Boundary Value Problems
a SpringerOpen Journal

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Boundary value conditions for linear differential equations with power degenerations

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Abstract

On the interval $[0, 1]$ we consider the nth order linear differential equation, the coefficient of the highest derivative of which is equivalent to the power function t^μ when $t \to 0$. The main aim of the paper is to pose "generalized" Cauchy conditions for the given equation at the point of singularity $t = 0$, which would be correct for any $\mu > 0$.

MSC: Primary 34B05; secondary 46N20

Keywords: Linear differential equation; Boundary value problem; Weight function; Multiweighted derivative; Point of singularity

1 Introduction

Let us consider the following nth order linear differential equation:

$$\sum_{i=0}^{n} a_i(t) y^{(i)}(t) = f(t), \quad t \in [0, 1], \tag{1}$$

where the coefficients $a_i(\cdot)$, $i = 0, 1, \ldots, n$, and the right-hand side $f(\cdot)$ are continuous functions on $[0, 1]$. Moreover, $a_n(t) > 0$ when $t \in (0, 1]$ and $a_n(t)$ is equivalent to t^μ when $t \to 0$, i.e., equation (1) has a singularity at the point $t = 0$ of order μ. It is known that if $0 < \mu < 1$, then all solutions of equation (1) belong to $C^n[0, 1]$. Hence, in this case we can pose the same boundary conditions for (1) as for a nonsingular equation. In particular, at the point of singularity the following Cauchy conditions can be posed:

$$y^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n - 1.$$

In the case $\mu \geq 1$, there are, in general, no finite limits $\lim_{t \to 0} y^{(i)}(t)$ for all $i = 0, 1, \ldots, n - 1$. Therefore, the Cauchy conditions have no meaning. The main aim of the paper is to pose "generalized" Cauchy conditions for (1) at $t = 0$, which would be correct for any $\mu > 0$.

The problem will be solved in the following way. Let $\gamma_i < 1$, $i = 0, 1, \ldots, n - 1$, be an arbitrary set of n real numbers and $\gamma_n = 1$. Suppose that the set of numbers $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$

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is such that

$$\alpha_i = \gamma_{i-1} - \gamma_i + 1, \quad i = 1, 2, \ldots, n, \tag{2}$$

and α_0 is calculated from the equality

$$\mu = \gamma_0 + \alpha_0 + n - 1.$$

That gives

$$\sum_{i=0}^{n} \alpha_i = \mu, \qquad \gamma_i = \alpha_n + \sum_{k=i+1}^{n-1} (\alpha_k - 1) < 1, \quad i = 0, 1, \ldots, n - 2, \text{ and } \gamma_{n-1} = \alpha_n < 1. \tag{3}$$

Using this set of numbers $\overline{\alpha}$ for $y(t) \in C^n(0, 1]$, we construct the following operations:

$$D_{\overline{\alpha}}^0 y(t) = t^{\alpha_0} y(t),$$

$$D_{\overline{\alpha}}^i y(t) = t^{\alpha_i} \frac{d}{dt} t^{\alpha_{i-1}} \frac{d}{dt} \cdots t^{\alpha_1} \frac{d}{dt} t^{\alpha_0} y(t), \quad i = 1, 2, \ldots, n.$$

We call this differential operation (or operator) $D_{\overline{\alpha}}^i$ the *multiweighted derivative of the function y of order i*, $i = 0, 1, \ldots, n$. At the point of singularity $t = 0$ of equation (1) we pose the boundary conditions

$$D_{\overline{\alpha}}^i y(0) = 0, \quad i = 0, 1, \ldots, n - 1, \tag{4}$$

where each $D_{\overline{\alpha}}^i y(0) = 0$, $i = 0, 1, \ldots, n - 1$, is understood in the sense of the existence of the finite limit $\lim_{t \to 0} D_{\overline{\alpha}}^i y(t) = D_{\overline{\alpha}}^i y(0)$. Conditions (4) are the required "generalized" Cauchy conditions, and in the paper we prove that problem (1) and (4) with (3) has a unique solution.

Let us note that the operator $D_{\overline{\alpha}}^n$ gives the basis for a space $W_{p,\overline{\alpha}}^n = W_{p,\overline{\alpha}}^n(I)$ of functions $y : I \to \mathbb{R}$ with the finite semi-norm

$$\|y\|_{W_{p,\overline{\alpha}}^n(I)} = \left\| D_{\overline{\alpha}}^n y \right\|_p,$$

where $1 < p < \infty$, $I = (0, 1)$ or $I = (1, \infty)$.

