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Existence and uniqueness of finite beam deflection on nonlinear non-uniform elastic foundation with arbitrary well-posed boundary condition

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Abstract

For arbitrary two-point boundary condition, which makes the corresponding linear uniform problem well-posed, we obtain an existence and uniqueness result for the boundary value problem of finite beam deflection resting on arbitrary nonlinear non-uniform elastic foundation. The difference between the desired solution and the corresponding linear uniform one in L^∞ sense is bounded explicitly in terms of given inputs of the problem. Our results seamlessly unify linear uniform and nonlinear non-uniform problems and lead to an iteration algorithm for uniformly approximating the desired deflection.

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1 Introduction

Suppose that a beam with finite length $2l$ is horizontally put on an elastic foundation. Let E and I be the Young's modulus and the mass moment of inertia of the beam respectively, so that EI is the flexural rigidity of the beam. Throughout this paper, we assume that E , I , and l are fixed positive constants. From the classical Euler–Bernoulli beam theory [16], we have the following governing equation, which we denote by $\text{NDE}(f, w)$, for the beam's vertical upward deflection $u(x)$:

$$\text{NDE}(f, w) : EI \cdot u^{(4)}(x) + f(u(x), x) = w(x), \quad x \in [-l, l].$$

Here, $w(x)$ is a vertical downward load density on the beam, and $-f(u(x), x)$ is the *nonlinear* and *non-uniform* elastic force density by the elastic foundation, which can depend on both the location x on the beam and the deflection $u(x)$ at x . Beam deflection is one of the basic and important problems in structural mechanics and mechanical engineering, and it has a lot of applications [1, 3, 4, 7–13, 15–18].

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For given f in $NDE(f, w)$, $f_u(u(x), x)$ corresponds to the nonlinear and non-uniform spring constant density of the elastic foundation. Let k be the *maximal spring constant density* of f at zero deflection, which is defined by

$$k = \max_{-l \leq x \leq l} f_u(0, x), \tag{1.1}$$

and is assumed to be positive. Then $NDE(f, w)$ is a generalization of the following linear equation, which we denote by $LDE(w)$:

$$LDE(w) : EI \cdot u^{(4)}(x) + k \cdot u(x) = w(x), \quad x \in [-l, l].$$

The elastic foundation represented by $LDE(w)$ has the elastic force density $-k \cdot u(x)$, which is *linear* in the sense that it strictly follows Hooke’s law and is *uniform* in the sense that its spring constant density k does not depend on the location x on the beam.

Let $gl(m, n, \mathbb{R})$ be the set of $m \times n$ matrices with real entries. For three times differentiable functions on $[-l, l]$, we define the following linear operator $\mathcal{B} : C^3[-l, l] \rightarrow gl(8, 1, \mathbb{R})$ by

$$\mathcal{B}[u] = \left(u(-l) \quad u'(-l) \quad u''(-l) \quad u^{(3)}(-l) \quad u(l) \quad u'(l) \quad u''(l) \quad u^{(3)}(l) \right)^T. \tag{1.2}$$

Then a two-point boundary condition, which we denote by $BC(\mathbf{M}, \mathbf{b})$, can be given with a 4×8 matrix $\mathbf{M} \in gl(4, 8, \mathbb{R})$ called a *boundary matrix*, and a 4×1 matrix $\mathbf{b} \in gl(4, 1, \mathbb{R})$ called a *boundary value* as follows:

$$BC(\mathbf{M}, \mathbf{b}) : \mathbf{M} \cdot \mathcal{B}[u] = \mathbf{b}.$$

For example, the boundary condition $u(-l) = u_-, u'(-l) = u'_-, u(l) = u_+, u'(l) = u'_+$ corresponds to

$$\mathbf{M} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right), \quad \mathbf{b} = \begin{pmatrix} u_- \\ u'_- \\ u_+ \\ u'_+ \end{pmatrix}. \tag{1.3}$$

The boundary value problem consisting of $LDE(w)$ and $BC(\mathbf{M}, \mathbf{b})$ is *well-posed* if it has a unique solution.

In this paper, we analyze the nonlinear non-uniform boundary value problem consisting of $NDE(f, w)$ and $BC(\mathbf{M}, \mathbf{b})$ for *arbitrary* boundary condition $BC(\mathbf{M}, \mathbf{b})$ which makes the corresponding linear uniform problem $LDE(w)$ and $BC(\mathbf{M}, \mathbf{b})$ well-posed. We will obtain an existence and uniqueness result for this problem under physically realistic and presumably minimal assumptions. Specifically, we have three Assumptions (F), (A), and (B) on the inputs $f, w, \mathbf{M}, \mathbf{b}$ of our problem. Assumption (F) is on the two-variable function f , which represents the elastic foundation in $NDE(f, w)$. It essentially ensures that f is modeling a physically realistic elastic foundation.

Assumption (F) $f(u, x)$ and $f_u(u, x)$ are continuous. $f(u, x) \cdot u \geq 0$ and $f_u(u, x) \geq 0$ for every $u \in \mathbb{R}$ and $x \in [-l, l]$. $k = \max_{-l \leq x \leq l} f_u(0, x) > 0$.

The condition $f(u, x) \cdot u \geq 0$ means that the elastic force by the elastic foundation is *restoring*. The condition $f_u(u, x) \geq 0$ means that the nonlinear non-uniform spring constant density $f_u(u, x)$ of the elastic foundation is nonnegative, so that the magnitude of the elastic force increases as the magnitude of the deflection u increases.

There have been various attempts [1, 3, 4, 7–13, 17] to generalize the classical linear uniform problem LDE(w) to nonlinear or non-uniform settings. For *infinitely long* beam, Choi and Jang [7] obtained an existence and uniqueness result for the following infinite version:

$$EI \cdot u^{(4)}(x) + f(u(x), x) = w(x), \quad x \in (-\infty, \infty) \tag{1.4}$$

of NDE(f, w) with assumptions similar to ours. Since the length of the beam they dealt with was infinite, it was sufficient for them to consider the boundary condition

$$\lim_{x \rightarrow \pm\infty} u(x) = 0. \tag{1.5}$$

They showed the existence and the uniqueness of the solution to the boundary value problem consisting of (1.4) and (1.5) in some regions of $C_0(\mathbb{R})$ around the zero function.

Following the framework of [7], we will construct a nonlinear operator $\Psi : L^\infty[-l, l] \rightarrow L^\infty[-l, l]$, whose fixed points are solutions of our boundary value problem NDE(f, w) and BC(\mathbf{M}, \mathbf{b}). We will find out appropriate regions in $L^\infty[-l, l]$ where Ψ is contractive, so that the desired nonlinear non-uniform deflection is guaranteed to exist in those regions by a generalization of Banach fixed point theorem [2]. Our results on finite beam are of more practical importance than those on infinite beam in [7] which are meaningful only in ideal situations.

A challenge with the finite beam problem is that there are a lot of possible well-posed boundary conditions BC(\mathbf{M}, \mathbf{b}). This is in sharp contrast to the infinite beam problem in [7], where it was sufficient to consider only *one* boundary condition (1.5). Note that Assumption (F), which is also assumed in [7] for $x \in (-\infty, \infty)$, implies that $f(0, x) = 0$ for every x . So the zero function is still a solution of the *homogeneous* boundary value problem consisting of (1.4) with $w = 0$ and (1.5). But the solution of the homogeneous boundary value problem NDE($f, 0$) and BC(\mathbf{M}, \mathbf{b}) is not the zero function in general. So the effect of the boundary condition BC(\mathbf{M}, \mathbf{b}) is already nontrivial even without a nontrivial loading w . In fact, the solution of the linear uniform homogeneous boundary value problem LDE(0) and BC(\mathbf{M}, \mathbf{b}) is nonzero unless $\mathbf{b} = \mathbf{0}$.

The boundary value problem LDE(w) and BC(\mathbf{M}, \mathbf{b}) is well-posed for any fixed $w \in L^\infty[-l, l]$ and $\mathbf{b} \in \mathfrak{gl}(4, 1, \mathbb{R})$ if and only if the boundary value problem LDE(0) and BC($\mathbf{M}, \mathbf{0}$) is well-posed, in which case we just call the boundary matrix $\mathbf{M} \in \mathfrak{gl}(4, 8, \mathbb{R})$ *well-posed*. Up to a natural equivalence relation, the set of well-posed boundary matrices in $\mathfrak{gl}(4, 8, \mathbb{R})$ is in one-to-one correspondence with a 16-dimensional algebra [6]. Hence, together with the 4-dimensional space $\mathfrak{gl}(4, 1, \mathbb{R})$ of boundary values \mathbf{b} , the set of different well-posed boundary conditions BC(\mathbf{M}, \mathbf{b}) forms a 20-dimensional space!^a

Starting from the inputs $f, w, \mathbf{M}, \mathbf{b}$ of our problem, we will successively derive various quantities in *effective* ways. All of these quantities are explicitly computable from given inputs *a priori*. See Fig. 4 in Sect. 7 for a map of these derivations. The following are a few important end results.

- The *intrinsic* L^2 -norm μ_M .
- The *non-uniformity ratio* $0 \leq \eta \leq 1$.
- The upper bound D on the magnitude of linear uniform deflection.
- The nonlinear operator $\Psi : L^\infty[-l, l] \rightarrow L^\infty[-l, l]$.
- The *elastic capacity* $\sigma > 0$.
- The *dual radii* $0 \leq r < R \leq \infty$.
- A *nonlinearity function* $\rho : [0, \infty) \rightarrow [0, \infty)$, which is continuous, strictly increasing, and $\rho(0) = 0$. It can be chosen with some freedom.

For $u_c \in L^\infty[-l, l]$ and $\delta \geq 0$, denote $\bar{B}(u_c, \delta) = \{u \in L^\infty[-l, l] \mid \|u - u_c\|_\infty \leq \delta\}$, which is the closed ball centered at u_c with radius δ in the Banach space $L^\infty[-l, l]$ with the norm $\|u\|_\infty = \sup_{-l \leq x \leq l} |u(x)|$. Here, $\delta = \infty$ is allowed, and $\bar{B}(u_c, \infty) = L^\infty[-l, l]$. For well-posed $M \in \text{gl}(4, 8, \mathbb{R})$, we denote the unique solution of the linear uniform boundary value problem $\text{LDE}(w)$ and $\text{BC}(M, \mathbf{b})$ by $\mathcal{L}_M[\mathbf{b}, w]$.

Assumption (A) $\mu_M \cdot \eta < 1$.

Assumption (B) $\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty < D$.

