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Zero Mach number limit of the compressible Euler–Korteweg equations

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Abstract

In this paper, we investigate the zero Mach number limit for the three-dimensional compressible Euler–Korteweg equations in the regime of smooth solutions. Based on the local existence theory of the compressible Euler–Korteweg equations, we establish a convergence–stability principle. Then we show that when the Mach number is sufficiently small, the initial-value problem of the compressible Euler–Korteweg equations has a unique smooth solution in the time interval where the corresponding incompressible Euler equations have a smooth solution. It is important to remark that when the incompressible Euler equations have a global smooth solution, the existence time of the solution for the compressible Euler–Korteweg equations tends to infinity as the Mach number goes to zero. Moreover, we obtain the convergence of smooth solutions for the compressible Euler–Korteweg equations towards those for the incompressible Euler equations with a convergence rate.

Keywords: Compressible Euler–Korteweg equations; Mach number limit; Convergence–stability principle; Incompressible Euler equations; Energy-type error estimates

1 Introduction

In this paper, we are concerned with the three-dimensional compressible Euler–Korteweg system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \kappa \rho \nabla \Delta \rho, \end{cases} \quad (1.1)$$

for $(x, t) \in \Omega \times [0, +\infty)$. Throughout this paper, Ω is assumed to be the three-dimensional torus. Here, the unknown functions are the density ρ and the velocity $u \in \mathbb{R}^3$, $p(\rho)$ is a given pressure function, and κ is the Weber number. This compressible Euler–Korteweg system results from a modification of the standard Euler equations governing the motion of compressible inviscid fluids through the adjunction of the Korteweg stress tensor, and arises as a mathematical model for a lot of phenomena in vortex dynamics, quantum hydrodynamics and hydrodynamics, e.g., flow of capillary fluids: liquid–vapor mixtures (for instance, highly pressurized and hot water in nuclear reactors cooling system), superfluids

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(for instance, helium near absolute zero), or even regular fluids at sufficiently small scales (for instance, ripples on shallow water or other thin films). We can see more details in [34] and [22] for the early developments of the theory of capillarity and, for instance, [21, 35] for the derivation of the equations of motion. Note that when $\kappa = 0$, (1.1) reduces to the compressible Euler equations.

Recently, some results regarding the well-posedness of the compressible Euler–Korteweg system have been obtained. Benzoni-Gavage, Danchin and Descombes [6] addressed the well-posedness of the Cauchy problem for the Euler–Korteweg model in the one-dimensional case by reformulating the equations in Lagrangian coordinates. Benzoni-Gavage, Danchin and Descombes [7] also considered the multidimensional case in Eulerian formulation and established a blow-up criterion. Audiard [2] constructed a Kreiss symmetrizer and obtained the well-posedness of the Euler–Korteweg system in a half-space. Audiard [3] obtained some dispersive smoothing effect of the Euler–Korteweg system both in one dimension and in higher dimensions. Audiard and Haspot [5] justified the global well-posedness of the multi-dimensional Euler–Korteweg equations for small irrotational initial data under a natural stability condition on the pressure. Giesselmann and Tzavaras [18] showed the weak-strong uniqueness, the large friction limit, and the vanishing capillarity limit of the Euler–Korteweg system under the relative energy framework. Audiard [4] obtained the existence of traveling waves for Euler–Korteweg equations with arbitrarily small energies in two dimensions and found that the standard for the linear instability of traveling waves implied nonlinear instability in one dimension.

Moreover, it is well known that the incompressible limit of compressible fluid dynamical equations is an important and challenging mathematical problem. Klainerman and Majda [23] first justified the convergence of the incompressible limit by using the partial differential equation method and singular limit approach of symmetric hyperbolic equations. Lin [26] proved the incompressible limit of the assumed weak solutions for the time-discretized compressible Navier–Stokes equations with big initial data by the uniform entropy-energy inequality. Hoff [19] proved that a compressible Navier–Stokes system with well-prepared initial data converged to an incompressible Navier–Stokes system as the Mach number goes to zero. Nevertheless, no smallness hypothesis is set on the external forces or on the initial data. Lions and Masmoudi [27] studied the incompressible limit of global weak solutions of compressible isentropic Navier–Stokes systems without size restrictions on the initial data. Desjardins and Grenier [13] researched the low Mach number limit for weak solutions of the compressible Navier–Stokes equations on the whole space by using a different method based on Strichartz’s estimates for the linear wave equation, and gained better convergence results and simpler proof than former similar papers. Desjardins, Grenier, Lions and Masmoudi [14] investigated the limit of global weak solutions of the compressible isentropic Navier–Stokes equations in a bounded domain. They stated that the velocity of the compressible equations converged weakly to the global weak solution of the incompressible Navier–Stokes equations as the Mach number approached to 0, and the convergence became strong under certain geometrical assumptions on the domain. These results have been extended or improved by many others, e.g., the authors of Refs. [1, 8, 12, 15, 16, 20, 25, 31, 33].

To the best of our knowledge, no results about the incompressible limit of this Euler–Korteweg model can be found, except for Giesselmann [17], who gave a low Mach asymptotic-preserving scheme for the Euler–Korteweg model. In this paper, we analyze

the incompressible limit of smooth solutions for the compressible Euler–Korteweg equations (1.1) with well-prepared initial data on the basis of the convergence-stability criterion, which was first formulated in [37]. Since then, this method has commonly been used in dealing with the singular limit of the partial differential equations. Yong [38] considered the zero Mach number limit of the smooth solution to the isentropic compressible Euler equations based on the convergence-stability principle. Li [24] presented the incompressible limit from the compressible MHD equations to ideal incompressible MHD equations under the framework of the convergence-stability principle. In fact, this approach is derived from singular perturbation theory [36], which is extensively used in the research of partial differential equations. For instance, Marin and Bhatti [29] studied the head-on collision model between capillary–gravity solitary waves using the singular perturbation method. This model was appropriate for shallow water waves and deep water waves and was investigated to find that the surface tension and the free parameter tended to remarkably decrease the solitary-wave profile. For similar methods, see [11, 30].

The main difficulty in the analysis of this model, a third-order system of conservation laws, is the absence of dissipative regularization since the viscosity is neglected. To overcome this difficulty, we need more refined treatments, which is different from the compressible Navier–Stokes–Korteweg equation in [25]. See the treatments of the terms I_4 and H_3 in Sect. 5 for the difference. In addition, our approach in this paper is simpler than that adopted in previous work [23, 28], and the requirements on the initial data and limit solution are fewer.

From a physical standpoint, when the density becomes almost constant, the velocity is very small, and we observe that at large time scales, the compressible fluid should act like the incompressible fluid. Therefore we introduce the following scaling:

$$\rho(x, t) = \rho^\varepsilon(x, \varepsilon t), \quad u(x, t) = \varepsilon u^\varepsilon(x, \varepsilon t)$$

and assume that the capillarity coefficient κ is small and scaled as

$$\kappa = \varepsilon \kappa'$$

with $\varepsilon \in (0, 1)$ a small parameter. With such scalings, the compressible Euler–Korteweg equations (1.1) take the form

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \frac{\nabla p(\rho^\varepsilon)}{\varepsilon^2} = \kappa' \frac{\rho^\varepsilon \nabla \Delta \rho^\varepsilon}{\varepsilon}, \end{cases} \tag{1.2}$$

with the initial data

$$\rho^\varepsilon(x, 0) = 1, \quad u^\varepsilon(x, 0) = u_0(x). \tag{1.3}$$

Formally, letting $\varepsilon \rightarrow 0$, we obtain from the momentum Eq. (1.2)₂ that ρ^ε converges to a positive constant ρ^* due to the periodic boundary conditions. Without loss of generality, let us assume that $\rho^* = 1$. Then, passing to the limit in the mass conservation equation of (1.2), we obtain $\operatorname{div} u^\varepsilon = 0$. Therefore, by denoting the formal limits of $\frac{\nabla p(\rho^\varepsilon) - \kappa' \varepsilon \rho^\varepsilon \nabla \Delta \rho^\varepsilon}{\varepsilon^2}$ and

u^ε by ∇p^0 and u^0 , respectively, we can formally obtain the incompressible Euler equations:

