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# Infinitely many positive solutions for a nonlocal problem with competing potentials

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**Abstract**

The present paper deals with a class of nonlocal problems. Under some suitable assumptions on the decay rate of the coefficients, we derive the existence of infinitely many positive solutions to the problem by applying reduction method. Comparing to the previous work, we encounter some new challenges because of competing potentials. By doing some delicate estimates for the competing potentials, we overcome the difficulties and find infinitely many positive solutions.

**MSC:** 35B40; 35J40

**Keywords:** Competing potentials; Infinitely many solutions; Lyapunov–Schmidt reduction

## 1 Introduction and main results

In this paper, we study the fractional Schrödinger problem

$$(-\Delta)^\sigma u + A(y)u = B(y)u^p, \quad y \in \mathbb{R}^n, \tag{1.1}$$

where $0 < \sigma < 1$, $n \geq 2$, $1 < p < \frac{n+2\sigma}{n-2\sigma}$ and $A(y)$, $B(y)$ are two radially symmetric potentials. Here the fractional Laplacian $(-\Delta)^\sigma$ is defined by

$$(-\Delta)^\sigma u = C_{n,\sigma} P.V. \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x - y|^{n+2\sigma}} \, dx,$$

where P.V. stands for the Cauchy principal value and $C_{n,\sigma}$ is a normalization constant.

Problem (1.1) has attracted considerable attention in the recent period and part of the motivation is due to looking for a standing wave $\psi = e^{-iht}u$ of the evolution equation

$$i\frac{\partial \psi}{\partial t} + (-\Delta)^\sigma \psi - \big(A(y) + h\big)\psi = |\psi|^{p-1}\psi, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{1.2}$$

since $\psi$ solves (1.2) if and only if $u$ solves (1.1), where $i$ is the imaginary unit and $h \in \mathbb{R}$. This class of Schrödinger-type equations is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modeled by Lévy processes. In recent years, there have been many investigations for the related fractional Schrödinger equation

$$(-\Delta)^{\sigma} u + V(y)u = f(y,u), \quad y \in \mathbb{R}^n$$

with $0 < \sigma < 1$ and $V : \mathbb{R}^n \to \mathbb{R}$ is an external potential function. A complete review of the available results in this context goes beyond the aim of this paper; we refer the interested reader to [4, 5, 9, 13–17, 19–22] and the references therein.

Especially, in [19] we studied (1.1) and infinitely many nonradial positive (sign-changing) solutions were established when $A(y) = 1$ and $B(y)$ satisfies some radial symmetry assumption by using Lyapunov–Schmidt reduction. In this paper, continuing our study in [19], we are concerned with the multiplicity of positive solutions for (1.1) in a situation in which there exist two competing potentials and even (1.1) may not have ground states.

To the best of our knowledge, not much is obtained for the existence of multiple solutions of Eq. (1.1) with competing potentials. So our purpose of this paper is to establish the existence of infinitely many nonradial positive solutions for (1.1) by constructing solutions with large number of bumps near the infinity under some assumptions for $A(y)$, $B(y)$ as follows:

$(A)$ there are constants $a > 0$, $m_1 > 0$, $\theta_1 > 0$ such that

$$A(|y|) = 1 + \frac{a}{|y|^{m_1}} + O\left(\frac{1}{|y|^{m_1+\theta_1}}\right) \quad \text{as } |y| \to +\infty;$$

$(B)$ there are constants $b \in \mathbb{R}$, $m_2 > 0$, $\theta_2 > 0$ such that

$$B(|y|) = 1 + \frac{b}{|y|^{m_2}} + O\left(\frac{1}{|y|^{m_2+\theta_2}}\right) \quad \text{as } |y| \to +\infty.$$

Our main results in this paper can be stated as follows.

**Theorem 1.1** *Suppose that* $n \geq 2$, $1 < p < \frac{n+2\sigma}{n-2\sigma}$, $\frac{n+2\sigma}{n+2\sigma+1} < \min\{m_1, m_2\} < n + 2\sigma$ *and the conditions* $(A)$ *and* $(B)$ *hold. If* $b < 0$ *or* $b > 0$ *and* $m_1 < m_2$, *then problem* (1.1) *has infinitely many nonradial positive solutions.*

To achieve our goal, we adopt a novel idea introduced in [23], by using $k$, the number of the bumps of the solutions, as the parameter in the construction of solutions for (1.1). In [23], the authors studied the following equation:

$$-\Delta u + V(y)u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n, u \in H^1(\mathbb{R}^n) \tag{1.3}$$

and applying the reduction method, they derived the existence of infinitely many solutions to (1.3) by exhibiting bumps at the vertices of the regular $k$-polygons for sufficiently large $k \in \mathbb{N}$ under some suitable conditions on $V(y)$ and $p$. But, in this paper, since the competing terms appear, we have to overcome many difficulties in the reduction process which involves some technical and careful computations. Furthermore, for more results on the existence of radial ground states, infinitely many bound states or nonradial solutions, higher energy bound states to (1.3), one can refer to [1–3, 6–8, 11, 12, 18] and the references therein.

In the end of this part, let us outline the main idea to prove our main results. For any integer $k > 0$, we define

$$y^i = \left( r\cos\frac{2(i-1)\pi}{k}, r\sin\frac{2(i-1)\pi}{k}, 0 \right), \quad i = 1,\dots,k,$$

where 0 is the zero vector in $\mathbb{R}^{n-2}$, $r \in [r_0 k^{\frac{n+2\sigma}{n+2\sigma-m}}, r_1 k^{\frac{n+2\sigma}{n+2\sigma-m}}]$ for some $r_1 > r_0 > 0$ with $m := \min\{m_1, m_2\}$. Also we denote by $H^\sigma(\mathbb{R}^n)$ the usual Sobolev space endowed with the standard norm

$$\|u\|_\sigma^2 = \int_{\mathbb{R}^n} \left( \left|(-\Delta)^{\frac{\sigma}{2}} u\right|^2 + u^2 \right).$$

Moreover, for $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, set

$$\begin{aligned} H_k = \Big\{ & u : u \in H^\sigma(\mathbb{R}^n), u \text{ is even in } y_j, j = 2,\dots,n, \\ & u(r\cos\theta, r\sin\theta, y'') = u\left( r\cos\left(\theta + \frac{2i\pi}{k}\right), r\sin\left(\theta + \frac{2i\pi}{k}\right), y'' \right) \Big\}. \end{aligned}$$

In what follows we will use the unique ground state $U$ of

$$(-\Delta)^\sigma u + u = u^p, \quad u > 0, y \in \mathbb{R}^n, \tag{1.4}$$

to build up the approximate solutions for (1.1). It is well known that in [16, 17], the authors have established the uniqueness and non-degeneracy of the ground state of (1.4) with

$$\frac{C_1}{1 + |y|^{n+2\sigma}} \le U(y) \le \frac{C_2}{1 + |y|^{n+2\sigma}}, \quad y \in \mathbb{R}^n, \tag{1.5}$$

and

$$\left|\partial_{y_j} U(y)\right| \le \frac{C}{1 + |y|^{n+2\sigma}}, \quad j = 1, 2, \dots, n. \tag{1.6}$$

Now if we define

$$W_r(y) = \sum_{i=1}^{k} U_{y^i}(y),$$

where $U_{y^i}(y) = U(y - y^i)$, then we will prove Theorem 1.1 by verifying the following result.