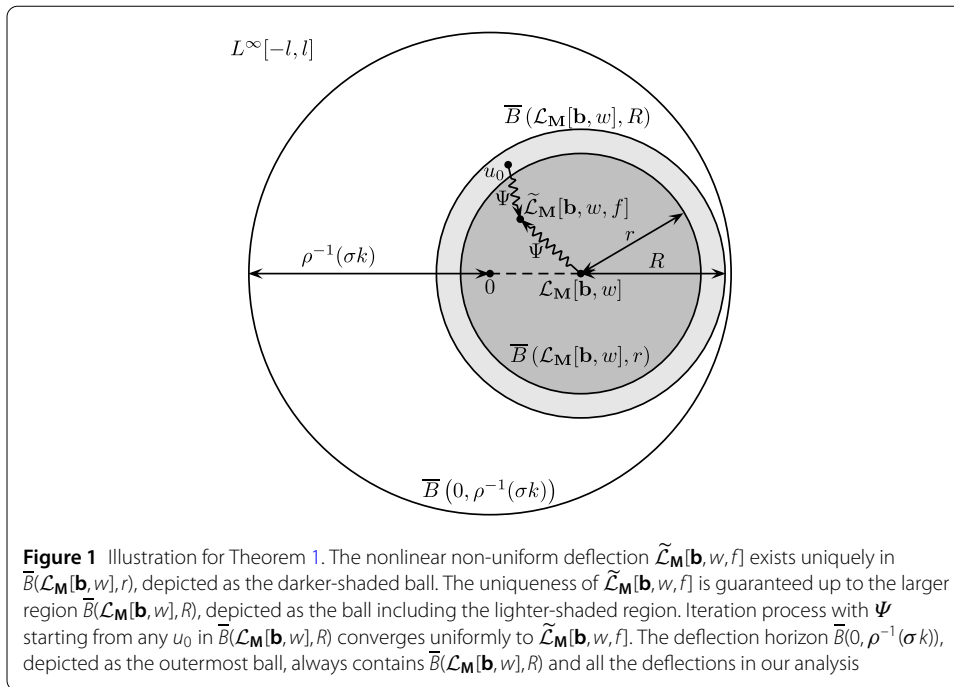
Theorem 1 *Suppose that the boundary value problem $\text{LDE}(w)$ and $\text{BC}(M, \mathbf{b})$ is well-posed, and $f, w \in L^\infty[-l, l]$, $M \in \text{gl}(4, 8, \mathbb{R})$, $\mathbf{b} \in \text{gl}(4, 1, \mathbb{R})$ satisfy Assumptions (F), (A), (B). Then the following (a), (b), (c) hold:*

- (a) *There exists a unique solution $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$ of the boundary value problem $\text{NDE}(f, w)$ and $\text{BC}(M, \mathbf{b})$ in $\bar{B}(\mathcal{L}_M[\mathbf{b}, w], R)$.*
- (b) *For every $u_0 \in \bar{B}(\mathcal{L}_M[\mathbf{b}, w], R)$, the sequence $\{u_n\}_{n=0}^\infty$ of functions defined by $u_n = \Psi[u_{n-1}]$, $n = 1, 2, 3, \dots$, converges uniformly to $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$, and $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f] \in \bar{B}(\mathcal{L}_M[\mathbf{b}, w], r)$.*
- (c) *$0 \leq r < R \leq R + \|\mathcal{L}_M[\mathbf{b}, w]\|_\infty \leq \rho^{-1}(\sigma k)$. r and R are increasing and decreasing with respect to $\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty$ respectively, and $\lim_{\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty \rightarrow 0} r = 0$, $\lim_{\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty \rightarrow 0} R = \rho^{-1}(\sigma k)$.*

Theorem 1 will be proved in Sect. 6. It is important to note that the statements in Theorem 1 are not *local* ones such as “there exists something in *some sufficiently small* neighborhood”. The constants $r, R, \rho^{-1}(\sigma k)$ are explicitly computable from the inputs, as will be demonstrated by examples.

See Fig. 1 for an illustration of Theorem 1. By Theorem 1(c), we always have $\bar{B}(\mathcal{L}_M[\mathbf{b}, w], r) \subset \bar{B}(\mathcal{L}_M[\mathbf{b}, w], R) \subset \bar{B}(0, \rho^{-1}(\sigma k))$. We justifiably call the region $\bar{B}(0, \rho^{-1}(\sigma k))$ the *deflection horizon* since all the deflections appearing in our analysis cannot escape from it. The deflection horizon conforms to, and is an explicit quantification of, the physical observation that the equations $\text{LDE}(w)$ and $\text{NDE}(f, w)$ are designed for small deflections originally.

The *dual radii* $r < R$ is an improved feature compared to [7]. $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$ is guaranteed to exist in the smaller region $\bar{B}(\mathcal{L}_M[\mathbf{b}, w], r)$, so that the smaller radius r provides a shaper bound on the location of $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$ relative to $\mathcal{L}_M[\mathbf{b}, w]$. The uniqueness of $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$ is guaranteed up to the larger region $\bar{B}(\mathcal{L}_M[\mathbf{b}, w], R)$, where the iteration process with Ψ is also guaranteed to converge to $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$. As the linear uniform deflection $\mathcal{L}_M[\mathbf{b}, w]$ gets smaller, $\bar{B}(\mathcal{L}_M[\mathbf{b}, w], r)$ gets smaller and $\bar{B}(\mathcal{L}_M[\mathbf{b}, w], R)$ gets larger. In the extreme case



when $\mathcal{L}_M[\mathbf{b}, w]$ vanishes, $r = 0$ so that $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f] = \mathcal{L}_M[\mathbf{b}, w] = 0$, and $R = \rho^{-1}(\sigma k)$ so that $\bar{B}(\mathcal{L}_M[\mathbf{b}, w], R)$ becomes the entire deflection horizon.

Theorem 1 does not exclude the possibility that $R = \infty$, in which case we have the global uniqueness of $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$ in $L^\infty[-l, l]$. This does happen when the nonlinearity of given elastic foundation is below a certain level. Especially for the linear uniform case when $f(u, x) = k \cdot u$, we can choose a nonlinearity function ρ which makes $R = \infty$. Since $\mathcal{L}_M[\mathbf{b}, w]$ is a solution of the boundary value problem NDE(f, w) and BC(\mathbf{M}, \mathbf{b}) in this case, Theorem 1(a) implies that $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f] = \mathcal{L}_M[\mathbf{b}, w]$ is the unique solution of the boundary value problem NDE(f, w) and BC(\mathbf{M}, \mathbf{b}) in the whole $L^\infty[-l, l]$. Moreover, Assumptions (A) and (B) will be shown to be satisfied for every $w, \mathbf{M}, \mathbf{b}$ in this case. Note that Assumption (F) is automatically satisfied by $f(u, x) = k \cdot u$. Thus Theorem 1 reproduces the well-known existence and uniqueness result for the linear uniform boundary value problem LDE(w) and BC(\mathbf{M}, \mathbf{b}) with no restriction on the inputs at all. This shows that Theorem 1 seamlessly covers the whole range of problems from linear uniform ones to nonlinear non-uniform ones.

Theorem 1(b) naturally leads to a numerical algorithm to approximate the nonlinear non-uniform deflection through iterations with the operator Ψ , which is also explicitly constructible from the inputs. The fact that our results are given in terms of the L^∞ -norm guarantees the approximation to be *uniform*.

The rest of the paper is organized as follows. In Sect. 2, quantities such as non-uniformity ratio η , nonlinearity function ρ , functional operator \mathcal{N} , which measure nonlinearity and non-uniformity of given elastic foundation, are derived from the two-variable function f in NDE(f, w). In Sect. 3, the effects of arbitrary well-posed boundary condition BC(\mathbf{M}, \mathbf{b}) are encoded in the linear integral operator \mathcal{K}_M and the linear homogeneous deflection $\mathcal{H}_M[\mathbf{b}]$, which arise naturally from the linear uniform problem LDE(w) and BC(\mathbf{M}, \mathbf{b}). Here, Assumption (A) is explained in detail, and the elastic capacity σ is defined. The nonlinear operator Ψ is defined and analyzed in Sect. 4. In Sect. 5, Assump-

tion (B) is explained in detail, and the radii r and R are derived explicitly from the inputs. The explicitness of these derivations is illustrated with concrete examples. In particular, it is shown that Theorem 1 can reproduce the classical existence and uniqueness result for the linear uniform problem. Theorem 1 is proved in Sect. 6, and some discussions on our results are given in Sect. 7.

2 Nonlinear non-uniform elastic foundation

For the rest of the paper, the function f in $NDE(f, w)$ is supposed to satisfy Assumption (F) in Sect. 1.

Definition 2.1 Given f , the *non-uniformity ratio* η at zero deflection is defined by

$$\eta = 1 - \frac{\min_{-l \leq x \leq l} f_u(0, x)}{\max_{-l \leq x \leq l} f_u(0, x)} = 1 - \frac{\min_{-l \leq x \leq l} f_u(0, x)}{k}.$$

η is dimensionless and has the range $0 \leq \eta \leq 1$. An elastic foundation which is uniform at zero deflection corresponds to the extreme case of $\eta = 0$. Non-uniformity of given elastic foundation increases as η increases.^b

The quantity $f_u(u, x) - k$ amounts to the deviation of the spring constant density $f_u(u, x)$ from the corresponding linear uniform density k . So $\hat{\rho}$, defined by

$$\hat{\rho}(t) = \max_{|\tau| \leq t, |x| \leq l} |f_u(\tau, x) - k|, \quad t \geq 0, \tag{2.1}$$

measures the nonlinearity in the spring constant density of given elastic foundation in terms of the magnitude t of deflection. Note that $\hat{\rho}$ becomes the zero function in the linear uniform case $f(u, x) = k \cdot u$. It is clear from its definition (2.1) that $\hat{\rho}$ is nondecreasing. By Definition 2.1, (1.1), and (2.1), we have $\hat{\rho}(0) = \max_{|x| \leq l} |f_u(0, x) - k| = \max_{|x| \leq l} \{k - f_u(0, x)\} = k - \min_{|x| \leq l} f_u(0, x) = \eta k$.

Definition 2.2 Given f , a strictly increasing continuous function $\rho : [0, \infty) \rightarrow [0, \infty)$ is called a *nonlinearity function* if $\rho(0) = 0$, and $\hat{\rho}(t) \leq \eta k + \rho(t)$ for $t \geq 0$, where $\hat{\rho}$ is defined by (2.1).

For a given f , there are infinitely many possibilities for choosing a nonlinearity function ρ . For a nonlinearity function ρ , denote

$$s_\rho = \sup_{t \geq 0} \rho(t). \tag{2.2}$$

$s_\rho = \infty$, when $\lim_{t \rightarrow \infty} \rho(t) = \infty$. Since ρ is strictly increasing, we have $s_\rho > 0$, and ρ always has the well-defined strictly increasing continuous inverse $\rho^{-1} : [0, s_\rho) \rightarrow [0, \infty)$ with $\rho^{-1}(0) = 0$ and $\lim_{s \rightarrow s_\rho} \rho^{-1}(s) = \infty$. In case $s_\rho < \infty$, we extend the domain of ρ^{-1} to $[0, \infty)$ by defining

$$\rho^{-1}(s) = \infty, \quad \text{if } s \in [s_\rho, \infty). \tag{2.3}$$

A nonlinearity function ρ and its inverse ρ^{-1} are used to convert between the deflection variable t and the spring constant density variable s in Sect. 5.

Since $f(u, x) = k \cdot u + \{f(u, x) - k \cdot u\}$, the quantity $f(u, x) - k \cdot u$ corresponds to the non-linear non-uniform part of the elastic force density $f(u, x)$. Thus the following functional operator \mathcal{N} embodies all the nonlinear non-uniform features of given elastic foundation.

Definition 2.3 Given f , define $\mathcal{N} : L^\infty[-l, l] \rightarrow L^\infty[-l, l]$ by

$$\mathcal{N}[u](x) = f(u(x), x) - k \cdot u(x), \quad x \in [-l, l], u \in L^\infty[-l, l].$$

As a consequence of Assumption (F), $f(0, x) = 0$ for $x \in [-l, l]$. It follows that $\mathcal{N}[0](x) = f(0, x) - k \cdot 0 = 0$ for $x \in [-l, l]$, hence we have

$$\mathcal{N}[0] = 0. \tag{2.4}$$

Lemma 2.1 Suppose that ρ is a nonlinearity function. Then $\|\mathcal{N}[u] - \mathcal{N}[v]\|_\infty \leq \{\eta k + \rho(\max\{\|u\|_\infty, \|v\|_\infty\})\} \cdot \|u - v\|_\infty$ for $u, v \in L^\infty[-l, l]$.