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = 0, \\ \operatorname{div} u^0 = 0, \end{cases} \tag{1.4}$$

with the initial data

$$u^0(x, 0) = u_0(x). \tag{1.5}$$

To justify the above formal procedure, we first follow [25, 38] and reformulate the compressible Euler–Korteweg equations (1.2) in terms of the pressure variable $p^\varepsilon = p(\rho^\varepsilon)$ and the velocity u^ε . Assume that $p(\rho^\varepsilon)$ is a smooth function with $p'(\rho^\varepsilon) > 0$ for $\rho^\varepsilon > 0$, then it has an inverse function $\rho^\varepsilon = \rho(p^\varepsilon)$. Set $q(p^\varepsilon) = [\rho(p^\varepsilon)p'(\rho(p^\varepsilon))]^{-1}$. Then the compressible Euler–Korteweg equations (1.2) for a smooth solution can be rewritten as

$$\begin{cases} q(p^\varepsilon)(p^\varepsilon_t + u^\varepsilon \cdot \nabla p^\varepsilon) + \operatorname{div} u^\varepsilon = 0, \\ \rho(p^\varepsilon)(u^\varepsilon_t + u^\varepsilon \cdot \nabla u^\varepsilon) + \varepsilon^{-2} \nabla p^\varepsilon = \varepsilon^{-1} \kappa' \rho(p^\varepsilon) \nabla \Delta \rho(p^\varepsilon), \end{cases} \tag{1.6}$$

with the initial data

$$p^\varepsilon(x, 0) = p_0, \quad u^\varepsilon(x, 0) = u_0(x), \tag{1.7}$$

with $p_0 = p(1) > 0$. Further, we introduce

$$\tilde{p}^\varepsilon = (p^\varepsilon - p_0)/\varepsilon, \quad \tilde{u}^\varepsilon = u^\varepsilon.$$

Then (1.6) can be rewritten as

$$\begin{cases} q(p_0 + \varepsilon \tilde{p}^\varepsilon)(\tilde{p}^\varepsilon_t + \tilde{u}^\varepsilon \cdot \nabla \tilde{p}^\varepsilon) + \varepsilon^{-1} \operatorname{div} \tilde{u}^\varepsilon = 0, \\ \rho(p_0 + \varepsilon \tilde{p}^\varepsilon)(\tilde{u}^\varepsilon_t + \tilde{u}^\varepsilon \cdot \nabla \tilde{u}^\varepsilon) + \varepsilon^{-1} \nabla \tilde{p}^\varepsilon = \varepsilon^{-1} \kappa' \rho(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla \Delta \rho(p_0 + \varepsilon \tilde{p}^\varepsilon), \end{cases} \tag{1.8}$$

with the initial data

$$\tilde{p}^\varepsilon(x, 0) = 0, \quad \tilde{u}^\varepsilon(x, 0) = u_0(x). \tag{1.9}$$

2 Main result

The main result can be stated as follows.

Theorem 2.1 *Suppose that $p(\rho^\varepsilon)$ is a smooth function with $p'(\rho^\varepsilon) > 0$ for $\rho^\varepsilon > 0$, and $u_0(x) \in H^6(\Omega)$ is divergence-free. Denote by $T_0 > 0$ the life-span of the unique classical solution $u^0(x, t) \in C([0, T_0], H^6(\Omega))$ to the initial-value problem (1.4)–(1.5). If $T_0 < \infty$, then, for a sufficiently small $\varepsilon > 0$, the initial-value problem (1.2)–(1.3) has a unique solution $(\rho^\varepsilon, u^\varepsilon)(x, t)$ satisfying*

$$\rho^\varepsilon - 1 \in C([0, T_0], H^5(\Omega)), \quad u^\varepsilon \in C([0, T_0], H^4(\Omega)).$$

Moreover, there exists a constant $K > 0$, independent of ε but dependent on T_0 , such that

$$\sup_{t \in [0, T_0]} \left(\left\| \frac{\rho^\varepsilon(\cdot, t) - 1}{\varepsilon} \right\|_4 + \sum_{|\alpha|=5} \left\| \varepsilon^{\frac{1}{2}} \partial_x^\alpha (\rho^\varepsilon - 1)(\cdot, t) \right\| + \left\| (u^\varepsilon - u^0)(\cdot, t) \right\|_4 \right) \leq K\varepsilon. \tag{2.1}$$

In case $T_0 = \infty$, the maximal existence time T_ε of $(\rho^\varepsilon, u^\varepsilon)(x, t)$ tends to infinity as ε goes to zero.

Remark 2.1 The initial data

$$\rho^\varepsilon(x, 0) = 1, \quad u^\varepsilon(x, 0) = u_0(x)$$

can be relaxed as

$$\rho^\varepsilon(x, 0) = 1 + O(\varepsilon^2), \quad u^\varepsilon(x, 0) = u_0(x) + O(\varepsilon)$$

without changing our arguments. Here, we do not know whether the convergence rate in (2.1) is optimal, in particular, the velocity convergence rate. Using the arguments in [9], we will try to address this topic in the future. However, with the method here, we can obtain the sharp convergence rate (2.1), and no smallness condition on the initial data is required.

Remark 2.2 Here, we only consider the zero-Mach limit of the smooth solutions for the compressible Euler–Korteweg equations with well-prepared initial data. It is more interesting to consider the similar problem of the compressible Euler–Korteweg equations for general initial data (ill-prepared initial data). That is, we should take into account acoustic waves that propagate with the high speed $\frac{1}{\varepsilon}$ in the space domain, as in [23, 28, 33]. Moreover, we hope that similar results can be obtained for the limit of the compressible Euler–Korteweg equations in critical space. These issues are what our efforts should aim at in the future.

Recalling a local-in-time existence theory due to Benzoni-Gavage, Danchin and Descombes [7] for (1.2), we have the local-in-time existence of the classical solution to the compressible Euler–Korteweg equations (1.2) as follows.

Lemma 2.1 (See [7]) *Let $p(\rho^\varepsilon)$ be a smooth function with $p'(\rho^\varepsilon) > 0$ for $\rho^\varepsilon > 0$. Assume that $\bar{u}(x) \in H^4$ and $\bar{\rho}(x) - 1 \in H^5$ with $\inf \bar{\rho}(x) > 0$. Then there exists a positive constant T such that equation (1.2) with initial data $(\bar{\rho}, \bar{u})(x)$ has a unique solution $(\rho^\varepsilon, u^\varepsilon) = (\rho^\varepsilon, u^\varepsilon)(x, t)$, satisfying $\rho^\varepsilon(x, t) > 0$ for all $(x, t) \in \Omega \times [0, T]$ and*

$$\begin{aligned} \rho^\varepsilon - 1 &\in C([0, T], H^5) \cap L^2([0, T], H^6), \\ u^\varepsilon &\in C([0, T], H^4) \cap L^2([0, T], H^5). \end{aligned}$$

For initial-value problem (1.8)–(1.9), we see immediately from Lemma 2.1 that

Corollary 2.1 *Under the assumptions of Lemma 2.1, there exists a positive constant $T_\varepsilon > 0$ such that Eqs. (1.8) with initial data $(\frac{p(\bar{\rho})-p_0}{\varepsilon}, \bar{u})$ have a unique classical solution $(\tilde{p}^\varepsilon, \tilde{u}^\varepsilon) =$*

$(\tilde{p}^\varepsilon, \tilde{u}^\varepsilon)(t, x)$, satisfying $\varepsilon \tilde{p}^\varepsilon(x, t) + p_0 > 0$ for all $(x, t) \in \Omega \times [0, T_\varepsilon]$ and

$$\begin{aligned} \tilde{p}^\varepsilon(t, x) &\in C([0, T_\varepsilon], H^5) \cap L^2([0, T_\varepsilon], H^6), \\ \tilde{u}^\varepsilon(t, x) &\in C([0, T_\varepsilon], H^4) \cap L^2([0, T_\varepsilon], H^5). \end{aligned}$$

Thus, for initial-value problem (1.8)–(1.9), Theorem 2.1 can be stated as follows.