**Theorem 1.2** *Under the assumption of Theorem 1.1, there is an integer $k_0 > 0$, such that, for any integer $k \ge k_0$, (1.1) has a solution $u_k$ of the form*

$$u_k = W_{r_k}(y) + \varphi_{r_k},$$

*where $\varphi_{r_k} \in H_k$, $r_k \in [r_0 k^{\frac{n+2\sigma}{n+2\sigma-m}}, r_1 k^{\frac{n+2\sigma}{n+2\sigma-m}}]$ for some constants $r_1 > r_0 > 0$ and as $k \to +\infty$,*

$$\int_{\mathbb{R}^n} \left( \left|(-\Delta)^{\frac{\sigma}{2}} \varphi_{r_k}\right|^2 + \varphi_{r_k}^2 \right) \to 0.$$

This paper is organized as follows. In Sect. 2, we will carry out a reduction procedure and then study the reduced one dimensional problem to prove Theorem 1.2 in Sect. 3. Some basic estimates and an energy expansion for the functional are left to the Appendix.

## 2 The reduction

In the following, we always assume that $k \in \mathbb{N}$ is a large number. Let

$$Z^j = \frac{\partial U_{y^j}}{\partial r}, \quad j = 1, \ldots, k,$$

where $y^j = (r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0)$ and

$$r \in S_k := \left[r_0 k^{\frac{n+2\sigma}{n+2\sigma-m}}, r_1 k^{\frac{n+2\sigma}{n+2\sigma-m}}\right],$$

where $r_0 = (\frac{h_0(n+2\sigma)}{h_1 m} - \alpha)^{\frac{1}{n+2\sigma-m}}$, $r_1 = (\frac{h_0(n+2\sigma)}{h_1 m} + \alpha)^{\frac{1}{n+2\sigma-m}}$, $\alpha > 0$ is a small constant and $h_0$, $h_1$ will be given in Sect. 3.

Define

$$E_r = \left\{ v : v \in H_k, \int_{\mathbb{R}^n} U_{y^1}^{p-1} Z^1 v = 0 \right\}.$$

Note that the variational functional corresponding to (1.1) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(\left|(-\Delta)^{\frac{\sigma}{2}} u\right|^2 + A(y)u^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} B(y)|u|^{p+1}.$$

Let

$$J(\varphi) = I(W_r + \varphi) = I\left(\sum_{j=1}^{k} U_{y^j} + \varphi\right), \quad \varphi \in E_r.$$

We can expand $J(\varphi)$ as follows:

$$J(\varphi) = J(0) + l(\varphi) + \frac{1}{2}\langle L(\varphi), \varphi\rangle + R(\varphi), \quad \varphi \in E_r, \tag{2.1}$$

where

$$l(\varphi) = \int_{\mathbb{R}^n} \sum_{j=1}^{k} U_{y^j}^p \varphi + \int_{\mathbb{R}^n} \left(A(|y|) - 1\right) W_r \varphi - \int_{\mathbb{R}^n} B(|y|) W_r^p \varphi,$$

$$\langle L(\varphi), \varphi\rangle = \int_{\mathbb{R}^n} \left(\left|(-\Delta)^{\frac{\sigma}{2}} \varphi\right|^2 + A(|y|)\varphi^2\right) - p\int_{\mathbb{R}^n} B(|y|) W_r^{p-1}\varphi^2$$

and

$$R(\varphi) = -\frac{1}{p+1}\int_{\mathbb{R}^n} B(|y|)\left((W_r + \varphi)^{p+1} - W_r^{p+1} - (p+1)W_r^p \varphi - \frac{1}{2}(p+1)p W_r^{p-1}\varphi^2\right).$$

In this part, we shall find a map $\varphi(r)$ from $S_k$ to $E_r$ such that $\varphi(r)$ is a critical point of $J(\varphi)$ under the constraint $\varphi(r) \in E_r$. Associated to the quadratic form $L(\varphi)$, we define $L$ to be a bounded linear map from $E_r$ to $E_r$ such that

$$\langle L\varphi, v\rangle = \int_{\mathbb{R}^n} \big((-\Delta)^{\frac{\sigma}{2}}\varphi(-\Delta)^{\frac{\sigma}{2}}v + A(|y|)\varphi v\big) - p\int_{\mathbb{R}^n} B(|y|)W_r^{p-1}\varphi v, \quad v \in E_r.$$

Then we have the following lemma, which shows the invertibility of $L$ in $E_r$.

**Lemma 2.1** *There is a constant $\rho > 0$ independent of $k$, such that, for any $r \in S_k$,*

$$\|L\varphi\| \geq \rho\|\varphi\|_\sigma, \quad \forall \varphi \in E_r.$$

*Proof* Arguing by contradiction, we suppose that there are $k \to +\infty$, $r_k \in S_k$, and $\varphi_k \in E_r$ such that

$$\|L\varphi_k\| = o(1)\|\varphi_k\|_\sigma \quad \text{with } \|\varphi_k\|_\sigma^2 = k.$$

Set

$$\Omega_i = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2} : \left\langle \frac{y'}{|y'|}, \frac{(y^i)'}{|(y^i)'|} \right\rangle \geq \cos\frac{\pi}{k} \right\}, \quad i = 1, 2, \ldots, k.$$

By symmetry, we have for $v \in E_r$

$$\begin{aligned} &\int_{\Omega_1} \big((-\Delta)^{\frac{\sigma}{2}}\varphi_k(-\Delta)^{\frac{\sigma}{2}}v + A(|y|)\varphi_k v\big) - p\int_{\Omega_1} B(|y|)W_{r_k}^{p-1}\varphi_k v \\ &\quad = \frac{1}{k}\langle L\varphi_k, v\rangle = o\left(\frac{1}{\sqrt{k}}\right)\|v\|_\sigma. \end{aligned} \tag{2.2}$$