Proof Define $N : \mathbb{R} \times [-l, l] \rightarrow \mathbb{R}$ by $N(u, x) = f(u, x) - k \cdot u$, so that

$$\mathcal{N}[u](x) = N(u(x), x), \quad x \in [-l, l], u \in L^\infty[-l, l] \tag{2.5}$$

by Definition 2.3. Suppose $u, v \in L^\infty[-l, l]$. By the mean value theorem, we have

$$N(u(x), x) - N(v(x), x) = N_u(\tau, x) \cdot \{u(x) - v(x)\}, \quad x \in [-l, l]$$

for some τ between $u(x)$ and $v(x)$, and hence for some τ such that

$$|\tau| \leq \max\{|u(x)|, |v(x)|\} \leq \max\{\|u\|_\infty, \|v\|_\infty\}.$$

So, for every $x \in [-l, l]$, we have

$$|N(u(x), x) - N(v(x), x)| \leq \left\{ \max_{|\tau| \leq \max\{\|u\|_\infty, \|v\|_\infty\}} |N_u(\tau, x)| \right\} \cdot |u(x) - v(x)|,$$

hence by (2.1) and (2.5),

$$\begin{aligned} \|\mathcal{N}[u] - \mathcal{N}[v]\|_\infty &= \sup_{|x| \leq l} |N(u(x), x) - N(v(x), x)| \\ &\leq \sup_{|x| \leq l} \left\{ \max_{|\tau| \leq \max\{\|u\|_\infty, \|v\|_\infty\}} |N_u(\tau, x)| \right\} \cdot \sup_{|x| \leq l} |u(x) - v(x)| \\ &\leq \left\{ \max_{|\tau| \leq \max\{\|u\|_\infty, \|v\|_\infty\}, |x| \leq l} |N_u(\tau, x)| \right\} \cdot \|u - v\|_\infty \\ &= \hat{\rho}(\max\{\|u\|_\infty, \|v\|_\infty\}) \cdot \|u - v\|_\infty, \end{aligned}$$

since $N_u(u, x) = f_u(u, x) - k$. Thus the result follows from Definition 2.2. □

Example 2.1 Suppose $l \geq \pi$, and let

$$f(u, x) = (1 + \epsilon \cos x) \left(\frac{k_0}{1 + \epsilon} \cdot u + au^3 \right), \quad 0 \leq \epsilon < 1, k_0 > 0, a > 0,$$

which satisfies Assumption (F) in Sect. 1. Since

$$f_u(u, x) = (1 + \epsilon \cos x) \left(\frac{k_0}{1 + \epsilon} + 3au^2 \right), \tag{2.6}$$

the maximal spring constant density k at zero deflection in (1.1) is

$$k = \max_{|x| \leq l} f_u(0, x) = \max_{|x| \leq l} \frac{1 + \epsilon \cos x}{1 + \epsilon} \cdot k_0 = k_0. \tag{2.7}$$

Since $l \geq \pi$, we have

$$\min_{|x| \leq l} f_u(0, x) = \min_{|x| \leq l} \frac{1 + \epsilon \cos x}{1 + \epsilon} \cdot k_0 = \frac{1 - \epsilon}{1 + \epsilon} \cdot k_0,$$

hence the non-uniformity ratio η at zero deflection in Definition 2.1 is

$$\eta = 1 - \frac{\frac{1 - \epsilon}{1 + \epsilon} \cdot k_0}{k_0} = \frac{2\epsilon}{1 + \epsilon}. \tag{2.8}$$

By (2.6) and (2.7),

$$f_u(u, x) - k = -\frac{\epsilon(1 - \cos x)}{1 + \epsilon} \cdot k + 3a(1 + \epsilon \cos x)u^2,$$

hence by (2.1) and (2.8),

$$\begin{aligned} \hat{\rho}(t) &\leq \max_{|x| \leq l} \left| \frac{\epsilon(1 - \cos x)}{1 + \epsilon} \cdot k \right| + \max_{|\tau| \leq t, |x| \leq l} |3a(1 + \epsilon \cos x)\tau^2| \\ &\leq \eta k + 3a(1 + \epsilon) \cdot t^2. \end{aligned}$$

So by Definition 2.2, we can take $\rho(t) = At^2$ for $t \geq 0$, where we put $A = 3a(1 + \epsilon)$. With this ρ , we have $s_\rho = \infty$, and its inverse $\rho^{-1} : [0, \infty) \rightarrow [0, \infty)$ is given by

$$\rho^{-1}(s) = \sqrt{s/A}, \quad s \geq 0. \tag{2.9}$$

Example 2.2 Let $f(u, x) = k \cdot u$, so that the elastic foundation is linear and uniform. Assumption (F) is clearly satisfied by f , and the non-uniformity ratio η at zero deflection is 0. Since $f_u(u, x) - k = k - k = 0$, we have $\hat{\rho}(t) = 0$ for $t \geq 0$ by (2.1), hence by Definition 2.2, we can take any strictly increasing continuous $\rho : [0, \infty) \rightarrow [0, \infty)$ such that $\rho(0) = 0$. The following choice of ρ will turn out to be useful.

$$\rho(t) = \sigma k \left(1 - \frac{1}{\sqrt{1 + ct}} \right), \quad t \geq 0.$$

Here, the constant $\sigma > 0$ is the elastic capacity defined in Definition 3.2, and $c > 0$ is the constant for converting deflection t to dimensionless ct . With this ρ , we have $s_\rho = \sigma k < \infty$, hence by (2.3), $\rho^{-1} : [0, \infty) \rightarrow [0, \infty)$ is given by

$$\rho^{-1}(s) = \begin{cases} \frac{1}{c} \left\{ \left(\frac{\sigma k}{\sigma k - s} \right)^2 - 1 \right\}, & \text{if } 0 \leq s < \sigma k, \\ \infty, & \text{if } s \geq \sigma k. \end{cases} \tag{2.10}$$

3 Boundary conditions and Assumption (A)

A boundary matrix $\mathbf{M} \in \text{gl}(4, 8, \mathbb{R})$ is called *well-posed* if the linear uniform boundary value problem $\text{LDE}(0)$ and $\text{BC}(\mathbf{M}, \mathbf{0})$ has the unique solution 0 . For the rest of the paper, we always assume that a boundary matrix \mathbf{M} is well-posed without mentioning. For any fixed loading density $w \in L^\infty[-l, l]$ and boundary value $\mathbf{b} \in \text{gl}(4, 1, \mathbb{R})$, the boundary value problem $\text{LDE}(w)$ and $\text{BC}(\mathbf{M}, \mathbf{b})$ has a unique solution if and only if \mathbf{M} is well-posed. Hence the following is well defined.

Definition 3.1 Let $w \in L^\infty[-l, l]$ and $\mathbf{b} \in \text{gl}(4, 1, \mathbb{R})$. The unique solution of the boundary value problem $\text{LDE}(w)$ and $\text{BC}(\mathbf{M}, \mathbf{0})$ is denoted by $\mathcal{K}_M[w]$. The unique solution of the boundary value problem $\text{LDE}(0)$ and $\text{BC}(\mathbf{M}, \mathbf{b})$ is denoted by $\mathcal{H}_M[\mathbf{b}]$. The unique solution of the boundary value problem $\text{LDE}(w)$ and $\text{BC}(\mathbf{M}, \mathbf{b})$ is denoted by $\mathcal{L}_M[\mathbf{b}, w]$.

Note that $\mathcal{K}_M[w]$ and $\mathcal{H}_M[\mathbf{b}]$ correspond to a particular solution and a homogeneous solution of $\text{LDE}(w)$ respectively. It is immediate from the elementary theory of linear ordinary differential equations that

$$\mathcal{L}_M[\mathbf{b}, w] = \mathcal{H}_M[\mathbf{b}] + \mathcal{K}_M[w], \quad \mathbf{b} \in \text{gl}(4, 1, \mathbb{R}), w \in L^\infty[-l, l]. \tag{3.1}$$

It is well known [14] that, for each well-posed boundary matrix \mathbf{M} , $\mathcal{K}_M[w]$ has the integral form

$$\mathcal{K}_M[w](x) = \int_{-l}^l G_M(x, \xi) w(\xi) d\xi, \quad x \in [-l, l], \tag{3.2}$$

where $G_M(x, \xi)$ is the *Green's function* corresponding to \mathbf{M} . The integral operator \mathcal{K}_M is a bounded linear operator on the Banach space $L^\infty[-l, l]$. See [6] for explicit construction of the Green's function $G_M(x, \xi)$ from arbitrary well-posed \mathbf{M} .

Two well-posed boundary matrices \mathbf{M}, \mathbf{N} are called *equivalent* if $\mathcal{K}_M = \mathcal{K}_N$. For example, it follows immediately from Definition 3.1 that the well-posed boundary matrix

$$\mathbf{N} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \end{array} \right)$$

is equivalent to \mathbf{M} in (1.3). It is shown in [6] that the space of equivalence classes of well-posed boundary matrices with respect to this relation, and hence the space of different integral operators \mathcal{K}_M , is in canonical one-to-one correspondence with the algebra $\text{gl}(4, \mathbb{R}) = \text{gl}(4, 4, \mathbb{R})$ whose dimension is 16.

We denote $\mu_M = k \cdot \|\mathcal{K}_M\|_\infty$, where $\|\mathcal{K}_M\|_\infty = \sup_{0 \neq u \in L^\infty[-l, l]} \frac{\|\mathcal{K}_M[u]\|_\infty}{\|u\|_\infty}$ is the L^∞ -norm of \mathcal{K}_M . The dimensionless quantity μ_M is called the *intrinsic L^∞ -norm* of \mathcal{K}_M . μ_M turns out to be an important quantity through which the boundary matrix \mathbf{M} affects our problem.

For the rest of the paper, Assumption (A) $\mu_M \cdot \eta < 1$ in Sect. 1 is supposed to be satisfied. Assumption (A) sets a mutual limit on the non-uniformity of the elastic foundation and the intrinsic L^∞ -norm of \mathcal{K}_M . Given an elastic foundation f , the possible boundary matrices \mathbf{M} are restricted by the condition $\mu_M < 1/\eta$. The restriction gets looser as η becomes smaller.

In the extreme case when $\eta = 0$, there is no restriction on \mathbf{M} . On the other hand, given a well-posed boundary matrix \mathbf{M} , the non-uniformity ratio η is restricted by $\eta < 1/\mu_{\mathbf{M}}$. This restriction gets looser as $\mu_{\mathbf{M}}$ becomes smaller. A critical phenomenon happens when $\mu_{\mathbf{M}}$ is less than 1, in which case there is no restriction on η since $0 \leq \eta \leq 1$.

We call $\mathcal{K}_{\mathbf{M}}$ *contractive* if $\mu_{\mathbf{M}} < 1$. The above observation shows that contractiveness of the operator $\mathcal{K}_{\mathbf{M}}$ for given boundary matrix \mathbf{M} is critical in our problem. There are cases that are contractive [5] and cases that are not [6].^c

Definition 3.2 Given f and \mathbf{M} , the constant σ , called the *elastic capacity*, is defined by $\sigma = \frac{1}{\mu_{\mathbf{M}}} - \eta$.

By Assumption (A) and the fact that $\mu_{\mathbf{M}} \neq 0$ for every well-posed \mathbf{M} , we always have $0 < \sigma < \infty$. σ is dimensionless, hence the quantity σk corresponds to spring constant density. Since $\mu_{\mathbf{M}} = 1/(\eta + \sigma)$ by Definition 3.2, we have

$$\|\mathcal{K}_{\mathbf{M}}[u]\|_{\infty} \leq \frac{1}{\eta k + \sigma k} \cdot \|u\|_{\infty}, \quad u \in L^{\infty}[-l, l]. \tag{3.3}$$

Being a part of $\mathcal{L}_{\mathbf{M}}[\mathbf{b}, w]$ as shown in (3.1), $\mathcal{H}_{\mathbf{M}}[\mathbf{b}]$ is another mean by which the boundary condition $\text{BC}(\mathbf{M}, \mathbf{b})$ affects our problem. So it is important to note that the linear operator $\mathcal{H}_{\mathbf{M}} : \mathfrak{gl}(4, 1, \mathbb{R}) \rightarrow L^{\infty}[-l, l]$ has the following explicit representation in terms of \mathbf{M} .

Lemma 3.1 Let $\mathbf{M} \in \mathfrak{gl}(4, 8, \mathbb{R})$ be well-posed, and let $\{y_1, y_2, y_3, y_4\}$ be a fundamental set of solutions of LDE(0). Then $\mathcal{H}_{\mathbf{M}}[\mathbf{b}] = \mathbf{y}^T \{\mathbf{M}^- \mathbf{W}(-l) + \mathbf{M}^+ \mathbf{W}(l)\}^{-1} \mathbf{b}$ for $\mathbf{b} \in \mathfrak{gl}(4, 1, \mathbb{R})$, where $\mathbf{y} = (y_1 \ y_2 \ y_3 \ y_4)^T$, $\mathbf{W} = (\mathbf{y} \ \mathbf{y}' \ \mathbf{y}'' \ \mathbf{y}''')^T$, and \mathbf{M}^- , \mathbf{M}^+ are the 4×4 minors of \mathbf{M} such that $\mathbf{M} = (\mathbf{M}^- \ \mathbf{M}^+)$.