Theorem 2.2 *Under the assumptions of Theorem 2.1, for a sufficiently small $\varepsilon > 0$, the initial-value problem (1.8)–(1.9) has a unique solution $(\tilde{p}^\varepsilon, \tilde{u}^\varepsilon)(x, t)$ satisfying*

$$\tilde{p}^\varepsilon(x, t) \in C([0, T_0], H^5), \quad \tilde{u}^\varepsilon(x, t) \in C([0, T_0], H^4).$$

Moreover, there exists a constant $K > 0$, independent of ε but dependent on T_0 , such that

$$\sup_{t \in [0, T_0]} \left(\|\tilde{p}^\varepsilon - \varepsilon p^0\|_4 + \sum_{|\alpha|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\alpha (\tilde{p}^\varepsilon - \varepsilon p^0)\| + \|\tilde{u}^\varepsilon - u^0\|_4 \right) \leq K\varepsilon. \tag{2.2}$$

In case $T_0 = \infty$, the maximal existence time T_ε of $(\tilde{p}^\varepsilon, \tilde{u}^\varepsilon)(x, t)$ tends to infinity as ε goes to zero. Here, $(p^0, u^0)(x, t)$ is given in Lemma 1.4 and satisfies (4.3).

From Theorem 2.2, we immediately have Theorem 2.1. In the following, we focus on the proof of Theorem 2.2.

Let us outline the idea of the proof as follows. On the basis of a local-in-time existence theory due to Benzoni-Gavage, Danchin and Descombes [7] for (1.2), we first establish a convergence-stability principle, which is similar to those developed in [36, 37] for singular limit problems of symmetrizable hyperbolic systems. Thus, instead of deriving ε -uniform *a priori* estimates, we only need to make the error estimate (2.1) in the common time interval $[0, \min\{T_0, T_\varepsilon\})$, where both solutions $(\rho^\varepsilon, u^\varepsilon)$ and (ρ^0, u^0) are regular. Due to the third-order term and the absence of dissipative regularization in (1.1) or (1.2), deriving the error estimate requires some elaborated treatments. This is the difference from the compressible Navier–Stokes–Korteweg equation in [25]. See the treatments of the terms I_4 and H_3 in Sect. 5 for the difference.

The rest of this paper is organized as follows. In the next section, we make some preliminaries. That is, we give some notations and Moser-type calculus inequalities. Then, we prove the convergence-stability principle in Sect. 4. Finally, all required (error) estimates are obtained in Sect. 5.

3 Preliminaries

In this section, we mainly make some preliminaries.

Notation $|U|$ denotes some norm of a vector or matrix U . For a nonnegative integer k , $H^k = H^k(\Omega)$ denotes the usual L^2 -type Sobolev space of order k . We write $\|\cdot\|_k$ for the standard norm of H^k and $\|\cdot\|$ for $\|\cdot\|_0$. When U is a function of another variable t as well as $x \in \Omega$, we write $\|U(\cdot, t)\|$ to recall that the norm is taken with respect to x while t is viewed as a parameter. In addition, we denote by $C([0, T], \mathbf{X})$ (resp. $L^2([0, T], \mathbf{X})$) the space of continuous (resp. square integrable) functions on $[0, T]$ with values in a Banach space \mathbf{X} .

In the subsequent proof process, we need the following Moser-type calculus inequalities in Sobolev spaces (refer to Proposition 2.1 in [28]).

Lemma 3.1

- (i) For $s \geq 2$, $H^s = H^s(\Omega)$ is an algebra. Namely, for all multi-indices α with $|\alpha| \leq s$ and $f(x), g(x) \in H^s(\Omega)$, we have $\partial_x^\alpha(fg) \in L^2(\Omega)$ and

$$\|\partial_x^\alpha(fg)\| \leq C_s \|f\|_s \|g\|_s.$$

- (ii) For $s \geq 3$, let $f(x) \in H^s(\Omega)$ and $g(x) \in H^{s-1}(\Omega)$. Then, for all multi-indices α with $|\alpha| \leq s$, we have the commutator $[\partial_x^\alpha, f]g \in L^2(\Omega)$ and

$$\|[\partial_x^\alpha, f]g\| \leq C_s \|\nabla f\|_{s-1} \|g\|_{s-1}.$$

- (iii) Assume that $g(u)$ is a smooth function on G , $u(x)$ is a continuous function with $u(x) \in G_1, \bar{G}_1 \subset\subset G$, and $u(x) \in L^\infty(\Omega) \cap H^s(\Omega)$. Then, for $s \geq 1$,

$$\|D^s g(u)\| \leq C_s \left| \frac{\partial g}{\partial u} \right|_{s-1, \bar{G}_1} \|u\|_{L^\infty}^{s-1} \|D^s u\|.$$

Here, $|\cdot|_{r, \bar{G}_1}$ is the C^r -norm on the set \bar{G}_1 and C_s is a generic constant depending only on s .

4 Convergence-stability principle

In fact, our proof of Theorem 2.2 is guided by the spirit of the convergence-stability principle developed in [36, 37] for singular limit problems of symmetrizable hyperbolic systems. Fix $\varepsilon > 0$ in (1.8). According to Corollary 2.1, there is a time interval $[0, T]$ such that the equations (1.8) with initial data $(\bar{p}, \bar{u})(x, \varepsilon)$ have a unique solution $(\tilde{p}^\varepsilon, \tilde{u}^\varepsilon)$ satisfying $\varepsilon \tilde{p}^\varepsilon + p_0 > 0$ for all $(x, t) \in \Omega \times [0, T]$ and

$$\tilde{p}^\varepsilon(x, t) \in C([0, T], H^5), \quad \tilde{u}^\varepsilon(x, t) \in C([0, T], H^4).$$

Define

$$T_\varepsilon = \sup \left\{ T > 0 : \tilde{p}^\varepsilon(x, t) \in C([0, T], H^5), \tilde{u}^\varepsilon(x, t) \in C([0, T], H^4); \right. \\ \left. -\frac{1}{2}p_0 \leq \tilde{p}^\varepsilon(x, t) \leq 2p_0, \forall (x, t) \in \Omega \times [0, T] \right\}. \tag{4.1}$$

(Here, 2 can be replaced with any positive number larger than 1.) Namely, $[0, T_\varepsilon)$ is the maximal time interval of $H^5 \times H^4$ -existence. Note that T_ε may tend to 0 as ε goes to 0.

To show that $\lim_{\varepsilon \rightarrow 0} T_\varepsilon > 0$, we follow the convergence-stability principle [37] and seek a formal approximation of $(\tilde{p}^\varepsilon, \tilde{u}^\varepsilon)(x, t)$. To this end, we consider the initial-value problem of the incompressible Euler equations:

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = 0, \\ \operatorname{div} u^0 = 0, \quad u^0(x, 0) = u_0(x). \end{cases} \tag{4.2}$$

Since $u_0 \in H^6$ and $\operatorname{div} u_0 = 0$, we have the following from [22, 32].

Lemma 4.1 *There exists $T_0 \in (0, +\infty)$ such that the IVP (4.2) of the incompressible Euler equation has a unique smooth solution*

$$(u^0, \nabla p^0) \in C([0, T_0], H^6) \times C([0, T_0], H^5)$$

satisfying

$$M_0 =: \sup_{0 \leq t \leq T_0} (\|u^0(\cdot, t)\|_6 + \|\nabla p^0(\cdot, t)\|_5 + \|\partial_t u^0(\cdot, t)\|_5 + \|\partial_t \nabla p^0(\cdot, t)\|_4) < \infty. \tag{4.3}$$

In the next section, we will prove the following theorem.