In particular,

$$\int_{\Omega_1} \big(\big|(-\Delta)^{\frac{\sigma}{2}}\varphi_k\big|^2 + A(|y|)\varphi_k^2\big) - p\int_{\Omega_1} B(|y|)W_{r_k}^{p-1}\varphi_k^2 = o_k(1)$$

and

$$\int_{\Omega_1} \big(\big|(-\Delta)^{\frac{\sigma}{2}}\varphi_k\big|^2 + \varphi_k^2\big) = 1.$$

Let $\tilde{\varphi}_k = \varphi(y + y^1)$. Since for any $R > 0$, $\operatorname{dist}(y^1, \partial\Omega_1) = r\sin\frac{\pi}{k}$, $B_R(y^1) \subset \Omega_1$. Thus

$$\int_{B_R(0)} \big(\big|(-\Delta)^{\frac{\sigma}{2}}\tilde{\varphi}_k\big|^2 + \tilde{\varphi}_k^2\big) \leq 1.$$

So, we may assume that there exists $\varphi \in H^\sigma(\mathbb{R}^n)$ such that, as $k \to +\infty$,

$$\tilde{\varphi}_k \rightharpoonup \varphi \quad \text{in } H^\sigma(\mathbb{R}^n), \qquad \tilde{\varphi}_k \to \varphi \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^n).$$

Moreover, $\tilde{\varphi}_k$ is even in $y_j$, $j = 2, \ldots, n$ and

$$\int_{\mathbb{R}^n} U^{p-1}\frac{\partial U}{\partial y_1}\tilde{\varphi}_k = 0.$$

We see that $\varphi$ is even in $y_j$, $j = 2,\dots,n$ and

$$\int_{\mathbb{R}^n} U^{p-1}\frac{\partial U}{\partial y_1}\varphi = 0. \tag{2.3}$$

Now, we claim that $\varphi$ solves the following linearized equation in $\mathbb{R}^n$:

$$(-\Delta)^\sigma\varphi + \varphi - pU^{p-1}\varphi = 0. \tag{2.4}$$

Indeed, define

$$\widetilde{E} = \left\{v : v \in H^\sigma(\mathbb{R}^n), \int_{\mathbb{R}^n} U^{p-1}\frac{\partial U}{\partial y_1}v = 0\right\}.$$

For any $R > 0$, let $v \in C_0^\infty(B_R(0)) \cap \widetilde{E}$ satisfying $v$ is even in $y_j$, $j = 2,\dots,n$. Then $v_1(y) = v(y - y^1) \in C_0^\infty(B_R(y^1))$. We may identify $v_1(y)$ as elements in $E_r$ by redefining the values outside $\Omega_1$ with the symmetry. By using (2.2) and Lemma A.2, we can find that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\sigma}{2}}\varphi(-\Delta)^{\frac{\sigma}{2}}v + \int_{\mathbb{R}^n}\left(\varphi v - pU^{p-1}\varphi v\right) = 0. \tag{2.5}$$

But (2.5) holds for $v = \frac{\partial U}{\partial y_1}$. Hence (2.5) is true for any $v \in H^\sigma(\mathbb{R}^n)$ and the claim holds. This being the nondegenerate result of $U$, we have $\varphi = c\frac{\partial U}{\partial y_1}$ since $\varphi$ is even in $y_j$, $j = 2,\dots,n$. So it follows from the orthogonal condition (2.3) that $\varphi = 0$ and thus

$$\int_{B_R(y^1)} \varphi_k^2 = o_k(1), \quad \forall R > 0.$$

Due to Lemma A.2, if $k > 0$ is large enough, we have, for $\eta$ satisfying $(n+2\sigma-\eta)(p-1) > n$,

$$\int_{\Omega_1\setminus B_{\frac{R}{2}}(y^1)} W_{r_k}^{p-1}\varphi_k^2 \le C\int_{\Omega_1\setminus B_{\frac{R}{2}}(y^1)} \frac{1}{(1+|y-y^1|)^{(n+2\sigma-\eta)(p-1)}}\varphi_k^2 = o_R(1).$$

So, taking $v = \varphi_k$ in (2.2), one has

$$\begin{aligned} o_k(1) &= \int_{\Omega_1}\left(\left|(-\Delta)^{\frac{\sigma}{2}}\varphi_k\right|^2 + A(|y|)\varphi_k^2\right) - p\int_{\Omega_1} B(|y|)W_{r_k}^{p-1}\varphi_k^2 \\ &= \int_{\Omega_1}\left(\left|(-\Delta)^{\frac{\sigma}{2}}\varphi_k\right|^2 + A(|y|)\varphi_k^2\right) - p\int_{B_{\frac{R}{2}}(y^1)} B(|y|)W_{r_k}^{p-1}\varphi_k^2 - o_R(1) \\ &\ge \frac{1}{2}\int_{\Omega_1}\left(\left|(-\Delta)^{\frac{\sigma}{2}}\varphi_k\right|^2 + A(|y|)\varphi_k^2\right) - o_k(1) - o_R(1). \end{aligned}$$

This shows a contradiction and our proof is finished. □

Next, we discuss the terms $R(\varphi)$ and $l(\varphi)$ in (2.1). We have

**Lemma 2.2** *There is a constant $C > 0$ independent of $k$, such that*

$$\left\|R'(\varphi)\right\| \le C\|\varphi\|_\sigma^{\min\{p,2\}}$$

*and*

$$\|R''(\varphi)\| \le C\|\varphi\|_\sigma^{\min\{p-1,1\}}$$

*for* $\varphi \in E_r$ *and* $\|\varphi\|_\sigma < 1$.

*Proof* It is clear that, for $v_1, v_2 \in E_r$,

$$\langle R'(\varphi), v_1\rangle = -\int_{\mathbb{R}^n} B(|y|)\big((W_r+\varphi)^p - W_r^p - pW_r^{p-1}\varphi\big)v_1$$

and

$$\langle R''(\varphi)v_1, v_2\rangle = -p\int_{\mathbb{R}^n} B(|y|)\big((W_r+\varphi)^{p-1} - W_r^{p-1}\big)v_1v_2.$$