Proof Since $\mathcal{H}_{\mathbf{M}}[\mathbf{b}]$ satisfies LDE(0) by Definition 3.1, we have

$$\mathcal{H}_{\mathbf{M}}[\mathbf{b}] = \sum_{j=1}^4 c_j y_j = \mathbf{y}^T \mathbf{c} \tag{3.4}$$

for some $\mathbf{c} = (c_1 \ c_2 \ c_3 \ c_4)^T \in \mathfrak{gl}(4, 1, \mathbb{R})$. By (1.2) and (3.4),

$$\begin{aligned} \mathcal{B}[\mathcal{H}_{\mathbf{M}}[\mathbf{b}]] &= \mathcal{B} \left[\sum_{j=1}^4 c_j y_j \right] = \sum_{j=1}^4 c_j \mathcal{B}[y_j] = \left(\mathcal{B}[y_1] \ \mathcal{B}[y_2] \ \mathcal{B}[y_3] \ \mathcal{B}[y_4] \right) \mathbf{c} \\ &= \left(\mathbf{y}(-l) \ \mathbf{y}'(-l) \ \mathbf{y}''(-l) \ \mathbf{y}'''(-l) \ \mathbf{y}(l) \ \mathbf{y}'(l) \ \mathbf{y}''(l) \ \mathbf{y}'''(l) \right)^T \mathbf{c} \\ &= \begin{pmatrix} \mathbf{W}(-l) \\ \mathbf{W}(l) \end{pmatrix} \mathbf{c}, \end{aligned}$$

hence

$$\begin{aligned} \mathbf{b} &= \mathbf{M} \cdot \begin{pmatrix} \mathbf{W}(-l) \\ \mathbf{W}(l) \end{pmatrix} \mathbf{c} = \begin{pmatrix} \mathbf{M}^- & \mathbf{M}^+ \end{pmatrix} \begin{pmatrix} \mathbf{W}(-l) \\ \mathbf{W}(l) \end{pmatrix} \mathbf{c} \\ &= \{\mathbf{M}^- \mathbf{W}(-l) + \mathbf{M}^+ \mathbf{W}(l)\} \mathbf{c}, \end{aligned}$$

since $\mathcal{H}_M[\mathbf{b}]$ satisfies $\text{BC}(M, \mathbf{b})$: $M \cdot \mathcal{B}[\mathcal{H}_M[\mathbf{b}]] = \mathbf{b}$ by Definition 3.1. Thus $\mathbf{c} = \{M^-W(-l) + M^+W(l)\}^{-1}\mathbf{b}$, hence the lemma follows by (3.4). \square

Since the entries of \mathbf{y} in Lemma 3.1 are linearly independent, $\mathcal{H}_M[\mathbf{b}] = 0$ if and only if $\mathbf{b} = \mathbf{0}$. Since the integral operator \mathcal{K}_M is injective for every well-posed M [6], $\mathcal{K}_M[w] = 0$ if and only if $w = 0$. By elementary theory of linear ordinary differential equations, $\mathcal{L}_M[\mathbf{b}, w] = 0$ if and only if $\mathcal{H}_M[\mathbf{b}] = 0$ and $\mathcal{K}_M[w] = 0$. It follows that $\mathcal{L}_M[\mathbf{b}, w] = 0$ if and only if $\mathbf{b} = \mathbf{0}$ and $w = 0$. Thus the linear uniform deflection $\mathcal{L}_M[\mathbf{b}, w]$ already becomes non-trivial unless both the boundary value \mathbf{b} and the loading density w are trivial, which is in contrast to the infinite beam situation in [7].

Since the space $\text{gl}(4, 1, \mathbb{R})$ of boundary values \mathbf{b} is 4-dimensional, the set of different boundary conditions $\text{BC}(M, \mathbf{b})$ forms a 20-dimensional space, combined with the 16-dimensionally different well-posed boundary matrices M .

4 The operator Ψ

Definition 4.1 Given f, w, M, \mathbf{b} , define $\Psi : L^\infty[-l, l] \rightarrow L^\infty[-l, l]$ by $\Psi[u] = \mathcal{H}_M[\mathbf{b}] + \mathcal{K}_M[w - \mathcal{N}[u]]$.

Note that the definition of Ψ involves all the inputs f, w, M, \mathbf{b} since \mathcal{N} is determined by f . By Definition 2.3, (3.1), (3.2), and Lemma 3.1, Ψ has the following explicit form, where \mathbf{y}, W, M^-, M^+ are defined as in Lemma 3.1:

$$\begin{aligned} \Psi[u](x) &= \mathbf{y}(x)^T \{M^-W(-l) + M^+W(l)\}^{-1}\mathbf{b} \\ &\quad + \int_{-l}^l G_M(x, \xi) \{w(\xi) - f(u(\xi), \xi) + k \cdot u(\xi)\} d\xi, \quad x \in [-l, l], u \in L^\infty[-l, l]. \end{aligned}$$

Lemma 4.1 shows that the solutions of the nonlinear non-uniform boundary value problem $\text{NDE}(f, w)$ and $\text{BC}(M, \mathbf{b})$ are exactly the fixed points of Ψ .

Lemma 4.1 *Let $u \in L^\infty[-l, l]$. Then $\Psi[u] = u$ if and only if u is a solution of the boundary value problem $\text{NDE}(f, w)$ and $\text{BC}(M, \mathbf{b})$.*

Proof Let $\alpha = \sqrt[4]{k/EI}$. Then $\text{LDE}(w)$ is equivalent to $u^{(4)} = -\alpha^4 u + \frac{\alpha^4}{k} \cdot w$. So, by Definition 3.1, we have

$$\mathcal{H}_M[\mathbf{b}]^{(4)} = -\alpha^4 \cdot \mathcal{H}_M[\mathbf{b}], \quad \mathbf{b} \in \text{gl}(4, 1, \mathbb{R}), \tag{4.1}$$

$$\mathcal{K}_M[w]^{(4)} = -\alpha^4 \cdot \mathcal{K}_M[w] + \frac{\alpha^4}{k} \cdot w, \quad w \in L^\infty[-l, l], \tag{4.2}$$

$$M \cdot \mathcal{B}[\mathcal{H}_M[\mathbf{b}]] = \mathbf{b}, \quad \mathbf{b} \in \text{gl}(4, 1, \mathbb{R}), \tag{4.3}$$

$$M \cdot \mathcal{B}[\mathcal{K}_M[w]] = \mathbf{0}, \quad w \in L^\infty[-l, l]. \tag{4.4}$$

$\text{NDE}(f, w)$ is equivalent to $u^{(4)} = \frac{\alpha^4}{k} \{w - f(u, x)\}$, and hence by Definition 2.3 is equivalent to

$$u^{(4)} = -\alpha^4 \cdot u + \frac{\alpha^4}{k} \cdot \{w - \mathcal{N}[u]\}. \tag{4.5}$$

By Definition 4.1 and (4.1), (4.2), (4.3), (4.4), we have

$$\begin{aligned} \Psi[u]^{(4)} &= \mathcal{H}_M[\mathbf{b}]^{(4)} + \mathcal{K}_M[w - \mathcal{N}[u]]^{(4)} \\ &= -\alpha^4 \cdot \mathcal{H}_M[\mathbf{b}] + \left[-\alpha^4 \cdot \mathcal{K}_M[w - \mathcal{N}[u]] + \frac{\alpha^4}{k} \{w - \mathcal{N}[u]\} \right] \\ &= -\alpha^4 \cdot \Psi[u] + \frac{\alpha^4}{k} \{w - \mathcal{N}[u]\}, \quad u \in L^\infty[-l, l], \end{aligned} \tag{4.6}$$

$$\begin{aligned} \mathbf{M} \cdot \mathcal{B}[\Psi[u]] &= \mathbf{M} \cdot \mathcal{B}[\mathcal{H}_M[\mathbf{b}]] + \mathbf{M} \cdot \mathcal{B}[\mathcal{K}_M[w - \mathcal{N}[u]]] = \mathbf{b} + \mathbf{0} \\ &= \mathbf{b}, \quad u \in L^\infty[-l, l]. \end{aligned} \tag{4.7}$$

Suppose $\Psi[u] = u$. Then, by (4.6),

$$u^{(4)} = \Psi[u]^{(4)} = -\alpha^4 \cdot \Psi[u] + \frac{\alpha^4}{k} \{w - \mathcal{N}[u]\} = -\alpha^4 \cdot u + \frac{\alpha^4}{k} \{w - \mathcal{N}[u]\},$$

hence u satisfies (4.5), which is equivalent to $\text{NDE}(f, w)$. By (4.7), $\mathbf{M} \cdot \mathcal{B}[u] = \mathbf{M} \cdot \mathcal{B}[\Psi[u]] = \mathbf{b}$, hence u satisfies $\text{BC}(\mathbf{M}, \mathbf{b})$. Thus u is a solution of the boundary value problem $\text{NDE}(f, w)$ and $\text{BC}(\mathbf{M}, \mathbf{b})$.

Conversely, suppose that u is a solution of the boundary value problem $\text{NDE}(f, w)$ and $\text{BC}(\mathbf{M}, \mathbf{b})$. Since $\text{NDE}(f, w)$ is equivalent to (4.5), we have $u^{(4)} = -\alpha^4 u + \frac{\alpha^4}{k} \{w - \mathcal{N}[u]\}$, hence by (4.6) and (4.7),

$$\begin{aligned} \{\Psi[u] - u\}^{(4)} &= \Psi[u]^{(4)} - u^{(4)} \\ &= \left[-\alpha^4 \cdot \Psi[u] + \frac{\alpha^4}{k} \{w - \mathcal{N}[u]\} \right] - \left[-\alpha^4 u + \frac{\alpha^4}{k} \{w - \mathcal{N}[u]\} \right] \\ &= -\alpha^4 \cdot \{\Psi[u] - u\}, \end{aligned}$$

and $\mathbf{M} \cdot \mathcal{B}[\Psi[u] - u] = \mathbf{M} \cdot \mathcal{B}[\Psi[u]] - \mathbf{M} \cdot \mathcal{B}[u] = \mathbf{b} - \mathbf{b} = \mathbf{0}$. It follows that $\Psi[u] - u$ is the unique solution of the boundary value problem $\text{LDE}(0)$ and $\text{BC}(\mathbf{M}, \mathbf{0})$, which is the zero function. Thus $\Psi[u] = u$, and the proof is complete. \square

In general, the nonlinear operator Ψ need not be *contractive* on the whole $L^\infty[-l, l]$. So it is crucial to find regions in $L^\infty[-l, l]$ where Ψ becomes contractive. The following result will be useful for that purpose.