The idea to study function spaces with the purpose to apply them to different problems concerning differential equations appeared in works by Sobolev in the thirties. From this time the theory of Sobolev spaces has been developed to be a very powerful instrument for solving boundary value problems of differential equations. Moreover, such concept as a "weight function" was introduced to take care of different problems connected to singularities. Correspondingly, Kudryavtsev presented a fairly complete theory of one-dimensional Sobolev spaces with power weights (see, e.g., [8–15] and the references given there). He considered a space $L_{p,\gamma}^n = L_{p,\gamma}^n(I)$ of functions $y : I \to \mathbb{R}$, which on the interval I have nth order derivative with the finite semi-norm

$$\|y\|_{L_{p,\gamma}^n} = \left\| t^{\gamma} y^{(n)} \right\|_p.$$

For the interval $I = (0, 1)$ it was shown that if $\gamma > n - \frac{1}{p}$, then the function y, in general, does not have finite limit value when $t \to 0$. Thus, the introduction of an additional function as "weight" does not always solve singularity problems. This argumentation served as a motivation for constructing the operator $D_{\overline{\alpha}}^n$ and defining the space $W_{p,\overline{\alpha}}^n$ with $n + 1$ weights. The theory of the function space $W_{p,\overline{\alpha}}^n$ generalizes the previous theory of the function space $L_{p,\gamma}^n$. In the series of works (see, e.g., [1–6]) there were considered similar problems for the $W_{p,\overline{\alpha}}^n$ as those considered by Kudryavtsev for the space $L_{p,\gamma}^n$. The results for the space $W_{p,\overline{\alpha}}^n$ cover such singularities that cannot be handled by one weight, but can be handled with many weights, and they give the basis for the main result of the present paper. More precisely, in the paper [5] it was shown that the properties of the space $W_{p,\overline{\alpha}}^n$ are dependent on the values γ_i, $i = 0, 1, \ldots, n - 1$, in accordance with which we have the following three cases of the degeneration of the weight functions t^{α_i}, $i = 1, 2, \ldots, n$:

1. $\gamma_{\max} = \max_{0 \le i \le n-1} \gamma_i < 1 - \frac{1}{p}$ (weak degeneration);
2. $\gamma_{\min} = \min_{0 \le i \le n-1} \gamma_i > 1 - \frac{1}{p}$ (strong degeneration);
3. $\gamma_{\min} < 1 - \frac{1}{p} < \gamma_{\max}$ (mixed degeneration).

Then in the paper [6] it was proved that, for any function $y \in W_{p,\overline{\alpha}}^n$, the condition $\gamma_{\max} < 1 - \frac{1}{p}$ (weak degeneration) is necessary and sufficient for the existence of limit values $\lim_{t \to 0} D_{\overline{\alpha}}^i y(t) = D_{\overline{\alpha}}^i y(0)$ for all $0 \le i \le n - 1$. If we resolve the equalities in (3) with respect to α_i, $i = 1, 2, \ldots, n$, we get the equalities in (2). Moreover, by the assumption the values γ_i, $i = 0, 1, \ldots, n - 1$, satisfy the condition $\gamma_i < 1$. Thus, we have a weak degeneration of the weight functions t^{α_i}, $i = 1, 2, \ldots, n$, which guarantees the existence of characteristics at the singular point $t = 0$.

The paper is organized as follows: In Sect. 2 we collect all the required notations, definitions, and statements; in Sect. 3 we state and prove our main result concerning the existence of a unique solution of problem (1) and (4).

2 Preliminaries

Let us introduce the following family of functions $K_k(x, t)$, $k = 0, 1, \ldots, n$, assuming that $K_n(x, t) \equiv 1$, $K_{n-1}(x, t) = \int_x^t s^{-\alpha_{n-1}} \, ds$, $K_{n-2}(x, t) = \int_x^t y^{-\alpha_{n-2}} \int_x^y s^{-\alpha_{n-1}} \, ds \, dy$ and, in general,

$$K_k(x, t) = \int_x^t t_{k+1}^{-\alpha_{k+1}} \int_x^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \cdots \int_x^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}} \, dt_{n-1} \, dt_{n-2} \cdots dt_{k+1}, \quad k = 0, 1, \ldots, n - 1,$$

for $t > x$. Moreover, we assume that $K_k(x, t) = 0$, $k = 0, 1, \ldots, n - 1$, for $t \le x$.