Theorem 4.1 *Under the conditions of Theorem 2.1, there exist constants $K = K(T_0)$ and $\varepsilon_0 = \varepsilon_0(T_0)$ such that, for all $\varepsilon \leq \varepsilon_0$,*

$$\|(\tilde{p}^\varepsilon - \varepsilon p^0)(\cdot, t)\|_4 + \sum_{|\alpha|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\alpha (\tilde{p}^\varepsilon - \varepsilon p^0)(\cdot, t)\| + \|(\tilde{u}^\varepsilon - u^0)(\cdot, t)\|_4 \leq K\varepsilon \tag{4.4}$$

for $t \in [0, \min\{T_0, T_\varepsilon\})$.

Having this theorem, we slightly modify the arguments in [25, 36] to prove a theorem.

Theorem 4.2 *Under the conditions of Theorem 2.1, there exists a constant $\varepsilon_0 = \varepsilon_0(T_0)$ such that, for all $\varepsilon \leq \varepsilon_0$,*

$$T_\varepsilon > T_0.$$

Proof Otherwise, there is a sequence $\{\varepsilon_k\}_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and $T_{\varepsilon_k} \leq T_0$. Thanks to the error estimate in Theorem 4.1, (4.3) and Sobolev’s inequality, there exists a k such that $4\tilde{p}^{\varepsilon_k}(x, t) \in (-3p_0, 5p_0)$ for all x and t . Next, we deduce from

$$\begin{aligned} \|\tilde{p}^{\varepsilon_k}(\cdot, t)\|_5 + \|\tilde{u}^{\varepsilon_k}(\cdot, t)\|_4 &\leq \|\tilde{p}^{\varepsilon_k}(\cdot, t) - \varepsilon_k p^0(\cdot, t)\|_5 + \|\varepsilon_k p^0(\cdot, t)\|_5 \\ &\quad + \|\tilde{u}^{\varepsilon_k}(\cdot, t) - u^0(\cdot, t)\|_4 + \|u^0(\cdot, t)\|_4, \end{aligned}$$

(4.3) in Lemma 4.1 and (4.4) in Theorem 4.1 that $\|\tilde{p}^{\varepsilon_k}(\cdot, t)\|_5 + \|\tilde{u}^{\varepsilon_k}(\cdot, t)\|_4$ is bounded uniformly with respect to $t \in [0, T_{\varepsilon_k})$. Now, we could apply Corollary 2.1, beginning at a time t less than T_{ε_k} (k is fixed here), to continue this solution beyond T_{ε_k} . This contradicts the definition of T_{ε_k} in (4.1).

Finally, Theorem 2.2 is proved by combining Theorem 4.1 and Theorem 4.2. □

We conclude this section with the following interesting remark, which is a by-product of our approach.

Remark 4.1 The proof of Theorem 4.1 requires that $T_0 < \infty$. However, when the initial-value problem (4.2) of the incompressible Euler equations has a global-in-time regular solution, T_0 can be any positive number. Hence, we have an almost global-in-time existence result for (1.8):

$$\lim_{\varepsilon \rightarrow 0} T_\varepsilon = +\infty.$$

5 Error estimate

In this section, we prove the error estimate in Theorem 4.1. We notice that, with u^0 and p^0 in Lemma 4.1,

$$(p_\varepsilon, u_\varepsilon) := (\varepsilon p^0, u^0)$$

satisfies

$$\begin{cases} q(p_0 + \varepsilon p_\varepsilon)(p_{\varepsilon t} + u_\varepsilon \cdot \nabla p_\varepsilon) + \varepsilon^{-1} \operatorname{div} u_\varepsilon = \varepsilon R_1, \\ \rho(p_0 + \varepsilon p_\varepsilon)(u_{\varepsilon t} + u_\varepsilon \cdot \nabla u_\varepsilon) + \varepsilon^{-1} \nabla p_\varepsilon = R_2, \end{cases} \tag{5.1}$$

with

$$\begin{aligned} R_1 &= q(p_0 + \varepsilon^2 p^0)(p_t^0 + u^0 \cdot \nabla p^0), \\ R_2 &= (\rho(p_0 + \varepsilon^2 p^0) - \rho(p_0))(u_t^0 + u^0 \cdot \nabla u^0). \end{aligned}$$

Note that Lemma 4.1 and Sobolev’s inequality imply $-\frac{1}{2}p_0 \leq \varepsilon p_\varepsilon (= \varepsilon^2 p^0) \leq p_0$ for $\varepsilon \ll 1$, which yields $\frac{1}{2}p_0 \leq p_0 + \varepsilon p_\varepsilon \leq 2p_0$. Further, $\rho(p_0 + \varepsilon p_\varepsilon)$ is strictly increasing, then we have

$$\rho\left(\frac{1}{2}p_0\right) \leq \rho(p_0 + \varepsilon p_\varepsilon) \leq \rho(2p_0). \tag{5.2}$$

From the definition of $q(p_0 + \varepsilon p_\varepsilon)$, we also have

$$\frac{1}{2p'(2)} \leq q(p_0 + \varepsilon p_\varepsilon) \leq \frac{2}{p'(\frac{1}{2})}. \tag{5.3}$$

Set

$$P = \tilde{p}^\varepsilon - p_\varepsilon, \quad U = \tilde{u}^\varepsilon - u_\varepsilon.$$

Then we deduce from (1.8) and (5.1) that

$$P_t + \tilde{u}^\varepsilon \cdot \nabla P + U \cdot \nabla p_\varepsilon + \varepsilon^{-1} q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon) \operatorname{div} U = f_1 \tag{5.4}$$

and

$$U_t + \tilde{u}^\varepsilon \cdot \nabla U + U \cdot \nabla u_\varepsilon + \varepsilon^{-1} \rho^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P = \varepsilon^{-1} \kappa' \nabla \Delta \tilde{\rho}^\varepsilon + f_2. \tag{5.5}$$

Here, we have used $\operatorname{div} u_\varepsilon = 0$ and $\tilde{\rho}^\varepsilon = \rho(p_0 + \varepsilon \tilde{p}^\varepsilon)$, and

$$\begin{aligned} f_1 &= -q^{-1}(p_0 + \varepsilon p_\varepsilon) \varepsilon R_1, \\ f_2 &= -\rho^{-1}(p_0 + \varepsilon p_\varepsilon) R_2 - (\rho^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon) - \rho^{-1}(p_0 + \varepsilon p_\varepsilon)) \nabla p^0. \end{aligned}$$

From Lemma 4.1 and Lemma 3.1, it follows that, for $t \in [0, T_0]$,

$$\|f_1\|_4 \leq C(M_0)\varepsilon, \quad \|f_2\| \leq C(M_0)(\varepsilon + \|P\|_4). \tag{5.6}$$

Here, and in the following, $C > 0$ is the generic constant and $C(\cdot) > 0$ stands for the generic constant depending on \cdot .

Let α be a multi-index with $|\alpha| \leq 4$. Differentiating the two sides of Eqs. (5.4) and (5.5) with ∂_x^α and setting

$$P_\alpha = \partial_x^\alpha P, \quad U_\alpha = \partial_x^\alpha U, \tilde{\rho}_\alpha^\varepsilon = \partial_x^\alpha \tilde{\rho}^\varepsilon,$$

we obtain

$$\begin{aligned} & \partial_t P_\alpha + \tilde{u}^\varepsilon \cdot \nabla P_\alpha + \varepsilon^{-1} q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon) \operatorname{div} U_\alpha \\ &= -[\partial_x^\alpha, \tilde{u}^\varepsilon] \nabla P - \partial_x^\alpha (U \cdot \nabla p_\varepsilon) - \varepsilon^{-1} [\partial_x^\alpha, q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)] \operatorname{div} U + \partial_x^\alpha f_1 \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} & \partial_t U_\alpha + \tilde{u}^\varepsilon \cdot \nabla U_\alpha + \varepsilon^{-1} \rho^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P_\alpha \\ &= -[\partial_x^\alpha, \tilde{u}^\varepsilon] \nabla U - \partial_x^\alpha (U \cdot \nabla u_\varepsilon) - \varepsilon^{-1} [\partial_x^\alpha, \rho^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)] \nabla P \\ & \quad + \varepsilon^{-1} \kappa' \nabla \Delta \tilde{\rho}_\alpha^\varepsilon + \partial_x^\alpha f_2. \end{aligned} \tag{5.8}$$