Lemma 4.2

(a) For $u, v \in L^\infty[-l, l]$,

$$\|\Psi[u] - \Psi[v]\|_\infty \leq \frac{\eta k + \rho(\|u - v\|_\infty + \|v\|_\infty)}{\eta k + \sigma k} \cdot \|u - v\|_\infty.$$

(b) Let $u_c \in L^\infty[-l, l]$ and $0 \leq \delta < \infty$. Then, for $u, v \in \bar{B}(u_c, \delta)$,

$$\|\Psi[u] - \Psi[v]\|_\infty \leq \frac{\eta k + \rho(\delta + \|u_c\|_\infty)}{\eta k + \sigma k} \cdot \|u - v\|_\infty.$$

Proof Let $u, v \in L^\infty[-l, l]$. By Definition 4.1, we have

$$\begin{aligned} \Psi[u] - \Psi[v] &= \{\mathcal{H}_M[\mathbf{b}] + \mathcal{K}_M[w - \mathcal{N}[u]]\} - \{\mathcal{H}_M[\mathbf{b}] + \mathcal{K}_M[w - \mathcal{N}[v]]\} \\ &= -\mathcal{K}_M[\mathcal{N}[u]] + \mathcal{K}_M[\mathcal{N}[v]] = -\mathcal{K}_M[\mathcal{N}[u] - \mathcal{N}[v]], \end{aligned}$$

since \mathcal{K}_M is linear. So, by (3.3) and Lemma 2.1, we have

$$\begin{aligned} \|\Psi[u] - \Psi[v]\|_\infty &= \|\mathcal{K}_M[\mathcal{N}[u] - \mathcal{N}[v]]\|_\infty \leq \frac{\|\mathcal{N}[u] - \mathcal{N}[v]\|_\infty}{\eta k + \sigma k} \\ &\leq \frac{\eta k + \rho(\max\{\|u\|_\infty, \|v\|_\infty\})}{\eta k + \sigma k} \cdot \|u - v\|_\infty. \end{aligned} \tag{4.8}$$

Since $\|u\|_\infty = \|(u - v) + v\|_\infty \leq \|u - v\|_\infty + \|v\|_\infty$, we have $\max\{\|u\|_\infty, \|v\|_\infty\} \leq \|u - v\|_\infty + \|v\|_\infty$. Thus (a) follows from (4.8) since ρ is increasing.

Suppose $u, v \in \bar{B}(u_c, \delta)$. Since $\|u - u_c\|_\infty \leq \delta$, we have $\|u\|_\infty = \|(u - u_c) + u_c\|_\infty \leq \|u - u_c\|_\infty + \|u_c\|_\infty \leq \delta + \|u_c\|_\infty$. Likewise, $\|v\|_\infty \leq \delta + \|u_c\|_\infty$, and hence we have $\max\{\|u\|_\infty, \|v\|_\infty\} \leq \delta + \|u_c\|_\infty$. Thus (b) follows from (4.8) since ρ is increasing. \square

5 Assumption (B) and the dual radii $r < R$

Let ρ be a nonlinearity function in Definition 2.2, and let $s_\rho > 0$ be defined by (2.2). Denote

$$\bar{s} = \min\{\sigma k, s_\rho\} > 0 \tag{5.1}$$

so that $[0, \bar{s}] = [0, \sigma k] \cap [0, s_\rho]$, where σ is the elastic capacity in Definition 3.2. Define the function $\varphi : [0, \bar{s}] \rightarrow [0, \infty)$ by

$$\varphi(s) = (\sigma k - s) \cdot \rho^{-1}(s). \tag{5.2}$$

φ is continuous, $\varphi(0) = 0$, and $\varphi(s) > 0$ for $0 < s < \bar{s}$ since ρ^{-1} is continuous, $\rho^{-1}(0) = 0$, and $\rho^{-1}(s) > 0$ for $0 < s < s_\rho$. For each $\hat{s} \in [0, s_\rho]$, define the function $\varphi_{\hat{s}} : [0, \bar{s}] \rightarrow [0, \infty)$ by

$$\varphi_{\hat{s}}(s) = (\sigma k - s + \eta k + \hat{s}) \cdot \rho^{-1}(\hat{s}). \tag{5.3}$$

Here, the variables s and \hat{s} represent spring constant densities. So $\varphi(s)$ and $\varphi_{\hat{s}}(s)$ represent elastic force densities since $\rho^{-1}(s)$ and $\rho^{-1}(\hat{s})$ represent deflections. See Figs. 2 and 3 for graphs of $\varphi(s)$ and $\varphi_{\hat{s}}(s)$ in Examples 5.1 and 5.2.

For each $\hat{s} \in [0, s_\rho]$, the graph of $\varphi_{\hat{s}}$ is a line segment with the slope $-\rho^{-1}(\hat{s})$, and $\varphi_{\hat{s}}(0) = (\sigma k + \eta k + \hat{s}) \cdot \rho^{-1}(\hat{s}) \geq 0$, $\lim_{s \rightarrow \bar{s}^-} \varphi_{\hat{s}}(s) = (\sigma k - \bar{s} + \eta k + \hat{s}) \cdot \rho^{-1}(\hat{s}) \geq 0$. For any fixed $s \in [0, \bar{s}]$, $\varphi_{\hat{s}}(s)$ is strictly increasing with respect to \hat{s} , and $\varphi_0(s) = 0$. It follows that the quantity \hat{s}_{\max} , defined by

$$\hat{s}_{\max} = \sup \{ \hat{s} \in [0, s_\rho] \mid \{s \in [0, \bar{s}] \mid \varphi(s) \geq \varphi_{\hat{s}}(s)\} \neq \emptyset \}, \tag{5.4}$$

is positive, and the set $\{s \in [0, \bar{s}] \mid \varphi(s) \geq \varphi_{\hat{s}}(s)\}$ is nonempty for every $\hat{s} \in [0, \hat{s}_{\max})$.

Definition 5.1 Given M, f , and ρ , denote $D = \rho^{-1}(\hat{s}_{\max})$.

Note that D does not involve the inputs w, \mathbf{b} in its definition. $0 < D \leq \rho^{-1}(s_\rho)$ since $0 < \hat{s}_{\max} \leq s_\rho$. By (2.3), $D = \infty$, when $\hat{s}_{\max} = s_\rho$. For the rest of the paper, Assumption (B) $\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty < D$ in Sect. 1 is supposed to be satisfied. Assumption (B), which involves all the inputs $f, w, \mathbf{M}, \mathbf{b}$, sets an upper bound on the magnitude of the linear uniform deflection $\mathcal{L}_M[\mathbf{b}, w]$. It would *not* be needed at all when $D = \infty$ or, equivalently, when $\hat{s}_{\max} = s_\rho$, as in Example 5.2. Denote

$$\hat{s}_0 = \rho(\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty), \tag{5.5}$$

which is equivalent to $\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty = \rho^{-1}(\hat{s}_0)$. Then Assumption (B) is equivalent to $0 \leq \hat{s}_0 < \hat{s}_{\max}$ by Definition 5.1, hence the set $\{s \in [0, \bar{s}] \mid \varphi(s) \geq \varphi_{\hat{s}_0}(s)\}$ is nonempty. Denoting

$$s_{\min} = \min \{s \in [0, \bar{s}] \mid \varphi(s) \geq \varphi_{\hat{s}_0}(s)\}, \tag{5.6}$$

$$s_{\max} = \sup \{s \in [0, \bar{s}] \mid \varphi(s) \geq \varphi_{\hat{s}_0}(s)\}, \tag{5.7}$$

we have

$$0 \leq s_{\min} < s_{\max} \leq \bar{s}. \tag{5.8}$$

Note that, as \hat{s}_0 gets smaller, the set $\{s \in [0, \bar{s}] \mid \varphi(s) \geq \varphi_{\hat{s}_0}(s)\}$ becomes larger until it becomes the whole $[0, \bar{s}]$ when $\hat{s}_0 = 0$. It follows that s_{\min} and s_{\max} are increasing and decreasing with respect to \hat{s}_0 respectively, and

$$\lim_{\hat{s}_0 \rightarrow 0^+} s_{\min} = 0, \quad \lim_{\hat{s}_0 \rightarrow 0^+} s_{\max} = \bar{s}. \tag{5.9}$$

Suppose that $s \in [0, \bar{s}]$ satisfies $\varphi(s) \geq \varphi_{\hat{s}_0}(s)$. Then, by (5.2) and (5.3), we have

$$\begin{aligned} (\sigma k - s) \cdot \rho^{-1}(s) &= \varphi(s) \geq \varphi_{\hat{s}_0}(s) = (\sigma k - s + \eta k + \hat{s}_0) \cdot \rho^{-1}(\hat{s}_0) \\ &= (\sigma k - s) \cdot \rho^{-1}(\hat{s}_0) + (\eta k + \hat{s}_0) \cdot \rho^{-1}(\hat{s}_0). \end{aligned}$$

Thus we have

$$(\eta k + \hat{s}_0) \cdot \rho^{-1}(\hat{s}_0) \leq (\sigma k - s) \{ \rho^{-1}(s) - \rho^{-1}(\hat{s}_0) \} \quad \text{if } \varphi(s) \geq \varphi_{\hat{s}_0}(s), \tag{5.10}$$

$$(\eta k + \hat{s}_0) \cdot \rho^{-1}(\hat{s}_0) = (\sigma k - s) \{ \rho^{-1}(s) - \rho^{-1}(\hat{s}_0) \} \quad \text{if } \varphi(s) = \varphi_{\hat{s}_0}(s). \tag{5.11}$$

Definition 5.2 Given $f, w, \mathbf{M}, \mathbf{b}$, and ρ , denote

$$r = \rho^{-1}(s_{\min}) - \rho^{-1}(\hat{s}_0), \quad R = \rho^{-1}(s_{\max}) - \rho^{-1}(\hat{s}_0).$$

Note that the quantities represented by r and R are deflections. By (2.3), $R = \infty$, when $s_{\max} = s_\rho$.

Lemma 5.1 r and R are increasing and decreasing with respect to $\hat{s}_0 = \rho(\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty)$ respectively. $0 \leq r < R \leq R + \rho^{-1}(\hat{s}_0) \leq \rho^{-1}(\sigma k)$, and $\lim_{\hat{s}_0 \rightarrow 0^+} r = 0, \lim_{\hat{s}_0 \rightarrow 0^+} R = \rho^{-1}(\sigma k)$.

Proof By (5.1) and (5.8), $s_{\min} < \bar{s} \leq \sigma k$. Since φ and $\varphi_{\hat{s}_0}$ are continuous, we have $\varphi(s_{\min}) = \varphi_{\hat{s}_0}(s_{\min})$ by (5.6). Hence, by (5.11) and Definition 5.2, we have

$$r = \frac{\eta k + \hat{s}_0}{\sigma k - s_{\min}} \cdot \rho^{-1}(\hat{s}_0).$$

It follows that r is increasing with respect to \hat{s}_0 since s_{\min} is increasing with respect to \hat{s}_0 . Since s_{\max} is decreasing with respect to \hat{s}_0 , R is decreasing with respect to \hat{s}_0 by Definition 5.2. Since $\lim_{\hat{s}_0 \rightarrow 0^+} \rho^{-1}(\hat{s}_0) = \rho^{-1}(0) = 0$, we have

$$\begin{aligned} \lim_{\hat{s}_0 \rightarrow 0^+} r &= \lim_{\hat{s}_0 \rightarrow 0^+} \rho^{-1}(s_{\min}) - \lim_{\hat{s}_0 \rightarrow 0^+} \rho^{-1}(\hat{s}_0) = \rho^{-1}(0) - 0 = 0, \\ \lim_{\hat{s}_0 \rightarrow 0^+} R &= \lim_{\hat{s}_0 \rightarrow 0^+} \rho^{-1}(s_{\max}) - \lim_{\hat{s}_0 \rightarrow 0^+} \rho^{-1}(\hat{s}_0) = \rho^{-1}(\bar{s}) - 0 = \rho^{-1}(\bar{s}) \end{aligned} \tag{5.12}$$

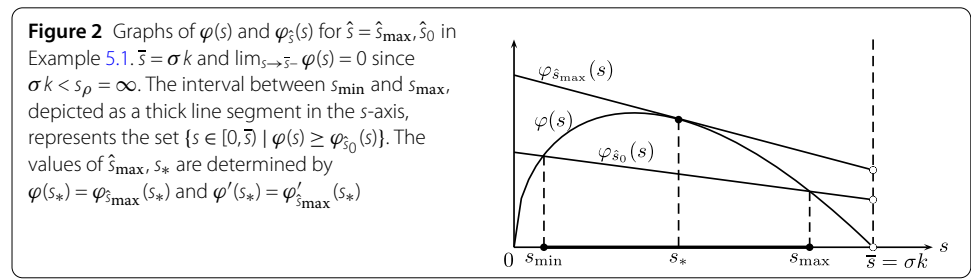
by (5.9) and Definition 5.2. If $\sigma k < s_\rho$, then $\bar{s} = \sigma k$ by (5.1). If $\sigma k \geq s_\rho$, then $\bar{s} = s_\rho$ by (5.1), and hence $\rho^{-1}(\bar{s}) = \infty = \rho^{-1}(\sigma k)$ by (2.3). So, by (5.12), we have $\lim_{\hat{s}_0 \rightarrow 0^+} R = \rho^{-1}(\sigma k)$. Thus $0 \leq r$ and $R \leq \rho^{-1}(\sigma k)$. The inequalities $r < R$ and $R + \rho^{-1}(\hat{s}_0) \leq \rho^{-1}(\sigma k)$ follow from (5.1), (5.8), and Definition 5.2. \square