First, if $p \ge 2$, it follows from Lemma A.2 that $W_r$ is bounded and then

$$\begin{aligned}
\big|\langle R'(\varphi), v_1\rangle\big| &\le C\int_{\mathbb{R}^n}\big(W_r^{p-2}|\varphi|^2|v_1| + |\varphi|^p|v_1|\big)\\
&\le C\left(\int_{\mathbb{R}^n}|\varphi|^{\frac{2(p+1)}{p}}\right)^{\frac{p}{p+1}}\left(\int_{\mathbb{R}^n}|v_1|^{p+1}\right)^{\frac{1}{p+1}}\\
&\quad + C\left(\int_{\mathbb{R}^n}|\varphi|^{p+1}\right)^{\frac{p}{p+1}}\left(\int_{\mathbb{R}^n}|v_1|^{p+1}\right)^{\frac{1}{p+1}}\\
&\le C\big(\|\varphi\|_\sigma^2 + \|\varphi\|_\sigma^p\big)\|v_1\|_\sigma
\end{aligned}$$

and

$$\begin{aligned}
\big|\langle R''(\varphi)v_1, v_2\rangle\big| &\le C\int_{\mathbb{R}^n}\big(W_r^{p-2}|\varphi| + |\varphi|^{p-1}\big)|v_1||v_2|\\
&\le C\left(\int_{\mathbb{R}^n}|\varphi|^3\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^n}|v_1|^3\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^n}|v_2|^3\right)^{\frac{1}{3}}\\
&\quad + C\left(\int_{\mathbb{R}^n}|\varphi|^{p+1}\right)^{\frac{p-1}{p+1}}\left(\int_{\mathbb{R}^n}|v_1|^{p+1}\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^n}|v_2|^{p+1}\right)^{\frac{1}{p+1}}\\
&\le C\big(\|\varphi\|_\sigma + \|\varphi\|_\sigma^{p-1}\big)\|v_1\|_\sigma\|v_2\|_\sigma.
\end{aligned}$$

As a result, if $p \ge 2$, we have

$$\|R'(\varphi)\| \le C\|\varphi\|_\sigma^2$$

and

$$\|R''(\varphi)\| \le C\|\varphi\|_\sigma.$$

With the same argument, if $1 < p < 2$, we find

$$\big|\langle R'(\varphi), v_1\rangle\big| \le C\int_{\mathbb{R}^n}|\varphi|^p|v_1| \le C\|\varphi\|_\sigma^p\|v_1\|_\sigma$$

and

$$\left|\left\langle R''(\varphi)v_1,v_2\right\rangle\right| \le C\int_{\mathbb{R}^n}|\varphi|^{p-1}|v_1||v_2| \le C\|\varphi\|_\sigma^{p-1}\|v_1\|_\sigma\|v_2\|_\sigma,$$

which completes this proof. □

**Lemma 2.3** *For any $\varphi \in E_r$, $r \in S_k$, there is a constant $C > 0$ and a small $\epsilon > 0$, independent of $k$, such that*

$$\left\|l(\varphi)\right\| \le C\frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}\|\varphi\|_\sigma,$$

*where $m = \min\{m_1, m_2\}$.*

*Proof* Recall that

$$\begin{aligned}\left|l(\varphi)\right| &= \left|\int_{\mathbb{R}^n}\sum_{j=1}^k U_{y^j}^p\varphi + \int_{\mathbb{R}^n}\bigl(A\bigl(|y|\bigr)-1\bigr)W_r\varphi - \int_{\mathbb{R}^n}B\bigl(|y|\bigr)W_r^p\varphi\right| \\ &\le \int_{\mathbb{R}^n}\left|\left(\sum_{j=1}^k U_{y^j}\right)^p - \sum_{j=1}^k U_{y^j}^p\right||\varphi| + \int_{\mathbb{R}^n}\left|\bigl(A\bigl(|y|\bigr)-1\bigr)W_r\varphi\right| \\ &\quad + \int_{\mathbb{R}^n}\left|\bigl(B\bigl(|y|\bigr)-1\bigr)W_r^p\varphi\right|.\end{aligned} \tag{2.6}$$

We are in a position to discuss the terms in (2.6). Using condition $(A)$, similar to (A.2), we compute that

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\bigl(A\bigl(|y|\bigr)-1\bigr)W_r\varphi\right| &\le \left(\int_{\mathbb{R}^n}\bigl(A\bigl(|y|\bigr)-1\bigr)^2W_r^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}\varphi^2\right)^{\frac{1}{2}} \\ &\le C\left(\frac{k^{\frac{1}{2}}}{r^{m_1}} + \frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}\right)\|\varphi\|_\sigma \\ &\le C\frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}\|\varphi\|_\sigma.\end{aligned} \tag{2.7}$$

With the same argument, having $m > \frac{n+2\sigma}{n+2\sigma+1}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\bigl(B\bigl(|y|\bigr)-1\bigr)W_r^p\varphi\right| &\le k\left(\int_{\mathbb{R}^n}\left|B\bigl(|y|\bigr)-1\right|^{\frac{p+1}{p}}U_{y_1}^{p+1}\right)^{\frac{p}{p+1}}\left(\int_{\mathbb{R}^n}\varphi^{p+1}\right)^{\frac{1}{p+1}} \\ &\le Ck\left(\frac{1}{r^{\frac{m_2(p+1)}{p}}}\int_{B_{\frac{r}{2}}(y^1)}U_{y_1}^{p+1} + \int_{\mathbb{R}^n\setminus B_{\frac{r}{2}}(y^1)}U_{y^1}^{p+1}\right)^{\frac{p}{p+1}}\|\varphi\|_\sigma \\ &\le C\frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}\|\varphi\|_\sigma.\end{aligned} \tag{2.8}$$

Finally, taking $\eta = n + 2\sigma$ in Lemma A.2, one has

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\left(\sum_{j=1}^{k} U_{y^j}\right)^p - \sum_{j=1}^{k} U_{y^j}^p\right||\varphi| &\le \left(\int_{\mathbb{R}^n}\left|\left(\sum_{j=1}^{k} U_{y^j}\right)^p - \sum_{j=1}^{k} U_{y^j}^p\right|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}\varphi^2\right)^{\frac{1}{2}} \\ &\le Ck^{\frac{1}{2}}\left(\int_{\Omega_1} U_{y^1}^{2(p-1)}\left(\sum_{j=2}^{k} U_{y^j}\right)^2\right)^{\frac{1}{2}}\|\varphi\|_\sigma \\ &\le Ck^{\frac{1}{2}}\left(\int_{\Omega_1}\frac{k^{2\eta}}{r^{2\eta}}U_{y^1}^{2(p-1)}\right)^{\frac{1}{2}}\|\varphi\|_\sigma \\ &\le C\frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}\|\varphi\|_\sigma. \end{aligned} \tag{2.9}$$