Let $n - 1 \ge k \ge 0$. If in the integrals of $K_k(x, t)$ we successively change the variables $t_{n-1} = x\tau_{n-1}$, $t_{n-2} = x\tau_{n-2}, \ldots, t_{k+1} = x\tau_{k+1}$, we get

$$x^{-\alpha_n} K_k(x, t) = x^{-\gamma_k} K_k\left(1, \frac{t}{x}\right)$$

and

$$x^{-\alpha_n} K_k(x, t) \ge x^{-\gamma_k} K_k(1, 2) \quad \text{for } 0 < x \le \frac{t}{2}, \tag{5}$$

i.e., the function $x^{-\alpha_n} K_k(x, t)$ has a singularity at zero of order $x^{-\gamma_k}$ for $0 < \gamma_k < 1$.

Let $n - 1 \geq k \geq 0$, $i = k, k+1, \ldots, n-1$. Changing the order of integration and using (3), for $t > 0$, we get

$$
\int_0^t s^{-\alpha_n} K_k(s,t)\, ds
$$

$$
= \int_0^t s^{-\alpha_n} \int_s^t t_{k+1}^{-\alpha_{k+1}} \int_s^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \cdots \int_s^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}}\, dt_{n-1}\, dt_{n-2} \cdots dt_{k+1}\, ds
$$

$$
= \int_0^t t_{k+1}^{-\alpha_{k+1}} \int_0^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \cdots \int_0^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}} \int_0^{t_{n-1}} s^{-\alpha_n}\, ds\, dt_{n-1} \cdots dt_{k+2}\, dt_{k+1}
$$

$$
= \frac{1}{1 - \gamma_{n-1}} \int_0^t t_{k+1}^{-\alpha_{k+1}} \int_0^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \cdots \int_0^{t_{n-2}} t_{n-1}^{-\gamma_{n-2}}\, dt_{n-1} \cdots dt_{k+2}\, dt_{k+1}
$$

$$
= \frac{1}{(1 - \gamma_{n-1})(1 - \gamma_{n-2})} \int_0^t t_{k+1}^{-\alpha_{k+1}} \int_0^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \cdots \int_0^{t_{n-3}} t_{n-2}^{-\gamma_{n-3}}\, dt_{n-2} \cdots dt_{k+2}\, dt_{k+1}
$$

$$
= \cdots = \frac{1}{\prod_{i=k+1}^{n-1}(1 - \gamma_i)} \int_0^t t_{k+1}^{-\gamma_k}\, dt_{k+1} = d_k t^{1-\gamma_k}, \tag{6}
$$

where $d_k = \frac{1}{\prod_{i=k}^{n-1}(1 - \gamma_i)}$.

Lemma 1 *Let $\gamma_{n-1} < 1$. Then, for $y \in C^n(0,1]$ satisfying the condition $\sup_{0 \leq t \leq 1} |D_{\overline{\alpha}}^n y(t)| < \infty$, there exists $D_{\overline{\alpha}}^{n-1} y(0)$ and the estimate*

$$
t^{\gamma_{n-1}-1} \left| D_{\overline{\alpha}}^{n-1} y(t) - D_{\overline{\alpha}}^{n-1} y(0) \right| \leq d_{n-1} \sup_{0 \leq t \leq 1} \left| D_{\overline{\alpha}}^n y(t) \right| \tag{7}
$$

holds.

Proof. Indeed, by assumption it follows that the function $t^{-\alpha_n} D_{\overline{\alpha}}^n y(t)$ is absolutely summable on the interval $(0,1]$. Therefore, by Newton–Leibniz formula for $0 < t \leq 1$, we have

$$
\int_0^t s^{-\alpha_n} D_{\overline{\alpha}}^n y(s)\, ds = \lim_{\varepsilon \to 0} \int_\varepsilon^t \frac{d}{ds} D_{\overline{\alpha}}^{n-1} y(s)\, ds
$$

$$
= D_{\overline{\alpha}}^{n-1} y(t) - \lim_{\varepsilon \to 0} D_{\overline{\alpha}}^{n-1} y(\varepsilon) = D_{\overline{\alpha}}^{n-1} y(t) - D_{\overline{\alpha}}^{n-1} y(0). \tag{8}
$$

Since $\alpha_n = \gamma_{n-1}$ and $\gamma_{n-1} < 1$, from (8) we get

$$
\left| D_{\overline{\alpha}}^{n-1} y(t) - D_{\overline{\alpha}}^{n-1} y(0) \right| \leq \sup_{0 \leq t \leq 1} \left| D_{\overline{\alpha}}^n y(t) \right| \int_0^t s^{-\alpha_n}\, ds = \sup_{0 \leq t \leq 1} \left| D_{\overline{\alpha}}^n y(t) \right| \cdot \frac{t^{1-\gamma_{n-1}}}{1 - \gamma_{n-1}}.
$$

The last gives (7). The proof of Lemma 1 is complete.