Taking the inner product of (5.7) and (5.8) with $q(p_0 + \varepsilon \tilde{p}^\varepsilon) P_\alpha$ and $\rho(p_0 + \varepsilon \tilde{p}^\varepsilon) U_\alpha$, respectively, and summing up the two resultant equalities gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega (q^\varepsilon P_\alpha^2 + \tilde{\rho}^\varepsilon |U_\alpha|^2) dx \\ &= \int_\Omega \left(\frac{1}{2} q_t^\varepsilon P_\alpha^2 + \frac{1}{2} \rho_t^\varepsilon |U_\alpha|^2 - q^\varepsilon P_\alpha (u^\varepsilon \cdot \nabla) P_\alpha - \tilde{\rho}^\varepsilon U_\alpha (u^\varepsilon \cdot \nabla) U_\alpha \right) dx \\ & \quad + \int_\Omega \left(-([\partial_x^\alpha, \tilde{u}^\varepsilon] \nabla P + \partial_x^\alpha (U \cdot \nabla p_\varepsilon)) q^\varepsilon P_\alpha - (\partial_x^\alpha (U \cdot \nabla u_\varepsilon) + [\partial_x^\alpha, \tilde{u}^\varepsilon] \nabla U) \rho^\varepsilon U_\alpha \right) dx \\ & \quad - \varepsilon^{-1} \int_\Omega (q^\varepsilon P_\alpha [\partial_x^\alpha, q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)] \operatorname{div} U + \rho^\varepsilon U_\alpha [\partial_x^\alpha, \rho^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)] \nabla P) dx \\ & \quad + \varepsilon^{-1} \kappa' \int_\Omega \tilde{\rho}^\varepsilon U_\alpha \nabla \Delta \tilde{\rho}_\alpha^\varepsilon dx + \int_\Omega (q^\varepsilon \partial_x^\alpha f_1 P_\alpha + \tilde{\rho}^\varepsilon \partial_x^\alpha f_2 U_\alpha) dx \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{5.9}$$

Here and below, we often use

$$\tilde{\rho}^\varepsilon = \rho(p_0 + \varepsilon \tilde{p}^\varepsilon), \quad q^\varepsilon = q(p_0 + \varepsilon \tilde{p}^\varepsilon).$$

To estimate the I_i , we first have the bounds of $\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)$ and $q'(p_0 + \varepsilon \tilde{p}^\varepsilon)$ as follows.

Lemma 5.1 *We have*

$$c_1 \leq \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \leq c_2, \quad c_3 \leq q'(p_0 + \varepsilon \tilde{p}^\varepsilon) \leq c_4,$$

where c_i ($i = 1, 2, 3, 4$) are positive constants independent of ε .

Proof Because $p(\rho^\varepsilon)$ is a smooth function with $p'(\rho^\varepsilon) > 0$ and has an inverse function $\rho^\varepsilon = \rho(p^\varepsilon)$, from the smoothness of ρ and $\frac{1}{2}p_0 \leq p_0 + \varepsilon\tilde{p}^\varepsilon \leq 2p_0$, it is easy to see that there are positive constants c_1 and c_2 such that

$$c_1 \leq \rho'(p_0 + \varepsilon\tilde{p}^\varepsilon) \leq c_2.$$

Moreover, from the definition of q , similarly, we have

$$c_3 \leq q'(p_0 + \varepsilon\tilde{p}^\varepsilon) \leq c_4.$$

This completes the proof. □

Remark 5.1 Analogously, we can obtain the boundedness of higher derivatives of $\rho(p_0 + \varepsilon\tilde{p}^\varepsilon)$ and $q(p_0 + \varepsilon\tilde{p}^\varepsilon)$.

Next, we follow [25, 36, 37] and formulate the following lemma.

Lemma 5.2 *Set*

$$D = D(t) = \sqrt{\|P(\cdot, t)\|_4^2 + \sum_{|\beta|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\beta P(\cdot, t)\|^2 + \|U(\cdot, t)\|_4^2} \tag{5.10}$$

for $t \in [0, \min\{T_0, T_\varepsilon\})$. Then, for multi-indices γ satisfying $|\gamma| \leq 2$, we have

$$\|\partial_x^\gamma \tilde{p}^\varepsilon\|_{L^\infty} + \|\partial_x^\gamma \tilde{u}^\varepsilon\|_{L^\infty} \leq C(M_0)(1 + D).$$

Proof It is obvious from (4.3) in Lemma 4.1, (5.10) and Sobolev’s inequality that

$$\|\partial_x^\gamma \tilde{u}^\varepsilon\|_{L^\infty} \leq \|\partial_x^\gamma (\tilde{u}^\varepsilon - u_\varepsilon)\|_{L^\infty} + \|\partial_x^\gamma u^0\|_{L^\infty} \leq C(M_0)(1 + D).$$

The other estimates can be shown similarly. This completes the proof. □

Now we turn to estimating the I_i in (5.9). Using integration by parts and Lemma 5.1 and Lemma 5.2, we deduce that

$$\begin{aligned} I_1 &= \frac{1}{2} \int_\Omega (q'(p_0 + \varepsilon\tilde{p}^\varepsilon)\varepsilon\tilde{p}_t^\varepsilon + q^\varepsilon \operatorname{div} \tilde{u}^\varepsilon + \tilde{u}^\varepsilon \cdot \nabla q^\varepsilon) P_\alpha^2 dx \\ &\quad + \frac{1}{2} \int_\Omega (\rho'(p_0 + \varepsilon\tilde{p}^\varepsilon)\varepsilon\tilde{p}_t^\varepsilon + \tilde{\rho}^\varepsilon \operatorname{div} \tilde{u}^\varepsilon + \tilde{u}^\varepsilon \cdot \nabla \tilde{\rho}^\varepsilon) |U_\alpha|^2 dx \\ &\leq C(\|\operatorname{div} \tilde{u}^\varepsilon\|_{L^\infty} + \|\tilde{u}^\varepsilon\|_{L^\infty} \|\nabla \tilde{p}^\varepsilon\|_{L^\infty})(\|P_\alpha\|^2 + \|U_\alpha\|^2) \\ &\leq C(M_0)(1 + D^2)(\|P_\alpha\|^2 + \|U_\alpha\|^2), \end{aligned}$$

with the help of (1.8)₁.

Thanks to Lemma 3.1, I_2 can be simply treated as

$$\begin{aligned} I_2 &\leq C\|P_\alpha\|(\|\partial_x^\alpha(U \cdot \nabla p_\varepsilon)\| + \|[\partial_x^\alpha, \tilde{u}^\varepsilon]\nabla P\|) \\ &\quad + C\|U_\alpha\|(\|\partial_x^\alpha(U \cdot \nabla u_\varepsilon)\| + \|[\partial_x^\alpha, \tilde{u}^\varepsilon]\nabla U\|) \\ &\leq C\|P_\alpha\|(\|\nabla p_\varepsilon\|_4\|U\|_4 + \|\nabla \tilde{u}^\varepsilon\|_3\|\nabla P\|_3) \\ &\quad + C\|U_\alpha\|(\|\nabla u_\varepsilon\|_4\|U\|_4 + \|\nabla \tilde{u}^\varepsilon\|_3\|\nabla U\|_3) \\ &\leq C(M_0)(1 + D)(\|U\|_4^2 + \|P\|_4^2). \end{aligned}$$

For I_3 , from Lemma 5.2, we first compute

$$\|\nabla \rho^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)\|_3, \quad \|\nabla q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)\|_3 \leq C(M_0)\varepsilon(1 + D^4). \tag{5.11}$$