Inserting (2.7)–(2.9) into (2.6), the conclusion follows. □

**Proposition 2.4** *There is an integer $k_0 > 0$, such that, for each $k \ge k_0$, there is a $C^1$ map from $S_k$ to $H_k$: $r \mapsto \varphi = \varphi(r)$, $r = |y^1|$, satisfying $\varphi(r) \in E_r$, and*

$$J'\big(\varphi(r)\big)|_{E_r} = 0.$$

*Moreover, there exists a small constant $\epsilon > 0$, such that, for some $C > 0$, independent of $k$,*

$$\big\|\varphi(r)\big\|_\sigma \le C\frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}. \tag{2.10}$$

*Proof* We will use the contraction theorem to prove it. It follows from Lemma 2.3 that $l(\varphi)$ is a bounded linear map in $E_r$. So applying the Reisz representation theorem there exists an $l_k \in E_r$ such that

$$l(\varphi) = \langle l_k, \varphi\rangle.$$

Thus, finding a critical point for $J(\varphi)$ is equivalent to solving

$$l_k + L\varphi + R'(\varphi) = 0. \tag{2.11}$$

By Lemma 2.1, $L$ is invertible and then (2.11) can be rewritten as

$$\varphi = T(\varphi) := -L^{-1}\big(l_k + R'(\varphi)\big).$$

Set

$$D_k := \left\{\varphi \in E_r : \|\varphi\|_\sigma \le C\frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}\right\},$$

where $\epsilon > 0$ is defined in Lemma 2.3.

From Lemmas 2.2 and 2.3, we have, for $\varphi \in E_r$,

$$\begin{aligned}\|T(\varphi)\|_\sigma &\le C\big(\|l_k\| + \|R'(\varphi)\|\big)\\ &\le C\|l_k\| + C\|\varphi\|_\sigma^{\min\{p,2\}} \le C\frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}.\end{aligned}$$

On the other hand, for any $\varphi_1, \varphi_2 \in D_k$, we can deduce that

$$\begin{aligned}\|T(\varphi_1) - T(\varphi_2)\|_\sigma &\le C\|R'(\varphi_1) - R'(\varphi_2)\|\\ &\le C\big(\|\varphi_1\|_\sigma^{\min\{p-1,1\}} + \|\varphi_2\|_\sigma^{\min\{p-1,1\}}\big)\|\varphi_1 - \varphi_2\|_\sigma\\ &\le \frac{1}{2}\|\varphi_1 - \varphi_2\|_\sigma.\end{aligned}$$

Therefore, $T$ maps $D_k$ to $D_k$ and is a contraction map. From the contraction map theorem, there exists $\varphi$ such that $\varphi = T(\varphi)$ and

$$\|\varphi\|_\sigma \le C\frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\epsilon}}.$$ □

## 3 Proof of the main result

Now we are ready to prove our Theorem 1.2. Let $\varphi_r := \varphi(r)$ be the map obtained in Proposition 2.4. Define

$$F(r) = I(W_r + \varphi_r), \quad \forall r \in S_k.$$

With the same argument in [10], we can check that, if $r$ is a critical point of $F(r)$, then $W_r + \varphi_r$ is a solution of (1.1).

*Proof of Theorem* 1.2 It follows from Propositions 2.4 and A.3 that

$$\begin{aligned}F(r) &= I(W_r) + O\big(\|l\|\|\varphi_r\|_\sigma + \|\varphi_r\|_\sigma^2\big)\\ &= k\left(d + \frac{ad_1}{r^{m_1}} - \frac{bd_2}{r^{m_2}} - \frac{q_0 k^{n+2\sigma}}{r^{n+2\sigma}} + O\left(\frac{1}{r^{m+\epsilon}}\right)\right).\end{aligned}$$

In the following, we only prove the case $b > 0$ and $m_1 < m_2$ since the case that $b < 0$ can be checked in similar way. If $b > 0$ and $m_1 < m_2$, then

$$F(r) = k\left(d + \frac{h_1}{r^m} - h_0\left(\frac{k}{r}\right)^{n+2\sigma} + O\left(\frac{1}{r^{m+\epsilon}}\right)\right)$$

for some $h_0, h_1 > 0$.

We next consider the following maximization problem:

$$\max_{r \in S_k} F(r). \tag{3.1}$$

Suppose that (3.1) is achieved by some $r_k$ in $S_k$ and then we can prove that $r_k$ is an interior point in $S_k$ by analyzing the following problem:

$$g(r) := \frac{h_1}{r^m} - h_0\left(\frac{k}{r}\right)^{n+2\sigma}.$$

By the direct computation, we find $g(r)$ admits a maximum point

$$r_k = \left(\frac{h_0(n+2\sigma)}{h_1 m}\right)^{\frac{1}{n+2\sigma-m}} k^{\frac{n+2\sigma}{n+2\sigma-m}}.$$

Now we claim that $r_k$ is an interior point of $S_k$. In fact, it is easy to see that

$$g(r_k) = \frac{h_1^{\frac{n+2\sigma}{n+2\sigma-m}}}{h_0^{\frac{m}{n+2\sigma-m}}} \frac{1}{(\frac{n+2\sigma}{m})^{\frac{n+2\sigma}{n+2\sigma-m}}} \left(\frac{n+2\sigma}{m} - 1\right) k^{-\frac{(n+2\sigma)m}{n+2\sigma-m}}.$$

On the other hand,

$$g\left(r_0 k^{\frac{n+2\sigma}{n+2\sigma-m}}\right) = \frac{h_1^{\frac{n+2\sigma}{n+2\sigma-m}}}{h_0^{\frac{m}{n+2\sigma-m}}} \frac{1}{(\frac{n+2\sigma}{m} - \frac{\alpha h_1}{h_0})^{\frac{n+2\sigma}{n+2\sigma-m}}} \left(\frac{n+2\sigma}{m} - \frac{\alpha h_1}{h_0} - 1\right) k^{-\frac{(n+2\sigma)m}{n+2\sigma-m}}$$

and

$$g\left(r_1 k^{\frac{n+2\sigma}{n+2\sigma-m}}\right) = \frac{h_1^{\frac{n+2\sigma}{n+2\sigma-m}}}{h_0^{\frac{m}{n+2\sigma-m}}} \frac{1}{(\frac{n+2\sigma}{m} + \frac{\alpha h_1}{h_0})^{\frac{n+2\sigma}{n+2\sigma-m}}} \left(\frac{n+2\sigma}{m} + \frac{\alpha h_1}{h_0} - 1\right) k^{-\frac{(n+2\sigma)m}{n+2\sigma-m}}.$$