Lemma 2 *Let $n - 1 \geq k \geq 0$ and $\gamma_i < 1$ for $i = k, k+1, \ldots, n-1$. Let a function $y \in C^n(0,1]$ satisfy the conditions: $\sup_{0 \leq t \leq 1} |D_{\overline{\alpha}}^n y(t)| < \infty$ and $D_{\overline{\alpha}}^i y(0) = 0$, $i = k+1, \ldots, n-1$. Then there exist $D_{\overline{\alpha}}^k y(0)$ and*

$$
t^{\gamma_k-1} \left| D_{\overline{\alpha}}^k y(t) - D_{\overline{\alpha}}^k y(0) \right| \leq d_k \sup_{0 \leq t \leq 1} \left| D_{\overline{\alpha}}^n y(t) \right|, \tag{9}
$$

$$
D_{\overline{\alpha}}^k y(t) = D_{\overline{\alpha}}^k y(0) + \int_0^t K_k(s,t) s^{-\alpha_n} D_{\overline{\alpha}}^n y(s)\, ds. \tag{10}
$$

Proof. Since $\sup_{0\le t\le 1}|D_{\overline{\alpha}}^n y(t)| < \infty$ and $\gamma_i < 1$, $i = k, k+1,\ldots,n-1$, from (6) we have that the function $K_i(s,t)s^{-\alpha_n}D_{\overline{\alpha}}^n y(s)$ is absolutely summable on $(0,t)$. Therefore, there exists

$$\int_0^t K_i(s,t)s^{-\alpha_n}D_{\overline{\alpha}}^n y(s)\,ds$$
$$= \int_0^t t_{i+1}^{-\alpha_{i+1}}\int_0^{t_{i+1}} t_{i+2}^{-\alpha_{i+2}}\cdots\int_0^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}}\int_0^{t_{n-1}} t_n^{-\alpha_n}D_{\overline{\alpha}}^n y(t_n)\,dt_n\,dt_{n-1}\cdots dt_{i+1} \tag{11}$$

for all $i = k, k+1,\ldots,n-1$.

Since $D_{\overline{\alpha}}^{n-1}y(0) = 0$, from (8) we have

$$D_{\overline{\alpha}}^{n-1}y(t) = \int_0^t t_n^{-\alpha_n}D_{\overline{\alpha}}^n y(t_n)\,dt_n, \quad t \in (0,1].$$

The last, together with (11) for $i = n-2$, gives

$$D_{\overline{\alpha}}^{n-2}y(t) - D_{\overline{\alpha}}^{n-2}y(0) = \int_0^t t_{n-1}^{-\alpha_{n-1}}\int_0^{t_{n-1}} t_n^{-\alpha_n}D_{\overline{\alpha}}^n y(t_n)\,dt_n\,dt_{n-1}.$$

By the condition $D_{\overline{\alpha}}^{n-2}y(0) = 0$, hence

$$D_{\overline{\alpha}}^{n-2}y(t) = \int_0^t t_{n-1}^{-\alpha_{n-1}}\int_0^{t_{n-1}} t_n^{-\alpha_n}D_{\overline{\alpha}}^n y(t_n)\,dt_n\,dt_{n-1}.$$

If we continue this process, using the fact that $D_{\overline{\alpha}}^i y(0) = 0$, $i = k+1,\ldots,n-1$, and the finiteness of (11), we get

$$D_{\overline{\alpha}}^{k+1}y(t) = \int_0^t t_{k+2}^{-\alpha_{k+2}}\int_0^{t_{k+2}} t_{k+3}^{-\alpha_{k+3}}\cdots\int_0^{t_{n-1}} t_n^{-\alpha_n}D_{\overline{\alpha}}^n y(t_n)\,dt_n\,dt_{n-1}\cdots dt_{k+2}.$$

That gives

$$D_{\overline{\alpha}}^k y(t) - D_{\overline{\alpha}}^k y(0) = \int_0^t t_{k+1}^{-\alpha_{k+1}}\int_0^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}}\cdots\int_0^{t_{n-1}} t_n^{-\alpha_n}D_{\overline{\alpha}}^n y(t_n)\,dt_n\,dt_{n-1}\cdots dt_{k+1}$$
$$= \int_0^t K_k(s,t)s^{-\alpha_n}D_{\overline{\alpha}}^n y(s)\,ds, \tag{12}$$

i.e., there exists $D_{\overline{\alpha}}^k y(0)$ and (10) holds. From (6) and (12) we have

$$\left|D_{\overline{\alpha}}^k y(t) - D_{\overline{\alpha}}^k y(0)\right| \le \sup_{0\le t\le 1}\left|D_{\overline{\alpha}}^n y(t)\right|\int_0^t K_k(s,t)s^{-\alpha_n}\,ds.$$
$$= \frac{1}{\prod_{i=k}^{n-1}(1-\gamma_i)}t^{1-\gamma_k}\sup_{0\le t\le 1}\left|D_{\overline{\alpha}}^n y(t)\right|.$$

Thus, we get (9). The proof of Lemma 2 is complete.