Then we have

$$\begin{aligned} I_3 &\leq \frac{1}{\varepsilon}(\|[\partial_x^\alpha, q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)] \operatorname{div} U\| \|q^\varepsilon P_\alpha\| + \|[\partial_x^\alpha, \rho^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)] \nabla P\| \|\tilde{\rho}^\varepsilon U_\alpha\|) \\ &\leq \frac{C}{\varepsilon}(\|\nabla q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)\|_3 \|\operatorname{div} U\|_3 \|P_\alpha\| + \|\nabla \rho^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)\|_3 \|\nabla P\|_3 \|U_\alpha\|) \\ &\leq C(M_0)(1 + D^4)(\|\operatorname{div} U\|_3 \|P_\alpha\| + \|\nabla P\|_3 \|U_\alpha\|) \\ &\leq C(M_0)(1 + D^4)(\|U\|_4^2 + \|P\|_4^2). \end{aligned}$$

To estimate I_4 , we first use integration by parts to obtain

$$\begin{aligned} I_4 &= -\varepsilon^{-1}\kappa' \int_\Omega \Delta \tilde{\rho}_\alpha^\varepsilon (\varepsilon \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) U_\alpha \nabla \tilde{p}^\varepsilon + \tilde{\rho}^\varepsilon \operatorname{div} U_\alpha) dx \\ &\leq C(M_0)(1 + D) \|\Delta \tilde{\rho}_\alpha^\varepsilon\| \|U_\alpha\| - \varepsilon^{-1}\kappa' \int_\Omega \tilde{\rho}^\varepsilon \Delta \tilde{\rho}_\alpha^\varepsilon \operatorname{div} U_\alpha dx. \end{aligned} \tag{5.12}$$

As

$$\tilde{\rho}^\varepsilon \operatorname{div} U_\alpha = \tilde{\rho}^\varepsilon \operatorname{div} \partial_x^\alpha \tilde{u}^\varepsilon = -(\partial_t \tilde{\rho}_\alpha^\varepsilon + \partial_x^\alpha (\tilde{u}^\varepsilon \cdot \nabla \tilde{\rho}^\varepsilon) + [\partial_x^\alpha, \tilde{\rho}^\varepsilon] \operatorname{div} \tilde{u}^\varepsilon),$$

the second term on the right-hand side of (5.12) can be estimated as

$$\begin{aligned} &-\varepsilon^{-1}\kappa' \int_\Omega \tilde{\rho}^\varepsilon \Delta \tilde{\rho}_\alpha^\varepsilon \operatorname{div} U_\alpha dx \\ &= \varepsilon^{-1}\kappa' \int_\Omega \Delta \tilde{\rho}_\alpha^\varepsilon \partial_t \tilde{\rho}_\alpha^\varepsilon dx + \varepsilon^{-1}\kappa' \int_\Omega \Delta \tilde{\rho}_\alpha^\varepsilon (\partial_x^\alpha (\tilde{u}^\varepsilon \cdot \nabla \tilde{\rho}^\varepsilon) + [\partial_x^\alpha, \tilde{\rho}^\varepsilon] \operatorname{div} \tilde{u}^\varepsilon) dx \\ &\leq -\frac{\kappa'}{2\varepsilon} \frac{d}{dt} \int_\Omega |\nabla \tilde{\rho}_\alpha^\varepsilon|^2 dx + C\varepsilon^{-1} \|\Delta \tilde{\rho}_\alpha^\varepsilon\| \|\tilde{u}^\varepsilon\|_4 \|\nabla \tilde{\rho}^\varepsilon\|_4. \end{aligned}$$

Since

$$\nabla \tilde{\rho}^\varepsilon = \varepsilon \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla \tilde{p}^\varepsilon = \varepsilon \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) (\nabla P + \nabla p_\varepsilon)$$

and

$$\begin{aligned} \Delta \tilde{\rho}_\alpha^\varepsilon &= \varepsilon \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \Delta P_\alpha + \varepsilon [\partial_x^\alpha, \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)] \Delta P + \varepsilon \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \Delta p_\varepsilon) \\ &\quad + \varepsilon^2 \partial_x^\alpha (\rho''(p_0 + \varepsilon \tilde{p}^\varepsilon) (\nabla \tilde{p}^\varepsilon)^2), \end{aligned} \tag{5.13}$$

we have

$$\|\nabla \tilde{\rho}^\varepsilon\|_4 \leq C(M_0) \left(\varepsilon \sum_{|\alpha|=4} \|\nabla P_\alpha\| + \varepsilon (1 + D^4) \|\nabla P\|_3 + \varepsilon^2 \right)$$

and

$$\|\Delta \tilde{\rho}_\alpha^\varepsilon\| \leq C(M_0) (\varepsilon \|\Delta P_\alpha\| + \varepsilon (1 + D^4) \|\Delta P\|_3 + \varepsilon (1 + D^5) \|P\|_4 + \varepsilon^2). \tag{5.14}$$

Moreover, due to

$$\begin{aligned} \nabla \tilde{\rho}_\alpha^\varepsilon &= \varepsilon \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla \tilde{p}^\varepsilon) \\ &= \varepsilon \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P_\alpha + \varepsilon [\partial_x^\alpha, \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)] \nabla P + \varepsilon \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla p_\varepsilon), \end{aligned}$$

a straightforward calculation yields

$$\begin{aligned} &-\frac{d}{dt} \int_\Omega |\nabla \tilde{\rho}_\alpha^\varepsilon|^2 dx \\ &= -\varepsilon^2 \frac{d}{dt} \int_\Omega |\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P_\alpha|^2 dx \\ &\quad - 2\varepsilon^2 \int_\Omega (\partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla p_\varepsilon) + [\partial_x^\alpha, \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)] \nabla P \\ &\quad + \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P_\alpha) \cdot \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla p_{\varepsilon t} + \varepsilon \rho''(p_0 + \varepsilon \tilde{p}^\varepsilon) \tilde{p}_t^\varepsilon \nabla p_\varepsilon) dx \\ &\quad - 2\varepsilon^2 \int_\Omega ([\partial_x^\alpha, \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)] \nabla P + \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P_\alpha + \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla p_\varepsilon)) \\ &\quad \cdot ([\partial_x^\alpha, \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)] \nabla P_t + [\partial_x^\alpha, \varepsilon \rho''(p_0 + \varepsilon \tilde{p}^\varepsilon) \tilde{p}_t^\varepsilon] \nabla P) dx \\ &\quad - 2\varepsilon^2 \int_\Omega (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P_{\alpha t} \\ &\quad + \varepsilon \rho''(p_0 + \varepsilon \tilde{p}^\varepsilon) \tilde{p}_t^\varepsilon \nabla P_\alpha) \cdot ([\partial_x^\alpha, \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)] \nabla P + \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla p_\varepsilon)) dx \\ &=: -\varepsilon^2 \frac{d}{dt} \int_\Omega |\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P_\alpha|^2 dx + I_{41} + I_{42} + I_{43}. \end{aligned} \tag{5.15}$$

Using (1.8)₂ and Lemma 4.1, Lemma 3.1, Lemma 5.1–5.2, and integration by parts, it is easy to obtain

$$\begin{aligned} I_{41} &= -2\varepsilon^2 \int_\Omega F \cdot \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla p_{\varepsilon t}) dx \\ &\quad + 2\varepsilon^2 \int_\Omega F \cdot \partial_x^\alpha (\rho''(p_0 + \varepsilon \tilde{p}^\varepsilon) (q(p_0 + \varepsilon \tilde{p}^\varepsilon))^{-1} \operatorname{div} u_\varepsilon \\ &\quad + \varepsilon (U + u_\varepsilon) \nabla (P + p_\varepsilon)) \nabla p_\varepsilon) dx \end{aligned}$$

$$\begin{aligned}
 & -2\varepsilon^2 \int_{\Omega} \nabla(q(p_0 + \varepsilon\tilde{p}^\varepsilon)^{-1}F) \cdot \partial_x^\alpha U \, dx + 2\varepsilon^2 \int_{\Omega} F \cdot [\partial_x^\alpha, q(p_0 + \varepsilon\tilde{p}^\varepsilon)^{-1}] \operatorname{div} U \, dx \\
 & \leq C(M_0)(1 + D^8)(\|P\|_4^2 + \|\varepsilon^{\frac{1}{2}}\nabla P_\alpha\|^2 + \|U\|_4^2) + \frac{\delta}{3}\|\varepsilon^{\frac{1}{2}}\Delta P_\alpha\|^2 + C(M_0)\varepsilon^2
 \end{aligned}$$