Since the function $f(t) = (\frac{1}{t})^{\frac{n+2\sigma}{n+2\sigma-m}}(t-1)$ attains its maximum at $t_0 = \frac{n+2\sigma}{m}$ when $t \in [\frac{n+2\sigma}{m} - \frac{\alpha h_1}{h_0}, \frac{n+2\sigma}{m} + \frac{\alpha h_1}{h_0}]$, we have $g(r_0 k^{\frac{n+2\sigma}{n+2\sigma-m}}) < g(r_k)$ and $g(r_1 k^{\frac{n+2\sigma}{n+2\sigma-m}}) < g(r_k)$. Thus, $r_k$ is an interior point of $S_k$ and $r_k$ is a critical point of $F(r)$. As a result,

$$u_k = W_{r_k} + \varphi_{r_k}$$

is a solution of (1.1). □

## Appendix: Energy expansion

In this section, we will give some basic estimates and the energy expansion for the approximate solutions. Recall that

$$y^i = \left(r\cos\frac{2(i-1)\pi}{k}, r\sin\frac{2(i-1)\pi}{k}, 0\right), \quad i = 1, \dots, k,$$

$$\Omega_i = \left\{y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2} : \left\langle \frac{y'}{|y'|}, \frac{(y^i)'}{|(y^i)'|}\right\rangle \geq \cos\frac{\pi}{k}\right\}, \quad i = 1, 2, \dots, k,$$

and

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(\left|(-\Delta)^{\frac{\sigma}{2}} u\right|^2 + A(y)u^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} B(y)|u|^{p+1}.$$

Now we introduce the following lemmas which have been proved in [24] and [19], respectively.

**Lemma A.1** *For any constant* $0 < \mu \le \min\{\alpha, \beta\}$, *there is a constant* $C > 0$, *such that*

$$\frac{1}{(1+|y-y^i|)^{\alpha}}\frac{1}{(1+|y-y^j|)^{\beta}} \le \frac{C}{|y^i-y^j|^{\mu}}\left(\frac{1}{(1+|y-y^i|)^{\alpha+\beta-\mu}}+\frac{1}{(1+|y-y^j|)^{\alpha+\beta-\mu}}\right).$$

**Lemma A.2** *For any* $y \in \Omega_1$ *and* $\eta \in (0, n+2\sigma]$, *there is a constant* $C > 0$, *such that*

$$\sum_{i=2}^{k} U_{y^i} \le \frac{C}{(1+|y-y^1|)^{n+2\sigma-\eta}}\frac{k^{\eta}}{|y^1|^{\eta}} \le C\frac{k^{\eta}}{|y^1|^{\eta}}.$$

**Proposition A.3** *There is a small constant* $\epsilon > 0$, *such that*

$$\begin{aligned} I(W_r) &= k\Bigg(d - \frac{1}{2}\sum_{j=2}^{k}\frac{q_0'}{|y^1-y^j|^{n+2\sigma}} + \frac{ad_1}{r^{m_1}} - \frac{bd_2}{r^{m_2}} \\ &\quad + O\left(\frac{1}{r^{m_1+\theta_1}}\right) + O\left(\frac{1}{r^{m_2+\theta_2}}\right) + O\left(\frac{k}{r}\right)^{n+2\sigma+\epsilon}\Bigg) \\ &= k\left(d + \frac{ad_1}{r^{m_1}} - \frac{bd_2}{r^{m_2}} - \frac{q_0k^{n+2\sigma}}{r^{n+2\sigma}} + O\left(\frac{1}{r^{m_1+\theta_1}}\right) + O\left(\frac{1}{r^{m_2+\theta_2}}\right) + O\left(\frac{k}{r}\right)^{n+2\sigma+\epsilon}\right), \end{aligned}$$

*where* $d = (\frac{1}{2}-\frac{1}{p+1})\int_{\mathbb{R}^n} U^{p+1}$, $d_1 = \frac{1}{2}\int_{\mathbb{R}^n} U^2$, $d_2 = \frac{1}{p+1}\int_{\mathbb{R}^n} U^{p+1}$ *and* $q_0 = \frac{1}{2}q_0'$ *with* $q_0, q_0'$ *are some positive constants.*

*Proof* Using the symmetry and Lemma A.1, we have

$$\begin{aligned} \int_{\mathbb{R}^n}\left(\left|(-\Delta)^{\frac{\sigma}{2}} W_r\right|^2 + W_r^2\right) &= \sum_{i=1}^{k}\sum_{j=1}^{k}\int_{\mathbb{R}^n} U_{y^j}^{p} U_{y^i} \\ &= k\left(\int_{\mathbb{R}^n} U_{y^1}^{p+1} + \sum_{j=2}^{k}\int_{\mathbb{R}^n} U_{y^1}^{p} U_{y^j}\right) \\ &= k\int_{\mathbb{R}^n} U^{p+1} + k\sum_{j=2}^{k}\int_{\mathbb{R}^n} U_{y^1}^{p} U_{y^j} \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} &\sum_{j=2}^{k}\int_{\mathbb{R}^n} U_{y^1}^{p} U_{y^j} \\ &\quad = C\int_{\mathbb{R}^n}\left(\frac{1}{1+|y-y^1|^{n+2\sigma}}\right)^{p}\sum_{j=2}^{k}\frac{1}{1+|y-y^j|^{n+2\sigma}} \\ &\quad \le C\sum_{j=2}^{k}\frac{1}{|y^1-y^j|^{n+2\sigma}}\left(\int_{\mathbb{R}^n}\frac{1}{(1+|y-y^1|)^{(n+2\sigma)p}} + \int_{\mathbb{R}^n}\frac{1}{(1+|y-y^j|)^{(n+2\sigma)p}}\right) \\ &\quad = \sum_{j=2}^{k}\frac{q_0'}{|y^1-y^j|^{n+2\sigma}}. \end{aligned}$$

On the other hand, we see that

$$\begin{aligned}\int_{\mathbb{R}^n}\big(A(|y|)-1\big)W_r^2 &= k\int_{\Omega_1}\big(A(|y|)-1\big)\Big(U_{y^1}+\sum_{j=2}^k U_{y^j}\Big)^2\\ &= k\bigg(\int_{\Omega_1}\big(A(|y|)-1\big)U_{y^1}^2+2\sum_{j=2}^k\int_{\Omega_1}\big(A(|y|)-1\big)U_{y^1}U_{y^j}\\ &\quad+\int_{\Omega_1}\big(A(|y|)-1\big)\Big(\sum_{j=2}^k U_{y^j}\Big)^2\bigg).\end{aligned}$$