Corollary 1 *Let $n-1 \ge k \ge 0$. Suppose that the conditions of Lemma 2 hold and $D_{\overline{\alpha}}^k y(0) = 0$. Then*

$$D_{\overline{\alpha}}^i y(t) = \int_0^t K_i(s,t)s^{-\alpha_n}D_{\overline{\alpha}}^n y(s)\,ds \tag{13}$$

and

$$\sup_{0 \le t \le 1} t^{\gamma_i - 1} \left| D_{\overline{\alpha}}^i y(t) \right| \le d_i \sup_{0 \le t \le 1} \left| D_{\overline{\alpha}}^n y(t) \right| \tag{14}$$

for $i = k, k+1, \ldots, n-1$.

Lemma 3 *Let $y : I \to \mathbb{R}$ be such that $y \in C^n(0,1]$. Then*

$$y^{(k)}(t) = \sum_{i=0}^{k} b_{k,i} t^{\gamma_i - \gamma_0 - \alpha_0 - k} D_{\overline{\alpha}}^i y(t), \quad k = 0, 1, \ldots, n, \tag{15}$$

where $b_{k,k} = 1$, $k = 0, 1, \ldots, n$, and coefficients $b_{k,i}$, $i = 1, 2, \ldots, k-1$, $k = 1, 2, \ldots n$, and $b_{k,0}$, $k = 1, 2, \ldots n$, are defined by the recurrent formulas:

$$b_{k,i} = b_{k-1,i-1} + b_{k-1,i} (\gamma_i - \gamma_0 - \alpha_0 - (k-1)) \quad and \quad b_{k,0} = -b_{k-1,0}(\alpha_0 + k - 1).$$

Proof. For $k = 0$, we have

$$y(t) = t^{-\alpha_0} D_{\overline{\alpha}}^0 y(t) = b_{0,0} t^{\gamma_0 - \gamma_0 - \alpha_0 - 0} D_{\overline{\alpha}}^0 y(t),$$

where $b_{0,0} = 1$.
  For $k = 1$, we have

$$D_{\overline{\alpha}}^1 y(t) = t^{\alpha_1} \left(\alpha_0 t^{-1} D_{\overline{\alpha}}^0 y(t) + t^{\alpha_0} y'(t) \right).$$

That, using $\alpha_1 = \gamma_0 - \gamma_1 + 1$, gives

$$y'(t) = -\alpha_0 t^{-\alpha_0 - 1} D_{\overline{\alpha}}^0 y(t) + t^{-\alpha_0 - \alpha_1} D_{\overline{\alpha}}^1 y(t)$$
$$= b_{1,0} t^{\gamma_0 - \gamma_0 - \alpha_0 - 1} D_{\overline{\alpha}}^0 y(t) + b_{1,1} t^{\gamma_1 - \gamma_0 - \alpha_0 - 1} D_{\overline{\alpha}}^1 y(t),$$

where $b_{1,0} = -b_{0,0}(\alpha_0 + 1 - 1)$ and $b_{1,1} = 1$.
  Now, we assume that

$$y^{(k-1)}(t) = \sum_{i=0}^{k-1} b_{k-1,i} t^{\gamma_i - \gamma_0 - \alpha_0 - (k-1)} D_{\overline{\alpha}}^i y(t)$$

is true. Then, using (2), we get

$$y^{(k)}(t) = \sum_{i=0}^{k-1} b_{k-1,i} t^{\gamma_i - \gamma_0 - \alpha_0 - (k-1) - \alpha_{i+1}} D_{\overline{\alpha}}^{i+1} y(t)$$

$$+ \sum_{i=0}^{k-1} b_{k-1,i} (\gamma_i - \gamma_0 - \alpha_0 - (k-1)) t^{\gamma_i - \gamma_0 - \alpha_0 - k} D_{\overline{\alpha}}^i y(t)$$

$$= b_{k-1,k-1} t^{\gamma_{k-1} - \gamma_0 - \alpha_0 - (k-1) - \alpha_k} D_{\overline{\alpha}}^k y(t)$$

$$+ \sum_{i=1}^{k-1} b_{k-1,i-1} t^{\gamma_{i-1} - \gamma_0 - \alpha_0 - (k-1) - \alpha_i} D_{\overline{\alpha}}^i y(t)$$