with $F = \partial_x^\alpha(\rho'(p_0 + \varepsilon\tilde{p}^\varepsilon)\nabla p_\varepsilon) + [\partial_x^\alpha, \rho'(p_0 + \varepsilon\tilde{p}^\varepsilon)]\nabla P + \rho'(p_0 + \varepsilon\tilde{p}^\varepsilon)\nabla P_\alpha$, here and in the following, δ is a proper positive constant, which is determined. Moreover, with the help of (5.1)₁ and using Lemma 4.1, Lemma 3.1, and Lemma 5.1–5.2, we can obtain

$$I_{42} \leq C(M_0)(1 + D^{10})(\|P\|_4^2 + \|\varepsilon^{\frac{1}{2}}\nabla P_\alpha\|^2 + \|U\|_4^2) + \frac{\delta}{3}\|\varepsilon^{\frac{1}{2}}\Delta P_\alpha\|^2 + C(M_0)\varepsilon^2.$$

Similarly, using (5.4), Lemma 4.1, Lemma 3.1, and Lemma 5.1–5.2, we can obtain

$$I_{43} \leq C(M_0)(1 + D^{10})(\|P\|_4^2 + \|\varepsilon^{\frac{1}{2}}\nabla P_\alpha\|^2 + \|U\|_4^2) + \frac{\delta}{3}\|\varepsilon^{\frac{1}{2}}\Delta P_\alpha\|^2 + C(M_0)\varepsilon^2.$$

Therefore, substitution of the above inequalities and (5.15) into (5.12) yields

$$\begin{aligned}
 I_4 \leq & -\frac{\kappa'\varepsilon}{2} \frac{d}{dt} \int_{\Omega} |\rho'(p_0 + \varepsilon\tilde{p}^\varepsilon)\nabla P_\alpha|^2 \, dx + 2\delta\|\varepsilon^{\frac{1}{2}}\Delta P_\alpha\|^2 \\
 & + C(M_0)(1 + D^{10})(\|P\|_4^2 + \|\varepsilon^{\frac{1}{2}}\nabla P_\alpha\|^2 + \|U\|_4^2) + \delta\|\varepsilon^{\frac{1}{2}}\Delta P_\alpha\|^2 + C(M_0)\varepsilon^2.
 \end{aligned}$$

Finally, from (5.6), we deduce that

$$I_5 \leq C(M_0)\varepsilon^2 + C(M_0)(1 + D)(\|P\|_4^2 + \|U_\alpha\|^2).$$

Hence, putting the estimates of I_i ($i = 1, 2, \dots, 5$) into (5.9), we have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (q^\varepsilon P_\alpha^2 + \tilde{\rho}^\varepsilon |U_\alpha|^2 + \kappa'\varepsilon |\rho'(p_0 + \varepsilon\tilde{p}^\varepsilon)\nabla P_\alpha|^2) \, dx \\
 & \leq C(M_0)(\varepsilon^2 + (1 + D^{10})(\|\varepsilon^{\frac{1}{2}}\nabla P_\alpha\|^2 + \|P\|_4^2 + \|U\|_4^2)) + \delta\|\varepsilon^{\frac{1}{2}}\Delta P_\alpha\|^2. \tag{5.16}
 \end{aligned}$$

To control the term with δ , we multiply (5.8) by $\varepsilon\rho(p_0 + \varepsilon\tilde{p}^\varepsilon)\nabla P_\alpha$ and integrate the resultant equality by parts over Ω to obtain

$$\begin{aligned}
 & \frac{d}{dt} \int_{\Omega} \varepsilon\tilde{\rho}^\varepsilon U_\alpha \nabla P_\alpha \, dx + \int_{\Omega} |\nabla P_\alpha|^2 \, dx \\
 & = \left(\varepsilon \int_{\Omega} \tilde{\rho}^\varepsilon U_\alpha \nabla P_{\alpha t} \, dx + \varepsilon \int_{\Omega} \tilde{\rho}_t^\varepsilon U_\alpha \nabla P_\alpha \, dx \right) \\
 & \quad - \varepsilon \int_{\Omega} \tilde{\rho}^\varepsilon \nabla P_\alpha (\tilde{u}^\varepsilon \cdot \nabla U_\alpha + \partial_x^\alpha(U \cdot \nabla u_\varepsilon)) \\
 & \quad + [\partial_x^\alpha, \tilde{u}^\varepsilon] \nabla U + \varepsilon^{-1} [\partial_x^\alpha, \rho^{-1}(p_0 + \varepsilon\tilde{p}^\varepsilon)] \nabla P \, dx \\
 & \quad + \kappa' \int_{\Omega} \tilde{\rho}^\varepsilon \nabla \Delta \tilde{\rho}_\alpha^\varepsilon \nabla P_\alpha \, dx + \varepsilon \int_{\Omega} \tilde{\rho}^\varepsilon \partial_x^\alpha f_2 \nabla P_\alpha \, dx \\
 & =: H_1 + H_2 + H_3 + H_4. \tag{5.17}
 \end{aligned}$$

We estimate the H_i as follows. By using integration by parts, it follows from (5.7), (1.8)₁, Lemma 3.1, and Lemma 5.1–5.2 that

$$\begin{aligned}
 H_1 &= \varepsilon \int_{\Omega} (\rho(p_0 + \varepsilon \tilde{p}^\varepsilon) \operatorname{div} U_\alpha + \varepsilon \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla \tilde{p}^\varepsilon U_\alpha) (\tilde{u}^\varepsilon \cdot \nabla P_\alpha + [\partial_x^\alpha, \tilde{u}^\varepsilon] \nabla P \\
 &\quad + \partial_x^\alpha (U \cdot \nabla p_\varepsilon) + \varepsilon^{-1} [\partial_x^\alpha, q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)] \operatorname{div} U - \partial_x^\alpha f_1) \, dx \\
 &\quad - \varepsilon \int_{\Omega} \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) (\varepsilon \tilde{u}^\varepsilon \nabla \tilde{p}^\varepsilon + q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon) \operatorname{div} \tilde{u}^\varepsilon) U_\alpha \nabla P_\alpha \, dx \\
 &\leq C\varepsilon (\|\operatorname{div} U_\alpha\| + \varepsilon \|\nabla \tilde{p}^\varepsilon U_\alpha\|) (\|\tilde{u}^\varepsilon \nabla P_\alpha\| + \|[\partial_x^\alpha, \tilde{u}^\varepsilon] \nabla P\| + \|\partial_x^\alpha (U \cdot \nabla p_\varepsilon)\| \\
 &\quad + \varepsilon^{-1} \|[\partial_x^\alpha, q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)] \operatorname{div} U\| + \|\partial_x^\alpha f_1\|) \\
 &\leq C\varepsilon (\|\operatorname{div} U_\alpha\| + \varepsilon \|\nabla \tilde{p}^\varepsilon\|_{L^\infty} \|U_\alpha\|) \\
 &\quad \times (\|\tilde{u}^\varepsilon\|_{L^\infty} \|\nabla P_\alpha\| + \|\nabla \tilde{u}^\varepsilon\|_3 \|\nabla P\|_3 + \|U\|_4 \|\nabla p_\varepsilon\|_4 \\
 &\quad + \varepsilon^{-1} \|\nabla q^{-1}(p_0 + \varepsilon \tilde{p}^\varepsilon)\|_3 \|\operatorname{div} U\|_3 + \|\partial_x^\alpha f_1\|) \\
 &\leq \frac{\delta}{3} \|\varepsilon^{\frac{1}{2}} \Delta P_\alpha\|^2 + C(M_0)\varepsilon^2 + C(M_0)(1 + D^4) (\|P\|_4^2 + \|\varepsilon^{\frac{1}{2}} \nabla P_\alpha\|^2 + \|U\|_4^2).
 \end{aligned}$$