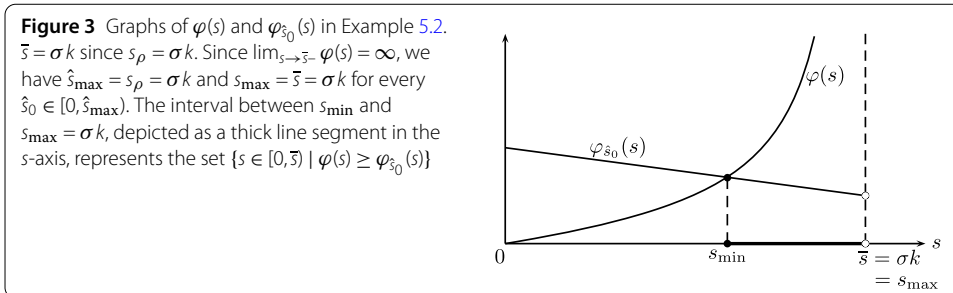
Example 5.1 Let f and ρ be given as in Example 2.1. Since $s_\rho = \infty$, we have $\bar{s} = \sigma k$ by (5.1). By (2.9), (5.2), and (5.3), $\rho^{-1}(\sigma k) = \sqrt{\sigma k/A}$ and $\varphi(s) = (\sigma k - s)\sqrt{s/A}$, $\varphi_{\hat{s}}(s) = (\sigma k - s + \eta k + \hat{s})\sqrt{\hat{s}/A}$. See Fig. 2. The system of equations $\varphi(s) = \varphi_{\hat{s}}(s)$ and $\varphi'(s) = \varphi'_{\hat{s}}(s)$ in s and \hat{s} has the unique solution $s = s_*$ and $\hat{s} = \hat{s}_{\max}$, where $0 < s_* < \sigma k$, $0 < \hat{s}_{\max} < s_\rho = \infty$. By (2.9) and Definition 5.1, we have $D = \sqrt{\hat{s}_{\max}/A} < \infty$. For each $\hat{s}_0 \in (0, \hat{s}_{\max})$, the equation $\varphi(s) = \varphi_{\hat{s}_0}(s)$ has exactly two solutions in $(0, \sigma k)$ which are $s_{\min} < s_{\max}$ and $\{s \in [0, \bar{s}] \mid \varphi(s) \geq \varphi_{\hat{s}_0}(s)\} = [s_{\min}, s_{\max}]$. When $\hat{s}_0 = 0$, we have $s_{\min} = 0$, $s_{\max} = \sigma k$, and $\{s \in [0, \bar{s}] \mid \varphi(s) \geq \varphi_{\hat{s}_0}(s)\} = [0, \sigma k]$. s_{\min} and s_{\max} are strictly increasing and strictly decreasing with respect to \hat{s}_0 respectively. By (2.9) and Definition 5.2, $r = \sqrt{s_{\min}/A} - \sqrt{\hat{s}_0/A}$ and $R = \sqrt{s_{\max}/A} - \sqrt{\hat{s}_0/A}$, which are strictly increasing and strictly decreasing with respect to \hat{s}_0 respectively. When $\hat{s}_0 = \rho(\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty) = 0$, we have $r = 0$ and $R = \sqrt{\sigma k/A} = \rho^{-1}(\sigma k)$.

Example 5.2 shows that Theorem 1 reproduces the well-known existence and uniqueness result for the linear uniform problem LDE(w) and BC(\mathbf{M}, \mathbf{b}).

Example 5.2 Let f and ρ be given as in Example 2.2. Since $s_\rho = \sigma k$, we have $\bar{s} = \sigma k$ by (5.1). Since $\eta = 0$, Assumption (A) imposes no restriction on μ_M , and

$$\varphi(s) = (\sigma k - s) \cdot \frac{1}{c} \left\{ \left(\frac{\sigma k}{\sigma k - s} \right)^2 - 1 \right\}, \tag{5.13}$$





$$\varphi_{\hat{s}}(s) = (\sigma k - s + \hat{s}) \cdot \frac{1}{c} \left\{ \left(\frac{\sigma k}{\sigma k - \hat{s}} \right)^2 - 1 \right\} \tag{5.14}$$

by (2.10), (5.2), and (5.3). See Fig. 3. Since $\lim_{s \rightarrow \bar{s}-} \varphi(s) = \lim_{s \rightarrow \sigma k-} \varphi(s) = \infty$, we have $\hat{s}_{\max} = s_\rho = \sigma k$ by (5.4). So, by (2.10) and Definition 5.1, we have $D = \rho^{-1}(\sigma k) = \infty$. Thus Assumption (B) imposes no restriction on $\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty$. Since $\lim_{s \rightarrow \bar{s}-} \varphi(s) = \infty$, we have $s_{\max} = \bar{s} = \sigma k$ by (5.7), and hence by (2.10) and Definition 5.2, $R = \infty$ for every $\hat{s}_0 \in [0, \hat{s}_{\max}) = [0, \sigma k)$. Thus, by Theorem 1, $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$, which coincides with $\mathcal{L}_M[\mathbf{b}, w]$ in this case, is unique in the whole $L^\infty[-l, l]$. Moreover, there are no restrictions on the inputs $w, \mathbf{M}, \mathbf{b}$ since Assumption (F) is also satisfied by $f(u, x) = k \cdot u$.

Given $\hat{s}_0 = \rho(\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty)$ in $[0, \hat{s}_{\max}) = [0, \sigma k)$, s_{\min} is the unique solution of the equation $\varphi(s) = \varphi_{\hat{s}_0}(s)$, which, by (5.13) and (5.14), is equivalent to

$$\begin{aligned} \frac{(\sigma k)^2}{\sigma k - s} &= \left(\frac{\sigma k}{\sigma k - \hat{s}_0} \right)^2 (\sigma k - s) + \hat{s}_0 \left\{ \left(\frac{\sigma k}{\sigma k - \hat{s}_0} \right)^2 - 1 \right\} \\ &= \frac{1}{(\sigma k - \hat{s}_0)^2} \{ (\sigma k)^2 \cdot (\sigma k - s) + \hat{s}_0^2 (2\sigma k - \hat{s}_0) \}, \end{aligned}$$

and hence to $(\sigma k)^2 \cdot (\sigma k - s)^2 + \hat{s}_0^2 (2\sigma k - \hat{s}_0) \cdot (\sigma k - s) - (\sigma k)^2 (\sigma k - \hat{s}_0)^2 = 0$. So we have

$$\begin{aligned} \sigma k - s_{\min} &= \frac{-\hat{s}_0^2 (2\sigma k - \hat{s}_0) + \sqrt{\{\hat{s}_0^2 (2\sigma k - \hat{s}_0)\}^2 + 4(\sigma k)^4 (\sigma k - \hat{s}_0)^2}}{2(\sigma k)^2} \\ &= \frac{2(\sigma k)^2 (\sigma k - \hat{s}_0)^2}{\hat{s}_0^2 (2\sigma k - \hat{s}_0) + \sqrt{\{\hat{s}_0^2 (2\sigma k - \hat{s}_0)\}^2 + 4(\sigma k)^4 (\sigma k - \hat{s}_0)^2}}, \end{aligned} \tag{5.15}$$

hence

$$s_{\min} = \sigma k - \frac{2(\sigma k)^2 (\sigma k - \hat{s}_0)^2}{\hat{s}_0^2 (2\sigma k - \hat{s}_0) + \sqrt{\{\hat{s}_0^2 (2\sigma k - \hat{s}_0)\}^2 + 4(\sigma k)^4 (\sigma k - \hat{s}_0)^2}}.$$

By (2.10), (5.15), and Definition 5.2,

$$\begin{aligned} r &= \rho^{-1}(s_{\min}) - \rho^{-1}(\hat{s}_0) \\ &= \frac{1}{c} \left\{ \left(\frac{\sigma k}{\sigma k - s_{\min}} \right)^2 - 1 \right\} - \frac{1}{c} \left\{ \left(\frac{\sigma k}{\sigma k - \hat{s}_0} \right)^2 - 1 \right\} \\ &= \frac{1}{c} \left[\frac{\{\hat{s}_0^2 (2\sigma k - \hat{s}_0) + \sqrt{\{\hat{s}_0^2 (2\sigma k - \hat{s}_0)\}^2 + 4(\sigma k)^4 (\sigma k - \hat{s}_0)^2}\}^2}{4(\sigma k)^2 (\sigma k - \hat{s}_0)^4} - \left(\frac{\sigma k}{\sigma k - \hat{s}_0} \right)^2 \right] \end{aligned}$$

$$= \frac{\{\hat{s}_0^2(2\sigma k - \hat{s}_0)\}^2 + \hat{s}_0^2(2\sigma k - \hat{s}_0)\sqrt{\{\hat{s}_0^2(2\sigma k - \hat{s}_0)\}^2 + 4(\sigma k)^4(\sigma k - \hat{s}_0)^2}}{2c(\sigma k)^2(\sigma k - \hat{s}_0)^4},$$

hence we have

$$\lim_{\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty \rightarrow 0} r = \lim_{\hat{s}_0 \rightarrow 0^+} r = 0, \quad \lim_{\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty \rightarrow \infty} r = \lim_{\hat{s}_0 \rightarrow \sigma k^-} r = \infty = R.$$

6 Proof of Theorem 1

By (2.4), (3.1), and Definition 4.1, we have

$$\Psi[0] = \mathcal{H}_M[\mathbf{b}] + \mathcal{K}_M[w - \mathcal{N}[0]] = \mathcal{L}_M[\mathbf{b}, w]. \tag{6.1}$$

Note that, if $\varphi(s) \geq \varphi_{\hat{s}_0}(s)$, then $s_{\min} \leq s < \bar{s} \leq s_\rho$ by (5.1) and (5.6), hence $\rho^{-1}(s) - \rho^{-1}(\hat{s}_0) \geq \rho^{-1}(s_{\min}) - \rho^{-1}(\hat{s}_0) = r \geq 0$ by Definition 5.2 and Lemma 5.1, and $\rho^{-1}(s) - \rho^{-1}(\hat{s}_0) < \infty$ since $s < s_\rho$.

Lemma 6.1 *Suppose that $s \in [0, \bar{s})$ satisfies $\varphi(s) \geq \varphi_{\hat{s}_0}(s)$, and let $\delta = \rho^{-1}(s) - \rho^{-1}(\hat{s}_0)$. Then the following (a) and (b) hold:*

- (a) $\Psi[u] \in \bar{B}(\mathcal{L}_M[\mathbf{b}, w], \delta)$ for $u \in \bar{B}(\mathcal{L}_M[\mathbf{b}, w], \delta)$.
- (b) $\|\Psi[u] - \Psi[v]\|_\infty \leq \frac{\eta k + s}{\eta k + \sigma k} \cdot \|u - v\|_\infty$ for $u, v \in \bar{B}(\mathcal{L}_M[\mathbf{b}, w], \delta)$.