First, we have

$$\begin{aligned}&\int_{\Omega_1}\big(A(|y|)-1\big)U_{y^1}^2\\ &\quad=\int_{B_{\frac{|y^1-y^2|}{2}}(y^1)}\Big(\frac{a}{r^{m_1}}+O\Big(\frac{1}{r^{m_1+\theta_1}}\Big)\Big)U_{y^1}^2+\int_{\Omega_1\setminus B_{\frac{|y^1-y^2|}{2}}(y^1)}\big(A(|y|)-1\big)U_{y^1}^2\\ &\quad=\Big(\frac{a}{r^{m_1}}+O\Big(\frac{1}{r^{m_1+\theta_1}}\Big)\Big)\Big(\int_{\mathbb{R}^n}U^2+\int_{\mathbb{R}^n\setminus B_{\frac{|y^1-y^2|}{2}}(y^1)}U_{y^1}^2\Big)+O\Big(\frac{k}{r}\Big)^{n+2\sigma+\epsilon}\\ &\quad=\frac{a}{r^{m_1}}\int_{\mathbb{R}^n}U^2+O\Big(\frac{1}{r^{m_1+\theta_1}}\Big)+O\Big(\frac{k}{r}\Big)^{n+2\sigma+\epsilon}.\end{aligned}$$

Second, using Lemma A.1,

$$\begin{aligned}&\sum_{j=2}^k\int_{\Omega_1}\big(A(|y|)-1\big)U_{y^1}U_{y^j}\\ &\quad=\sum_{j=2}^k\int_{B_{\frac{|y^1-y^2|}{2}}(y^1)}\Big(\frac{a}{r^{m_1}}+O\Big(\frac{1}{r^{m_1+\theta_1}}\Big)\Big)U_{y^1}U_{y^j}+\sum_{j=2}^k\int_{\Omega_1\setminus B_{\frac{|y^1-y^2|}{2}}(y^1)}U_{y^1}U_{y^j}\\ &\quad\le\frac{a}{r^{m_1}}\sum_{j=2}^k\frac{C}{|y^1-y^j|^{n+2\sigma}}\int_{B_{\frac{|y^1-y^2|}{2}}(y^1)}\frac{1}{(1+|y-y^1|)^{n+2\sigma}}+O\Big(\frac{1}{r^{m_1+\theta_1}}\Big)\\ &\qquad+\sum_{j=2}^k\frac{C}{|y^1-y^j|^{n+2\sigma}}\int_{\Omega_1\setminus B_{\frac{|y^1-y^2|}{2}}(y^1)}\frac{1}{(1+|y-y^1|)^{n+2\sigma}}\\ &\quad\le\frac{C}{r^{m_1+\theta_1}}+\sum_{j=2}^k\frac{C}{|y^1-y^j|^{n+2\sigma}}\Big(\frac{k}{r}\Big)^{\tau}\int_{\Omega_1\setminus B_{\frac{|y^1-y^2|}{2}}(y^1)}\frac{1}{(1+|y-y^1|)^{n+2\sigma-\tau}}\\ &\quad=C\Big(\frac{k}{r}\Big)^{n+2\sigma+\epsilon}+\frac{C}{r^{m_1+\theta_1}},\end{aligned}$$

where $\tau>0$ satisfies $n+2\sigma-\tau>n$ and we used the fact that $|y-y^j|\ge|y-y^1|$ for $y\in\Omega_1$.

But, by Lemma A.2, we find

$$
\begin{aligned}
&\sum_{j=2}^{k}\int_{\Omega_1}\big(A\big(|y|\big)-1\big)\Bigg(\sum_{j=2}^{k}U_{y^j}\Bigg)^2\\
&\quad\le C\int_{B_{\frac{|y^1-y^2|}{2}}(y^1)}\bigg(\frac{a}{r^{m_1}}+O\bigg(\frac{1}{r^{m_1+\theta_1}}\bigg)\bigg)\bigg(\frac{k}{r}\bigg)^{n+2\sigma}\frac{1}{(1+|y-y^1|)^{n+2\sigma}}\\
&\qquad+C\int_{\Omega_1\setminus B_{\frac{|y^1-y^2|}{2}}(y^1)}\bigg(\frac{k}{r}\bigg)^{n+2\sigma}\frac{1}{(1+|y-y^1|)^{n+2\sigma}}\\
&\quad=C\bigg(\frac{k}{r}\bigg)^{n+2\sigma+\epsilon}+\frac{C}{r^{m_1+\theta_1}}.
\end{aligned}
$$

As a result,

$$
\int_{\mathbb{R}^n}\big(A\big(|y|\big)-1\big)W_r^2=k\bigg(\frac{a}{r^{m_1}}\int_{\mathbb{R}^n}U^2+O\bigg(\bigg(\frac{k}{r}\bigg)^{n+2\sigma+\epsilon}+\frac{1}{r^{m_1+\theta_1}}\bigg)\bigg) \tag{A.2}
$$

and then

$$
\begin{aligned}
\int_{\mathbb{R}^n}\big(\big|(-\Delta)^{\frac{\sigma}{2}}W_r\big|^2+A\big(|y|\big)W_r^2\big)&=k\Bigg(\int_{\mathbb{R}^n}U^{p+1}+\sum_{j=2}^{k}\frac{q_0'}{|y^1-y^j|^{n+2\sigma}}+\frac{a}{r^{m_1}}\int_{\mathbb{R}^n}U^2\\
&\quad+O\bigg(\bigg(\frac{k}{r}\bigg)^{n+2\sigma+\epsilon}+\frac{1}{r^{m_1+\theta_1}}\bigg)\Bigg).
\end{aligned} \tag{A.3}
$$

Now, from the symmetry, we also find

$$
\begin{aligned}
\int_{\mathbb{R}^n}B\big(|y|\big)W_r^{p+1}&=k\int_{\Omega_1}B\big(|y|\big)U_{y^1}^{p+1}+k(p+1)\int_{\Omega_1}B\big(|y|\big)\sum_{j=2}^{k}U_{y^1}^{p}U_{y^j}\\
&\quad+k\begin{cases}O(\int_{\Omega_1}U_{y^1}^{\frac{p+1}{2}}(\sum_{j=2}^{k}U_{y^j})^{\frac{p+1}{2}}), & \text{if } 1<p<2,\\ O(\int_{\Omega_1}U_{y^1}^{p-1}(\sum_{j=2}^{k}U_{y^j})^2), & \text{if } p\ge 2.\end{cases}
\end{aligned} \tag{A.4}
$$

