsilon_0+1)(\nu+1)}{\nu} > 1$. Recall the definitions of the functions G_i, $i = 1,\dots,5$:

$$\begin{aligned} &G_1(t) = a_1(t^{1-\lambda} - 1), \qquad G_2(t) = a_2 t^\lambda, \quad G_3(t) = a_3 t^{\frac{\lambda}{\lambda-1}}, \qquad G_4(t) = a_4 t^\lambda, \\ &G_5(t) = a_5(1+t)^{\frac{1}{1-\lambda}}, \end{aligned} \tag{98}$$

where a_i, $i = 1,2,3,4,5$, are positive constants depending on a, ν, and ε_0. As in [30], we consider

$$m_0(1+t)^r \leq 1 + \|\triangle u_0\|^2 \leq m_1(1+t)^r, \tag{99}$$

where $r < \nu - 1$ and $m_0, m_1 > 0$. Then for some positive constants a_i $(i = 6,7)$ depending only on a, ν, m_0, m_1, r, we have

$$a_6(1+t)^{-\nu+1+r} \leq h_1(t) \leq a_7(1+t)^{-\nu+1+r}, \tag{100}$$

where $-\nu + 1 + r < 0$. Recalling the definitions of the functions h_1, h_2, and q, we have

$$q(t)h_2(t) = (1 + t)^{\frac{-\nu+r}{1+\varepsilon_0}}.$$

It is clear that condition (58) is satisfied if

$$q^{\lambda}(t)h_2^{\lambda}(t)\chi^{\lambda}(t) + (1 + t)^{\frac{-\lambda}{\lambda-1}}\chi^{\lambda-1}(t) \leq (1 + t)^{\frac{-\lambda}{\lambda-1}}. \tag{101}$$

Choosing $\chi(t) = (1+t)^m$, where $m < \min(0, \frac{-1}{\lambda-1} + \frac{(\nu-r)(\nu+1)}{\lambda\nu})$, we have the following two cases depending on r.

Case 1: If $0 < r < \nu - 1$, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that we have the following decay rate estimate of E (60):

$$E(t) \leq C_\varepsilon(1 + t)^{-(\nu-r-1)+\varepsilon}. \tag{102}$$

Case 2: If $r \leq 0$, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that the decay rate estimate of E (60) is given by:

$$E(t) \leq C_\varepsilon(1 + t)^{-(\nu-1)+\varepsilon}. \tag{103}$$

Thus estimates (102) and (103) give $\lim_{t\to+\infty} E(T) = 0$.

5 Conclusion

As far as we know, there are no decay results in the literature known for logarithmic plate equation with infinite memory and a wider class of relaxation functions. Our work extends the works for some wave equations treated in the literature to the plate equation with logarithmic nonlinearity. Also, we succeed to extend some general decay results, known for the case of finite history, to the case of infinite history, where the relaxation function satisfies a wider class of relaxation functions. Furthermore, we dropped the boundedness assumption on the history data considered in earlier results in the literature.

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