Proof By (6.1), we have

$$\begin{aligned} \Psi[u] - \mathcal{L}_M[\mathbf{b}, w] &= \Psi[u] - \Psi[0] \\ &= \{\Psi[u] - \Psi[\mathcal{L}_M[\mathbf{b}, w]]\} + \{\Psi[\mathcal{L}_M[\mathbf{b}, w]] - \Psi[0]\}, \quad u \in L^\infty[-l, l]. \end{aligned} \tag{6.2}$$

Suppose $u \in \bar{B}(\mathcal{L}_M[\mathbf{b}, w], \delta)$. Then $\|u - \mathcal{L}_M[\mathbf{b}, w]\|_\infty \leq \delta$, hence by Lemma 4.2(a), (5.5), and (6.2),

$$\begin{aligned} &\|\Psi[u] - \mathcal{L}_M[\mathbf{b}, w]\|_\infty \\ &\leq \|\Psi[u] - \Psi[\mathcal{L}_M[\mathbf{b}, w]]\|_\infty + \|\Psi[\mathcal{L}_M[\mathbf{b}, w]] - \Psi[0]\|_\infty \\ &\leq \frac{\eta k + \rho(\|u - \mathcal{L}_M[\mathbf{b}, w]\|_\infty + \|\mathcal{L}_M[\mathbf{b}, w]\|_\infty)}{\eta k + \sigma k} \cdot \|u - \mathcal{L}_M[\mathbf{b}, w]\|_\infty \\ &\quad + \frac{\eta k + \rho(\|\mathcal{L}_M[\mathbf{b}, w]\|_\infty)}{\eta k + \sigma k} \cdot \|\mathcal{L}_M[\mathbf{b}, w]\|_\infty \\ &\leq \frac{\eta k + \rho(\delta + \rho^{-1}(\hat{s}_0))}{\eta k + \sigma k} \cdot \delta + \frac{\eta k + \hat{s}_0}{\eta k + \sigma k} \cdot \rho^{-1}(\hat{s}_0). \end{aligned}$$

So we have

$$\|\Psi[u] - \mathcal{L}_M[\mathbf{b}, w]\|_\infty \leq \frac{\eta k + s}{\eta k + \sigma k} \cdot \delta + \frac{\eta k + \hat{s}_0}{\eta k + \sigma k} \cdot \rho^{-1}(\hat{s}_0) \tag{6.3}$$

since

$$\rho(\delta + \rho^{-1}(\hat{s}_0)) = \rho(\rho^{-1}(s)) = s. \tag{6.4}$$

Since $\varphi(s) \geq \varphi_{\hat{s}_0}(s)$, we have $(\eta k + \hat{s}_0) \cdot \rho^{-1}(\hat{s}_0) \leq (\sigma k - s)\{\rho^{-1}(s) - \rho^{-1}(\hat{s}_0)\} = (\sigma k - s) \cdot \delta$ by (5.10). So, by (6.3), we have

$$\|\Psi[u] - \mathcal{L}_M[\mathbf{b}, w]\|_\infty \leq \frac{\eta k + s}{\eta k + \sigma k} \cdot \delta + \frac{\sigma k - s}{\eta k + \sigma k} \cdot \delta = \delta.$$

Thus $\Psi[u] \in \overline{B}(\mathcal{L}_M[\mathbf{b}, w], \delta)$, which shows (a).

Suppose $u, v \in \overline{B}(\mathcal{L}_M[\mathbf{b}, w], \delta)$. Then, by Lemma 4.2(b), (5.5), and (6.4), we have

$$\begin{aligned} \|\Psi[u] - \Psi[v]\|_\infty &\leq \frac{\eta k + \rho(\delta + \|\mathcal{L}_M[\mathbf{b}, w]\|_\infty)}{\eta k + \sigma k} \cdot \|u - v\|_\infty \\ &\leq \frac{\eta k + \rho(\delta + \rho^{-1}(\hat{s}_0))}{\eta k + \sigma k} \cdot \|u - v\|_\infty = \frac{\eta k + s}{\eta k + \sigma k} \cdot \|u - v\|_\infty. \end{aligned}$$

This shows (b), and the proof is complete. □

Proposition 1 (Banach fixed point theorem [2]) *Let X be a complete metric space with the metric $d(\cdot, \cdot)$. Suppose that the map $\Phi : X \rightarrow X$ satisfies $d(\Phi[u], \Phi[v]) \leq L \cdot d(u, v)$ for every $u, v \in X$ for some constant $0 \leq L < 1$. Then Φ has a unique fixed point in X . For any $u_0 \in X$, the sequence $\{u_n\}_{n=0}^\infty$, defined by $u_n = \Phi[u_{n-1}]$, $n = 1, 2, 3, \dots$, converges to the unique fixed point of Φ with respect to d .*

With the metric $\|\cdot\|_\infty$, $\overline{B}(u_c, \delta)$ for any $u_c \in L^\infty[-l, l]$ and $0 \leq \delta \leq \infty$ is a complete metric space since it is closed in the complete metric space $L^\infty[-l, l]$. Thus Lemma 6.1 indicates that Theorem 1 would follow by applying Proposition 1 to the nested spaces $\overline{B}(\mathcal{L}_M[\mathbf{b}, w], r) \subset \overline{B}(\mathcal{L}_M[\mathbf{b}, w], R)$ with the map Ψ . However, there are exceptional cases which do not fit into this picture. When $s_{\max} = \bar{s}$ so that s_{\max} is *not* in the domain $[0, \bar{s})$ of φ and $\varphi_{\hat{s}_0}$, then Lemma 6.1 cannot be applied to $s = s_{\max}$. Consequently, Proposition 1 cannot be applied directly to $\overline{B}(\mathcal{L}_M[\mathbf{b}, w], R)$ in these cases. To deal with these exceptional situations, we devise Lemma 6.2, which is a generalization of Proposition 1.

Lemma 6.2 *Let $u_c \in L^\infty[-l, l]$, $0 \leq r' < R' \leq \infty$, and let $\Phi : L^\infty[-l, l] \rightarrow L^\infty[-l, l]$ be continuous. Suppose that there exists a strictly increasing sequence $r' = \delta_0 < \delta_1 < \delta_2 < \dots \nearrow R'$ such that the following (a) and (b) are satisfied for each $i = 0, 1, 2, \dots$*

- (a) $\Phi[u] \subset \overline{B}(u_c, \delta_i)$ for $u \in \overline{B}(u_c, \delta_i)$.
- (b) There exists $0 \leq L_i < 1$ such that $\|\Phi[u] - \Phi[v]\|_\infty \leq L_i \cdot \|u - v\|_\infty$ for $u, v \in \overline{B}(u_c, \delta_i)$.

Then there exists a unique fixed point u_ of Φ in $\overline{B}(u_c, R')$. For any $u_0 \in \overline{B}(u_c, R')$, the sequence $\{u_n\}_{n=0}^\infty$, defined by $u_n = \Phi[u_{n-1}]$, $n = 1, 2, 3, \dots$, converges uniformly to u_* , and $u_* \in \overline{B}(u_c, r')$.*

Proof By Proposition 1, there exists a unique fixed point u_* of Φ in $\overline{B}(u_c, r') = \overline{B}(u_c, \delta_0)$. Suppose that there exists another fixed point u_{**} of Φ in the open ball $B(u_c, R') = \{u \in L^\infty[-l, l] \mid \|u - u_c\|_\infty < R'\}$. Since $\delta_i \nearrow R'$ as $i \rightarrow \infty$, we have

$$B(u_c, R') = \bigcup_{i=0}^\infty \overline{B}(u_c, \delta_i), \tag{6.5}$$

hence $u_{**} \in \overline{B}(u_c, \delta_{i_1})$ for some i_1 . Then, by Proposition 1, u_{**} is the unique fixed point of Φ in $\overline{B}(u_c, \delta_{i_1})$. It follows that $u_{**} = u_*$ since $\overline{B}(u_c, r') \subset \overline{B}(u_c, \delta_{i_1})$. Thus u_* is the unique

fixed point of Φ in the open ball $B(u_c, R')$. Suppose $u_0 \in B(u_c, R')$, so that $u_0 \in \bar{B}(u_c, \delta_{i_2})$ for some i_2 by (6.5). Then, by Proposition 1, the sequence $\{u_n\}_{n=1}^\infty$ defined by $u_n = \Phi[u_{n-1}]$, $n = 1, 2, 3, \dots$, converges to u_* . Thus it is sufficient to assume that $R' < \infty$ since $\bar{B}(u_c, R') = L^\infty[-l, l] = B(u_c, R')$ if $R' = \infty$.

Suppose $R' < \infty$. By (6.5) and condition (a) for each i ,

$$\Phi[B(u_c, R')] = \Phi\left[\bigcup_{i=0}^\infty \bar{B}(u_c, \delta_i)\right] = \bigcup_{i=0}^\infty \Phi[\bar{B}(u_c, \delta_i)] \subset \bigcup_{i=0}^\infty \bar{B}(u_c, \delta_i) = B(u_c, R').$$

So, by the continuity of Φ , we have

$$\Phi[\bar{B}(u_c, R')] \subset \bar{B}(u_c, R'). \tag{6.6}$$

Since $0 \leq L_i < 1$ for $i = 0, 1, 2, \dots$, we have

$$\|\Phi[u] - \Phi[v]\|_\infty \leq \|u - v\|_\infty, \quad u, v \in \bar{B}(u_c, R') \tag{6.7}$$

by (6.5), condition (b) for each i , and the continuity of Φ . Suppose that there exists another fixed point u_{**} of Φ in the closed ball $\bar{B}(u_c, R')$. Since u_* , u_{**} are fixed points of Φ , we have $\Phi[u_*] = u_*$ and $\Phi[u_{**}] = u_{**}$, hence

$$\begin{aligned} \|u_* - u_{**}\|_\infty &= \|\Phi[u_*] - \Phi[u_{**}]\|_\infty \\ &= \|\Phi[u_*] - \Phi[(u_* + u_{**})/2] + \Phi[(u_* + u_{**})/2] - \Phi[u_{**}]\|_\infty \\ &\leq \|\Phi[u_*] - \Phi[(u_* + u_{**})/2]\|_\infty + \|\Phi[(u_* + u_{**})/2] - \Phi[u_{**}]\|_\infty. \end{aligned} \tag{6.8}$$

Note that $(u_* + u_{**})/2$ is always contained in the open ball $B(u_c, R')$. So, by (6.5), $(u_* + u_{**})/2 \in \bar{B}(u_c, \delta_{i_3})$ for some i_3 , hence we have

$$\begin{aligned} \|\Phi[u_*] - \Phi[(u_* + u_{**})/2]\|_\infty &\leq L_{i_3} \cdot \|u_* - (u_* + u_{**})/2\|_\infty \\ &< \frac{1}{2} \|u_* - u_{**}\|_\infty \end{aligned} \tag{6.9}$$

since $u_* \in \bar{B}(u_c, R') \subset \bar{B}(u_c, \delta_{i_3})$ and $0 \leq L_{i_3} < 1$. By (6.7),

$$\|\Phi[(u_* + u_{**})/2] - \Phi[u_{**}]\|_\infty \leq \|(u_* + u_{**})/2 - u_{**}\|_\infty = \frac{1}{2} \|u_* - u_{**}\|_\infty \tag{6.10}$$

since $(u_* + u_{**})/2, u_{**} \in \bar{B}(u_c, R')$. It follows from (6.8), (6.9), and (6.10) that $\|u_* - u_{**}\|_\infty < \|u_* - u_{**}\|_\infty$, which is a contradiction. Thus we conclude that u_* is the unique fixed point of Φ in the closed ball $\bar{B}(u_c, R')$.

Let $u_0 \in \bar{B}(u_c, R')$, and let the sequence $\{u_n\}_{n=0}^\infty$ be defined by $u_n = \Phi[u_{n-1}]$, $n = 1, 2, 3, \dots$. By (6.6), $u_n \in \bar{B}(u_c, R')$ for $n = 0, 1, 2, \dots$. Suppose that u_n is in the sphere $S(u_c, R') = \{u \in L^\infty[-l, l] \mid \|u - u_c\|_\infty = R'\}$ for every $n = 0, 1, 2, \dots$. Then there exist u_{**} in $S(u_c, R')$ and a subsequence $\{u_{n_k}\}_{k=0}^\infty$ of $\{u_n\}_{n=0}^\infty$ converging to u_{**} since $S(u_c, R')$ is compact. It follows that u_{**} is a fixed point of Φ , which is a contradiction. So there exists n_0 such that u_{n_0} is in the open ball $B(u_c, R')$, and $u_{n_0} \in \bar{B}(u_c, \delta_{i_4})$ for some i_4 by (6.5). Thus $\{u_n\}_{n=n_0}^\infty$, hence $\{u_n\}_{n=0}^\infty$ converges uniformly to u_* by Proposition 1. \square

Proof of Theorem 1. (c) immediately follows from Lemma 5.1. By Lemma 4.1, the desired solution $\tilde{\mathcal{L}}_{\mathbf{M}}[\mathbf{b}, w, f]$ is a fixed point of Ψ . So, for (a) and (b), it is sufficient to show that the conditions in Lemma 6.2 are satisfied with $u_c = \mathcal{L}_{\mathbf{M}}[\mathbf{b}, w]$, $r' = r$, $R' = R$, and $\Phi = \Psi$.