Observe that $|y-y^j|\ge|y-y^1|$ and $|y-y^j|\ge\frac{1}{2}|y^j-y^1|$ if $y\in\Omega_1$. So we have

$$
\begin{aligned}
&\int_{\Omega_1}U_{y^1}^{\frac{p+1}{2}}\Bigg(\sum_{j=2}^{k}U_{y^j}\Bigg)^{\frac{p+1}{2}}\\
&\quad\le C\int_{\Omega_1}\frac{1}{(1+|y-y^1|)^{\frac{(n+2\sigma)(p+1)}{2}}}\Bigg(\sum_{j=2}^{k}\frac{1}{|y^j-y^1|^{n+2\sigma-\kappa}}\Bigg)^{\frac{p+1}{2}}\frac{1}{(1+|y-y^1|)^{\frac{p+1}{2}\kappa}}\\
&\quad=C\Bigg(\sum_{j=2}^{k}\frac{1}{|y^j-y^1|^{n+2\sigma-\kappa}}\Bigg)^{\frac{p+1}{2}}\int_{\Omega_1}\frac{1}{(1+|y-y^1|)^{\frac{(n+2\sigma)(p+1)}{2}+\kappa}}\\
&\quad\le C\Bigg(\sum_{j=2}^{k}\frac{1}{|y^j-y^1|^{n+2\sigma-\kappa}}\Bigg)^{\frac{p+1}{2}}\le C\bigg(\frac{k}{r}\bigg)^{n+2\sigma+\epsilon}
\end{aligned}
$$

and similarly

$$\int_{\Omega_1} U_{y^1}^{p-1}\left(\sum_{j=2}^{k} U_{y^j}\right)^2 \le C\left(\frac{k}{r}\right)^{n+2\sigma+\epsilon}$$

with $\kappa > 0$ satisfying $\min\{\frac{p+1}{2}(n+2\sigma-\kappa), 2(n+2\sigma-\kappa)\} > n+2\sigma$.

Note that

$$\int_{\Omega_1} B(|y|)\sum_{j=2}^{k} U_{y^1}^{p} U_{y^j} = \int_{\Omega_1} U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} + \int_{\Omega_1}\bigl(B(|y|)-1\bigr)U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j}. \tag{A.5}$$

By Lemma A.1, we can deduce that

$$\begin{aligned}
&\int_{\Omega_1} U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} \\
&= \int_{\mathbb{R}^n} U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} - \int_{\mathbb{R}^n\setminus\Omega_1} U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} \\
&\le \int_{\mathbb{R}^n} U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} + C\left(\frac{k}{r}\right)^{\tau}\sum_{j=2}^{k}\int_{\mathbb{R}^n\setminus\Omega_1}\frac{1}{(1+|y-y^1|)^{p(n+2\sigma)-\tau}}\frac{1}{(1+|y-y^j|)^{n+2\sigma}} \\
&= \int_{\mathbb{R}^n} U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} + \left(\frac{k}{r}\right)^{\tau}\sum_{j=2}^{k}\frac{1}{|y^j-y^1|^{n+2\sigma}}\int_{\mathbb{R}^n\setminus\Omega_1}\biggl(\frac{1}{(1+|y-y^1|)^{p(n+2\sigma)-\tau}} \\
&\quad + \frac{1}{(1+|y-y^j|)^{p(n+2\sigma)-\tau}}\biggr) \\
&= \sum_{j=2}^{k}\frac{q_0'}{|y^1-y^j|^{n+2\sigma}} + O\left(\left(\frac{k}{r}\right)^{n+2\sigma+\epsilon}\right),
\end{aligned}$$

since $y \in \mathbb{R}^n\setminus\Omega_1$, $|y-y^1| \ge c\frac{r}{k}$ for some $c>0$ and we could choose $p(n+2\sigma)-\tau \ge n+2\sigma$.

Furthermore,

$$\begin{aligned}
&\int_{\Omega_1}\bigl|B(|y|)-1\bigr|U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} \\
&= \int_{\mathbb{R}^n}\bigl|B(|y|)-1\bigr|U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} - \int_{\mathbb{R}^n\setminus\Omega_1}\bigl|B(|y|)-1\bigr|U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} \\
&= \int_{B_{\frac{r}{2}}(x^1)}\bigl|B(|y|)-1\bigr|U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} + \int_{\mathbb{R}^n\setminus B_{\frac{r}{2}}(x^1)} U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} + O\left(\left(\frac{k}{r}\right)^{n+2\sigma+\epsilon}\right) \\
&\le \left(\frac{C}{r^{m_2}} + O\left(\frac{1}{r^{m_2+\theta_2}}\right)\right)\int_{\mathbb{R}^n} U_{y^1}^{p}\sum_{j=2}^{k} U_{y^j} + O\left(\left(\frac{k}{r}\right)^{n+2\sigma+\epsilon}\right) \\
&= O\left(\left(\frac{k}{r}\right)^{n+2\sigma+\epsilon} + \frac{1}{r^{m_2+\theta_2}}\right).
\end{aligned}$$

Finally,

$$
\begin{aligned}
&\int_{\Omega_1} B(|y|)U_{y^1}^{p+1} \\
&= \int_{\mathbb{R}^n} B(|y|)U_{y^1}^{p+1} - \int_{\mathbb{R}^n\setminus B_{\frac{2\pi r}{k}}(y^1)} B(|y|)U_{y^1}^{p+1} + \int_{\Omega_1\setminus B_{\frac{2\pi r}{k}}(y^1)} B(|y|)U_{y^1}^{p+1} \\
&= \int_{B_{\frac{r}{2}}(y^1)} B(|y|)U_{y^1}^{p+1} + \int_{\mathbb{R}^n\setminus B_{\frac{r}{2}}(y^1)} B(|y|)U_{y^1}^{p+1} + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{n+2\sigma+\epsilon}\biggr) \\
&= \int_{B_{\frac{r}{2}}(y^1)} \biggl(1+\frac{b}{r^{m_2}}+O\Bigl(\frac{1}{r^{m_2+\theta_2}}\Bigr)\biggr)U_{y^1}^{p+1} + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{n+2\sigma+\epsilon}\biggr) \\
&= \biggl(1+\frac{b}{r^{m_2}}+O\Bigl(\frac{1}{r^{m_2+\theta_2}}\Bigr)\biggr)\int_{\mathbb{R}^n} U^{p+1} + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{n+2\sigma+\epsilon}\biggr).
\end{aligned}
$$

Thus, we have proved

$$
\begin{aligned}
\int_{\mathbb{R}^n} B(|y|)W_r^{p+1} = k\Biggl(\biggl(1+\frac{b}{r^{m_2}}\biggr)\int_{\mathbb{R}^n} U^{p+1} + \sum_{j=2}^{k}\frac{(p+1)q_0'}{|y^1-y^j|^{n+2\sigma}} \\
+ O\biggl(\Bigl(\frac{k}{r}\Bigr)^{n+2\sigma+\epsilon} + \frac{1}{r^{m_2+\theta_2}}\biggr)\Biggr),
\end{aligned}
$$

which, combining with (A.3), completes our proof. □

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