Next, as for I_2, I_3 and I_5 , it is easy to compute

$$H_2 \leq \frac{\delta}{3} \|\varepsilon^{\frac{1}{2}} \Delta P_\alpha\|^2 + C(M_0)(1 + D^8) (\|P\|_4^2 + \|\varepsilon^{\frac{1}{2}} \nabla P_\alpha\|^2 + \|U\|_4^2)$$

and

$$H_4 \leq C(M_0)\varepsilon^2 + C(M_0) \|\varepsilon^{\frac{1}{2}} \nabla P_\alpha\|^2.$$

Finally, noting (5.13) and (5.14), we have

$$\begin{aligned}
 H_3 &= -\kappa' \int_{\Omega} (\rho(p_0 + \varepsilon \tilde{p}^\varepsilon) \Delta \tilde{\rho}_\alpha^\varepsilon \Delta P_\alpha + \varepsilon \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \Delta \tilde{\rho}_\alpha^\varepsilon \nabla \tilde{p}^\varepsilon \nabla P_\alpha) \, dx \\
 &= -\kappa' \varepsilon \int_{\Omega} \rho(p_0 + \varepsilon \tilde{p}^\varepsilon) \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) (\Delta P_\alpha)^2 \, dx \\
 &\quad - \kappa' \varepsilon \int_{\Omega} \rho(p_0 + \varepsilon \tilde{p}^\varepsilon) \Delta P_\alpha ([\partial_x^\alpha, \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)] \Delta P \\
 &\quad + \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \Delta p_\varepsilon) + \varepsilon \partial_x^\alpha (\rho''(p_0 + \varepsilon \tilde{p}^\varepsilon) (\nabla \tilde{p}^\varepsilon)^2)) \, dx \\
 &\quad - \kappa' \varepsilon \int_{\Omega} \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla \tilde{p}^\varepsilon \nabla P_\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \Delta P_\alpha + [\partial_x^\alpha, \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon)] \Delta P \\
 &\quad + \varepsilon \partial_x^\alpha (\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \Delta p_\varepsilon) + \varepsilon \partial_x^\alpha (\rho''(p_0 + \varepsilon \tilde{p}^\varepsilon) (\nabla \tilde{p}^\varepsilon)^2)) \, dx \\
 &\leq -\kappa' \varepsilon \int_{\Omega} \rho(p_0 + \varepsilon \tilde{p}^\varepsilon) \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) (\Delta P_\alpha)^2 \, dx + \frac{\delta}{3} \|\varepsilon^{\frac{1}{2}} \Delta P_\alpha\|^2 \\
 &\quad + C(M_0)(1 + D^{10}) (\|P\|_4^2 + \|\varepsilon^{\frac{1}{2}} \nabla P\|_4^2) + C(M_0)\varepsilon^2.
 \end{aligned}$$

Hence, inserting the above estimates of the H_i into (5.17) yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \varepsilon \rho(p_0 + \varepsilon \tilde{p}^\varepsilon) U_\alpha \nabla P_\alpha \, dx + \int_{\mathbb{R}^3} (|\nabla P_\alpha|^2 + \kappa' \varepsilon \rho(p_0 + \varepsilon \tilde{p}^\varepsilon) \rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) (\Delta P_\alpha)^2) \, dx \\ & \leq \delta \|\varepsilon^{\frac{1}{2}} \Delta P_\alpha\|^2 + C(M_0)(1 + D^{10})(\|P\|_4^2 + \|\varepsilon^{\frac{1}{2}} \nabla P\|_4^2 + \|U\|_4^2) + C(M_0)\varepsilon^2. \end{aligned} \tag{5.18}$$

Finally, choose the proper constant $\lambda > 0$, which satisfies

$$\frac{1}{2}(\tilde{\rho}^\varepsilon |U_\alpha|^2 + \kappa' \varepsilon |\rho'(p_0 + \varepsilon \tilde{p}^\varepsilon) \nabla P_\alpha|^2) + \lambda \varepsilon \rho(p_0 + \varepsilon \tilde{p}^\varepsilon) U_\alpha \nabla P_\alpha \geq \frac{1}{4}(|U_\alpha|^2 + \kappa' \varepsilon |\nabla P_\alpha|^2).$$

Further, let us take $\delta > 0$ with $\delta < \frac{\lambda}{2(\lambda+1)} \kappa' \rho(\frac{1}{2}p_0)c_1$. Therefore, combining (5.16) and (5.18), we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\|P\|_4^2 + \|U\|_4^2 + \sum_{|\beta|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\beta P\|^2 \right) + \left(\|\nabla U\|_4^2 + \|\nabla P\|_4^2 + \sum_{|\gamma|=4} \|\varepsilon^{\frac{1}{2}} \Delta \partial_x^\gamma P\|^2 \right) \\ & \leq C(M_0)(1 + D^{10}) \left(\|P\|_4^2 + \|U\|_4^2 + \sum_{|\beta|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\beta P\|^2 \right) + C(M_0)\varepsilon^2. \end{aligned} \tag{5.19}$$

Then we integrate (5.19) from 0 to T with $[0, T] \subset [0, \min\{T_\varepsilon, T_0\})$ to obtain

$$\begin{aligned} & \|P\|_4^2 + \|U\|_4^2 + \sum_{|\beta|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\beta P\|^2 + \int_0^T \left(\|\nabla U\|_4^2 + \|\nabla P\|_4^2 + \sum_{|\gamma|=4} \|\varepsilon^{\frac{1}{2}} \Delta \partial_x^\gamma P\|^2 \right) dt \\ & \leq C(M_0)T\varepsilon^2 + C(M_0) \int_0^T (1 + D^{10}) \left(\|P\|_4^2 + \|U\|_4^2 + \sum_{|\beta|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\beta P\|^2 \right) dt. \end{aligned} \tag{5.20}$$

Here, we have used the fact that the initial data are in equilibrium. Furthermore, we apply Gronwall's inequality (refer to Theorem 6.2 in [10]) to (5.20) to obtain

$$\|P\|_4^2 + \|U\|_4^2 + \sum_{|\beta|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\beta P\|^2 \leq C(M_0)T_0\varepsilon^2 \exp \left[C(M_0) \int_0^T (1 + D^{10}) dt \right]. \tag{5.21}$$

Denote by $\varepsilon Q(T)$ the right-hand side of (5.21), that is,

$$Q(T) = C(M_0)T_0\varepsilon \exp \left[C(M_0) \int_0^T (1 + D^{10}) dt \right].$$

Recall that $\|P\|_4^2 + \|U\|_4^2 + \sum_{|\beta|=5} \|\varepsilon^{\frac{1}{2}} \partial_x^\beta P\|^2 = D^2$; then, due to (5.21) and $\varepsilon \in (0, 1)$, it follows that

$$D(T)^2 \leq \varepsilon Q(T) \leq Q(T). \tag{5.22}$$

Moreover,

$$Q'(t) = C(M_0)(1 + D^{10})Q(t) \leq C(M_0)Q(t) + C(M_0)Q^6(t), \quad t \in [0, T].$$

Applying the nonlinear Gronwall-type inequality (refer to Lemma 6.3 in [36]) to the last inequality yields

$$Q(t) \leq e^{CT_0}$$

for $t \in [0, T] \subset [0, \min\{T_0, T_\varepsilon\})$ if we choose ε so small that

$$Q(0) = C(M_0)T_0\varepsilon \leq e^{-C(M_0)T_0}.$$

That is, $Q(T)$ is uniformly bound. Because of (5.22), there exists a constant c , independent of ε , such that

$$D(T) \leq c, \tag{5.23}$$

for $T \in [0, \min\{T_0, T_\varepsilon\})$. Finally, estimate (4.4) holds from (5.21) and (5.23). This completes the proof of Theorem 4.1.

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Competing interests

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Authors' contributions

Both authors declare that they have made equal contributions in this article. Both authors have read and approved the final manuscript.

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