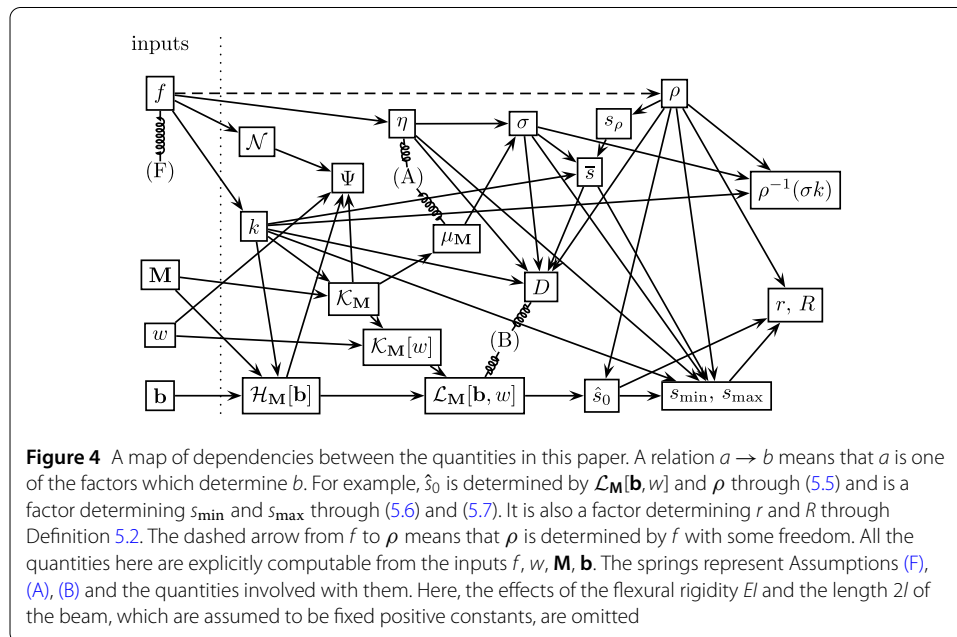
By Lemma 5.1, we have $0 \leq r' < R' \leq \infty$. Note that the continuity of Φ follows from Lemma 4.2. By (5.6) and (5.7), there exists a strictly increasing sequence $s_{\min} = s_0 < s_1 < s_2 < \dots \nearrow s_{\max}$ in $[0, \bar{s}]$ such that $\varphi(s_i) \geq \varphi_{\hat{s}_0}(s_i)$ for $i = 0, 1, 2, \dots$. Take $\delta_i = \rho^{-1}(s_i) - \rho^{-1}(\hat{s}_0)$ for $i = 0, 1, 2, \dots$. Since ρ^{-1} is strictly increasing, the sequence $\{\delta_i\}_{i=0}^{\infty}$ is strictly increasing, hence by Definition 5.2 we have $r' = \delta_0 < \delta_1 < \delta_2 < \dots \nearrow R'$. Take $L_i = \frac{\eta k + s_i}{\eta k + \sigma k}$ for $i = 0, 1, 2, \dots$. Since $s_{\max} \leq \bar{s} \leq \sigma k$ by (5.1) and (5.7) and $s_i < s_{\max}$ for $i = 0, 1, 2, \dots$, we have $0 \leq L_i < 1$ for $i = 0, 1, 2, \dots$. Thus, by Lemma 6.1, the conditions in Lemma 6.2 are satisfied, and the proof is complete. \square

7 Discussions

From the inputs $f, w, \mathbf{M}, \mathbf{b}$ of our nonlinear non-uniform boundary value problem $\text{NDE}(f, w)$ and $\text{BC}(\mathbf{M}, \mathbf{b})$, we derived various quantities, including r, R , and $\rho^{-1}(\sigma k)$ in Theorem 1. All of these quantities are explicitly computable, as was demonstrated by examples. Fig. 4 will be helpful in navigating through the various dependencies between them.

7.1 Effects of boundary conditions

As we mentioned in Sect. 1, the boundary conditions usually dealt with in the literature for the finite beam problem, are strikingly few in number. At each end of the beam, the two-point boundary conditions typically considered correspond to one of the types such as ‘free’, ‘clamped’, or ‘hinged’. Fig. 4 in particular shows the effects of *all* the 20-dimensionally different boundary conditions $\text{BC}(\mathbf{M}, \mathbf{b})$ we are dealing with in this paper. Note that the diversity of boundary conditions is encoded in $\mathcal{K}_{\mathbf{M}}$ and $\mathcal{H}_{\mathbf{M}}[\mathbf{b}]$. The constants E, I, l in $\text{LDE}(w)$ also contribute to the determination of $\mathcal{K}_{\mathbf{M}}$ and $\mathcal{H}_{\mathbf{M}}[\mathbf{b}]$, which is deliberately omitted for the sake of simplicity.



The integral operator \mathcal{K}_M , the set of which amounts to 16-dimensional space, is a major mean by which the boundary matrix \mathbf{M} affects our problem. Together with the other inputs, \mathcal{K}_M is used to construct the nonlinear operator Ψ . Its intrinsic L^∞ -norm μ_M should satisfy Assumption (A) and determines the elastic capacity σ together with the non-uniformity ratio η of the elastic foundation. In particular, the contractiveness of \mathcal{K}_M is critical in Assumption (A).

The boundary matrix \mathbf{M} also determines the linear operator \mathcal{H}_M , which in turn determines the linear uniform deflection $\mathcal{L}_M[\mathbf{b}, w]$ together with the boundary value \mathbf{b} and the loading density w . $\mathcal{L}_M[\mathbf{b}, w]$ should satisfy Assumption (B) and determines the dual radii r and R .

7.2 Assumptions (F), (A), (B)

Assumption (F) can be considered as a minimal restriction on f in order to model physically realistic elastic foundations.

It is intuitively natural to imagine that too much of (i) or (ii) below would break the neat behavior, such as Theorem 1, of the resulting deflection.

- (i) Nonlinearity and non-uniformity of given elastic foundation.
- (ii) Loading density w and boundary value \mathbf{b} .

After all, the linear uniform equation $LDE(w)$ and its nonlinear non-uniform generalization $NDE(f, w)$ themselves would become physically unrealistic for too much of (i) or (ii). The introduction of Assumptions (A) and (B) is natural in this regard since these assumptions keep (i) and (ii) small enough to guarantee Theorem 1. What is important to note is that Assumptions (A), (B) provide explicit bounds which tell *how small is enough*. They also tell *exactly which should be small* among the various quantities that can be derived from the inputs $f, w, \mathbf{M}, \mathbf{b}$.

In fact, there are situations where Assumptions (A), (B) are not needed at all. Assumption (A) would not be needed for the following cases:

- The non-uniformity ratio η is 0. Note from Definition 2.1 that $\eta = 0$ does not necessarily imply that the given elastic foundation is uniform.
- The integral operator \mathcal{K}_M is contractive, *i.e.*, the intrinsic L^∞ -norm μ_M of \mathcal{K}_M is less than 1.

Assumption (B) would not be needed for the following cases:

- The constant $D = \rho^{-1}(\hat{s}_{\max})$ in Definition 5.1 becomes ∞ or, equivalently, $\hat{s}_{\max} = s_\rho$.
- The linear uniform deflection $\mathcal{L}_M[\mathbf{b}, w]$ is 0 or, equivalently, $\mathbf{b} = \mathbf{0}$ and $w = 0$.

7.3 Nonlinearity function ρ

The nonlinearity function ρ is the only object which can be chosen with some freedom. As the nonlinearity of given elastic foundation is small, we can take smaller ρ , which would result in better bounds in general. Suppose that the nonlinearity of given elastic foundation is small enough so that we can choose ρ such that $\lim_{s \rightarrow \bar{s}^-} \varphi(s) = \infty$, which is possible only if $\bar{s} = s_\rho \leq \sigma k$ by the definition (5.2) of φ . Then we have $\hat{s}_{\max} = s_\rho$ by (5.4) and $s_{\max} = \bar{s} = s_\rho$ by (5.7), hence $D = \infty$ by Definition 5.1 and $R = \infty$ by Definition 5.2. Thus we have the global uniqueness of $\tilde{\mathcal{L}}_M[\mathbf{b}, w, f]$ in $L^\infty[-l, l]$ by Theorem 1(a), while Assumption (B) imposes no restriction on $\mathcal{L}_M[\mathbf{b}, w]$. In this case, the deflection horizon $\bar{B}(0, \rho^{-1}(\sigma k))$ becomes the whole $L^\infty[-l, l]$ since $\rho^{-1}(\sigma k) = \infty$ by (2.3).

7.4 Iteration algorithm with Ψ

Theorem 1(b) leads to an algorithm for uniformly approximating the nonlinear non-uniform deflection $\tilde{\mathcal{L}}_{\mathbf{M}}[\mathbf{b}, w, f]$ with iterations by the operator Ψ . The linear uniform deflection $\mathcal{L}_{\mathbf{M}}[\mathbf{b}, w]$ would be an obvious choice for the initial guess u_0 .^d Let $u_0 = \mathcal{L}_{\mathbf{M}}[\mathbf{b}, w]$. Then, by Lemma 6.1(a) with $s = s_{\min}$, we have $u_n \in \bar{B}(\mathcal{L}_{\mathbf{M}}[\mathbf{b}, w], r)$ for $n = 0, 1, 2, \dots$ since $r = \rho^{-1}(s_{\min}) - \rho^{-1}(\hat{s}_0)$ by Definition 5.2. Since $\tilde{\mathcal{L}}_{\mathbf{M}}[\mathbf{b}, w, f] \in \bar{B}(\mathcal{L}_{\mathbf{M}}[\mathbf{b}, w], r)$ by Theorem 1(b) and $\Psi[\tilde{\mathcal{L}}_{\mathbf{M}}[\mathbf{b}, w, f]] = \tilde{\mathcal{L}}_{\mathbf{M}}[\mathbf{b}, w, f]$ by Lemma 4.1, we have the following approximation speed by Lemma 6.1(b):

$$\begin{aligned} \|u_n - \tilde{\mathcal{L}}_{\mathbf{M}}[\mathbf{b}, w, f]\|_{\infty} &= \|\Psi[u_{n-1}] - \Psi[\tilde{\mathcal{L}}_{\mathbf{M}}[\mathbf{b}, w, f]]\|_{\infty} \\ &\leq \frac{\eta k + s_{\min}}{\eta k + \sigma k} \cdot \|u_{n-1} - \tilde{\mathcal{L}}_{\mathbf{M}}[\mathbf{b}, w, f]\|_{\infty}, \quad n = 1, 2, 3, \dots \end{aligned}$$

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Endnotes

- ^a See [4] for more general functional boundary conditions.
- ^b The non-uniformity ratio η in this paper corresponds to $1 - \eta$ in [7].
- ^c The integral operator $\mathcal{K}_{\mathbf{M}}$ can also be regarded as a bounded linear operator on $L^2[-l, l]$. For every well-posed \mathbf{M} , the intrinsic L^{∞} -norm $\mu_{\mathbf{M}}$ is equal to the intrinsic L^2 -norm.
- ^d Another obvious choice $u_0 = 0$ would result in $u_1 = \Psi[0] = \mathcal{L}_{\mathbf{M}}[\mathbf{b}, w]$ by (6.1).

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