$$+ \sum_{i=1}^{k-1} b_{k-1,i}\big(\gamma_i - \gamma_0 - \alpha_0 - (k-1)\big) t^{\gamma_i - \gamma_0 - \alpha_0 - k} D_{\overline{\alpha}}^i y(t)$$

$$+ b_{k-1,0}\big(\gamma_0 - \gamma_0 - \alpha_0 - (k-1)\big) t^{\gamma_0 - \gamma_0 - \alpha_0 - k} D_{\overline{\alpha}}^0 y(t)$$

$$= -b_{k-1,0}(\alpha_0 + k - 1) t^{\gamma_0 - \gamma_0 - \alpha_0 - k} D_{\overline{\alpha}}^0 y(t)$$

$$+ \sum_{i=1}^{k-1} (b_{k-1,i-1} + b_{k-1,i}\big(\gamma_i - \gamma_0 - \alpha_0 - (k-1)\big) t^{\gamma_i - \gamma_0 - \alpha_0 - k} D_{\overline{\alpha}}^i y(t)$$

$$+ b_{k-1,k-1} t^{\gamma_k - \gamma_0 - \alpha_0 - k} D_{\overline{\alpha}}^k y(t).$$

The last implies (15). The proof of Lemma 3 is complete.

3 Main result

Theorem 1 *Let* (3) *hold. Let the coefficients* $a_i(t)$, $i = 0, 1, \ldots, n$, *of equation* (1) *be continuous functions on* $(0, 1]$ *and satisfy the conditions*

$$c_1 t^\mu \le a_n(t) \le c_2 t^\mu, \quad \mu > 0, t \in (0, 1], \tag{16}$$

$$a_i(t) = o\big(t^{\mu - n + i}\big), \quad i = 0, 1, \ldots, n-1, \text{ for } t \to 0, \tag{17}$$

where the constants $c_1 > 0$ *and* $c_2 > 0$ *do not depend on* $t \in (0, 1]$. *Then, for any* $\mu > 0$ *and for any function* $f(\cdot)$ *continuous on* $[0, 1]$, *there exists a unique solution of problem* (1) *and* (4), *and the following estimate*

$$\sum_{k=0}^{n} \sup_{0 \le t \le 1} t^{\mu - n + k} \big| y^{(k)}(t) \big| \le c \max_{0 \le t \le 1} \big| f(t) \big| \tag{18}$$

holds, where $c > 0$ *does not depend on* f.

Proof. If we substitute (15) in (1), then we have

$$\sum_{k=0}^{n} \left(\sum_{i=k}^{n} a_i(t) b_{i,k} t^{\gamma_k - \gamma_0 - \alpha_0 - i} \right) D_{\overline{\alpha}}^k y(t) = f(t).$$

By introducing the notations $\widetilde{a}_k(t) = \sum_{i=k}^{n} a_i(t) b_{i,k} t^{\gamma_k - \gamma_0 - \alpha_0 - i}$, $k = 0, 1, \ldots, n$, we get

$$\sum_{k=0}^{n} \widetilde{a}_k(t) D_{\overline{\alpha}}^k y(t) = f(t). \tag{19}$$

From conditions (16) and (17) we have

$$c_1 \le \widetilde{a}_n(t) \le c_2, \quad t \in (0, 1], \tag{20}$$

$$\widetilde{a}_k(t) = o\big(t^{\gamma_k - 1}\big), \quad k = 0, 1, \ldots, n-1, \text{ for } t \to 0. \tag{21}$$

By condition (3) we have that $\gamma_i < 1$, $i = 0, 1, \ldots, n-1$, therefore (14) and (13) are valid for $k = 0$.

Let $z(t) = \widetilde{a}_n(t)D^n_{\overline{\alpha}}y(t)$, then from (4), (13), and (19) we obtain

$$z(t) + \int_0^t \sum_{k=0}^{n-1} \widetilde{a}_k(t)K_k(s,t)s^{-\alpha_n}\widetilde{a}_n^{-1}(s)z(s)\,ds = f(t). \tag{22}$$

Now, we prove that integral equation (22) has a unique solution continuous on $[0,1]$, for which the estimate

$$\max_{0\le t\le 1}\left|z(t)\right| \le \overline{c}\max_{0\le t\le 1}\left|f(t)\right| \tag{23}$$

holds, where $\overline{c} > 0$ does not depend on f.

Despite the fact that integral equation (22) has the form of a Volterra equation, (5) implies that the kernel of the integral operator in (22) is unbounded if, for some $0 \le k \le n-1$, we have $0 < \gamma_k < 1$. Therefore, we cannot apply the regular theory of Volterra integral equations in the space of continuous functions (see, e.g., [16]). Hence, let us first solve equation (22) on some interval $[0,\delta]$, $0 < \delta < 1$, using the method of contraction mapping (see, e.g., [7]).

By (21) there exists $1 > \delta > 0$ such that

$$c_1^{-1}\sup_{0\le t\le\delta}\sum_{k=0}^{n-1}\frac{1}{\prod_{i=k}^{n-1}(1-\gamma_i)}\left|\widetilde{a}_k(t)\right|t^{1-\gamma_k} = q < 1. \tag{24}$$

In $C[0,\delta]$ we consider the integral operator

$$Kz(t) = \sum_{k=0}^{n-1}\widetilde{a}_k(t)\int_0^t K_k(s,t)s^{-\alpha_n}\widetilde{a}_n^{-1}(s)z(s)\,ds, \quad t\in[0,\delta].$$

Due to (6), (21), and (24) we have

$$\sup_{0\le t\le\delta}\left|Kz(t)\right| \le \sup_{0\le t\le\delta}\sum_{k=0}^{n-1}\left|\widetilde{a}_k(t)\right|\int_0^t K_k(s,t)s^{-\alpha_n}\left|\widetilde{a}_n(s)\right|^{-1}\left|z(s)\right|ds$$

$$\le \sup_{0\le t\le\delta}c_1^{-1}\sum_{k=0}^{n-1}\left|\widetilde{a}_k(t)\right|\sup_{0\le t\le\delta}\left|z(t)\right|\int_0^t K_k(s,t)s^{-\alpha_n}\,ds$$

$$= c_1^{-1}\sup_{0\le t\le\delta}\sum_{k=0}^{n-1}\frac{1}{\prod_{i=k}^{n-1}(1-\gamma_i)}\left|\widetilde{a}_k(t)\right|t^{1-\gamma_k}\sup_{0\le t\le\delta}\left|z(t)\right| = q\sup_{0\le t\le\delta}\left|z(t)\right|.$$

Therefore, K is a contraction operator in $C[0,\delta]$. Applying the method of contraction mapping to integral equation (22) ([7], pp. 88–89), we have that equation (22) has a unique solution $\overline{z}_0 \in C[0,\delta]$; in addition, $\max_{0\le t\le\delta}|\overline{z}_0(t)| \le \overline{c}_0\max_{0\le t\le 1}|f(t)|$. The successive approximations $z_0, z_1, \ldots, z_n, \ldots$ to this solution are of the form

$$z_n(t) + \int_0^t \sum_{k=0}^{n-1}\widetilde{a}_k(t)K_k(s,t)s^{-\alpha_n}\widetilde{a}_n^{-1}(s)z_{n-1}(s)\,ds = f(t),$$

where any function from $C[0,\delta]$ can be chosen as the first approximation $z_0(t)$.

Thus, we have found the solution $\bar{z}_0 \in C[0, \delta]$ of equation (22) on the interval $[0, \delta]$. From equation (22) we have

$$\bar{z}_0(\delta) = f(\delta) - \sum_{k=0}^{n-1} \widetilde{a}_k(\delta) \int_0^\delta K_k(s, \delta) s^{-\alpha_n} \widetilde{a}_n^{-1}(s) \bar{z}_0(s) \, ds.$$

Let us now solve equation (22) on the interval $[\delta, 1]$ with the condition

$$z(\delta) = \bar{z}_0(\delta). \tag{25}$$

On $[\delta, 1]$ we present equation (22) in the form

$$z(t) + \sum_{k=0}^{n-1} \widetilde{a}_k(t) \left(\int_0^\delta K_k(s, t) s^{-\alpha_n} \widetilde{a}_n^{-1}(s) z(s) \, ds + \int_\delta^t K_k(s, t) s^{-\alpha_n} \widetilde{a}_n^{-1}(s) z(s) \, ds \right)$$
$$= f(t).$$

Since $z = \bar{z}_0$ on the interval $[0, \delta]$, we have

$$z(t) + \sum_{k=0}^{n-1} \widetilde{a}_k(t) \int_\delta^t K_k(s, t) s^{-\alpha_n} \widetilde{a}_n^{-1}(s) z(s) \, ds$$
$$= f(t) - \sum_{k=0}^{n-1} \widetilde{a}_k(t) \int_0^\delta K_k(s, t) s^{-\alpha_n} \widetilde{a}_n^{-1}(s) \bar{z}_0(s) \, ds. \tag{26}$$

It means that the kernel of the integral operator K is continuous on the bounded set $\delta \le s \le t \le 1$. Hence, equation (22) is a regular Volterra integral equation on the interval $[\delta, 1]$. Therefore, it has a unique continuous solution $\bar{z}_1$ on the interval $[\delta, 1]$ (see, e.g., [16]), for which the estimate $\max_{\delta \le t \le 1} |\bar{z}_1(t)| \le \bar{c}_1 \max_{0 \le t \le 1} |f(t)|$ holds.

From (26) we have

$$\bar{z}_1(\delta) = f(\delta) - \sum_{k=0}^{n-1} \widetilde{a}_k(\delta) \int_0^\delta K_k(s, \delta) s^{-\alpha_n} \widetilde{a}_n^{-1}(s) \bar{z}_0(s) \, ds.$$

The last gives that $\bar{z}_0(\delta) = \bar{z}_1(\delta)$, i.e., (25) holds. Hence, the function

$$z(t) = \begin{cases} \bar{z}_0(t), & 0 \le t \le \delta, \\ \bar{z}_1(t), & \delta \le t \le 1, \end{cases}$$

belongs to $C[0, 1]$ and it is a unique solution of (22). From the estimates for $\bar{z}_0$ and $\bar{z}_1$, we get (23) with $\bar{c} = \max\{\bar{c}_0, \bar{c}_1\}$.

Thus, problem (1) and (4) is reduced to the problem

$$\begin{cases} D_\alpha^n y(t) = \widetilde{a}_n^{-1}(t) z(t), \\ D_\alpha^i y(0) = 0, \quad i = 0, 1, \dots, n - 1, \end{cases} \tag{27}$$

with condition (3), where $z \in C[0, 1]$ is a unique solution of (22).

Since $z \in C[0,1]$, the function $\widetilde{a}_n^{-1}(t) = (a_n(t)t^{-\mu})^{-1}$ is continuous on $(0,1]$ and by (20) we have that $\frac{1}{c_2} \leq \widetilde{a}_n^{-1}(t) \leq \frac{1}{c_1}$, $t \in (0,1]$. Then from (27) we get $D_{\overline{\alpha}}^n y(t) \in C(0,1]$ and $\sup_{0 \leq t \leq 1} |D_{\overline{\alpha}}^n y(t)| < \infty$. Moreover, by condition (3) we have $\gamma_i < 1$, $i = 0,1,\ldots,n-1$. Then, on the basis of Lemmas 1 and 2 and from (13) and (14), we get

$$
\begin{aligned}
D_{\overline{\alpha}}^i y(t) &= \int_0^t K_i(s,t) s^{-\alpha_n} D_{\overline{\alpha}}^n y(s)\,ds \\
&= \int_0^t K_i(s,t) s^{-\alpha_n} \widetilde{a}_n^{-1}(s) z(s)\,ds, \quad i = 0,1,\ldots,n-1,
\end{aligned}
\tag{28}
$$

and the estimate

$$
\sup_{0 \leq t \leq 1} t^{\gamma_i - 1} \left| D_{\overline{\alpha}}^i y(t) \right| \leq d_i \sup_{0 \leq t \leq 1} \left| D_{\overline{\alpha}}^n y(t) \right| \leq \frac{d_i}{c_1} \max_{0 \leq t \leq 1} \left| z(t) \right|, \quad i = 0,1,\ldots,n.
\tag{29}
$$

From (29) the uniqueness of problem (27) follows.

From (23) and (29) we obtain

$$
\sup_{0 \leq t \leq 1} t^{\gamma_i - 1} \left| D_{\overline{\alpha}}^i y(t) \right| \leq \frac{\overline{c} d_i}{c_1} \max_{0 \leq t \leq 1} \left| f(t) \right|, \quad i = 0,1,\ldots,n.
\tag{30}
$$

Since $\mu = \gamma_0 + \alpha_0 + n - 1$, from (15) we have

$$
t^{\mu - n + k} y^{(k)}(t) = \sum_{i=0}^{k} b_{k,i} t^{\gamma_i - 1} D_{\overline{\alpha}}^i y(t), \quad k = 0,1,\ldots,n.
$$

The last, together with (30), gives the estimate

$$
\sup_{0 \leq t \leq 1} \left| t^{\mu - n + k} y^{(k)}(t) \right| \leq \widetilde{c} \max_{0 \leq t \leq 1} \left| f(t) \right|, \quad k = 0,1,\ldots,n,
$$

where $\widetilde{c} > 0$ depends on $\overline{c}$, c_1, d_i and $b_{k,i}$, $k,i = 0,1,\ldots,n$, and does not depend on f. The last implies the validity of (18). The proof of Theorem 1 is complete.

Remark 1 Under the conditions of Theorem 1 problem (1) and (4) is solvable for any set $\alpha_0, \alpha_1, \ldots, \alpha_n$ satisfying condition (3).

Availability of data and materials
Not applicable.

Competing interests
The author declares that she has no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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